<u>April 5, 1973</u> (more stability)

To understand Serre's theorem:

E vector bundle over X (affine, say, non-singular). Want to construct a section s of E transversal to the zero section. Transversal means $\forall x$ either $s(x) \neq 0$ or $s(x) = 0$ and the image of s in

$$m_x E / m_x^2 E = (m_x/m_x^2) \otimes_{k(x)} E(x)$$

gives rise to a surjective map

$$(m_x/m_x^2)^* \twoheadrightarrow E(x).$$

If this is the case then for each x where $s(x) = 0$, we have $\dim(m_x/m_x^2) \geq \operatorname{rank}(E)$, so $\operatorname{codim}(x) \geq \operatorname{rank}(E)$.

Start by constructing sections $s_1, s_2, \dots s_n$, inductively so that for all j:

$$1 \leq j \leq n \qquad \underbrace{\operatorname{codim}}_{W_j(s_1,\dots,s_n)} \{x \mid \operatorname{rank}(s_1(x), \dots, s_n(x)) \leq n-j\} \geq j$$

<u>n=1</u>: Choose s_1 ~~to~~ be $\neq 0$ at generic points

<u>n=2</u>: Having chosen s_1 we choose s_2 so that

$$x \in X_0 \Rightarrow \operatorname{rank}(s_1(x), s_2(x)) \geq 2$$

$$x \in X_1 \Rightarrow \operatorname{rank}(s_1(x), s_2(x)) \geq 1.$$

<u>n=3</u>:

$$x \in X_0 \Longrightarrow \operatorname{rank}(s_1(x), s_2(x), s_3(x)) \geq 3$$

$$x \in X_1 \Longrightarrow \quad " \quad \geq 2$$

$$x \in X_2 \Longrightarrow \quad\quad \geq 1$$

Inductive step: Assume $s_1, \dots, s_{n-1}$ chosen so that

$$\forall p \geq 0 \quad x \in X_p \Longrightarrow \operatorname{rank}(s_1(x), \dots, s_{n-1}(x)) \geq n-1-p.$$

Consider closed set Z_p where $\operatorname{rank}(s_1(x), \dots, s_{n-1}(x)) \leq n-1-p$. It doesn't contain any $x \in X_j$, $j < p$ so $Z_p \cap X_p$ is finite and if s_n is chosen independent of $s_1 \cdots s_{n-1}$ at these points, then

$$x \in X_p \Longrightarrow \operatorname{rank}(s_1(x), \dots, s_n(x)) \geq n-p$$

This gives you a finite set of open conditions at a finite set $\bigcup_p Z_p \cap X_p$, so s_n can be chosen as desired.

But actually if we are working with a fixed space of sections over an alg. closed field, then one should think of the preceding as a flag, because replacing s_i by any combination $s_i + \sum_{j<i} \alpha_j s_j$ doesn't not affect the conclusion.

The next step is to take a suitable linear combination.

Rank $(E) \geq 2$. We have chosen s_1, s_2 so that

$$\begin{aligned} \operatorname{rank}(s_1, s_2) &\geq 2 \quad \text{on } X_0 \\ &\geq 1 \quad \text{on } X_1 . \end{aligned}$$

Then $s_1 + a s_2$ is non-zero on X_0 and finitely many points of X_1. At the bad points we can choose a so that it doesn't vanish, because s_1, s_2 do not simultaneously vanish on X_1.

Rank $(E) \geq 3$. We have chosen s_1, s_2, s_3 so that

$$\begin{aligned} \operatorname{rank}(s_1, s_2, s_3) &\geq 3 \quad \text{on } X_0 \\ &\geq 2 \quad \text{on } X_1 \\ &\geq 1 \quad \text{on } X_2 \end{aligned}$$

By preceding step, we can assume $s_1 \neq 0$ on X_1. Then where $(s_1, s_2 + a s_3)$ are dependent contains a finite subset of X_1 and because s_2 and s_3 are not simultaneously dep. on s_1 we can choose a so that $(s_1, s_2 + a s_3)$ are ind. on $X_{\leq 1}$. Thus we can find s_1, s_2 ind. on $X_{\leq 1}$ such that $s_1 \neq 0$ on $X_{\leq 1}$. Now then $s_1 + b s_2$ vanishes on finitely many points of X_2 but it might vanish identically.

So go back to the choice of a. We have $s_1 \neq 0$ on X_1 and so can arrange a so that $s_2 + a s_3$ is ind. of s_1 on X_1. But $s_1 = 0$ on X_2, we know s_2, s_3 are not both zero, so we can arrange a so that $s_2 + a s_3 \neq 0$ on those points of X_2 where $s_1 = 0$. So can assume that s_1, s_2 are ind. on X_1 and that they have rank ≥ 1 on X_2. Then we can consider $s_1 + b s_2$, which can vanish only at finitely many points of X_2.

So the result is to replace (s_1, s_2, s_3) by

$$(s_1', s_2', s_3') = (s_1 + a_{12} s_2 + a_{13} s_3,\ s_2 + a_{23} s_3,\ s_3)$$

so that s_1' non-zero on $X_{\leq 2}$

s_1', s_2' independent on $X_{\leq 1}$

s_1', s_2', s_3' ind. on $X_{\leq 0}$.

April 7, 1973 (more stability)

Given a vector bundle E of rank n over a variety X over an alg. closed field k, suppose E generated by a space $V \to \Gamma(E)$. $X(E)$ the building (= simp. complex associated to the ordered set of proper subbundles of E – assume X irreducible). Suppose K is a finite subcomplex of $X(E)$.

Better fix an integer d and consider $X_d(E)$ the subcomplex of $X(E)$ consisting of chains $0 < F_0 < \cdots < F_g \subset E$ where $\operatorname{rank}(F_g) \leq d$. Let K be a finite subcomplex of $X(E)$. Then for each vertex F of K, the vector bundle E/F is gen. by V, and so for a Zariski dense subset of $v \in V$, v spans a trivial subbundle of E "transversal" to F. (Forgot to say that we are assuming $d < \operatorname{rank}(E) - \dim(X)$ so that $\operatorname{rank}(E/F) \geq \operatorname{rank}(E) - d > \dim X$). So it also clear that we can find $v_1, v_2, \ldots, v_r$ spanning a trivial subbundle of rank r of E transversal to all F in K provided $d + r \leq \operatorname{rank}(E) - \dim(X)$.

Now having chosen $v_1, v_2, \ldots, v_r$ let me consider the subcomplex of $X_d(E)$ consisting of those F which are well placed with respect to v_i in some sense.

Let $L \subset E$ be a line bundle. Call a subbundle F well placed with respect to L if either $L \subset F$ or if $L \to E/F$ is a subbundle. Thus want

$$\begin{tikzcd}[column sep=small, row sep=tiny] & & L \arrow[dr, phantom, "\subset"] & & \\ 0 \arrow[r, phantom, "\subset"] & L \cap F \arrow[ur, phantom, "\subset"] \arrow[dr, phantom, "\subset"] & & L+F \arrow[r, phantom, "\subset"] & E \\ & & F \arrow[ur, phantom, "\subset"] & & \end{tikzcd}$$

to be a diagram of subbundles. Makes sense ~~for~~ any subbundle.

~~Let Y be the subcomplex of X consisting of well-placed subbundles with respect to ... Claim Y is connected~~

$Y =$ subcomplex of $X_d(E)$ consisting of F such that F well-placed with respect to $v_1, \ldots, v_r$. Try to prove Y is contractible. First want to use $F \mapsto \langle v_1 \rangle + F$. Let H be a subbundle of rank d such that $\langle v_1 \rangle + H$ is a subbundle of rank $d+1$. Is it possible to use induction on d? So

?

A local ring, V a free module of rank n,

$$0 \subset F_1 \subset \cdots \subset F_{n-1} \subset V$$

a full flag. Call W well placed with respect to the flag if $F_i \cap W$ and $F_i + W$ are subbundles of V. (since

$$0 \to F_i \cap W \to F_i \oplus W \to V \to V/F_i + W \to 0$$

is exact, this means that $F_i + W$ has to be a subbundle.)

Let $X(V)'$ denote the subcomplex of F in $X(V)$ well placed with respect to the flag. To show $X(V)'$ is a bouquet of $(n-2)$-spheres. Use induction. True for $n=2$. Now let $Y \subset X(V)'$ consist of those W such that $F_1 + W < V$. Thus we are concerned with forgeting hyperplanes trans. to F_1. So we have to compute the link of H. Consists of $0 < W < H$ well placed wrt F

April 8, 1973 (more stability)

Let E be a vector bundle of rank n over X connected and let

$$0 \subset E_1 \subset \cdots \subset E_n = E$$

be a full flag for E. Call a subbundle F of E adapted to the flag if for each i, $E_i \cap F$ and $E_i + F$ are subbundles of E, equivalently E/E_i+F is loc. free (since one has exact sequences

$$0 \to E_i \cap F \to E_i \oplus F \longrightarrow E \longrightarrow E/E_i+F \longrightarrow 0\text{)}$$

Let $X(E)$ be the simplicial complex whose simplices are chains of subbundles $F_0 < \cdots < F_q$ with $0 \neq F_0$, $F_q \neq E$, and $X(E)'$ the full subcomplex whose vertices are those F adapted to the flag. I want to show that $X(E)'$ is a bouquet of $(n-2)$-spheres.

First point: suppose F is adapted to $\{E_i\}$, and F' is a subbundle of F. Claim F' adapted to $\{E_i\}$ iff F' adapted to $\{F \cap E_i\}$. Proof.

$$F \cap (F' + E_i) = F' + (F \cap E_i)$$

$$0 \to F/F \cap (F'+E_i) \longrightarrow E/F'+E_i \longrightarrow E/F \to 0$$

shows that $F' + (F \cap E_i)$ is a sub-bundle of F iff $F' + E_i$ is a subbundle of E.

Now let $\mathcal{H}$ be the set of F in $X(E)'$ such that $F + E_1$ is not in $X(E)'$. Note that if F is adapted to $\{E_i\}$ so is $F + E_a$ because

$$F + E_a + E_i = \begin{cases} F + E_i & i \geq a \\ F + E_a & a \geq i \end{cases}$$

are subbundles. Thus if $F + E_1$ is not in $X(E)'$ it must be that $F + E_1 = E$, and since E_1 is a line bundle and $F < E$, we must have $F \oplus E_1 = E$. Thus $\mathcal{H}$ is the set of subbundles F of E of rank $n-1$ which are adapted to $\{E_i\}$ and such that $F \oplus E_1 = E$.

Let $Y \subset X(V)'$ be the full subcomplex having the vertices not in $\mathcal{H}$. Then for $F \in Y$ we have the retraction

$$F \leq F + E_1 \geq E_1$$

so Y is contractible.

Given $H \in \mathcal{H}$, what is its link? The ordered set of $0 < F < H$ which are adapted to $\{E_i\}$, or equivalently (by the preceding point) which are adapted to $\{E_i \cap H\}$. Now

$$0 = E_1 \cap H \subset E_2 \cap H \subset \cdots \subset E_n \cap H = H$$

is a full flag since H has rank $n-1$. Thus the link of H is the complex of proper subbundles in H adapted to the flag $\{E_{i+1} \cap H\}_{i=1}^{n-1}$. By induction the links will be a bouquet of $(n-3)$-spheres, so I can conclude as before that $X(E)'$ is a bouquet of $(n-2)$-spheres.

Application: Let A be a local ring with an infinite residue field k, and suppose A is a k-alg. $E = A\otimes_k V$. To prove $X(E)$ is a bouquet of $(n-2)$-spheres $n = \dim_k(V)$. Suffices to show any finite subset S of $X(E)$ is adapted to some flag in E, for then have

$$S \subset X(E)'_{\{E_i\}} \subset X(E)$$

showing that $X(E)$ has no non-trivial homotopy groups in degrees $< n-2$.

So let $F \subset A\otimes_k V$ be a subbundle with quotient Q. Let $0 < V_1 \subsetneq \cdots \subsetneq V_n = V$ be a full flag in V. The generic situation is where the composite

$$V_q \subset V \longrightarrow k\otimes_A Q$$

is an isomorphism, $q = \mathrm{rank}(Q)$. If this is the case, then I claim F is adapted to the flag $\{A\otimes_k V_i\}$. Recall that if elements $z_1, \ldots, z_j \in Q$ are such that there images in $k\otimes_A Q$ are independent, then $A^j \to Q$ is a subbundle (because can extend to a map $A^q \to Q$ which is an isom after $k\otimes_A ?$, hence before). Thus it follows that for $i \le q$

$$A\otimes_k V_i \longrightarrow A\otimes_k V \longrightarrow Q = A\otimes_k V/F$$

is a subbundle injection, so $(A\otimes_k V_i + F)$ is a subbundle of $A\otimes_k V$. For $i \ge q$ it is onto, so $A\otimes_k V_i + F = A\otimes_k V$.

Thus we can consider for each $F \in S$ the set of flags in V such that $V_q \oplus k\otimes_A F = V$, $q = n - (\mathrm{rank}\, F)$, and these form a Zariski dense subset of all flags. Since

k is infinite, there exists a flag such that each $F \in S$ is adapted with respect to it.

But suppose now that A is local with residue field k infinite, ~~finitely many~~ and let E be a vector bundle of rank n. Given a finite set S of subbundles F of E, we can since k is infinite, find a full flag $\{V_i\}$ in $k \otimes E$ such that

$$V_{g(F)} \oplus k \otimes F = k \otimes E$$

$$g(F) = n - \text{rank}(F)$$

for each F in S. Now lift the flag $\{V_i\}$ to a flag $\{E_i\}$ in E. Again it follows that

$$E_i + F \quad \text{is a subbundle of } E$$

for each i, so F is adapted to $\{E_i\}$. Thus I seem to have proved

<u>Proposition</u>. If A is a local ring with <u>infinite</u> residue field (not nec. noeth. or commutative), then $X(A^n)$ has homotopy type of a bouquet of $(n-2)$-spheres.

As before this gives a stability result for the Q-category.

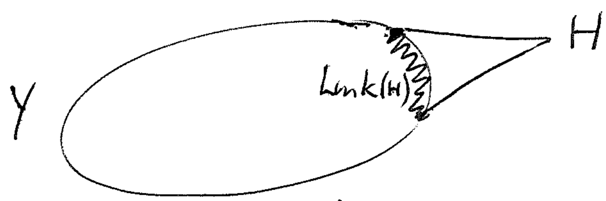
April 9, 1973. (more stability)

A local ring residue field k, E a free A-module of rank n, $X(E)$ the ordered set of proper subbundles F of E. To show $X(E) \sim \bigvee S^{n-2}$.

$n=2$, $X(E)$ discrete

$n=3$, have to check $X(E)$ connected. But given two lines L_1, L_2 we know that either L_1 and L_2 are independent or that we can find L_3 independent of L_1, L_2 separately. Precisely, look at the lines $L_1 \otimes k$, $L_2 \otimes k$ in $E \otimes k$, choose an independent line and lift it to L_3.

$n \geq 4$. Fix a line L and let $\mathcal{H}_L$ be the set of complementary "hyperplanes," ~~[illegible]~~ and let Y be the ~~[illegible]~~ full subcategory of $X(E)$ consisting of F not in $\mathcal{H}_L$. Then ~~[illegible]~~ we have the picture



where $\mathrm{Link}(H) = X(H)$. By induction I know that $\mathrm{Link}(H) \sim \bigvee S^{n-3}$ whence from

$$\underbrace{\coprod_H H_{n-2}(\mathrm{Link}(H))}_{0} \to H_{n-2}(Y) \longrightarrow H_{n-2}(X) \xrightarrow{\partial} \coprod_H \tilde{H}_{n-3}(\mathrm{Link}(H)) \longrightarrow$$

Thus $X \sim \bigvee S^{n-2}$ $\Longleftarrow Y \sim \bigvee S^{n-2}$ ~~[illegible]~~ (at least ignoring π_1).

Now let $Y' \subset Y$ consist of F such that F is not independent of L, i.e. $F \otimes k \supset L \otimes k$.

(Observe that $\mathcal{H}_L$ depends only on $L\otimes k \subset E\otimes k$. Thus $H\otimes k$ is complementary to $L\otimes k \iff H$ is complementary to L.) The same is true for Y'. Now we can retract Y to Y' by sending F

$$F \leq F \qquad \text{if} \quad F \in Y'$$
$$F \leq F+L \qquad \text{if} \quad F \in Y - Y'$$

This is well-defined because if $F_1 \leq F_2$?

Doesn't work, because we can have F_2 dependent on L, F_1 independent, and $F_1 + L \not\subset F_2$.

Wait: $\mathrm{Link}(H) = X(H)$ contracts within Y because $F \leq F+L$ ($F \subset H \Rightarrow E/F+L \cong H/F$ so $F+L$ is a subbundle). Thus we know that $\mathrm{Link}(H) \longrightarrow Y$ is null-homotopic, and so

$$X \sim \bigvee S^{n-2} \iff Y \sim \bigvee S^{n-2}$$

April 10, 1973 (more stability)

Example: Let S be a set and [redacted] consider the simplicial complex $K(S,n)$ whose simplices are [redacted]

$$\left((s_0, i_0), \dots, (s_q, i_q) \right)$$

$K(S,n) = \underbrace{S * \cdots * S}_{n\text{-times}}$

with $0 \le i_0 < i_1 < \cdots < i_q \le n$. Claim $K(S,n)$ is $(n-1)$-connected (begins in dim. n).

Use induction on n. For $n=0$, [redacted] it is clear. For $n=1$ we have a connected graph so it is also clear.

Fix $s_0 \in S$. The link of $(s,0)$ is clearly $K(S,n-1)$. The result of removing all the vertices $(s,0)$ for $s \neq s_0$ is a cone with vertex $(s_0,0)$. Thus

$$K(S,n[redacted]) = \bigvee_{S-s_0} \operatorname{Susp} K(S,n-1)$$

so the induction works.

$$\operatorname{rank} \tilde{H}_n(K(S,n)) = (m-1) \cdot \operatorname{rank} \tilde{H}_n(K(S,n-1)) \qquad m = \operatorname{card} S$$

$$\therefore \quad \operatorname{rank} \tilde{H}_n(K(S,n)) = (m-1)^{n+1}$$

Check Euler chars:

no. of $i_0 < \cdots < i_q$ $\qquad \dfrac{(n+1)\cdots(q)}{(q+1)!} = \dbinom{n+1}{q+1}$

no. of q-simplices is $\dbinom{n+1}{q+1}(m[redacted])^{q+1}$

so

$$\chi = -\sum_{q=0}^{n} (-1)^{q+1} \binom{n+1}{q+1} (m[redacted])^{q+1} = +1 - (1-m)^{n+1}$$

$-1+1$

$$= 1 + (-1)^{n[redacted]}(m-1)^{n+1}$$

OK

April 22, 1973. More stability.

Let $M = \coprod_{n \geq 0} BG_n$ be a top monoid associated to the family $Gl_n(A)$ or Σ_n say, and ε the base point of BG_1. Multiplying by ε on the left or right defines an embedding $BG_{n-1} \longrightarrow BG_n$ unique up to homotopy (more or less) and so we can speak of the cofibre BG_n/BG_{n-1}.

Have spectral sequence

$$E^1_{pq} = H_{p+q}(BG_p/BG_{p-1}) \Longrightarrow H_n(BG_\infty)$$

which results from filtering BG_∞ via BG_p. This spectral sequence has products because the H-space structure on BG_∞ induces maps

$$(BG_p/BG_{p-1}) \wedge (BG_q/BG_{q-1}) \longrightarrow BG_{p+q}/BG_{p+q-1}$$

Example: $BG_p = BU_p$. Then

$$BU_p/BU_{p-1} = MU_p$$

since BU_{p-1} is the canonical sphere bundle over BU_p.

In general it does not seem to be the case that BG_p/BG_{p-1} forms a spectrum in a natural way. However it does once one fixes a map

$$S^j \longrightarrow BG_e/BG_{e-1}$$

for some e, j. The question becomes whether one gets any interesting cohomology theories in this way.

<u>Question</u>: From the calculations for a finite field, one ~~is~~ lead to conjecture that the fibres of

$$BG_{n-1}^{+} \longrightarrow BG_n^{+}$$

is a Moore space of type $\mathbb{Z}/(q^n-1)\mathbb{Z},\ 2n-1$?

Recall that the Q category is an H-space with multiplication given by direct sum. Clearly we get

$$Q_p \times Q_q \longrightarrow Q_{p+q} \qquad V, W \longmapsto V \oplus W$$

for each p, q hence we get maps

$$(Q_p/Q_{p-1}) \wedge (Q_q/Q_{q-1}) \longrightarrow (Q_{p+q}/Q_{p+q-1}).$$

Now recall that $Q_1/Q_0 = \mathrm{Susp}(BG_1)$, hence there is a canonical map

$$S^1 \longrightarrow Q_1/Q_0$$

so that $\{Q_p/Q_{p-1}\}$ is a spectrum in a <u>canonical way</u>, in fact a ring spectrum.

Now recall that we have a cocart. square

$$\begin{tikzcd} (\Sigma X_n)_{G_n} \arrow[r] \arrow[d] & G_n \arrow[d] \\ Q_{n-1} \arrow[r] & Q_n \end{tikzcd}$$

so that Q_n/Q_{n-1} is the Thom space of the bundle over BG_n with fibre the suspension of X_n, which is a wedge of $(n-1)$-spheres.

Now it should be possible to exhibit a $G_p \times G_q$ equivariant map

$$\Sigma X_p * \Sigma X_q \longrightarrow \Sigma X_{p+q}.$$

In fact given a vector space V let $J(V)$ be the ordered set of proper layers in V, and $J'(V)$ the ordered set of all layers. Then

$$J'(V) \times J'(W) \longrightarrow J'(V \oplus W)$$
$$(V_0, V_1) \,,\; (W_0, W_1) \longrightarrow (V_0 \oplus W_0,\, V_1 \oplus W_1)$$

carries $J(V) \times J'(W) \cup J'(V) \times J(W)$ into $J(V \oplus W)$, so it induces a map

$$\frac{J'(V)}{J(V)} \wedge \frac{J'(W)}{J(W)} \longrightarrow \frac{J'(V \oplus W)}{J(V \oplus W)}.$$

~~And~~ Since $J'(V)$ is contractible, this is a map

$$\Sigma J(V) \wedge \Sigma J(W) \longrightarrow \Sigma J(V \oplus W).$$

Better one has only to note that

$$\boxed{J(V) \times J'(W) \underset{J(V)\times J(W)}{\cup} J'(V) \times J(W) = J(V) * J(W)}$$

up to homotopy.

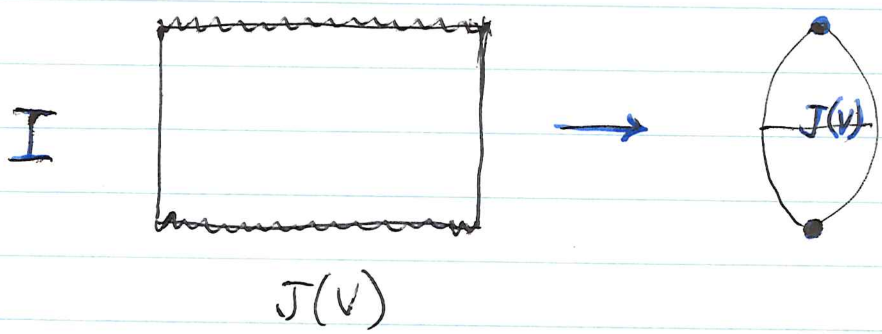
In particular we have

$$J(V) \times J'(k) \underset{J(V) \times J(k)}{\cup} J'(V) \times J(k)$$

$$\parallel$$

$$J(V) \times I \underset{J(V) \times \dot{I}}{\cup} CJ(V) \times \dot{I}$$

is heq to $\Sigma J(V)$:



The Q-category in classical K-theory: I recall that it is a simplicial groupoid:

$$M \times M \equiv M = pt$$

where $$M = \coprod_{n\geq 0} BG_n$$

so it is

$$\coprod_{a,b\geq 0} BG_{a,b} \equiv \coprod_a BG_a = pt$$

Filtering by the total degree as before we see that Q_p/Q_{p-1} is the simplicial space

$$\begin{array}{ccc} pt & pt & \\ \amalg & \amalg & pt \\ \coprod_{a+b=p} BG_{a,b} & BG_p & \end{array}$$

whose homology we can compute dimensionwise as before. Thus

$$\bigoplus_p H_*(Q_p/Q_{p-1}) \Leftarrow \mathrm{Tor}^{R}(\mathbb{Z},\mathbb{Z})$$

where $$R = \bigoplus_{n\geq 0} H_*(BG_n).$$

Now take $G_n = U_n$, and recall

$$\bigoplus H_*(BU_n) = \mathbb{Z}[b_0, b_1, \dots]$$

where $b_i \in H_{2i}(BU_1)$ is the dual basis to c_1^i.
Thus

$$\mathrm{Tor}_q^R(\mathbb{Z},\mathbb{Z}) = \Lambda[\tilde{b}_0, \tilde{b}_1, \ldots]$$

where $\tilde{b}_i \in \mathrm{Tor}_1^R(\mathbb{Z},\mathbb{Z})_i$ is the image of b_i in the indecomposable space of R. So this means that

$$Q_1/Q_0 = \Sigma BU_1$$

has the generators. Precisely we can say that

$$\bigoplus_{p\geq 0} H_*(Q_p/Q_{p-1})$$

is an exterior algebra with generators $\tilde{b}_0, \tilde{b}_1, \ldots\ldots$
where $\tilde{b}_i \in H_{2i+1}(Q_1/Q_0)$.

Note that the least degree element of $H_*(Q_p/Q_{p-1})$ is

$$\tilde{b}_0 \cdots \tilde{b}_{p-1}$$

which has degree $\displaystyle\sum_0^{p-1}(2i+1) = 2\frac{p(p-1)}{2} + p = p^2$.

The spectrum $\{Q_p/Q_{p-1}\}$ has homology

$$\varinjlim_p H_{*+p}(Q_p/Q_{p-1}) = 0$$

and so it represents ~~the~~ trivial gen. homology theory.

<u>April 14, 1973</u> $GL_2(\mathbb{C})$.

Let k be a field $G = PGL_2(k)$ = group of automorphism of P^1_k. I want to compute the low dimensional homology of G.

Let G act on P^1_k and consider the complex of chains on P^1_k considered as a simplicial complex in which every finite non-empty subset is a simplex. We get an exact sequence of G-modules.

$$\longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

$$C_0 = \mathbb{Z}[G/B] \qquad B : \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \text{ fixes } z=\infty.$$

modulo scalars

$$C_1 = \mathbb{Z}[G] \otimes_{\mathbb{Z}[N]} \mathbb{Z}^{\text{sign}}$$

N = normalizer of torus $T = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ modulo scalars

$$C_2 = \mathbb{Z}[G] \otimes_{\mathbb{Z}[\Sigma_3]} \mathbb{Z}^{\text{sign}}$$

G acts transitively on triples of points in $\mathbb{P}^1$. The stabilizer of $0, 1, \infty$ is Σ_3 generated by the transpositions $z \longmapsto 1-z$, $z \longmapsto \frac{1}{z}$

$$C_3 = \coprod_{\{z_0, \dots, z_3\}} \mathbb{Z}(z_0, \dots, z_3)$$

G doesn't act transitively on 3-simplices. Let $\tilde{C}_3$ be the group of linear combinations of ordered 3-simplices. Thus

$$\tilde{C}_3 \otimes_{\mathbb{Z}[\Sigma_4]} \mathbb{Z}^{\text{sign}} \xrightarrow{\sim} C_3$$

$$\tilde{C}_3 = \bigoplus_{z \neq 0,1} \mathbb{Z}[G]$$

because any (z_0, z_1, z_2, z_3) is uniquely G-conjugate to one of the form $(0, 1, \infty, z)$.

Now I will work with coefficients $\dot{\mathbb{Z}}$ such that 2, 3 are invertible. Look at coefficients such that B, T have same homology. Then

$$H_*(G, C_0) = H_*(T)$$

$$H_*(G, C_1) = H_*(N, \dot{\mathbb{Z}}^{\text{sgn}})$$

$$= \left(H_*(T) \otimes \dot{\mathbb{Z}}^{\text{sgn}}\right)^{\mathbb{Z}_2} = \{x \in H^*(T) \mid \underset{wx=-x}{}\}.$$

and so the map

$$H_*(G, C_1) \longrightarrow H_*(G, C_0)$$

is the inclusion of the anti-invariant elements of $H_*^*(T)$, and so the cokernel must be the coinvariants.

$$H_*(G, C_2) = H_*(\Sigma_3, \mathbb{Z}^{sgn}) = 0.$$

What are the orbits of G on 3-simplices $\{z_1, z_2, z_3, z_4\}$. On ordered sets get $(0, 1, \infty, z)$ with $z \neq 0, 1, \infty$. Have then an action of Σ_4

$$z \longmapsto \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}$$

so then $z_4 \longmapsto \dfrac{z_4 - z_1}{z_4 - z_3} \cdot \dfrac{z_2 - z_3}{z_2 - z_1} = \gamma$

Now permute z_1, z_2, z_3, z_4 and see how γ changes

$(0, 1, \infty, z) \longmapsto z$

$(0, 1, z, \infty) \longmapsto 1 - z$ $\qquad \dfrac{x}{x - z} \quad 1 - z$

$(0, \infty, 1, z) \longmapsto \dfrac{z}{z - 1}$ $\qquad \dfrac{x}{x-1}$

$(0, z, 1, \infty) \longmapsto \dfrac{z - 1}{z}$ $\qquad \dfrac{x}{x-1} \quad \dfrac{z-1}{z}$

$(0, \infty, z, 1) \longmapsto \dfrac{1}{1 - z}$ $\qquad \dfrac{x}{x - z}$

$(0, z, \infty, 1) \longmapsto \dfrac{1}{z}$ $\qquad \dfrac{x}{z}$

$(1\ 0\ \infty\ z) \longmapsto 1-z$ $\qquad \dfrac{\cancel{z}-1}{-1}$

$(1\ 0\ z\ \infty) \longmapsto z$ $\qquad \dfrac{x-1}{x-z}\cdot z$

$(\infty\ 0\ 1\ z) \longmapsto$

$(z\ 0\ 1\ \infty) \longmapsto$

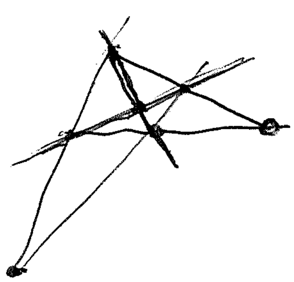
$(\infty\ 0\ z\ 1) \longmapsto$

$(z\ 0\ \infty\ 1) \longmapsto$

$(z\ \infty\ 1\ 0) \longmapsto z$ $\qquad \dfrac{x-z}{x-1}$

$(\infty\ z\ 0\ 1) \longmapsto z$ $\qquad \dfrac{z}{x}$

Thus the Klein group acts trivially. (This well-known, ~~[illegible]~~ I think. Thus Atiyah told me one gets a surjection $\Sigma_4 \to \Sigma_3$ with kernel the Klein group by ~~letting~~ Σ_4 act on line pairs



).

Thus the cross-ration changes into

$$z,\ \frac{1}{z},\ 1-z,\ \frac{1}{1-z},\ 1-\frac{1}{z},\ 1-\frac{1}{1-z} = \frac{z}{z-1}$$

and there should be an ~~~~ invariant rational function of degree 6 in z: (at least one by Luroth, and all others are related by G)

$$(z-\lambda)\left(\frac{1}{z}-\lambda\right)(1-z-\lambda)\left(1-\frac{1}{z}-\lambda\right)\left(\frac{1}{1-z}-\lambda\right)\left(\frac{z}{z-1}-\lambda\right)$$

I don't know if there is a particularly simple one.

In any case let's go back to $H_*(G, C_3)$. We have that this is a direct sum over $k-\{0,1\}$ modulo Σ_3 of the cohomology of the stabilizer of $(0,1,\infty,z)$ which is at least the Klein group. The bad points for the Σ_3 action are z

$z = \frac{1}{z}$ $\qquad z = \pm 1$ $\qquad \boxed{-1}$

$z = 1-z$ $\qquad \boxed{\frac{1}{2}}$ $\quad \boxed{2}$

$z = \frac{1}{1-z}$ $\qquad z - z^2 = 1$

$z^2 - z + 1 = 0 \qquad z = \frac{1 \pm \sqrt{-3}}{2} = \sqrt[3]{-1}$ $\qquad \boxed{\frac{1 \pm \sqrt{-3}}{2}}$

$z = 1 - \frac{1}{z}$

$z^2 = z - 1$

$z^2 - z + 1 = 0.$

$z = \frac{z}{z-1}$ $\qquad z^2 - z = z$

$z = 2z \qquad z = 2.$

---

one bad orbit is $\boxed{-1, \frac{1}{2}, 2}$ stabilizer $\mathbb{Z}/2$.

other is $\boxed{\frac{1+\sqrt{-3}}{2}, \frac{1-\sqrt{-3}}{2}}$ stabilizer is $\mathbb{Z}/3$.

also $\boxed{0, 1, \infty}$ stabilizer $\mathbb{Z}/2$.

Simplest perhaps is

$$w = \frac{(z^2 - z + 1)^3}{z^2(z-1)^2}$$

has triple zeroes at $\frac{1+\sqrt{-3}}{2}$ $\frac{1-\sqrt{-3}}{2}$ and double poles at $0, 1, \infty$

Clearly w is unchanged under $z \mapsto 1 - z$, $z \mapsto \frac{1}{z}$ hence under Σ_3 as these transpositions generate.

By sending $z \mapsto w$ I get a bijection of the Σ_3 orbits on $k - \{0, 1\}$ with k. Now when $z = 2$, $w = \frac{(4-2+1)^3}{4} = \frac{27}{4}$ the stabilizer is Σ_2 which acts non-trivially on the sign representation. So this doesn't contribute to the cohomology. When $z = \frac{1+\sqrt{-3}}{2}$, $w = 0$ and the stabilizer is $\mathbb{Z}/3$ which acts trivially on the sign rep. Thus $w = 0$ contributes along with all $w \neq \frac{27}{4}$. So this suggests we change w to

$$\bar{w} = \frac{(z+1)^2 \left(z - \frac{1}{2}\right)^2 (z-2)^2}{z^2(z-1)^2}$$

Thus it seems that

$$H_*(G, C_3) = \coprod_{\bar{w} \in k - 0} A[0]$$

where A is the coefficient group (assuming 2,3 invertible)

5 tuples. $(z_1, z_2, z_3, z_4, z_5)$ and try to understand relations between cross-ratios.

$$(0, 1, \infty, a, b)$$

$$(1, \infty, a, b) \longmapsto \frac{b-1}{b-a}$$

$$(0, \infty, a, b) \longmapsto \frac{b}{b-a}$$

$$(0, 1, a, b) \longmapsto \frac{b}{b-a} \cdot \frac{1-a}{1}$$

$$(0, 1, \infty, b) \longmapsto b$$

$$(0, 1, \infty, a) \longmapsto a$$

$$(a, 0, 1, \infty, b)$$

$$(0, 1, \infty, b) \longmapsto b$$

$$(a, 1, \infty, b) \longmapsto \frac{b-a}{1-a}$$

$$(a, 0, \infty, b) \longmapsto \frac{b-a}{-a}$$

$$(a, 0, 1, b) \longmapsto \frac{b-a}{b-1} \cdot \frac{1}{a}$$

$$(a, 0, 1, \infty) \longmapsto \frac{\infty-a}{\infty-1}\, \frac{1}{a} = \frac{1}{a}$$

TOO COMPLICATED.

Explore more abstractly:

The point perhaps to keep in mind is that what I am trying to do is to understand the cohomology of $GL_2(k)$ via that of the algebraic group GL_2 which is known. Thus suppose $k = \mathbb{C}$ and we have mod ℓ coefficients. Then the subgroups B, T have the same cohomology as the corresponding algebraic groups. This takes care of the dimensions 0, 1, 2 but once one hits dim. 3 there appears to be a problem.

?

April 16, 1973. $\quad GL_n(k[t])$ & vector bundles on curves

Let k be a field, $\Gamma = GL_n(k[t]) = \mathrm{Aut}(M)$, $M = k[t]^n$, and let X be the building at ∞ of $k(t) \otimes_{k[t]} M = V$. Thus X is the simplicial complex whose q-simplices are chains of lattices

$$L_0 < \cdots < L_q$$

in V for the d.v.r. $\mathcal{O}_\infty = k[\frac{1}{t}]_{\mathfrak{m}}$, $\mathfrak{m} = (\frac{1}{t})$, such that $\mathfrak{m}(L_q/L_0) = 0$ (equivalently $\cancel{t}^{-1} L_q \subset L_0$). Can also think of such a lattice as an extension of M to a vector bundle E of rank n on $\mathbb{P}^1$.

We know X is contractible, hence the chains on X form a Γ-resolution of $\mathbb{Z}$, and we obtain a spectral sequence relating the homology of Γ with the homology of X/Γ with coefficients in the local system of isotropy homology. We now have to compute the Γ-orbits on the q-simplices, and the stabilizers.

~~Can think of L as a pair $(E, \alpha: M \xrightarrow{\sim} j^*E)$~~
~~$j: \mathbb{A}^1 \subset \mathbb{P}^1$. Then (E,α) (E',α') are Γ conjugate~~
~~means that~~

$$\begin{array}{ccccc} M & \xrightarrow[\sim]{\alpha} & j^*E & & E_\infty \\ \wr\big\downarrow \gamma & & & & \\ M & \xrightarrow[\alpha']{\sim} & j^*E' & & E'_\infty \end{array}$$

We think of $L \subset V$ as $E_\infty \subset V$ with $M = \Gamma(\mathbb{A}^1, E) \subset E_\eta = V$. For L, L' to be Γ conjugate means the vector bundles E, E' are isom, and the stabilizer of L is simply the group of automorphisms of the bundle E.

We know every vector bundle E on $\mathbb{P}^1$ is isomorphic to

$$\mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_n)$$

with $k_1 \leq \cdots \leq k_n$. Let

$$\alpha_i(E) = k_{i+1} - k_i$$

for $i = 1, \ldots, n-1$; these are roots in some sense. I want now to compute the group of autos of this bundle. Now I know that if I write

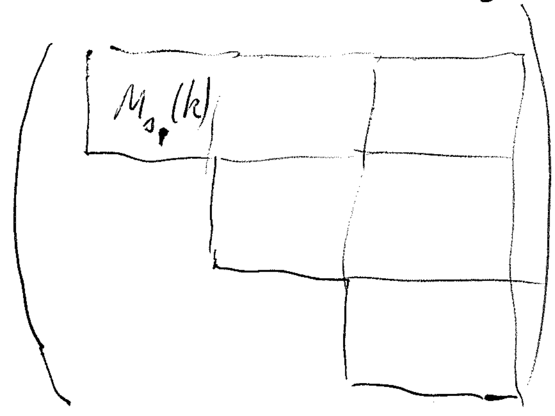
$$E \simeq \bigoplus_k \mathcal{O}(k)^{n_k} \qquad \sum n_k = n$$

then the subbundle $\displaystyle\bigoplus_{k \geq p} \mathcal{O}(k)^{n_k} = F_p E$ is intrinsic

Better

$$\mathrm{End}\left(\mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_n)\right) = \text{ring of matrices } \{\mathrm{Hom}(\mathcal{O}(k_j), \mathcal{O}(k_i))\}$$

$$= \quad '' \quad '' \quad '' \quad \{\Gamma(\mathcal{O}(k_i - k_j))\}$$

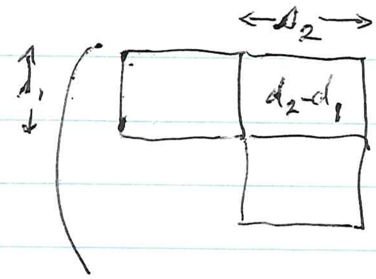
$=$ 

I need some notation. Thus given $k_1 \le \cdots \le k_n$ I need to know the size of the blocks, and the jumps. Thus I want to know the ~~jumps~~ blocks and the degrees. So ~~[illegible]~~ suppose we put

$$\underbrace{d_1 = \cdots = d_1}_{s_1} < \underbrace{d_2 = \cdots = d_2}_{s_2} < \cdots = d_n = k_1 \le \cdots \le k_n$$

so that

$$\mathrm{End}\,(\mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_n)) =$$



where (i,j)-th block consists of homogeneous polynomials of degrees $d_j - d_i$ and of size $s_i \times s_j$

and the auto group is the set of matrices such that the diagonal entries are invertible.

~~Observe that~~

$$\cancel{\mathrm{Aut}(E) \longrightarrow \mathrm{Aut}(E \otimes k(\infty)) = GL_n(k)}$$

Next we want to compute the homo

$$\mathrm{Aut}\,(E) \longrightarrow \mathrm{Aut}\,(E \otimes k(\infty))$$

so we first have to understand

$$\Gamma(\mathcal{O}(k)) \longrightarrow \Gamma(\mathcal{O}(k) \otimes k(\infty))$$

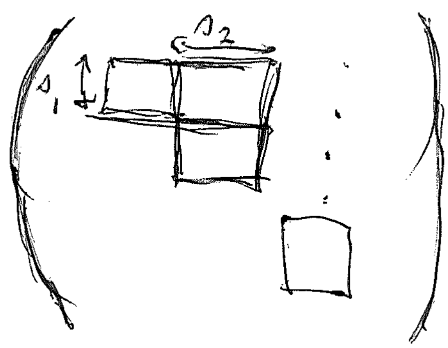
The former is homog. polys of degree k in t_0, t_1 where $t = t_1/t_0$. At ∞, t_0 is zero and we ~~[illegible]~~ t_1^k as the base of $\Gamma(\mathcal{O}(k) \otimes k(\infty))$. Thus if we think of $\Gamma(\mathcal{O}(k))$ as polys in $z = t^{-1}$ of degrees $\leq k$, the above map takes the constant term. So it is now clear that

$$\mathrm{Aut}(E) \longrightarrow \mathrm{Aut}(E \otimes k(\infty)) \simeq Gl_n(k)$$

simply evaluate the polynomial matrices at $z = 0$. It is therefore clear that

$$\mathrm{Im}\left(\mathrm{Aut}(E) \to \mathrm{Aut}(E \otimes k(\infty))\right) = \begin{array}{l}\text{the parabolic subgroup fixing}\\ \text{the canonical filtration of } E \otimes k(\infty)\end{array}$$

i.e. matrices



At this point we understand the Γ-classes of vertices and their stabilizers. Now I recall that a q-simplex $L_0 < \cdots < L_q$ is simply a vertex L_q together with the flag

$$0 \leq \bar{L}_0 < \bar{L}_1 < \cdots < \bar{L}_q = L/t^{-1}L.$$

Thus a q-simplex is simply a vector bundle E with a flag $F: 0 \leq \bar{E}_0 < \cdots < \bar{E}_\delta = E \otimes k(\infty)$.

We are therefore interested in determining the classes of such flags under $\mathrm{Aut}(E)$. This leads to

<u>Problem</u>: Given an n-dimensional vector space V over k with a filtration

$$V > W_1 > W_2 > \cdots > W_r = 0$$

having jumps $\quad s_i = \dim (W_{i-1}/W_i) \qquad i = 1, \ldots, r$

let P be the corresponding parabolic subgroups of $\mathrm{Aut}(V)$. Classify the classes of flags

$$0 \leq V_0 < V_1 < \cdots < V_\delta = V$$

under the action of <u>P</u>.

Change notation. Start with

$$(*) \qquad 0 < V_1 < \cdots < V_r = V \qquad \dim V = n$$

given and fixed, and $P = \mathrm{Aut}(0 < V_1 < \cdots < V_{r-1} < V)$. Now suppose given a subspace W of dimension p. To determine its P-class one has only to give the dimensions of the filtration

$$0 \leq V_1 \cap W \leq \cdots \leq V_r \cap W = W$$

Thus if the jumps in $(*)$ are $s_1, s_2, \ldots, s_r$

then the P-class of a subspace is a sequence of jumps $t_1, \ldots, t_r$ with $0 \le t_i \le s_i$.

To simplify things take the case where all $s_i = 1$. Then a subspace is determined by a sequence $t_i = 0$ or 1, and a flag

$$0 \le W_0 < W_1 < \cdots < W_q = V$$

is determined by an increasing family

$$t(0) \le t(W_0) < t(W_1) < \cdots < t(W_q) = t(V)$$

where

$$t(W) = \text{the sequence } \left(\dim (W \cap V_1), \dim \left(\frac{W \cap V_2}{W \cap V_1} \right), \ldots \right)$$

Thus it seems that what we are getting is that any Γ class of simplices may be identified with a simplex in the following simplicial complex: It has for vertices sequences $\vec{k} : k_1 \le \cdots \le k_n$ and a simplex is an increasing sequence

$$\vec{k}_0 < \cdots < \vec{k}_n$$

for the product ordering such that each component of $\vec{k}_n - \vec{k}_0$ is either 0 or 1.

Thus what this seems to be is the n-fold product of the ordered simplicial complex

divided out by the action of Σ_n.

So the next point is to understand the stabilizers. Thus given a q-simplex I want to understand its stabilizer.

Again consider the case where all $s_i = 1$. The stabilizer Aut(E) maps onto the Borel subgroup of Aut $(E \otimes k(\infty))$. Now one knows that the mod l cohomology of Aut(E) is the same as that of the torus. Thus it seems that all of the simplices with top vertex E have same mod l stabilizer homology. But now when we come to higher s_i the situation is even messier.

<u>Proposition</u>: Let C be a complete n.s. curve of genus g over k alg. closed, and let E be a vector bundle of rank n over C. Assume E has a full flag

$$(*) \qquad 0 < E_1 < \cdots < E_n = E$$

with quotients $L_i = E_i / E_{i-1}$ satisfying

$$\deg(L_{i-1}) - \deg(L_i) > 0$$

Then $(*)$ is the unique maximal flag in E.

<u>Proof</u>. Recall that a maximal flag is one such that E_1 is a line bundle of maximum possible degree in E, E_2/E_1 is of max. deg in E/E_1, etc.

Suffices to show that if L is a sub line bundle of E of maximum degree, then $L = E_1$. (Induction on n).

But then $\deg(L) \geq \deg(E_1) > \deg(L_i)$ for all $i \geq 2$. so one sees that the map

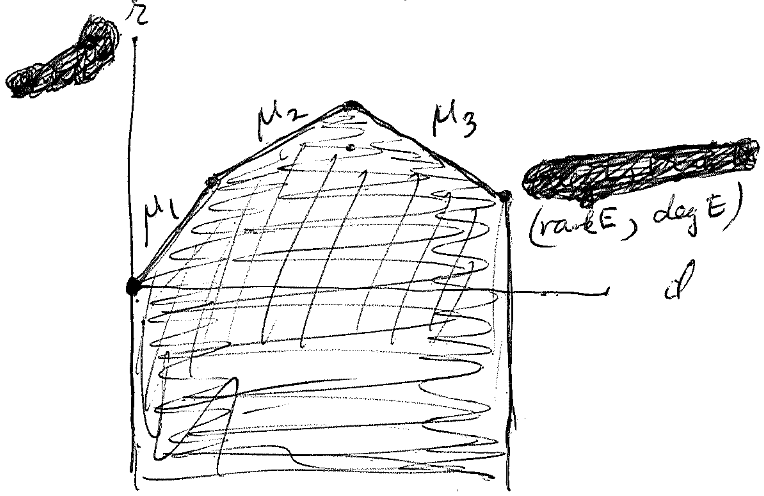
$$L \longrightarrow E_n \longrightarrow E_n / E_{n-1} \quad \text{is zero}$$

so $L \subset E_{n-1}$, etc. until finally that $L \subset E_1$, whence $L = E_1$ as the degrees are equal. DONE

Given a vector bundle E over C, we consider

$$\mu_1(E) = \sup\left\{ \frac{\deg(F)}{\operatorname{rank}(F)} \;\middle|\; 0 < F \leq E \right\}$$

where F runs over subbundles of E. Actually we shall maybe eventually want to consider the polygon obtained by plotting the points $(\deg(F), \operatorname{rank}(F))$ in the plane and taking the shaded area



So we get a sequence of slopes.

$$\mu_1 > \mu_2 > \cdots\cdots$$

Suppose that we consider two subbundles (non-zero) F_1, F_2 with slope $= \mu_1(E)$. Then have an exact sequence of vector bundles

$$0 \longrightarrow F_1 \cap F_2 \longrightarrow F_1 \oplus F_2 \longrightarrow F_1 + F_2 \longrightarrow 0$$

hence

$$d(F_1 \cap F_2) + d(F_1 + F_2) = d(F_1) + d(F_2)$$
$$r(F_1 \cap F_2) + r(F_1 + F_2) = r(F_1) + r(F_2)$$

Let $\overline{F_1+F_2}$ be the smallest subbundle of E containing F_1+F_2, whence

$$d(F_1+F_2) \leq d(\overline{F_1+F_2})$$

with equality iff the two are equal, and

$$r(F_1+F_2) = r(\overline{F_1+F_2}).$$

Then by assumption

$$d(F_1 \cap F_2) \leq \mu_1(E)\, r(F_1 \cap F_2)$$

$$d(F_1+F_2) \leq d(\overline{F_1+F_2}) \leq \mu_1(E)\, r(F_1+F_2)$$

so adding get

$$d(F_1)+d(F_2) \leq \mu_1(E)\left(r(F_1)+r(F_2)\right)$$

$$= d(F_1)+d(F_2)$$

since F_1, F_2 have slope $\mu_1(E)$. Thus all the preceding inequalities must be equalities and so we ~~see~~ that

$$d(F_1 \cap F_2) = \mu_1(E)\, r(F_1 \cap F_2)$$

$F_1+F_2 = \overline{F_1+F_2}$ is a subbundle of E

F_1+F_2 has slope $\mu_1(E)$.

So if F_1 is a subbundle with slope $\mu_1(E)$ having the maximum rank, then $F_2 \subset F_1$. Thus get

<u>Proposition:</u> There is a unique maximal subbundle of E of slope $\mu_1(E)$, and it is semi-stable of that slope

Proposition: If E is semi-stable and $\deg(E) < 0$, then $H^0(E) = 0$.

Proof: If $H^0(E) = 0$, then E has a sub-line bdle of degree ≥ 0 contradicting
$$\frac{\deg(L)}{1} \leq \frac{\deg(E)}{rg(E)} < 0.$$

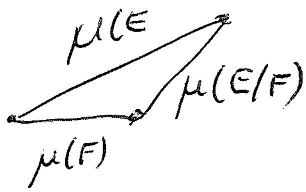
Cor: E semi-stable and $\deg(E) > rg(E)\cdot(2g-2)$ $\Rightarrow H^1(E) = 0$.

Proof: $\Omega \otimes E^\vee$ is also semi-stable. (Check: Any subbundle of $E^\vee$ is of the form $F^\perp$ for some subbundle F of E, and $F^\perp = (E/F)^\vee$.
$$\deg(F^\perp) = \deg(F) - \deg(E)$$
$$rg(F^\perp) = rg(E) - rg(F)$$

$$\mu(E) = \frac{\deg(E)}{rg(E)} \qquad \mu(E^\vee) = \frac{-\deg(E)}{rg(E)} = -\mu(E).$$

$$\mu(F^\perp) = \frac{\deg(F^\perp)}{rg(F^\perp)} = \frac{\deg(F) - \deg(E)}{rg(F^\perp)}$$
$$\leq \frac{\mu(E)\,rg(F) - \mu(E)\,rg(E)}{rg(E) - rg(F)} = -\mu(E) = \mu(E^\vee)$$

so OK. Actually the way to see this is to note that E semi-stable is equivalent to $\mu(E/F) \geq \mu(E)$ for any proper quotient bundle.



and hence $\mu(F^\vee) = -\mu(E/F) \leq -\mu(E) = \mu(E^\vee)$

as desired.) So $\Omega \otimes E^\vee$ is semi-stable with

$$\deg(\Omega \otimes E^\vee) = rg(E)(2g-2) - \deg(E) < 0$$

so

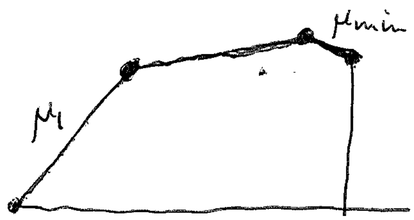
$$H^1(E) \text{ dual to } H^0(\Omega \otimes E^\vee) = 0.$$

<u>Proposition:</u> If $\mu_1(E) < 0$, then $H^0(E) = 0$. and if $\mu_{min}(E) > 2g-2$, then $H^1(E) = 0$.

Recall that $\mu_1(E) \geq \mu_2(E) > \cdots > \mu_{min}(E)$. The same proof works as before. Thus by definition for any line bundle $L \subset E$, we have

$$\deg(L) \leq \mu_1(E)$$

so $\deg(L) < 0$ if $\mu_1(E) < 0$. Similarly



for any F we have $\dfrac{\deg(E/F)}{rg(E/F)} \geq \mu_{min}(E)$

so if this is $> \deg(\Omega) = 2g-2$, then can't have $H^0(\check{E} \otimes \Omega) \neq 0$.

For a consistent notation put
$$\mu_{max}(E) = \mu_1(E).$$

so that
$$\mu_{max}(E) \geq \mu(E) \geq \mu_{min}(E)$$
with equalities for semi-stable bundles.

Recall that $H^1(E)=0$ and E is gen. by $H^0(E)$ iff $H^1(E(-P))=0$ for all points P. Thus we see that if
$$\mu_{min}(E(-P)) = \mu_{min}(E) - 1 > 2g-2$$
then $H^1(E)=0$ and E is gen by $H^0(E)$. Thus we get

<u>Proposition</u>: If $\mu_{min}(E) \geq 2g$, then $H^1(E)=0$ and E is generated by $H^0(E)$.

<u>Corollary</u>: If the ground field is finite, then there are only finitely many isomorphism classes of vector bundles with given rank, μ, and μ_{min}.

<u>Proof.</u> Tensoring with a line bundle we can suppose μ_{min} large enough so the preceding proposition applies. Thus we know $H^0(E)$ by Riemann-Roch, and so it suffices to show that Grassmannians over C have finitely many rational points of given degrees. This must be usual Hilbert scheme nonsense.

Can do directly as follows. Choose a very ample line bundle L e.g. $\mathcal{O}((2g+1)P)$.

~~Then if $H^1(E\otimes L^{-1})=0$ we have by general "regularity" considerations an exact sequence~~

$$\cancel{L^{-1}\otimes_k T_1(E) \xrightarrow{\alpha} \mathcal{O}\otimes_k T_0(E) \longrightarrow E \longrightarrow 0}$$

~~with~~

$$\cancel{T_0(E) = H^0(E)}$$

$$\cancel{T_1(E) = H^0(Hom(L^{-1}, Ker(\mathcal{O}\otimes T_0(E)\to E)))}$$

~~And if E is sufficiently positive~~

So suppose $H^1(E\otimes L^{-1})=0$. Then E is regular with respect to the embedding defined by L, so we have

$$0 \longrightarrow Z \longrightarrow \mathcal{O}\otimes H^0(E) \longrightarrow E \longrightarrow 0$$

~~where~~

$$\mathcal{O}\otimes H^0(Z\otimes L) \longrightarrow Z\otimes L \longrightarrow 0$$

exact. Moreover we know the dimensions of $H^0(E)$ and $H^0(Z\otimes L)$ from the dimensions of $H^0(E\otimes L^{\otimes k})$ which can be determined by R-R. Thus we have a presentation

$$L^{-1}\otimes_k T_1(E) \xrightarrow{\alpha} \mathcal{O}\otimes_k H^0(E) \longrightarrow E \longrightarrow 0$$

where the dimensions of $T_1(E)$ and $H^0(E)$ are known. Since there are only finitely many possibilities for α, the result is now clear.

<u>Corollary</u>: There are only finitely many stable vector bundles of given rank and degree, when the ground field is finite.

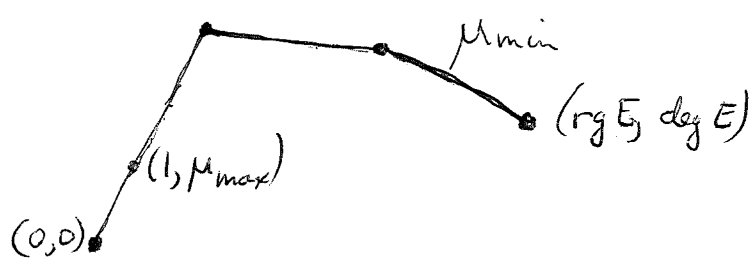
<u>Proposition</u>: Let E_ν be a sequence of vector bundles of the same rank and degree. TFAE

a) $\mu_{max}(E_\nu) \longrightarrow \infty$

b) $\mu_{min}(E_\nu) \longrightarrow -\infty$

c) Let $\delta(E_\nu)$ be the maximal degree of a sub-line bundle of E_ν. Then $\delta(E_\nu) \longrightarrow +\infty$.

<u>Proof</u>: From the picture



one sees that the polygon contains the point $(1, \mu_{max})$, hence

$$\mu_{min} \leq \frac{\deg E - \mu_{max}}{rg\, E - 1}$$

whence a) $\Rightarrow$ b). Converse similar.

c) $\Rightarrow$ a) because $\delta(E) \leqslant \mu_{max}(E)$.

a) $\Rightarrow$ c). Recall from Serre's course the

<u>Lemma</u>: If $L_1, \ldots, L_n$ are the quotients of a maximal flag in E, then

$$\deg(L_{i+1}) - \deg(L_i) \leq 2g.$$

Proof: Enough to consider the case $n=2$, $i=1$. ~~With~~ ~~[illegible]~~ Tensoring E with a line bundle doesn't change $\deg(L_2)-\deg(L_1)$, so can suppose

$$\deg(E) = 2g-1+\varepsilon \qquad \varepsilon = 0 \text{ or } 1$$

Then RR $\Rightarrow$

$$h^0(E) \geq \deg(E) + 2(1-g) = 1+\varepsilon \geq 1$$

so E has a sub-line-bundle of degree ≥ 0, so

$$\deg(L_1) \geq 0$$

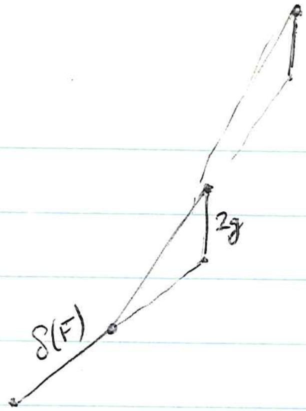
Thus

$$\begin{aligned}\deg(L_2)-\deg(L_1) &= \deg E - 2\deg(L_1) \\ &\leq \deg(E) = 2g-1+\varepsilon \leq 2g\end{aligned}$$

as claimed.

~~Thus once we give $\deg(L_1)$ in a maximal flag~~

So suppose we get $\delta(E)$. Then for any subbundle F, $\delta(F) \leq \delta(E)$, and so if F has a max flag with quotients $L_1, \ldots, L_r$, the best the degree of F can be is



best :

$$\cancel{\deg(F_i) = \deg(F_{i-1}) + \deg(L_i)}$$
$$\cancel{\leq \deg(F_{i-1}) + \deg(L_{i-1}) + 2g}$$

$$\deg(L_i) = [\deg(L_i) - \deg(L_{i-1})] + \cdots + [\deg(L_2) - \deg(L_1)] + \delta(F).$$
$$\leq (i-1)\,2g + \delta(F).$$

$$\deg(F) = \sum_{i=1}^{r} \deg(L_i) \leq 2g \sum_{i=1}^{r} (i-1) + r\,\delta(F)$$

$$\deg(F) \leq 2g\,\frac{r(r-1)}{2} + r\,\delta(F)$$

$$\boxed{\frac{\deg(F)}{r} \leq g(r-1) + \delta(F)}$$

which shows that if $\delta(E_\nu)$ remains bdd so does $\mu_{max}(E_\nu)$, whence a) $\Longrightarrow$ c).

~~Suppose C is of genus 1. The~~

Now I want to understand the limiting behavior of a sequence E_ν of the same rank and degree which go to infinity.

<u>Proposition:</u> Given an exact sequence

$$0 \to E' \to E \longrightarrow E'' \longrightarrow 0.$$

if $\qquad \mu_{min}(E') \geq \mu_{max}(E'') + 4g$ see page 19

then the sequence splits.

<u>Proof</u>: Recall $\mu_{min}(E' \otimes L) = \mu_{min}(E') + \deg L$, and whether the sequence splits or not is unchanged by tensoring with a line bundle. Since

$$\mu_{min}(E') - 2g \geq \mu_{max}(E'') + 2g$$

we can by tensoring with a suitable line bundle assume that

$$\mu_{min}(E') \geq 2g$$

$$\mu_{max}(E'') + 2g - 2 < 0$$

$\exists$ integer m
$\mu_{min}(E') - 2g \geq m \geq \mu_{max}(E'') + 2g - 1$
now $\otimes$ with L of degree $-m$

The first ~~equation~~ inequality implies that E' is generated by $H^0(E')$, hence $\exists$ $\mathcal{O}^n \twoheadrightarrow E''$ so

$$\mathrm{Ext}^1(E'', \mathcal{O}^n) \longrightarrow\!\!\!\!\to \mathrm{Ext}^1(E'', E').$$

But

$$\mathrm{Ext}^1(E'', \mathcal{O}) = H^1(E''^{\vee}) \text{ dual to } H^0(E'' \otimes \Omega)$$

and

$$\mu_{max}(E'' \otimes \Omega) = \mu_{max}(E'') + 2g - 2 < 0$$

so $H^0(E'' \otimes \Omega) = 0$ and the sequence splits as claimed.

Suppose now that E is indecomposable of rank n with slopes $\mu_1 > \mu_2 > \cdots > \mu_r$. Then

$$\mu_1 - \mu_2 \leq 4g-1$$

$$\vdots$$

$$\mu_{r-1} - \mu_r \leq 4g-1$$

$$\mu_1 - \mu_r \leq r(4g-1) \leq n(4g-1)$$

$$\mu_1 \leq n(4g-1) + \mu_r \leq n(4g-1) + \frac{\deg E}{n}$$

Thus concludes:

<u>Proposition</u>: The set of iso. classes of indecomposable vector bundles of a given rank and degree form a limited family. So $\exists$ a finite no. when the ground field is finite.

The $4g$ can be improved to $2g-2$. The point: $\operatorname{Ext}^1(E'', E') \neq 0 \iff \operatorname{Hom}(E', E'' \otimes \Omega) \neq 0 \implies$ $\exists\ f : E' \to E'' \otimes \Omega \neq 0$ so

$$\mu_{min}(E') \leq \mu(\operatorname{Coim} f) \leq \mu(\operatorname{Im} f) \leq \mu_{max}(E'' \otimes \Omega)$$

$$= \mu_{max}(E'') + 2g - 2.$$

April 18, 1973. Localization

Let A be a Dedekind domain with quotient field F, let $\mathfrak{m}$ be a maximal ideal, and let B be the Dedekind ring obtained by removing $\mathfrak{m}$ from A. Then $B = \bigcap_{\mathfrak{m}' \neq \mathfrak{m}} A_{\mathfrak{m}'} \subset F$.

Let M be a vector bundle over B, let X be the building consisting of $A_{\mathfrak{m}}$-lattices L in $F \otimes_B M$. Equivalently, X is the building of extensions of M to a vector bundle E over A. Formulas:

$$E = M \cap L \subset F \otimes_B M$$

$$L = E_{\mathfrak{m}} = A_{\mathfrak{m}} \otimes_A E$$

$$M = B \otimes_A E$$

Then $\Gamma = \mathrm{Aut}(M)$ acts on X which is contractible.

The Γ-classes of vertices of X are the same as iso. classes of E extending M. One knows that a vector bundle over a Dedekind domain is determined by its rank and first Chern class. Thus

$$\text{iso. classes of } E = \{\alpha \in \mathrm{Pic}(A) \mid \alpha \mapsto cl(M) \in \mathrm{Pic}\, B\}$$

In virtue of the exact sequence

$$0 \longrightarrow A^* \longrightarrow B^* \xrightarrow{\ \mathrm{ord}_{\mathfrak{m}}\ } \mathbb{Z} \xrightarrow{\ \cdot[\mathfrak{m}]\ } \mathrm{Pic}\, A \longrightarrow \mathrm{Pic}\, B \longrightarrow 0$$

the iso classes are the cosets in $\mathrm{Pic}\, A$ for the cyclic group generated by $[\mathfrak{m}]$. This cyclic group is finite iff $\exists\ f \in \mathfrak{m} \cap B^*$, which is the case in the standard arithmetic examples.

A q-simplex in X is the same as a lattice L_q together with a filtration

$$0 \le \bar{L}_0 < \bar{L}_1 < \cdots < \bar{L}_q$$

where $\bar{L}_q = L_q \otimes_{A_m} k$, $k = A/m$. Thus a Γ-class of ~~lattices is the~~ q-simplices is the same as an isom. class of pairs $(E, \mathcal{F})$ where E is a bundle over A extending M, and where $\mathcal{F}$ is a filtration

$$0 \le \bar{E}_1 < \cdots < \bar{E}_{q-1} < \bar{E} = A/m \otimes E$$

Thus to determine the Γ-classes we have to determine the action of $\mathrm{Aut}(E)$ upon the set of such flags. Now we know that

$$\mathrm{Aut}(\bar{E}) \simeq Gl_n(k)$$

and that $Sl_n(k)$ acts transitively on flags with same dimensions, so what we want is

<u>Lemma:</u> $\mathrm{Aut}(E) \longrightarrow \mathrm{Aut}(\bar{E})$ is onto the elementary subgroup.

Assuming this it follows that a Γ-class of q-simplices is described by an $\alpha \in \mathrm{Pic}(A)$ over $cl(M)$ together with a sequence of positive integers

$$n_1, \ldots, n_{q-1} \qquad E_1 \overset{n_{q-1}}{<} \cdots < E_{q-1} \overset{n_1}{<} \bar{E}$$

such that $\sum_i n_i \le n = \mathrm{rank}(M)$.

To prove the lemma suppose given an exact sequence

$$(1) \qquad 0 \longrightarrow k \xrightarrow{\cdot x} \bar{E} \longrightarrow W \longrightarrow 0$$

Now modifying slightly the Serre theorem, one can find a section s of E such that $s(m) = x$ and such that s is unimodular whence we get an exact sequence

$$(2) \qquad 0 \longrightarrow A \longrightarrow E \longrightarrow E' \longrightarrow 0$$

which reduces to (1) modulo m. Splitting (2) we have

$$\begin{array}{ccc} \mathrm{Aut}(2) & \xrightarrow{\text{red. mod } m} & \mathrm{Aut}(1) \\ \| & & \\ \left(\begin{array}{c|c} A^* & 0 \\ \hline E' & \mathrm{Aut}(E') \end{array}\right) & \longrightarrow & \left(\begin{array}{c|c} k^* & 0 \\ \hline W & \mathrm{Aut}(W) \end{array}\right) \end{array}$$

and since $E' \twoheadrightarrow W$, it is clear that every elementary automorphism in $\mathrm{Aut}(\bar{E})$ lifts to one in $\mathrm{Aut}(E)$. DONE.

So now have an explicit description of the Γ-orbits on the simplices of X. It is clear that the stabilizer of

$$E_0 < \cdots < E_q$$

is simply the corresponding <u>parahorique</u> subgroup of $\mathrm{Aut}(E)$.

Now this ~~set~~ situation to which we arrive I considered before when trying to compute the mod p cohomology of GL_n of a local field of res. char. p. Instead of getting a clear relation between the cohomology of $GL_n(F)$, $GL_n(k)$, $GL_n(A)$ as we do in the localization theorem, we get this confused picture with the parhorique groups.

Splitting thm.

A ring, W fixed f.t. A-module, $\mathcal{E}_W$ = the groupoid of epis $P \twoheadrightarrow W$, $P \in \mathcal{P}(A)$, with isos over W. Then have a forgetful functor

$$k : \mathcal{E}_W \longrightarrow \mathcal{E}_0 \qquad (P \twoheadrightarrow W) \mapsto P.$$

Also have an action of $\mathcal{E}_0$ on $\mathcal{E}_W$ compatible with k

$$Q \# (P \overset{u}{\twoheadrightarrow} W) = (Q \oplus P \xrightarrow{0+u} W).$$

Let I be the monoid $\pi_0 \mathcal{E}_0$; it acts on $H_*(\mathcal{E}_W)$ and $H_*(\mathcal{E}_0)$. To prove:

<u>Thm</u>: $k_* : I^{-1} H_*(\mathcal{E}_W) \xrightarrow{\sim} I^{-1} H_*(\mathcal{E}_0)$.

Put $\pi_0 \mathcal{E}_W = J$, so

$$H_*(\mathcal{E}_W) = \coprod_{j \in J} H_*(\mathrm{Aut}(E_j))$$

where E_j is a rep for $j \in J$. The action of I on $H_*(\mathcal{E}_W)$ is as follows. If i is rep by Q_i, then multip. by i (denoted λ_i) is

$$\begin{tikzcd}
H_*(\mathrm{Aut}(E_j)) \arrow[r] \arrow[dr, "\lambda_i"'] & H_*(\mathrm{Aut}(Q_i \# E_j)) \arrow[d, equal] \\
& H_*(\mathrm{Aut}(E_{i\#j}))
\end{tikzcd}$$

where last map induced by any isom of $Q_i \# E_j$ with $E_{i\#j}$. Thus to invert I what we are doing is this: Take direct limit over the translation cat

$$I^{-1} H_*(\mathcal{E}_W) = \varinjlim_{i \in \mathrm{Trans}(I)} \left\{ i \longmapsto H_*(\mathcal{E}_W),\ (i + i_0 = i') \mapsto \lambda_{i_0} \right\}$$

But we can form over $\mathrm{Trans}(I) = \langle I, I\rangle$ the cofibred cat $\langle I, I\times J\rangle$, and we have

$$I^{-1}H_*(\mathcal{E}_W) = \varinjlim_{(i,j)\in\langle I, I\times J\rangle} \{ (i,j) \longmapsto H_*(\mathrm{Aut}(E_j)) \}.$$

Let $I^{-1}J = \pi_0\langle I, I\times J\rangle$ = set of couples $(i,j) = (i_0+i, i_0\# j)$.

Then clearly

$$I^{-1}H_*(\mathcal{E}_W) = \coprod_{\alpha\in I^{-1}J} \varinjlim_{(i,j)\in\langle I, I\times J\rangle_\alpha} \{(i,j)\mapsto H_*\mathrm{Aut}\,E_j\}$$

Recall $\perp$ operations in $\mathcal{E}_W$.

<u>Lemma:</u> $j_1 + j_2 = kj_1 \# j_2$

<u>Proof:</u> j_i rep by $u_i : P_i \twoheadrightarrow W$. Then

j_1+j_2 rep. by $P_1\oplus P_2 \xrightarrow{u_1+u_2} W$

kj_1+j_2 ——— $P_1\oplus P_2 \xrightarrow{0+u_2} W$

so want

$$\begin{tikzcd}
P_1\oplus P_2 \arrow[d, "{(id+0,\ \varphi+id)}"'] \arrow[dr, "u_1+u_2"] & \\
P_1\oplus P_2 \arrow[r, "0+u_2"'] & W
\end{tikzcd}$$

$u_2\varphi = u_1$

$$\begin{tikzcd}
P_1 \arrow[d, "\varphi"'] \arrow[dr, "u_1"] & \\
P_2 \arrow[r, "u_2"'] & W
\end{tikzcd}$$

so φ exists as P_1 is proj and u_2 onto.

The problem now is this: Given $j_0 + j = j'$ there are two maps

$$H_*(\mathrm{Aut}\, E_j) \overset{\lambda_{j_0}}{\rightrightarrows} H_*(\mathrm{Aut}(E_{j'}))$$

The former is induced by $\# k_{j_0}$, the latter with $\perp j_0$.

$$\begin{tikzcd} \mathrm{Aut}(P \xrightarrow{u} E) \arrow[r] \arrow[dr] & \mathrm{Aut}(Q \oplus P \xrightarrow{0+u} \overset{W}{E}) \\ & \mathrm{Aut}(Q \oplus P \xrightarrow{u'+u} \overset{W}{E_j}) \end{tikzcd}$$

and although I see that the objects $(Q \oplus P \xrightarrow{0+u} E)$ and $(Q \oplus P \xrightarrow{u'+u} W)$ are isom., I don't see that these representatitions are conjugate, no matter how big Q is. In fact they aren't, since the former has no invariants mapping onto the latter

---

Therefore your generalization doesn't work.

April 19, 1973. ζ functions.

Take a vector bundle M over $\mathbb{Z}$ of rank n and form the formal series

$$\sum_{L\subset M} \frac{1}{(\text{card } M/L)^s}$$

where L runs over all lattices contained in M.
Problem: Compute this series.

Now the first thing to notice is that M/L can be split into its primary components, hence we get an Euler product:

$$\sum_{L\subset M} \frac{1}{\text{card}(M/L)^s} = \prod_p \sum_{\substack{M/L \\ \text{a } p\text{-group}}} \frac{1}{\text{card}(M/L)^s}$$

and that lattices such that M/L is a p-group may be identified with lattices in $M\otimes\mathbb{Q}_p$ contained within $M\otimes\mathbb{Z}_p$. Thus we are down to a local problem.

Local problem: Let A be a discrete valuation ring with quotient field F and residue field k. Assume k has q elements. Calculate the sum:

$$\sum_{L\subset A^n} \frac{1}{\text{card}(A^n/L)^s}$$

To avoid biasing things, fix a lattice M in F^n. Want to compute the no. of lattices $L \subset M$ with given card(M/L). Let π generate the maximal ideal of A. ~~Any lattice $L \subset M$ determines a~~

Try $n = 2$. Then let p be greatest such that $L \subset \pi^p M$ and r least such that

$$\pi^{p+r} M \subset L$$

The integers p, r being given, one sees that L is completely determined by giving a line in $M/\pi^r M$, that is, the line

$$\pi^{-p} L / \pi^r M \subset M/\pi^r M. \simeq (A/\pi^r A)^2$$

How many such lines? No of unimodular vectors is

$$(q^2-1)(q^2)^{r-1}$$

Number of units is

$$(q-1)\, q^{r-1} \qquad \text{if } r \geq 1$$

Thus get

$$\frac{q^2-1}{q-1}\, q^{r-1} = q^r + q^{r-1} \qquad \text{if } r \geq 1$$

and it seems I want

$$\sum_{\substack{p \geq 0 \\ r \geq 1}} \frac{q^2-1}{q-1} q^{r-1} \cdot (q^{2p} \cdot q^r)^{-s} + \sum_{p \geq 0} (q^{2p})^{-s}$$

$$= \left[(1-q^{-s})(1-q^{1-s}) \right]^{-1}$$

Better method: Let M have the basis e_1, e_2 and consider the trace of the filtration

$$0 \subset Ae_1 \subset A^2$$

on L: $\quad 0 \subset L \cap Ae_1 \subset L$. Then we get a basis for L of the form

$$\pi^j e_1 \,,\ \alpha e_1 + \pi^k e_2$$

where the class of α in $A/\pi^j A$ is unique. Thus we are interested in the sum

$$\sum_{j,k\geq 0} q^j \left(q^{j+k}\right)^{-s} = \left[(1-q^{-s})(1-q^{1-s})\right]^{-1}$$

as before.

In general if $M = Ae_1 + \cdots + Ae_n$ one filters:

$$0 \subset Ae_1 \subset Ae_1 + Ae_2 \subset \cdots \subset M$$

$$0 \subset L_1 \subset L_2 \subset \cdots \subset L$$

and finds a basis

$$\pi^{k_1} e_1$$

$$\pi^{k_2} e_2 + \alpha_{21} e_1$$

$$\pi^{k_n} e_n + \alpha_{n,n-1} e_{n-1} + \cdots + \alpha_{n1} e_1$$

where α_{ij} is determined in $A/\pi^{k_j} A$, and one can compute that the sum is $\prod_{i=0}^{n-1} (1-q^{i-s})^{-1}$. (Weil's book p. 197)

April 20, 1973. Cohomology computations

k finite field, $q = \text{card } k$, ℓ prime $\ell \nmid q$. I want to compute

$$H_*(GL_n(k), st(k^n))$$

where $st(k^n)$ is the Steinberg module mod ℓ.

If X is the building of k^n we have an exact sequence

$$0 \longrightarrow st(k^n) \longrightarrow C_{n-2}(X) \longrightarrow \cdots \longrightarrow C_0(X) \longrightarrow \mathbb{F}_\ell \to 0$$

with $C_p(X)$ = mod p chains on X. This exact sequence holds for $n=1$ if we define $st(k^1) = \mathbb{F}_\ell$. Since

$$C_{p-1}(X) = \coprod_{0 < W_1 < \cdots < W_p < V} \mathbb{F}_\ell$$

we have

$$H_*(GL_n, C_{p-1}(X)) = \coprod_{\substack{\Sigma d_i = n \\ d_i > 0}} H_*(GL_{d_1, \ldots, d_{p+1}})$$

where $GL_{d_1,\ldots,d_{p+1}}$:

$$\begin{array}{|c|c|c|}\hline * & & \\ \hline * & * & \\ \hline * & * & * \\ \hline\end{array} \qquad \begin{array}{c} d_1 \\ d_2 \\ \vdots \end{array}$$

Since $\ell \nmid q$ we have

$$H_*(GL_{d_1,\ldots,d_{p+1}}) = H_*(GL_{d_1}) \otimes \cdots \otimes H_*(GL_{d_{p+1}}).$$

Thus on applying $H_*(GL_n, ?)$ to the complex

$$K_n : 0 \longrightarrow C_{n-2}(X) \longrightarrow \cdots\cdots \longrightarrow C_0(X) \longrightarrow \mathbb{F}_\ell \longrightarrow 0$$

degrees: n $\qquad$ 1 $\qquad$ 0

we get ~~the degrees~~

$$\cdots\cdots \longrightarrow \bigoplus_{\substack{a+b=n \\ a>0}} H_*(GL_a) \otimes H_*(GL_b) \to H_*(GL_n) \to 0$$

which is the degree n part of the bar ~~resolution~~ construction

$$\cdots \longrightarrow \bar{R} \otimes \bar{R} \otimes \bar{R} \longrightarrow \bar{R} \otimes \bar{R} \longrightarrow \bar{R} \longrightarrow 0$$

where

$$R = \bigoplus_{n \geq 0} H_*(GL_n) \quad , \quad \bar{R} = \bigoplus_{n>0} H_*(GL_n)$$

Now we have seen

$$R = P[\varepsilon, \xi_1, \ldots] \otimes \Lambda[\eta_1, \ldots]$$

where ε base for $H_0(GL_1)$

ξ_j base for $H_{2jr}(GL_r)$ $\qquad j \geq 1$

η_j $\qquad H_{2jr-1}(GL_r)$ $\qquad j \geq 1$

The homology of the above bar construction is

$$\mathrm{Tor}^R(k, k) = \Lambda[\bar{\varepsilon}, \bar{\xi}_1, \ldots] \otimes \Gamma[\bar{\eta}_1, \ldots]$$

where $\bar{\varepsilon}, \bar{\xi}_1, \ldots, \bar{\eta}_1, \ldots\ldots$ is the obvious base for

$$\mathrm{Tor}_1^R(k,k) = \bar{R}/\bar{R}^2$$

If $r = 1$, any monomial

$$\bar{\xi}^{\alpha}\ \bar{\eta}^{\beta}$$

is of degree $(|\alpha| + |\beta|)r$. Thus Tor_1 occurs only in degree 1. The point is that R is a graded algebra and its generators are homogeneous of degree 1. Thus Tor_q is homogeneous of degree q, so we find that given n, the complex

$\cancel{H_*(G, K_n)}$

$$0 \longrightarrow \underset{n}{H_*(GL_n, C_{n-2})} \longrightarrow \cdots\cdots \longrightarrow \underset{1}{H_*(GL_n)} \longrightarrow \underset{0}{0}$$

has exactly one homology group which is in degree n. But we have

$$K_n \underset{\text{quis}}{\sim} st(k^n)[n]$$

so there is a spectral sequence

$$E^1_{pq} = \underset{\substack{\| \\ 0 \text{ for } p \neq n}}{H_q(GL_n, (K_n)_p)} \Longrightarrow H_{p+q-n}(GL_n, st(k^n))$$

so the spectral sequence degenerates yielding an isomorphism

$$H_*(GL_n, st(k^n)) = \mathrm{Tor}_n^R(k,k)$$

monomials of degree n in $\bar{\xi}, \bar{\eta}$.

<u>Example:</u> $n=1$, where

$$H_*(GL_1, st(k^1)) = H_*(GL_1) \quad \text{has base} \quad \begin{array}{ll} \bar{\xi}_i & 2i \\ \bar{\eta}_i & 2i-1 \end{array}$$

Then one has a product

$$H_*(GL_1, st(k^1)) \otimes H_*(GL_1, st(k^1)) \longrightarrow H_*(GL_2, st(k^2))$$

and maybe divided powers whereby one ~~generates~~ generates the latter.

April 24, 1973

Localization via Serre's methods
(Incomplete, mistake p.10)

Let A be a complete discrete valuation ring with residue field k and quotient field F, & let l be a prime no. $\neq$ char(k). I want to understand the continuous homology of $GL_n(F)$ mod l.

If V is a vector space over F, let ~~$X(F)$ be the building of its~~ $J(V)$ be the ordered set of layers (L_0, L_1) in the ordered set of lattices in V such that $mL_1 < L_0$. Let $\mathrm{Aut}(V)$ act on $J(V)$ and form the associated cofibred category over $\mathrm{Aut}(V)$. We may identify this with the category of pairs (L_0, L_1) of ~~lattices~~ free A-modules such that L_1/L_0 is a k-mod with maps ~~~~

$$(L_0, L_1) \xrightarrow{\phi} (L_0', L_1')$$

defined to be an ~~~~ embedding $L_1 \overset{\phi}{\hookrightarrow} L' \ni$

$$L_0' \subset \phi L_0 \subset \phi L_1 \subset L' .$$

Category to be denoted $(J(V), \mathrm{Aut}(V))$. Now there is an evident functor

$$\begin{aligned} (J(V), \mathrm{Aut}(V)) &\longrightarrow Q_n \subset Q(k\text{-mods}) \\ (L_0, L_1) &\longmapsto L_1/L_0 \end{aligned}$$

where $n = \mathrm{rank}(V)$. This functor is fibred, the fibre over a k-module W being the groupoid of surjections

$$L \twoheadrightarrow W$$

where L is a free A-module of rank n, and their isomorphisms. Thus we get a spectral sequence

$$E^2_{pq} = H_p\left(Q_n,\ W^d \longmapsto H_q\left(\begin{array}{cc} \overset{d}{\longleftrightarrow} & \overset{n-d}{\longleftrightarrow} \\ \multicolumn{2}{c}{\left(\begin{array}{c|c} \equiv 1 & \equiv 0 \\ \hline * & * \end{array}\right)} \end{array}\right)\right) \Longrightarrow H_{p+q}(GL_n(F))$$

where $\left(\begin{array}{c|c} \equiv 1 & \equiv 0 \\ \hline * & * \end{array}\right) \subset GL_n(A)$ is the subgroup of autos of A^n which induce the identity on k^n/k^{n-d}. Actually it should [redacted] be the group

$$\cancel{\left(\begin{array}{c|c} * & * \\ \hline \equiv 0 & \equiv 1 \end{array}\right)} \quad \begin{array}{cc} \multicolumn{2}{c}{\left(\begin{array}{c|c} * & \equiv 0 \\ \hline * & \equiv 1 \end{array}\right)} \\ \underset{n-d}{\longleftrightarrow} & \underset{d}{\longleftrightarrow} \end{array} \subset GL_n(A).$$

So now the question to ask is whether it might be the case that as $n \longrightarrow \infty$ the group

$$\begin{array}{cc} \multicolumn{2}{c}{\left(\begin{array}{c|c} \equiv 1 & * \\ \hline \equiv 0 & * \end{array}\right)} \\ \underset{d}{\longleftrightarrow} & \underset{n}{\longleftrightarrow} \end{array}$$

has the same homology as $GL_n(A)$.

Fix a k-module W and let $\mathcal{L}_W$ denote the groupoid consisting of surjections

$$E \xrightarrow{p} W$$

where E is a free A-module (f.g.). Let $\mathcal{L}$ be the groupoid of free A-modules. $\mathcal{L}_W$ has operation

$$(E \twoheadrightarrow W) + (E' \twoheadrightarrow W) = (E \times_W E' \twoheadrightarrow W)$$

(Also one has $E \oplus E' \longrightarrow W$). The kernel functor

$$k : \mathcal{L}_W \longrightarrow \mathcal{L}$$

$$E \overset{p}{\twoheadrightarrow} W \longmapsto \operatorname{Ker}(p)$$

is compatible with the operations. In addition $\mathcal{L}$ acts on $\mathcal{L}_W$ by

$$L * (E \overset{p}{\twoheadrightarrow} W) = (L \oplus E \xrightarrow{p\,pr_2} W)$$

and

$$k(L * E) = L \oplus kE.$$

We have the basic identity

$$(E) + (E) = (E \times_W E \longrightarrow W)$$

$$= kE * E$$

Hence if we fix E_0, we have

$$
\begin{aligned}
E \perp E \perp E_0 &\cong (kE * E) \perp E_0 \\
&\cong kE * (E \perp E_0) \\
&\cong kE * (E_0 \perp E) \\
&\cong (kE * E_0) \perp E
\end{aligned}
$$

So if θ is an exponential char. class for representations over $\mathcal{L}_W$ we have

$$\theta(E)\,\theta(E)\,\theta(E_0) = \theta(kE * E_0)\,\theta(E)$$

and so if $\theta(E)$ is invertible, then

$$\theta(E) = \theta(kE * E_0)\,\theta(E_0)^{-1}$$

So it is now clear that k induces a map

$$H_*(\mathcal{L}_W) \longrightarrow H_*(\mathcal{L})$$

which becomes an isomorphism after localization.

~~But the connected components are easily seen to be~~

$$\cancel{\begin{pmatrix} \cong 1 & * \\ \cong 0 & GL_\infty(A) \end{pmatrix} \longrightarrow GL_\infty(A).}$$

$$L \longmapsto L * E_0$$

$$H_*(\mathcal{L})[\pi_0\mathcal{L}^{-1}] \longrightarrow H_*(\mathcal{L}_W)[(\pi_0\mathcal{L})^{-1}] \xrightarrow{\sim} H_*(\mathcal{L})[\pi_0\mathcal{L}^{-1}] \quad ?$$

$$L * E_0 \longmapsto L \oplus kE_0$$

(over)

Thus we can conclude that the inclusion

$$GL_n(A) \hookrightarrow \left[\begin{array}{c|c} \equiv 1 & * \\ \hline \equiv 0 & GL_n(A) \end{array}\right]$$

induces an isomorphism in the limit as $n \to \infty$.

---

alternative proof. Define operation on $\mathcal{L}_W$

$$(E \overset{p}{\twoheadrightarrow} W) \oplus (E' \overset{p'}{\twoheadrightarrow} W) = E \oplus E' \xrightarrow{p+p'} W$$

and the functor

$$t : \mathcal{L}_W \longrightarrow \mathcal{L}$$
$$(E \twoheadrightarrow W) \longmapsto E$$

compatible with operation and with the action

$$L * (E \overset{p}{\twoheadrightarrow} W) = L \oplus E \xrightarrow{p\, pr_2} W.$$

Have

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r, "{(id,-id)}"] \arrow[dr, "0"'] & E \oplus E \arrow[r, "+"] \arrow[d, "p+p"] & E \arrow[r] \arrow[dl, "p"] \arrow[l, bend right, dashed, "{(id,0)}"'] & 0 \\ & & W & & \end{tikzcd}$$

giving a canonical isom. in $\mathcal{L}_W$

$$\boxed{E \oplus E \cong tE * E}$$

It follows that we have for any invertible exponential char. class θ that

$$E \oplus E \oplus E_0 \cong (tE * E) \oplus E_0$$
$$\cong E \oplus (tE * E_0)$$

so

$$\boxed{\theta(E) = \theta(tE * E_0)\,\theta(E_0)^{-1}}$$

To finish one notes that

$$\theta \longmapsto \left(L \mapsto \theta(L * E_0)\,\theta(E_0)^{-1}\right)$$

maps inv. exp. classes for $\mathcal{L}_W$ to those for $\mathcal{L}$ and that

$$\varphi \longmapsto \left(E \longmapsto \varphi(tE)\right)$$

goes the other way, and clearly these are inverses of each other. (Check:

$$\cancel{(L_1 \oplus L_2) * E_0 = (L_1 * E_0) \oplus (L}$$

$$\theta(L_1 * E_0)\,\theta(E_0)^{-1} \cdot \theta(L_2 * E_0)\,\theta(E_0)^{-1}$$
$$= \theta\big((L_1 * E_0) \oplus (L_2 * E_0)\big)\;\theta(E_0 \oplus E_0)^{-1}$$
$$= \theta\big((L_1 \oplus L_2) * E_0 \oplus E_0\big)\;\theta(E_0)^{-2} = \theta\big((L_1 \oplus L_2) * E_0\big)\,\theta(E_0)^{-1}$$

OKAY.)

It would seem that we also have a new proof of the splitting theorem for exact sequences.

Now given a k-module W we consider the groupoid of surjections

$$E \longrightarrow W$$

and all automorphisms including autos. of W. We can operate:

$$L * (E \xrightarrow{p} W) = (L \oplus E \xrightarrow{p\,pr_1} W)$$

as before, and hence stabilize, getting the group

$$\Gamma_{d,\infty} = \begin{array}{c} \left(\begin{array}{c|c} * & * \\ \hline \equiv 0 & * \end{array}\right) \\ \leftarrow d \rightarrow \; \leftarrow \infty \rightarrow \end{array}$$

Now I have a homomorphism over $GL_d(k)$

$$\begin{tikzcd} \Gamma_{d,\infty} \arrow[r] \arrow[dr] & GL_{d+\infty}(A) \times GL_d(k) \arrow[d] \\ & GL_d(k) \end{tikzcd}$$

, hence to show the horizontal arrow induces a heq, it suffices to show the inclusion

$$\begin{array}{c} \left(\begin{array}{c|c} \equiv 1 & * \\ \hline \equiv 0 & * \end{array}\right) \\ d \qquad \infty \end{array} \subset GL_{d+\infty}(A)$$

is a homology isomorphism, which I have proved above.

Thus it is clear that we can define the transfer in this situation, namely, we lift the representation in $GL_d(k)$ to $\Gamma_{d,\infty}$ so as to be trivial in $GL_{d+\infty}(A)$, and then look at the map to the kernel.

So in the limit we get a spectral sequence

$$E^2_{pq} = H_p(Q(k), H_q(GL(A))) \Longrightarrow H_{p+q}(GL(F))$$

which is what one expects. Can you make a proof out of this construction for the localization theorem?

April 28, 1973

Problem: Given a commutative ring A, I know how to decompose $K_i A \otimes \mathbb{Q}$ $i \geq 0$ into eigenspaces for the Adams operations. The problem is to explicitly construct a space representing the K-theory of A of a given weight.

Let $k = \overline{\mathbb{F}}_p$ and let l be a prime number $\neq p$. Then Ψ^p = base extension by Frobenius in characteristic p. We can consider the effect of Frobenius on the different spaces we have been lead to consider in the K-theory of k. In this situation we have that $B(k) \simeq BGL(k)^+$ is the $\mathbb{Q}/\mathbb{Z}$-version of $BU[\frac{1}{p}]$

$$B(k) \longrightarrow BU[\tfrac{1}{p}] \longrightarrow BU_{\mathbb{Q}} . \qquad\qquad \mathbb{Z}' = \mathbb{Z}[\tfrac{1}{p}]$$

So we expect that $gr_i B(k)$ (weight i), should be an Eilenberg-Maclane space of type $(\mathbb{Q}/\mathbb{Z}', 2i-1)$. Thus its mod l homology should be fairly complicated, and not so easy to recognize.

~~Let $k = \overline{\mathbb{F}}_p$~~

Classical approach: form connected K-theory with periodicity operator β, then take the relative term of multiplying by β.

April 30, 1973. Becker-Gottlieb proof of Adams conj.

Suppose $f: E \longrightarrow B$ is a proper ~~map~~ submersion of smooth manifolds. The key point is to define a transfer map

$$(1) \qquad h^0(E) \longrightarrow h^0(B)$$

for any GCT h. In the case where f is orientable for h, this transfer coincides with the map

$$x \longmapsto f_*(e(\tau_f)\cdot x) \qquad x \in h^0(E).$$

Definition of (1). Choose an embedding

$$\begin{tikzcd} E \arrow[rr, hook, "i"] \arrow[dr, "f"'] & & B \times \mathbb{R}^N \arrow[dl, "p = pr_1"] \\ & B & \end{tikzcd}$$

and form

$$\begin{tikzcd} & E \times \mathbb{R}^N \arrow[dr, "f \times id"] & \\ E \arrow[ur, hook, "j"] \arrow[rr, hook, "i"] \arrow[dr, "f"'] & & B \times \mathbb{R}^N \arrow[dl, "p = pr_1"] \\ & B & \end{tikzcd}$$

and choose tubular nbds for j, i so that we get a diagram

$$\begin{tikzcd}
 & E\times\mathbb{R}^N \arrow[dr, "\substack{\text{vector}\\ \text{bundle}\\ \text{surjection}}"] & & \\
E \arrow[ur, hook, bend left, "0\text{-section}"] \arrow[rr, hook] \arrow[dr, "f"'] & & \nu \arrow[dl, "\text{framed}"] & \subset B\times\mathbb{R}^N \\
 & B & &
\end{tikzcd}$$

Choose a splitting of the vector bundle surjection whence we have

$$\begin{tikzcd}
E \arrow[r, hook, "0\text{-section}"] & E\times\mathbb{R}^N & \nu \arrow[l, hook', "\text{subbundle}", "\text{injection}"'] \arrow[r, "\text{framed}"] & B
\end{tikzcd}$$

and thus

$$\begin{tikzcd}
h^0(E) \arrow[r, "\text{susp}", "\sim"'] & h^N_{p/B}(E\times\mathbb{R}^N) \arrow[r, "\text{res.}"] & h^N_{p/B}(\nu) \arrow[r] & h^N_{p/N}(B\times\mathbb{R}^N) \arrow[d, phantom, "\cong"] \\
 & & & h^0(B)
\end{tikzcd}$$

When the map f is orientable for h^0, then we have

$$\begin{tikzcd}
h^0(E) \arrow[r, "\sim"] \arrow[dr, bend right, "\cdot e(\tau)"'] & h^d_{p/B}(\tau) \arrow[r, "\sim"] \arrow[d] & h^N_{p/B}(\tau\oplus\nu) \arrow[d] & \\
 & h^d(E) \arrow[r, "\sim"] \arrow[rr, bend right, "f_*"'] & h^N_{p/B}(\nu) \arrow[r] & h^0(B)
\end{tikzcd}$$

$\tau\oplus\nu = E\times\mathbb{R}^N$

Another version: Suppose to simplify that $f : E \longrightarrow B$ is a differentiable fibre bundle with compact fibres. Let s be a generic section of the tangent bundle along the fibres and Z its zero submanifold. Then have

$$\begin{tikzcd} Y \arrow[rr, hook, "i"] \arrow[dr, "g"'] & & E \arrow[dl, "f"] \\ & B & \end{tikzcd} \qquad \nu_i = \tau_f$$

and so g is canonically framed. Hence we get

$$tr : \quad h^q(E) \xrightarrow{i^*} h^q(Y) \xrightarrow{g_*} h^q(B).$$

and we have the formula

$$g_* i^*(f^* b) = g_* 1 \cdot b \qquad \forall b \in h^*(B)$$

where $g_* 1 \in h^0(B)$ is a class which augments to $\chi(F)$, F the fibre of f.

Now for the Adams conjecture one considers a ~~vector bundle~~ principal G-bundle (G compact Lie group) $P \longrightarrow \cancel{B}$, and forms the associated bundle

$$P/N \longrightarrow B$$

where N is the normalizer of a maximal torus T in G. One knows (classically) $\chi(G/T) = $ order of W

hence $\chi(G/N) = 1$, and so applying the preceding ~~principle~~ transfer theory we find that

$$h^0(B) \hookrightarrow h^0(P/N) \qquad \text{image is a direct summand}$$

for any GCT. Now since spherical fibrations lead to a GCT ~~then~~ by Boardman–Vogt, this reduces the Adams conjecture to the case of a bundle with axes, where it can be done by Adams' methods.

<u>Strong splitting principle</u>: Given a complex vector bundle E over X, there exists a space $f: Y \to X$ such that in the S-category X is a direct factor of Y, and such that $f^*(E)$ has axes.

General case: Suppose we have $f: E \to B$ proper and we choose an embedding

$$\begin{tikzcd} E \arrow[r, hook] \arrow[dr, "f"'] & B \times \mathbb{R}^N \arrow[d] \\ & B \end{tikzcd}$$

Then we have defined $f_!: h^0(E) \to h^0(B)$ which is $h^0(B)$-linear, hence

$$f_! f^*(b) = f_! 1 \cdot b \qquad f_! 1 \in h^0(B).$$

and it would seem from the definition that $f_!$ would be compatible with transversal basechange, which implies that $f_! 1$ augments to $\chi(f^{-1}(b))$ for every regular point $b \in B$. This implies that the Euler classes of the different fibres of f are the same, which one knows isn't the case.

Conclude this construction makes sense only for fibre bundles and not for a general proper map between manifolds. What is missing is that we need to take a generic section ~~along the fibres~~ of the tangent bundle along the fibres. For a general map this bundle is only a virtual bundle, so it doesn't have an Euler class (except mod 2).

April 26, 1973 K-theory for $\mathbb{Z} \cup \infty$

Recall that we decided long ago while looking at the Γ function that a vector bundle E over $\tilde{\mathbb{Z}} = \mathbb{Z} \cup \infty$ should be a vector bundle M over $\mathbb{Z}$ together with a positive definite quadratic form q on $M_{\mathbb{R}}$. One sets

$$\theta_E = \sum_{x \in M} e^{-\pi q(x)}$$

to measure the "number" of sections of E.

Poisson summation formula:

$$\sum_{m \in M} f(x+m) = \sum_{\lambda \in M'} a_\lambda e^{2\pi i \langle x, \lambda \rangle}$$

where

$$a_\lambda = \frac{1}{\mathrm{vol}(V/M)} \int_{V/M} \sum_{m \in M} f(x+m) e^{-2\pi i \langle x, \lambda \rangle} dx$$

$$= \frac{1}{\mathrm{vol}(V/M)} \underbrace{\int_V f(x) e^{-2\pi i \langle x, \lambda \rangle} dx}_{\hat{f}(\lambda)}$$

so

$$\sum_{m \in M} f(x+m) = \frac{1}{\mathrm{vol}(V/M)} \sum_{\lambda \in M'} \hat{f}(\lambda) e^{2\pi i \langle x, \lambda \rangle}$$

Now taking $f(x) = e^{-\pi q(x)}$ and $dx = dx_1 \cdots dx_n$ where $q(x) = \sum x_i^2$

we know that $\hat{f}(\lambda) = e^{-\pi g(\lambda^*)}$ where if $g(x) = b(x,x)$ then $b(x,\lambda^*) = \langle x, \lambda \rangle$

So we get

$$\sum_{m \in M} e^{-\pi g(m)} = \frac{1}{\mathrm{vol}_g(V/M)} \sum_{\lambda \in M'} e^{-\pi g(\lambda^*)}$$

which is the analogue of the ~~[illegible]~~ Riemann-Roch formula:

$$\theta_E / \theta_{E^\vee} = d_E$$

$$E = (M, g) \qquad E^\vee = (M', g^*)$$

$$d_E = \frac{1}{\mathrm{vol}(V/M)} \quad (\nearrow \infty \quad \text{as} \quad \underset{\text{box at } \infty \text{ gets larger}}{\underset{\Updownarrow}{g \downarrow 0}}).$$

Exact sequence of vector bundles over $\tilde{\mathbb{Z}}$:

An exact sequence of vector bundles over $\tilde{\mathbb{Z}}$

$$(1) \qquad 0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

is by definition an exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of vector bundles over $\mathbb{Z}$ together with an exact sequence of quadratic spaces

$$(*) \qquad 0 \longrightarrow M'_{\mathbb{R}} \longrightarrow M_{\mathbb{R}} \longrightarrow M''_{\mathbb{R}} \longrightarrow 0$$

which means that q on $M_{\mathbb{R}}$ induces q' and q'' in the evident way. Motivation for the definition is as follows. We know that

$$GL_n(\mathbb{Z}) \backslash GL_n(\mathbb{R}) / O_n$$

is the set of isomorphism classes of rank n vector bundles. Thus

$$GL_{a,b}(\mathbb{Z}) \backslash GL_{a,b}(\mathbb{R}) / O_a \times O_b$$

should be the set of iso. classes of exact sequences with ranks a, b. Notice that

$$GL_{a,b}(\mathbb{R}) / O_a \times O_b \simeq GL_{a+b}(\mathbb{R}) / O_{a+b}$$

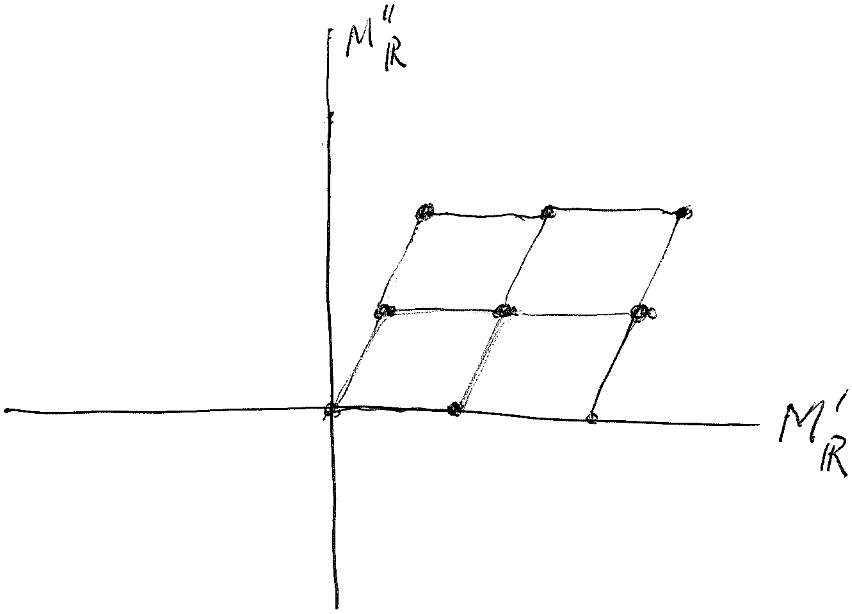
(think of triangular matrices with positive diagonal entries) hence an exact sequence $(*)$ really amounts to giving $M_{\mathbb{R}}$ as the orthogonal direct sum of $M'_{\mathbb{R}}$ and $M''_{\mathbb{R}}$.

Now suppose with are given an exact sequence[1] of $\tilde{\mathbb{Z}}$-bundles of rank n and choose an isomorphism

$$M_{\mathbb{R}} = \mathbb{R}^n \qquad q = \sum_{i=1}^{n} x_i^2$$
$$\cup$$
$$M'_{\mathbb{R}} = \mathbb{R}^a \qquad q' = \sum_{i=1}^{a} x_i^2$$

Then I would like to compare θ_E with $\theta_{E'}, \theta_{E''}$ and hopefully prove

$$\theta_E \leq \theta_{E'} \theta_{E''} .$$



Any $m \in M$ determines $pm \in M''$. $\quad p: M \to M''$.

$$\theta_E = \sum_{m\in M} e^{-\pi q(m)} = \sum_{m''\in M''} \sum_{m\in p^{-1}\{m''\}} e^{-\pi q(m)}$$

Given m'' fix $s(m'') \in M \ni ps(m'') = m''$. Then

$$\theta_E = \sum_{m''\in M''} \sum_{m'\in M'} e^{-\pi q(s(m'')+m')}$$

$$= \sum_{m''\in M''} e^{-\pi q(m'')} \sum_{m'\in M'} e^{-\pi q(s(m'')-m''+m')}$$

Since $s(m'')-m'' \in M'_{\mathbb{R}}$ what you want to know therefore is that $\forall z \in M'_{\mathbb{R}}$

$$\sum_{m'\in M'} e^{-\pi q(z+m')} \leq \sum_{m'\in M'} e^{-\pi q(m')}$$

(with equality iff $z \in M'$ maybe)

What happens on the line: $\quad M = \mathbb{Z}$.

$$f(x) = \sum_{m \in \mathbb{Z}} e^{-\pi \alpha (x+m)^2} \qquad \alpha > 0$$

$$= \sum_{n} a_n e^{2\pi i \langle x, n \rangle}$$

$$a_n = \int_0^1 \sum_m e^{-\pi\alpha(x+m)^2} e^{-2\pi i n x}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-\pi \alpha x^2} e^{-2\pi i n x}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-\pi\left[\left(\sqrt{\alpha}\,x + \frac{in}{\sqrt{\alpha}}\right)^2\right] - \pi \frac{n^2}{\alpha}}\, dx$$

$$= \frac{e^{-\pi \frac{n^2}{\alpha}}}{\sqrt{\alpha}} \int_{-\infty}^{\infty} e^{-\pi (\sqrt{\alpha})^2 x^2} \sqrt{\alpha}\, dx = \frac{e^{-\pi \frac{n^2}{\alpha}}}{\sqrt{\alpha}}$$

$$\sum_{m \in \mathbb{Z}} e^{-\pi\alpha(x+m)^2} = \sum_{n \in \mathbb{Z}} \frac{e^{-\frac{\pi n^2}{\alpha}}}{\sqrt{\alpha}} \cos(2\pi n x)$$

From this we see that

$$f''(0) = -\sum \frac{e^{-\pi n^2/\alpha}}{\sqrt{\alpha}} (2\pi n)^2 < 0$$

and so f has a local maximum at $x = 0$, which lends support to our contention.

Curiosity: Differentiate the Fourier expansion

$$\sum_{m\in\mathbb{Z}}' e^{-\pi(x+m)^2} = \sum_{\lambda\in\mathbb{Z}}' e^{-\pi\lambda^2} e^{2\pi i\lambda x}$$

$$\sum' e^{-\pi(x+m)^2}(-2\pi)(x+m) = \sum e^{-\pi\lambda^2} 2\pi i\lambda e^{2\pi i\lambda x}$$

$$\sum' e^{-\pi(x+m)^2}\left[4\pi^2(x+m)^2 - 2\pi\right] = \sum e^{\pi\lambda^2}(2\pi i\lambda)^2 e^{2\pi i\lambda x}$$

let $x=0$

$$\sum_{m\in\mathbb{Z}}' e^{-\pi m^2}\left(4\pi^2 m^2 - 2\pi\right) = \sum_{\lambda\in\mathbb{Z}} e^{\pi\lambda^2}\left(-4\pi^2\lambda^2\right)$$

so

$$\boxed{\sum_{m\in\mathbb{Z}}' 4\pi^2 m^2 e^{-\pi m^2} = \pi \sum_{m\in\mathbb{Z}}' e^{-\pi m^2}}$$

Probably it is possible by this method to determine

$$\sum' P(m)\, e^{-\pi m^2}$$

for any polynomial P.

<u>Lemma:</u> $$\sum_{m\in M}' e^{-\pi q(x+m)} \le \sum_{m\in M}' e^{-\pi q(m)} \qquad \forall x\in M_{\mathbb{R}}$$

with equality iff $x\in M$.

<u>Proof:</u> We have the Fourier expansion

$$\sum_{m\in M} e^{-\pi q(x+m)} = \frac{1}{vol_q(M_{\mathbb{R}}/M)} \sum_{\lambda\in M'}' e^{-\pi q(\lambda^*)} e^{2\pi i\langle x,\lambda\rangle}$$

Now take real parts

$$\sum_{m \in M} e^{-\pi q(x+m)} = \frac{1}{\mathrm{vol}_q(M_{\mathbb{R}}/M)} \sum_{\lambda \in M'} e^{-\pi q(\lambda^*)} \cos(2\pi \langle x,\lambda \rangle)$$

~~But~~ Now use the fact that $\cos(2\pi\langle x,\lambda\rangle) \leq 1$ with equality for all $\lambda \iff \langle x,\lambda\rangle \in \mathbb{Z}$ all $\lambda \iff x \in M'' = M$.

So returning to page 4 we find that

$$\theta_E \leq \sum_{m'' \in M''} e^{-\pi q''(m'')} \sum_{m' \in M'} e^{-\pi q^*(s(m'')-m''+m')}$$

$$\leq \sum_{m'' \in M''} e^{-\pi q''(m'')} \sum_{m' \in M'} e^{-\pi q'(m')} = \theta_{E''}\theta_{E'}$$

with equality iff $s(m'')-m'' \in M'$, i.e. we could take $s(m'') = m''$ which means that we can find for each $m'' \in M''$ a rep. $s(m'') \in M$ with $q(s(m'')) = q''(m'')$. Thus the sequence actually splits as an orthogonal direct sum.

Thus have proved

<u>Prop</u>: For any exact sequence (1) of $\tilde{\mathbb{Z}}$-bundles we have $\theta_E \leq \theta_{E'}\theta_{E''}$ with equality iff the sequence splits, i.e. $M \cap (M'_{\mathbb{R}})^{\perp} \xrightarrow{\sim} M''$.

<u>Remark</u>: The above proposition is somewhat surprising from the finite field viewpoint, where

$$\theta_E = q^{h^0(E)}$$

and it is quite easy to have $\theta_E = \theta_{E'}\theta_{E''}$ without the sequence splitting. ∎

The preceding proposition ought to be true for a number field.

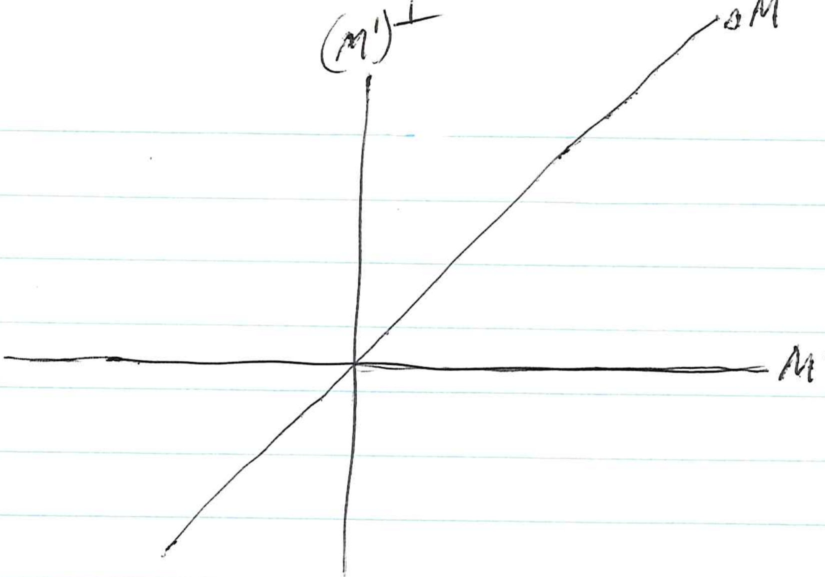
Consider now what happens when we remove a prime p from $\tilde{\mathbb{Z}}$. Bundles over $\tilde{\mathbb{Z}} - \{p\}$ should be pairs consisting of a $\mathbb{Z}[\frac{1}{p}]$-module M (free f.t.) + a q on $M_{\mathbb{R}}$. The notion of exact sequence should be the same as before.

<u>Questions</u>: To what extent do exact sequences of bundles over $\tilde{\mathbb{Z}} - \{p\}$ split, and to what extent is a vector bundle determined by its rank and first Chern class?

Given

$$0 \longrightarrow M' \longrightarrow M \xrightarrow{p} M'' \longrightarrow 0$$

vector bundles over $\mathbb{Z}[\frac{1}{p}]$ and q on M, we ~~get~~ can choose a splitting $s: M'' \longrightarrow M$, $ps = id$. Then we have the picture:



and $(M')^{\perp}$ can be interpreted as the graph of a map from $M'' \longrightarrow M'_{\mathbb{R}}$. Because we are over $\mathbb{Z}[\frac{1}{p}]$ which is dense in $\mathbb{R}$, this map can be approximated by a map $M'' \longrightarrow M'$ as close as one wants. Thus we can approximate the given exact sequence by split exact sequences, but not every sequence splits.

Similarly to any vector bundle E we can associate

$$\Lambda^n E = (\underset{\substack{\wr\| \\ \mathbb{Z}[\frac{1}{p}]}}{\Lambda^n M}, \Lambda^n q)$$

thus getting a line bundle whose isomorphism class is an element of

$$\mathrm{Pic} = \mathbb{R}^{+}/\{p^n \mid n \in \mathbb{Z}\}.$$

Now choosing in M a vector of length close to 1, (i.e. a line which is close to being a trivial line bundle), then continuing the process to get a flag, we see that E is approximately an orthogonal direct sum of trivial line bundles + $\Lambda^n E$. Put another way

$GL_n(\mathbb{Z}[\frac{1}{p}])$ acts densely on the fibres of the map

$$GL_n(\mathbb{R})/O(\mathbb{R}) \xrightarrow{\sqrt{disc}} \mathbb{R}^+ \longrightarrow \mathbb{R}^+/\{p^n\},$$

which is as close as we can get to having that a vector bundle up to isomorphism is determined by its rank and first Chern class.

In same sense then we get a family of "virtual" subgroups of $GL_n(\mathbb{Z}[\frac{1}{p}])$ in Mackey's sense, since it is probably true that $GL_n(\mathbb{Z}[\frac{1}{p}])$ acts ergodically on the fibres. The meaning of all this, especially the relation with $L_2(G/\Gamma)$ deserves elaboration.

Real problem: If $M = \mathbb{Z}[\frac{1}{p}]^n$, I know that $\Gamma = \mathrm{Aut}(M)$ acts "pseudo-transitively" on the set of possible extensions of M to a vector bundle on $\tilde{\mathbb{Z}} - \{p\}$ with prescribed first Chern class. Can you find what might be thought of as the cohomology of the stabilizer of this "transitive" action.

<u>Problems</u>: If I believe that the correct gadget is a $\mathbb{Z}$-bundle M with pos. def. form q, then I want a localization situation:

NO, wrong relative term

$$\begin{pmatrix}\text{pos. def.}\\ \text{real quad}\\ \text{forms}\end{pmatrix} \longrightarrow (\tilde{\mathbb{Z}}\text{-bundles}) \longrightarrow (\mathbb{Z}\text{-bundles})$$

What I lack at the moment is a way of going from a $\mathbb{Z}$-bundle M to Q(pos. def. real quad forms) $= Q(\infty)$. Thus we can consider the symmetric space X of all forms q on $M_{\mathbb{R}}$. The problem is to modify this so as to get a map to $Q(\infty)$.

Actually it may be unreasonable to expect there to be a $Q(\infty)$. Thus we have a cartesian situation

$$\begin{tikzcd}
\tilde{\mathbb{Z}} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\
\mathcal{O}_{disc} \arrow[r] & \mathbb{R}_{disc}
\end{tikzcd}$$

and there is no obvious reason why the bottom horizontal arrow is a localization, and hence has an identifiable relative term.

<u>Question</u>: Given q_1 and q_2 on a real vector space one can simultaneously diagonal them. Is the simplicial complex whose simplices are

chains $g_0 \leq \cdots \leq g_n$ of simultaneously diagonalizable forms a contractible complex?
