March 1973

Useful ideas not incorporated in "Higher alg K-theory I"

<u>relatively filtering</u> $f: \mathcal{C} \to \mathcal{C}'$ meaning

f/Y filtering for each Y.

It implies $(f_!, f^*)$ is a map of topoi from

$$\mathrm{Funct}(\mathcal{C}', \mathrm{Sets}) \longrightarrow \mathrm{Funct}(\mathcal{C}, \mathrm{Sets})$$

This is the same as a functor $f: \mathcal{C} \to \mathcal{C}'$ having an ind-representable left adjoint.

Closely related is what happens in the localization th.

$$Q(\mathcal{A}) \xrightarrow{f} Q(\mathcal{A}/\mathcal{B})$$

given $u: V \to V'$ in $Q(\mathcal{A}/\mathcal{B})$, $\exists$ a base change functor $u^*: \mathcal{F}_{V'} \longrightarrow \mathrm{Ind}(\mathcal{F}_V)$ so f is very close to being a fibred functor. ($\mathcal{F}_V$ consists of pairs $(M, u: fM \simeq V)$.)

---

original proof of resolution thm based on the contractibility of cat. of $P \twoheadrightarrow M$ with arrows

$$\begin{tikzcd} P \arrow[r, tail] \arrow[dr, two heads] & P' \arrow[d, two heads] \\ & M \end{tikzcd}$$

March 8, 1973

Suppose X is a regular scheme whose local rings satisfy Gersten's conjecture. Better to **start** we might as well consider K_1 where there is no problem:

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \coprod_{x\in X_0} i_{x*} k(x)^* \longrightarrow \coprod_{x \in X_1} i_{x*}\mathbb{Z} \longrightarrow 0$$

I want analogue for the etale topology. To simplify, suppose X is a curve, with generic point η, and let

$$i : \operatorname{Spec} k(\eta) \longrightarrow X$$

be the canonical map. To understand the map

$$\mathbb{G}_{m,X} \longrightarrow i_* \mathbb{G}_{m,\eta}.$$

If $U \longrightarrow X$ is etale, then

$$\Gamma(U, i_*\mathbb{G}_{m,\eta}) = \Gamma(k(\eta)\times_X U, \mathbb{G}_m)$$
$$= \coprod_{x\in U_0} k(x)^*$$

But we have ~~that~~ locally for the Zariski top on U that

$$0 \longrightarrow \Gamma(U,\mathcal{O}^*) \longrightarrow \coprod_{x\in U_0} k(x)^* \longrightarrow \coprod_{x\in U_1} \mathbb{Z} \longrightarrow 0$$

is exact. It might be true that $U \mapsto \coprod_{x\in U_1}\mathbb{Z}$ is an etale sheaf.

Take $\coprod_{x \in X_1} (i_x)_*(\mathbb{Z})$. Then

$$\Gamma\Big(U, \coprod_{x\in X_1} (i_x)_*(\mathbb{Z})\Big) = \coprod_{x \in X_1} \Gamma(k(x) \times_X U, \mathbb{Z})$$

$$= \coprod_{x \in X_1} \coprod_{\substack{u \in U_1 \\ \text{over } x}} \mathbb{Z} = \coprod_{u \in U_1} \mathbb{Z}.$$

So it seems to be OK. Thus it seems to be the case that we have an exact sequence of sheaves for the etale top

$$0 \longrightarrow \mathbb{G}_{m,X} = K_1(\mathcal{O}_X^{et}) \longrightarrow \coprod_{x \in X_0} (i_x)_*(\mathbb{G}_{m,k(x)}) \longrightarrow \coprod_{x \in X_1} (i_x)_* \mathbb{Z} \to 0$$

This should be OK <u>for any</u> regular scheme.

<u>Problem</u>: Suppose X regular + essentially of finite type over a field. Then we have a sequence of presheaves

$$(*) \quad 0 \longrightarrow K_n(U) \longrightarrow \coprod_{y \in U_0} K_n k(y) \longrightarrow \coprod_{y \in U_1} K_{n-1} k(y) \longrightarrow \cdots$$

for the etale top. This sequence is Zariski exact, hence the associated etale sheaves will be exact. The <u>conjectures</u> are as follows.

1.) If $(*)$ is tensored with $\mathbb{Q}$ it becomes an exact sequence of etale sheaves.

2) For any scheme X let $\mathcal{K}^{et}_{n,X}$ be the étale sheaf assoc. to $U \mapsto K_n(U)$. Then the conjecture says we should have a resolution

$$0 \longrightarrow \mathcal{K}^{et}_{n,X} \longrightarrow \coprod_{x \in X_0} (\iota_x)_* \mathcal{K}^{et}_{n,k(x)} \longrightarrow \coprod_{x \in X_1} (\iota_x)_* \mathcal{K}^{et}_{n-1,k(x)} \longrightarrow$$

Note that for a field F, $\mathcal{K}^{et}_{n,F}$ is the $Gal(F^s/F)$-module $K_n(F^s)$.

The rationale behind 1) is that after $\otimes \mathbb{Q}$ we have

$$K_n(F) \otimes \mathbb{Q} \xrightarrow{\sim} \left(K_n(F^s) \otimes \mathbb{Q}\right)^{Gal(F^s/F)}$$

which implies that K-groups of a field $\otimes\mathbb{Q}$ do form an étale sheaf.

Assuming this conjecture 1) $(*)$ should be a flask étale resolution, so we get

$$H^p(X_{et}, \mathcal{K}_p \otimes \mathbb{Q}) = H^p(X_{Zar}, \mathcal{K}_p \otimes \mathbb{Q}).$$

March 10, 1973. Projective line and the Lang problem

k alg closed field of char p.
$k_\phi[F]$ twisted algebra with $Fa = a^\delta F$

Define the twisted projective line to have for its modules the 2-fibre product

$$\begin{array}{ccc} \mathrm{Mod}(P^1_\phi) & \longrightarrow & \mathrm{Mod}(k_\phi[F^{-1}]) \\ \downarrow & \overset{\cong}{\nearrow} & \downarrow \\ \mathrm{Mod}(k_\phi[F]) & \longrightarrow & \mathrm{Mod}(k_\phi[F,F^{-1}]). \end{array}$$

Problem: Compute K-groups of $\mathrm{Modf}(P^1_\phi)$.

Introduce the graded ring

$$B = \bigoplus F_n(k_\phi[F])t^n = \bigoplus_{i<n} kF^i t^n = k_\phi[Ft, t]$$

where t is central and of degree 1. Then it should be the case that $\mathrm{Modf}(P^1_\phi)$ is the quotient category of graded f.t. B-modules by the subcategory of finite length modules.

Thus ~~we~~ we should have an exact seq

$$\begin{array}{ccccccc} \longrightarrow K_i(\mathrm{Modfgr}(k)) & \xrightarrow{i_*} & K_i(\mathrm{Modfgr}(B)) & \longrightarrow & K_i(\mathrm{Modf}(P^1_\phi)) & \xrightarrow{\partial} \\ \| & & \| & & & \\ K_i k[T] & & K_i k[T] & & & \end{array}$$

To compute i_* enough by projection formula to compute $i_* 1$. But

$$0 \to B(-2) \xrightarrow{(\cdot F, \cdot -t)} B(-1) \oplus B(-1) \xrightarrow{(\cdot t, \cdot F)} B \longrightarrow k$$

so if V is a k-module, we get pretty clearly an exact sequence

$$0 \to B(-2)\otimes \phi V \to B(-1)\otimes V \oplus B(-1)\otimes \phi V \longrightarrow B\otimes V \to 0$$

so the relation should be multiplication by

$$1-(T+\phi_*)+\phi_* T^2$$

$$= (1-T)(1-\phi_* T).$$

Thus it should be clear that

$$K_i(P^1_\phi) = K_i k \oplus K_i k$$

as usual, and moreover this is apt to be true for a general twisted Laurent polynomial ring.

My original reason for wanting to consider this problem ~~was~~ derived from the exact seq.

$$\longrightarrow K_{i+1}(k_\phi[F,F^{-1}]) \xrightarrow{\partial} K_i(k) \longrightarrow K_i(k_\phi[F]) \longrightarrow$$

which leads to

$$K_{i+1}(k_\phi(F,F^{-1})) \xrightarrow{\sim} K_i(\mathbb{F}_q) \qquad i \geq 0$$

$$K_0(k_\phi(F,F^{-1})) = \mathbb{Z}$$

at least when $k=\mathbb{F}_q$. Therefore it would seem that we have a canonical . heq

$$BGL(k_\phi[F,F^{-1}])^+ \xrightarrow{\sim} Q(\mathcal{P}(\mathbb{F}_q))$$

whose nature is very mysterious to me.

[redacted]

For example, suppose I take a matrix $A \in GL_n(k_\phi[F,F^{-1}])$, and I suppose it is linear

$$A = a_0 + a_1 F \qquad a_i \in M_n(k)$$

with inverse

$$\sum_{|n|\le N} b_n F^n$$

Then do the Karoubi calculation:

$$(a_0 + a_1 F)\sum b_n F^n = 1$$

$$a_0 b_n + a_1 b_{n-1}^{\sigma} = \begin{cases} 1 & n=0 \\ 0 & n\neq 0 \end{cases}$$

$$\left(\sum b_n F^n\right)(a_0 + a_1 F) = 1$$

$$b_n a_0^{\sigma^n} + b_{n-1} a_1^{\sigma^{n-1}} = \begin{cases} 1 & n=0 \\ 0 & n\neq 0 \end{cases}$$

and

$$\begin{aligned} b_0 a_0 b_0 &= -b_0 a_1 b_{-1}^{\sigma} + b_0 \\ &= b_1 a_0^{\sigma} b_{-1}^{\sigma} + b_0 \\ &= -b_1 (a_1 b_{-2}^{\sigma})^{\sigma} + b_0 \\ &= b_2 a_0^{\sigma^2} b_{-2}^{\sigma^2} + b_0 = \cdots = b_0 \end{aligned}$$

so that $a_0 b_0$ and $b_0 a_0$ are projectors.

$$a_0 b_0 a_1 = - a_0 b_1 a_0^{\sigma} = a_1 (b_0 a_0)^{\sigma}$$

so that

$$a_0 b_0 (a_0 + a_1 F) = (a_0 + a_1 F) b_0 a_0$$

Thus if one thinks of A as an isomorphism

$$k_\phi [F, F^{-1}] \otimes_k V \longrightarrow k_\phi [F, F^{-1}] \otimes_k W$$

we can decompose

$$V = V_0 \oplus V_1 \qquad V_0 = Im(b_0 a_0)$$
$$W = W_0 \oplus W_1 \qquad W_0 = Im(a_0 b_0)$$

so that A preserves this decomposition, and we know that on the zero part A is invertible over $k_\phi[F]$ and on the 1 part A is invertible over $k_\phi[F^{-1}]$.

What this calculation amounts to is the fact that a bundle on the projective line is a sum of copies of $\mathcal{O}(n)$.

<u>Question</u>: Let X be a curve. Consider inside the category $P(X)$ the full subcategory $P_{r,s}$ of those vector bundles E such that E has a filtration whose quotients are line bundles L with $r \le deg(L) \le s$. Can you find r, s so that $P_{r,s}$ has the same K-theory as $P(X)$?

March 11, 1972. Curves.

X complete non-sing. curve (irreducible) over k alg. closed. Consider the full subcat of $\mathcal{P}(X)$ consisting of vector bundles E for which there exists a flag

$$0 \subset E_1 \subset \cdots \subset E_r = E$$

such that E_i/E_{i-1} is a line bundle of degree

$$-2g \leq \deg(E_i/E_{i-1}) \leq 0$$

I want to show this subcategory has the same K-theory as $\mathcal{P}(X)$.

~~[illegible]~~

I know the K-theory can be computed using sufficiently positive bundles. The first thing to note:

<u>Lemma 1</u>: E a v. bundle generated by $H^0(E)$. Then $\exists$ a flag $0 \subset E_1 \subset \cdots \subset E_r = E$ such that each E_i/E_{i-1} is a line bundle of degree ≥ 0.

<u>Proof</u>: Induction on rank r. If $r > 0$, then $\exists$ $s \in H^0(E)$ $s \neq 0$. Let $L \subset E$ be the sub line bundle gen. by s. Then $H^0(L) \neq 0$ so $\deg L \geq 0$. But $E/L \twoheadleftarrow E \twoheadleftarrow \mathcal{O} \otimes H^0(E)$, so induction applies to E/L.

~~Recall by R-R that $\deg(L) > 2g-2 \Rightarrow h^1(L) = 0$. since $h^1(L) = h^0(L^\vee \otimes \Omega) = 0$ as $\deg(L^\vee \otimes \Omega) < 0$. Now define E to be sufficiently positive to mean As a consequence, we have $\deg(L) >$~~

Lemma 2: 1) $\deg(L) \geq 2g-1 \implies h^1(L)=0$

2) $\deg(L) \geq 2g \implies L$ gen. by $H^0(L)$.

3) $\deg(L) \geq 2g+1 \implies L$ very ample.

Proof: $h^1(L) = h^0(L^\vee \otimes \Omega) = 0$ because $\deg(L^\vee\otimes\Omega) = \deg \Omega - \deg L = (2g-2) - \deg L < 0$.

Given $P \in X$ we have

$$0 \longrightarrow \mathcal{O}(-P) \longrightarrow \mathcal{O} \longrightarrow k(P) \longrightarrow 0$$

$$0 \longrightarrow L(-P) \longrightarrow L \longrightarrow L\otimes k(P) \longrightarrow 0$$

$$H^0(L) \longrightarrow L\otimes k(P) \longrightarrow \underset{\overset{\shortparallel}{0}}{H^1(L(-P))}$$

and $\deg(L(-P)) = \deg L - 1 \geq 2g-1$, so

and $H^0(L) \twoheadrightarrow L\otimes k(P) \implies L$ gen. by $H^0(L)$.

(Last steps uses $k = \bar{k}$). Proof of 3) below.

Now define E to be sufficiently positive if it has a flag whose quotients are line bundles of degree $\geq 2g$. Clearly closed under extensions in $\mathcal{P}(X)$.

Claim it has the same K-theory as X. Call this category $\mathcal{P}_{\geq 2g}(X)$, and more generally introduce the subcategory $\mathcal{P}_{\geq n}(X)$ consisting of vector bundles having a flag with degrees $\geq n$.

---

3): To show L very ample, enough to show $H^0(L)$ generates $L\otimes \mathcal{O}_P/m_P^2$ for every P. Thus want

$$H^0(L) \to H^0(L\otimes \mathcal{O}_P/m_P^2) \longrightarrow H^1(L\otimes \mathcal{O}(-2P))$$

$\therefore$ Clear. $\qquad H^1(L\otimes\mathcal{O}(-2P)) = 0$ for $\deg L - 2 \geq 2g-1$

~~[illegible]~~

<u>Lemma 1'</u>: $\mathcal{P}_{\geq n}$ is closed under quotients.

Proof. $0 \subset E_1 \subset \cdots \subset E_r = E$ each $\deg(E_i/E_{i-1}) \geq n$ and let $E \twoheadrightarrow F$. Consider the induced filtration of F in the vector bundle sense, i.e. extends what happens at the generic point. Proceed by induction on $r = \operatorname{rank}(E)$. Look at E_1. If it goes to zero in F, replace E by E/E_1. Otherwise it generates a sub-line-bundle F_1 of F. Then $E_1 \subset F_1 \Rightarrow$ $n \leq \deg(E_1) \leq \deg(F_1)$ and also $E/E_1 \twoheadrightarrow F/F_1$, so can induct.

<u>Lemma 3</u>: $K_i \mathcal{P}_{\geq n} \xrightarrow{\sim} K_i \mathcal{P}(X)$.

Suffices to show $K_i \mathcal{P}_{\geq n} \xrightarrow{\sim} K_i \mathcal{P}_{\geq n-1}$ for any n. But choose an exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow L_0^k \longrightarrow B \longrightarrow 0$$

where L_0 has positive degree. Then you have the characteristic sequence

$$0 \longrightarrow E \longrightarrow E \otimes L_0^k \longrightarrow E \otimes B \longrightarrow 0.$$

and $E \otimes L_0^k$, $E \otimes B \in \mathcal{P}_{\geq n}$ if $E \in \mathcal{P}_{\geq n-1}$ (Note that by the above lemma B has a flag with quotients of $\deg \geq 1$).

<u>Note</u>. Put $\mathcal{P}_{\geq n} = \mathcal{P}_{n,\infty}$. Then have

$$\mathcal{P}_{r,s} \times \mathcal{P}_{r',s'} \longrightarrow \mathcal{P}_{r+r',s+s'}.$$

<u>*Notation*</u> : $P_{[r,s]}$ smallest subcat closed under extensions in $P(X)$ containing line bundles of degrees $r \leq \deg \leq s$.

Next suppose we have $E \in P_{[2g,\infty]}$. So by definition $\exists$ flag with E_i/E_{i-1} a line bundle of degree $\geq 2g$. Thus we know $h^1(E_i/E_{i-1}) = 0$ and that E_i/E_{i-1} gen. by its sections. Same must be true for E. In effect given

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

with $h^1(E') = h^1(E'') = 0$ and E', E'' gen. by sections. Then clearly $h^1(E) = 0$ and

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & R(E') \arrow[r] & \mathcal{O}\otimes H^0(E') \arrow[r] \arrow[d] & E' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R(E) \arrow[r] & \mathcal{O}\otimes H^0(E) \arrow[r] \arrow[d] & E \arrow[d] & \\
0 \arrow[r] & R(E'') \arrow[r] & \mathcal{O}\otimes H^0(E'') \arrow[r] \arrow[d] & E'' \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

so E gen by $H^0(E)$, and moreover $E \mapsto R(E)$ is exact on this category.

To show $E \in P_{[2g,\infty]} \Longrightarrow R(E) \in P_{[-2g,0]}$: Enough by exactness to do it for a line bundle L. If $\deg(L) > 2g$, say $2g+e$, then we can find $L' \subset L$ with L' of degree $2g$ ($L' = L \otimes \mathcal{O}(-P)^e$). Then get following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & R(L') \arrow[r] \arrow[d] & \mathcal{O}\otimes H^0(L') \arrow[r] \arrow[d] & L' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R(L) \arrow[r] \arrow[d] & \mathcal{O}\otimes H^0(L) \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R(L/L') \arrow[r] \arrow[d] & \mathcal{O}\otimes H^0(L/L') \arrow[r] \arrow[d] & L/L' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where one should note that R makes good sense for all coherent sheaves F such that $H^1(F)=0$ and F gen. by $H^0(F)$. And the point is that for $F = k(P)$, $R(k(P)) = \mathcal{O}(-P)$ has degree -1. So we are now reduced to the case where L has degree $2g$.

$\deg(L) = 2g \Rightarrow L$ gen. by $H^0(L)$. Serre's argument $\Rightarrow L$ generated by a 2 dimensional subspace of $H^0(L)$. (Recall the argument. Assume E a v.b. of rank r gen. by $V \to H^0(E)$, with $\dim V = m > r + \dim(X)$. Consider G = Grassmannian of $r+d$ dim'l subspaces $W \subset V$. Compute the bad W. Must consider triples ($x \in X$, H hyperplane in $E(x)$, W^{r+d} cont. in inv. image of H in V

$$d \qquad\qquad r-1 \qquad\qquad (r+d)(m-1-r-d)$$

so want

$$d + r - 1 + (r+d)(m-1-r-d) \overset{?}{<} (r+d)(m-r-d)$$

$$d+r-1 < (r+d) \qquad \text{YES.})$$

So let $W^2 \subset H^0(L)$ span.

Thus we get

$$
\begin{tikzcd}
& & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & \mathcal{O}\otimes W \arrow[r] \arrow[d] & L \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & R(L) \arrow[r] \arrow[d] & \mathcal{O}\otimes H^0(L) \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & m \arrow[r, "\simeq"] \arrow[d] & \mathcal{O}\otimes H^0(L)/W \arrow[r] \arrow[d] & 0 & \\
& 0 & 0 & &
\end{tikzcd}
$$

which shows that we get

$$0 \to K \longrightarrow R(L) \longrightarrow \mathcal{O}\otimes H^0(L)/W \longrightarrow 0$$

But K is a line bundle of degree $= -\deg(L) = -2g$. Thus we have proved.

<u>Lemma</u>: As E ranges over $\mathcal{P}_{[2g,\infty]}$ we have the exact resolution

$$0 \longrightarrow R(E) \longrightarrow \mathcal{O}\otimes H^0(E) \longrightarrow E \longrightarrow 0$$

where $R(E)$ has values in $\mathcal{P}_{[-2g,0]}$.

~~Corollary~~

$$
\begin{tikzcd}
K_i\, \mathcal{P}_{[2g,0]} \arrow[rr] & & K_i\, \mathcal{P}_X \\
& K_i\, \mathcal{P}_{[2g,\infty]} \arrow[ul, "{[\mathcal{O}\otimes H^0(E)] - [R(E)]}"] \arrow[ur, "\int"'] &
\end{tikzcd}
$$

$$E \mapsto [\mathcal{O}\otimes H^0(E)] - [R(E)]$$

Remark: The preceding proof shows that we can do a bit better for the surjectivity. Thus observe in the resolution

$$0 \longrightarrow R(E) \longrightarrow \mathcal{O} \otimes H^0(E) \longrightarrow E \longrightarrow 0$$

we showed that $R(E)$ for $E \in \mathcal{P}_{[2g,\infty]}$ has a flag whose quotients are line bundles of degrees between $-2g$ and 0, but the trivial bundle was the only line bundle of degree zero which occurred.

~~On the other hand suppose I work with E which are even more positive. Fix a point P_0 and then work with line bundles L of degree so high that ∃ an embedding $\mathcal{O}(2g P_0) \hookrightarrow L$ always. The point of the $\mathcal{O}(2gP_0)$ is so we could find~~

$$\cancel{0 \to K \to \mathcal{O} \otimes W \to \mathcal{O}(2g P_0) \to 0}$$

~~with $W \subset H^0(\mathcal{O}(2gP_0))$ of d ?~~

Let X be an elliptic curve to fix the ideas. Recall that line bundles of degree 1 are of the form $\mathcal{O}(P)$ for a unique point P. Fix a line bundle I of degree 2. It has 2 ind. section so

$$0 \longrightarrow J \longrightarrow \mathcal{O} \otimes H^0(L) \longrightarrow I \longrightarrow 0$$

where J is of degree -2 in fact

$$J \otimes I = \Lambda^2 (\mathcal{O} \otimes H^0(L)) \simeq \mathcal{O}$$

so $J \simeq I^\vee$. Now suppose L is an line bundle of degree ≥ 3. Then $h^0(I^\vee \otimes L) = 1$, so we have a non-zero homom. $I \hookrightarrow L$. Thus

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & J \arrow[r] \arrow[d] & \mathcal{O} \otimes H^0(I) \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R(L) \arrow[r] \arrow[d] & \mathcal{O} \otimes H^0(L) \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R(L/I) \arrow[r] \arrow[d] & \mathcal{O} \otimes H^0(L/I) \arrow[r] \arrow[d] & L/I \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

and we have seen that $R(L/I)$ has a flag with quotients of degree -1. Thus from this example we have managed to get an exact resolution in the category closed under extensions generated by $\mathcal{O}$, J, and all line bundles of degree -1.

But recall that semi-stable vector bundles of a given slope form an abelian category. Just consider semi-stable bundles of degree 0, i.e. ones such that every subbundle has degree ≤ 0. Any successive extension of line bundles of degree 0 is such a thing, for if

$$F \subset E \quad \text{where} \quad 0 \subset E_1 \subset \cdots \subset E_r = E$$

then

$$F \cap E_i / F \cap E_{i-1} \subset E_i / E_{i-1}$$

so $F \cap E_i / F \cap E_{i-1}$ is either zero or a line bundle of degree $\leq \deg(E_i/E_{i-1}) = 0$. Suppose $\theta: E \to E'$ is a homomorphism of successive extensions of line bundles of degree zero. Then $\operatorname{Im}\theta \subset \operatorname{Coim}(\theta)$ and we can consider a flag in $\operatorname{Coim}(\theta)$ and its traces on $\mathrm{Im}\,\theta$. Then as before the quotients for $\mathrm{Coim}(\theta)$ have degrees ≤ 0, those for $\mathrm{Im}(\theta)$ have degrees ≥ 0, so all must be of degree 0 and $\mathrm{Im}\,\theta = \mathrm{Coim}(\theta)$. So it's clear one ends up with the following

<u>Lemma</u>: The additive category of vector bundles which are successive extensions of line bundles of degree 0 is an abelian artinean <u>category</u>.

What I know now is that for an elliptic curve I can generate the K-groups using only vector bundles which are successive extensions of line bundles of the following types: $\mathcal{O}$, any line bundle of degree $+1$, a fixed line bundle of degree 2

Call this category $\mathcal{E}$

Now the subcategory of successive extensions of line bundles of degree 1 is an Artinean cat. with one simple object for each line bundle of degree 1, which up to isom is $\mathcal{O}(P)$. Thus we get a homom.

$$K_i(k) \otimes D \longrightarrow K_i(\mathcal{E}) \longrightarrow\!\!\!\!\!\rightarrow K_i X .$$

which is induced by multiplying:

$$V \otimes P \longmapsto V \otimes \mathcal{O}(P).$$

Now it seems desirable to show this map factors through $K_i(k) \otimes \mathrm{Pic}(X)$ in some sense. Thus I want to see at least the relations coming from tensor product of line bundles if I can. (what I want is some evidence for the map $K_i\mathcal{E} \twoheadrightarrow K_i X$ to be an isom.) Let I be the fixed line bundle of degree 2. Then in the category $\mathcal{E}$ we have exact sequences of the form

$$(*) \qquad 0 \longrightarrow \mathcal{O} \longrightarrow E \longrightarrow I \longrightarrow 0$$

where E is a successive extension of bundles of degree 1. For example, given a section $\mathcal{O} \hookrightarrow I$, the cokernel has degree 2, so often is of the form $k(P) \oplus k(Q)$ for $P \neq Q$. Thus we get an exact sequence

$$(**) \qquad 0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(P) \oplus \mathcal{O}(Q) \longrightarrow I \longrightarrow 0.$$

(i.e. I with the given section will be isomorphic to $\mathcal{O}(P+Q)$.)

Observe that $(*)$ implies $\Lambda^2 E = I$, so if E is an extension of L_1 by L_2 we have $L_1 \otimes L_2 = I$. On the other hand if

$$\mathcal{O}(P) \otimes \mathcal{O}(Q) = \mathcal{O}(P+Q) \simeq I$$

then at least for $P \neq Q$ we get the exact sequence $(**)$. And for $I = \mathcal{O}(2P)$ take the non-trivial extension ?

The point is that the iso classes of extensions $(*)$ is $\mathrm{Ext}^1(I, \mathcal{O}) = H^1(I^\vee)$ which is 2 diml. The ones we are after are the non-splits ones, and modulo scalars we are interested in the lines in $H^1(I^\vee)$. Also one should note that mapping X to bundles by sending P to $\mathcal{O}(P) \oplus \mathcal{O}(*P)$ ($*$ = inverse for group law) factors through $X/x=-x = \mathbb{P}^1_k$.

Unfortunately these exact sequences only relate P with $*P$

Try to prove for an elliptic curve that
$$K_i\, P_{[-2,0]} \xrightarrow{\sim} K_i\, P(X).$$

Choose
$$0 \longrightarrow \mathcal{O} \longrightarrow F_1 \longrightarrow F_2 \longrightarrow 0$$
with F_i suff. positive, say in $P_{[4,\infty]}$, so that $E \otimes F_i \in P_{[2,\infty]}$ for E in $P_{[-2,0]}$. Then one has

$$\begin{tikzcd}
0 \arrow[r] & R(E\otimes F_1) \arrow[r] \arrow[d] & \mathcal{O}\otimes H^0(E\otimes F_1) \arrow[r] \arrow[d] & E\otimes F_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R(E\otimes F_2) \arrow[r] & \mathcal{O}\otimes H^0(E\otimes F_2) \arrow[r] & E\otimes F_2 \arrow[r] & 0
\end{tikzcd}$$

exact rows. Diagram chasing shows there are exact sequences

$$0 \longrightarrow Z(E) \longrightarrow R(E\otimes F_2) \oplus \mathcal{O}\otimes H^0(E\otimes F_1) \twoheadrightarrow \mathcal{O}\otimes H^0(E\otimes F_2) \longrightarrow 0$$

$$0 \longrightarrow R(E\otimes F_1) \longrightarrow Z(E) \longrightarrow E \longrightarrow 0$$

which shows that $Z(E)$ is exact is exact from $P_{[-2,0]}$ to $P_{[-2,0]}$. This should show that going from $P_{[-2,0]}$ into P_X then back to $P_{[2,\infty]}$ then into $P_{[-2,0]}$

$E \qquad E \qquad E\otimes F_1 \leftarrow E\otimes F_2$

is the identity. Check tomorrow.

<u>Question:</u> For a curve of genus g can one find a line bundle S of degree $2g$ such that in the sequence

$$0 \to R(S) \to \mathcal{O} \otimes \underset{g+1}{H^0(S)} \to S \to 0$$

$R(S)$ is a successive extension of g line bundles of degree -2? If so, then could get $K_i X$ down into $K_i \mathcal{P}_{[-2,0]}$ in general. In fact, it appears to be enough to have an S of large degree so that $R(S)$ is in $\mathcal{P}_{[-2,0]}$.

March 13, 1973. curves continued.

The problem is to show for a curve X that its K-theory can be computed using those vector bundles which are successive extensions of line bundles of degrees between -2 and 0. The idea is to show for any sufficiently positive bundle E that the resolution

$$0 \to R(E) \longrightarrow \mathcal{O}\otimes H^0(E) \longrightarrow E \longrightarrow 0$$

is such that $R(E)$ has a full flag with quotients of degrees between -2 and 0. Immediate reduction to the case where E is a line bundle of high degree.

Lemma: Assume that there exists a line bundle L generated by its sections such that $R(L)$ has a full flag with quotients of degrees between $-d$ and 0, where $R(L)$ is defined by the exact seq

$$0 \to R(L) \longrightarrow \mathcal{O}\otimes H^0(L) \longrightarrow L \longrightarrow 0$$

(makes sense for any coh sheaf on X). Assume also $H^1(L)=0$. Then for any suff. positive E, $R(E)$ has a full flag with quotients of degrees between $-d$ and 0.

Proof. Can suppose E is a line bundle of high degree, whence there is an embedding $L \hookrightarrow E$. Then

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & R(L) \arrow[r] \arrow[d] & \mathcal{O}\otimes H^0(L) \arrow[r] \arrow[d] & L \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & R(E) \arrow[r] & \mathcal{O}\otimes H^0(E) \arrow[r] & E \arrow[r] & 0
\end{tikzcd}$$

so from serpent lemma get $R(L) \subset R(E)$ and

$$0 \to R(E)/R(L) \to \mathcal{O} \otimes H^0(E)/H^0(L) \to E/L \to 0.$$

~~[illegible]~~ Because $H^1(L) = 0$, we have $H^0(E)/H^0(L) = H^0(E/L)$ so $R(E)/R(L) = R(E/L)$ ~~has~~ full flag with quotients of degree (-1). Thus $R(E)$ has a full flag with quotients of degree $(-d)$ and (-1).

<u>Lemma</u>: For any vector bundle E

$$\forall x, \quad H^1(E(-x)) = 0 \iff E \text{ gen. by } H^0(E) \text{ and } H^1(E) = 0.$$

<u>Proof</u>.
$$0 \to E(-x) \to E \to E \otimes k(x) \to 0$$
$$0 \to H^0(E(-x)) \to H^0(E) \to H^0(E \otimes k(x))$$
$$\hookrightarrow H^1(E(-x)) \to H^1(E) \to 0$$

<u>Lemma</u>: ~~[illegible]~~ E line bundle

1) E gen by $H^0(E)$ and $H^1(E) = 0$ and $E \not\cong \mathcal{O}$ $\Longrightarrow \deg E \geq g+1$

2) ~~[illegible]~~ The set of E of degree d such that E gen by $H^0(E)$ and $H^1(E) = 0$ is open in $\mathrm{Pic}^d(X)$ and in fact is the complement of a subvariety of dimension $\leq 2g-d$. In particular, it is dense for $d \geq g+1$.

<u>Proof</u>. E gen by $H^0(E)$ and $H^1(E) = 0 \iff H^1(E(-x)) = 0$ all x.
$\iff H^0(\Omega \otimes E^\vee(x)) = 0$ for all x. ~~[illegible]~~

$\left(\deg\left(\Omega\otimes \check{E}(x)\right) = 2g-2-d+1 = 2g-1-d\right)$. But

$$H^0\left(\Omega\otimes \check{E}(x)\right) > 0 \iff \exists\, y_1,\dots,y_{2g-1-d} \ni$$

$$\Omega\otimes \check{E}(x) = \mathcal{O}\left(\textstyle\sum y_i\right)$$

or equiv. $\ni$ $\quad E = \Omega(\underbrace{x - \textstyle\sum y_i}_{2g-d \text{ points}})$

Thus the bad E of degree d is a variety of dimension $2g-d$ ($\therefore$ empty if $d \geq 2g$), proving 2). ~~But $d \leq g$ ∴ then $2g-d \geq g$ and one knows that every line bundle of degree $\geq g$ is represented by a positive divisor, proving 1).~~

~~A better way: $d \leq g \Longrightarrow \Omega\otimes E^{\vee}(x)$ has degree $\geq 2g-1-g = g-1 \Longrightarrow H^0(\Omega\otimes E^{\vee}(x)) \geq$~~

Suppose $d = g$. Then how many E can we represent in the form $\Omega(x - \sum_{i=1}^{g-1} y_i)$. Clearly one gets a map of $X^g \longrightarrow J_{2g-2+1-g+1} = J_g$ in this way. Anyway, it seems unlikely that this map could fail to be dominant, so it would be onto. In fact, all I have to do to make this work it to consider the generic line bundle E of degree g, and show $h^0(E)=1$, $h^1(E)=0$, which is a well-known fact used in the const. of the Jacobian (Serre Grps alg. & corps de class page 56, Lemma 4). So this proves 1).

But 1) is really trivial, i.e. $H^1(E)=0$ $\Rightarrow h^0(E) = \deg(E)+1-g \leq 1$ if $\deg E \leq g$, so E can't be generated by $H^0(E)$ unless E is trivial.

<u>Example</u>: Let us suppose there exists a line bundle L generated by $H^0(L)$ of degree d. Let $f: X \to \mathbb{P}(H^0(L))$ be the canonical map so that $f^*\mathcal{O}(1) = L$. Then for k large we have $H^1(L^k) = 0$ and L^k is generated by its sections. Thus from the exact sequence on $\mathbb{P}(H^0(L))$

$$\mathcal{O}(-1) \otimes T_1 \longrightarrow \mathcal{O} \otimes S_k \longrightarrow \mathcal{O}(k) \longrightarrow 0$$

we get the exact sequence

$$L^{-1} \otimes T_1 \longrightarrow \mathcal{O} \otimes S_k \longrightarrow L^k \longrightarrow 0$$

on X which shows that $R(L^k)$ is a quotient of $L^{-1} \otimes T_1$ and hence we know it has a full flag with quotients of degrees $\geq \deg(L^{-1}) = -d$. NO

<u>Better</u>: Take 2 sections of L which generate, whence we have a map $f: X \to \mathbb{P}^1$ which is finite, hence <u>flat</u>. Thus when I take

$$0 \to \mathcal{O}(-1) \otimes T_1(\mathcal{O}(k)) \longrightarrow \mathcal{O} \otimes S_k \longrightarrow \mathcal{O}(k) \longrightarrow 0$$

on $\mathbb{P}^1$ and pull it back to X, I get the <u>exact</u> sequence

$$0 \longrightarrow L^{-1} \otimes_k T_1 \longrightarrow \mathcal{O} \otimes S_k \longrightarrow L^k \longrightarrow 0$$

~~which shows that $R(L^k)$ has~~ for any $k \geq 0$. ~~Now consider~~ Now $H^0(L^{-1}) = 0 \implies S_k \subset H^0(L^k)$, so now consider

$$\begin{tikzcd}
0 \arrow[r] & L^{-1}\otimes T_1 \arrow[r] \arrow[d] & \mathcal{O}\otimes S_k \arrow[r] \arrow[d] & L^k \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & R(L) \arrow[r] & \mathcal{O}\otimes H^0(L^k) \arrow[r] & L^k \arrow[r] & 0
\end{tikzcd}$$

and apply serpent to get

$$0 \longrightarrow L^{-1}\otimes T_1 \longrightarrow R(L) \longrightarrow \mathcal{O}\otimes H^0(L^k)/S_k \longrightarrow 0$$

which does show that $R(L)$ has the good composition series. Thus have proved

<u>Proposition</u>: Let L be a line bundle of degree $d>0$ which is generated by $H^0(L)$. Then for any sufficiently positive coherent sheaf F, $R(F)$ has a full flag with quotients having degrees between $-d$ and 0.

Observe the d above is the least integer such that X is a d-fold covering of $\mathbb{P}^1$. Put another way, it is the least d for which $\exists$ a divisor $\mathfrak{a}\geq 0$ of degree d such that $\dim H^0(\mathcal{O}(\mathfrak{a}))\geq 2$.

Thus if $g=2$, take a Weierstrass point P, whence $h^0(2P)\geq 2$. Then X is a double covering of $\mathbb{P}^1$, and so we win. Better: Take $L=\Omega$ which has degree 2; then Ω is generated by $H^0(\Omega)$ because $\dim H^1(\Omega(-x)) = \dim H^0(\mathcal{O}(x)) = 1$ for all x.

According to Mumford (pg. 58) can always find

L gen. by $H^0(L)$ of degree g, if X not hyperelliptic.
Better: in general can take $L = gP$ where P is a Weierstrass point.

At the moment we are down to g when $g \geq 2$.

March 15, 1973 Education in curves.

Fact: L is very ample $\Leftrightarrow$ for all pos. div. $\mathfrak{a}$ of degree 2
$$H^1(L(-\mathfrak{a})) \xrightarrow{\sim} H^1(L).$$

In effect, ~~if~~ $\mathfrak{a} = P+Q$, ~~and~~ and if $P \neq Q$ we get
$$0 \to L(-P-Q) \to L \to L\otimes(k(P)\oplus k(Q)) \to 0$$
$$H^0(L) \to H^0(L\otimes k(P) \oplus L\otimes k(Q)) \to H^1(L(-P-Q)) \to H^1(L) \to 0$$
and so can find sections separating $P, Q \Leftrightarrow H^1(L(-P-Q)) \xrightarrow{\sim} H^1(L)$. Similarly for $P=Q$.

~~Example~~ Def: X is hyperelliptic $\Leftrightarrow$ X is a double covering of $\mathbb{P}^1 \Longleftrightarrow \exists\ \mathfrak{a} \geq 0$ of degree 2 with ~~~~ $\dim H^0(\mathcal{O}(\mathfrak{a})) > 1$.

Cor. X is not hyperelliptic $\Leftrightarrow$ Ω is very ample

Proof: $H^1(\Omega(-\mathfrak{a})) \xrightarrow{\sim} H^1(\Omega) \Longleftrightarrow H^0(\mathcal{O}(\mathfrak{a})) \xleftarrow{\sim} H^0(\Omega)$

i.e. iff $\exists$ no fn. $\neq$ constant with polar divisor of degree 2.

Thus for a non-hyperelliptic curve X one has a canonical proj. Imbedding $X \hookrightarrow \mathbb{P}^{g-1}$ as a curve of degree $2g-2$.

Example: $g=3$, so get a plane curve of degree 4. (plane quartic)

~~~~ Conversely let X be a plane curve of degree 4 which is non-singular. Then we have

(1) $$0 \to \mathcal{O}_P(-4) \to \mathcal{O}_P \to \mathcal{O}_X \to 0$$

$P = \mathbb{P}^2$

$f: X \to P$
~~~~

(2) $$0 \longrightarrow \Omega_P \longrightarrow \mathcal{O}_P(-1)^3 \longrightarrow \mathcal{O}_P \longrightarrow 0.$$

(3) $$0 \longrightarrow I/I^2 \longrightarrow f^*\Omega_P \longrightarrow \Omega_X \longrightarrow 0 \qquad I = \mathcal{O}(-4).$$

Thus if $L = f^*(\mathcal{O}(1))$ we get $I/I^2 = L^{-4}$

$$L^{-4} \otimes \Omega_X \simeq \Lambda^3 f^*\Omega_P \simeq L^{-3}$$

so $\Omega_X \cong L = f^*(\mathcal{O}(1))$. Thus a non-hypelliptic curve of genus three is the same thing as a plane curve (non-sing) of degree 4. In addition, notice that from

$$0 \longrightarrow \Omega_P(1) \longrightarrow \mathcal{O}_P^3 \longrightarrow \mathcal{O}_P(1) \longrightarrow 0$$

we get

$$0 \longrightarrow f^*\Omega_P \otimes \Omega_X \longrightarrow \mathcal{O}_X^3 \longrightarrow \Omega_X \longrightarrow 0$$

so that
$$R(\Omega_X) = f^*\Omega_P \otimes \Omega_X.$$

I want to show this rank 2 bundle of degree -4 is an extension of line bundles of degree -2.

<u>Won't work</u>: In effect if we get

$$0 \longrightarrow L \longrightarrow R(\Omega_X) \longrightarrow L' \longrightarrow 0$$

with degree $L = -2$, then L is a subbundle of $\mathcal{O}^3$ so $L^\vee$ is a bundle of degree 2 which is a quotient of $\mathcal{O}^3$, and so we are in the hyper-elliptic case.

<u>Remark</u>: If E is a subbundle of $\mathcal{O}^n$ and E has a sub-line-bundle L of degree -2, then $L^\vee$ is a quotient of $\mathcal{O}^n$, so X is hyperelliptic.

<u>Back to elliptic curves.</u> Denote by K a fixed bundle of degree -2, and denote by $\mathcal{E}$ the category of vector bundles which are successive extensions of K, $\mathcal{O}$, and any line bundle of degree -1. Let $\mathcal{E}'$ be the full subcategory consisting of the ones which are successive extensions of K, and any deg(-1) bundle. I want to show $\mathcal{E}'$ and $\mathcal{E}$ have the same K-theory using the resolution theorem.

First suppose E is an extension of two line bundles

$$0 \to L' \to E \to L'' \to 0$$

If $\deg(L') > \deg(L'')$, then $\deg(L''^{-1} \otimes L') > 0$ and so

$$H^1(L''^{-1} \otimes L') = \mathrm{Ext}^1(L'', L') = 0$$

because we are on an elliptic curve (so $2g-2=0$). Thus the sequence splits.

~~Lemma: If $0 \subset E_1 \subset \cdots \subset E_r = E$ is a maximal flag in a vector bundle over an elliptic curve, then~~

$$\cancel{\deg(E_1) \leq \deg(E_2/E_1) \leq \cdots\cdots}$$

So we see that any E in our special class $\mathcal{E}$ has a filtration $0 \subset E_1 \subset E_2 \subset E$ where E_1 is a succ. ext of K's, E_2/E_1 is a succ. ext. of deg(-1) bundles, and E/E_2 is a succ. ext. of $\mathcal{O}$'s.

Let V be an arbitrary succ. extension of $\mathcal{O}$'s.

Then for any L of degree -1 we have

$$(*) \qquad H^1(L^{-1}) = 0 \implies H^1(L^{-1} \otimes V) = 0.$$

But if L_1, L_2 are two bundles of degree -1 with $L_1 \otimes L_2 = K$, then we have an exact sequence

$$0 \longrightarrow K \longrightarrow L_1 \oplus L_2 \longrightarrow \mathcal{O} \longrightarrow 0$$

(e.g. can take $K = \mathcal{O}(-P-Q)$ for some $P \neq Q$ and take $L_1 = \mathcal{O}(-P)$, $L_2 = \mathcal{O}(-Q)$.)

Thus for $V = \mathcal{O}$ we have an exact sequence

$$(**) \qquad 0 \to \bar{R}(V) \longrightarrow L_1 \otimes Hom(L_1, V) \oplus L_2 \otimes Hom(L_2, V) \longrightarrow V \longrightarrow 0$$

where $\bar{R}(V)$ and since $(*)$ implies the middle functors are exact in V, it follows that we get an exact sequence $(**)$ for any V, and that $\bar{R}(V)$ is a succ. ext. of K's.

Now given E in $\mathcal{E}$ we have an exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow V \longrightarrow 0$$

with E' in $\mathcal{E}'$ and V a successive extension of $\mathcal{O}$'s. From $(**)$ we get

$$\begin{tikzcd}
 & & & 0 & \\
0 \arrow[r] & E' \arrow[r] & E \arrow[r] & V \arrow[r] \arrow[u] & 0 \\
 & & Q \arrow[r] \arrow[u] & (L_1 \oplus L_2)^{\sigma} \arrow[u] & \\
 & & \bar{R}(V) \arrow[r, equal] \arrow[u] & \bar{R}(V) \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Thus we see that E is a quotient of Q which is an extension of $E' \in \mathcal{E}'$ by $(L_1+L_2)^\sigma \in \mathcal{E}'$. Thus we see every E is a quotient of same member of $\mathcal{E}'$, with kernel in $\mathcal{E}'$.

Last we have to check that if we have an exact sequence in $\mathcal{E}$

$$0 \to E' \to E \to E'' \to 0$$

with E in $\mathcal{E}'$, then E' is in $\mathcal{E}'$. But let E_i be a full flag in E with quotients K or of deg -1, and let F_1/F_2 be a line bundle subquotient of E' with $F_1/F_2 \simeq \mathcal{O}$. choose E. ?

<u>Elliptic curves again:</u> Let K be a fixed bundle of degree -2 say $K = \mathcal{O}(-x_1-x_2)$ where $x_1 \neq x_2$, so that we have an exact sequence

$$0 \to K \to L_1 \oplus L_2 \to \mathcal{O} \to 0$$

with L_1 and L_2 of degree -1 ($L_i = \mathcal{O}(-x_i)$). Let E be any line bundle of degree ≥ 3. Then E is gen. by $H^0(E)$, so we have an exact sequence

$$0 \to R(E) \to \mathcal{O} \otimes H^0(E) \to E \to 0$$

Also $\exists$ injection $K^{-1} \hookrightarrow E$ so we get an exact sequence

$$0 \to H^0(K^{-1}) \to H^0(E) \to H^0(E/K^{-1}) \to 0$$

since $H^1(K^{-1}) = 0$. Thus applying serpent lemma to

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O} \otimes H^0(K^{-1}) \arrow[r] \arrow[d] & \mathcal{O} \otimes H^0(E) \arrow[r] \arrow[d] & \mathcal{O} \otimes H^0(E/K^{-1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K^{-1} \arrow[r] & E \arrow[r] & E/K^{-1} \arrow[r] & 0
\end{tikzcd}$$

one sees that

$$0 \to R(K^{-1}) \to R(E) \to R(E/K^{-1}) \to 0$$

But $R(K^{-1}) = K$ as $H^0(K^{-1})$ is 2 diml, and $R(E/K^{-1})$ is an extension of line bundles of degree -1.

So now define $\tilde{R}(E)$ by the exact sequence

$$0 \to \tilde{R}(E) \to (L_1 \oplus L_2) \otimes H^0(E) \to E \to 0$$

when the latter is explained by:

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & & \\
 & K\otimes H^0(E) \arrow[d] \arrow[rr, equal] & & K\otimes H^0(E) \arrow[d] & & \\
0 \arrow[r] & \tilde{R}(E) \arrow[rr] \arrow[d] & & (L_1\oplus L_2)\otimes H^0(E) \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & R(E) \arrow[rr] \arrow[d] & & \mathcal{O}\otimes H^0(E) \arrow[r] \arrow[d] & E \arrow[r] & 0 \\
 & 0 & & 0 & &
\end{tikzcd}$$

Thus we see $\tilde{R}(E)$ is an extension of K's by line bundles of degree -1. Moreover we can extend the definition of $R(E)$ and $\tilde{R}(E)$ to bundles which are extensions of line bundles of degree ≥ 3. Then we have the exact resolution

$$0 \to \tilde{R}(E) \to (L_1\oplus L_2)\otimes H^0(E) \to E \to 0$$

which we can use as before to prove that the K-theory of X is the same as the subcategory $\mathcal{E}'$ of bundles which are extensions of K's and line bundles of degree -1. Translating the inclusion $\mathcal{E}' \to P(X)$ by tensoring with K^{-1} and then dualizing (both these operations induce isos on K-groups), we obtain

<u>Theorem</u>: Let $\mathcal{C}$ be the full subcategory of $P(X)$ consisting of bundles which are extensions of line bundles of degree -1 and $\mathcal{O}$. Then $K_i(\mathcal{E}) \xrightarrow{\sim} K_i(X)$.

Simpler proof for elliptic curves:

Let V denote the unique non-trivial extension

$$0 \longrightarrow \mathcal{O} \longrightarrow V \longrightarrow \mathcal{O} \longrightarrow 0$$

(recall $H^1(\mathcal{O}) = k$). Claim that if E is any vector bundle which is a successive extension of line bundles of positive degree, then in the sequence

$$0 \longrightarrow S(E) \longrightarrow V \otimes \mathrm{Hom}(V,E) \xrightarrow{\ c\ } E$$

c is onto, and $S(E)$ is a succ. ext. of degree-1 line bundles and trivial line bundles. First point is that

$$\mathrm{Ext}^1(V,E) = 0$$

so $E \mapsto \mathrm{Hom}(V,E)$ is exact. But let L be a line bundle of degree $+1$, and choose $\mathcal{O} \hookrightarrow L$ non-zero

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[d] & V \arrow[r] \arrow[d, dashed] & \mathcal{O} \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & \mathcal{O} \arrow[r] & L \arrow[r] & \underset{\wr\, k}{\mathcal{O}/L} \arrow[r] & 0
\end{tikzcd}$$

the dotted arrows exists since $H^1(L) = 0$. It is clear that $\mathcal{O} \longrightarrow \mathcal{O}/L = k$ must be onto, otherwise $V \to L$ would factor thru $\mathcal{O}$ and $V \simeq \mathcal{O}^2$. Thus L is a quotient of L, which proves c is onto for a line bundle of degree 1. Also we have

$$\begin{tikzcd}
0 \arrow[r] & S(L) \arrow[r] & V \otimes \mathrm{Hom}(V,L) \arrow[r] & L \arrow[r] & 0 \\
0 \arrow[r] & L^{-1} \arrow[r] \arrow[u] & V \arrow[r] \arrow[u] & L \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

so by serpent $S(L)$ is an extension of L^{-1} by V.
Now for a line bundle E of degree > 1 have

$$0 \to L \to E \to E/L \to 0$$

and so by exactness

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & S(L) \arrow[r] \arrow[d] & V \otimes \mathrm{Hom}(V,L) \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S(E) \arrow[r] \arrow[d] & V \otimes \mathrm{Hom}(V,E) \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S(E/L) \arrow[r] \arrow[d] & V \otimes \mathrm{Hom}(V,E/L) \arrow[r] \arrow[d] & E/L \arrow[r] \arrow[d] & {} \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

So it remains to see that $S(E/L)$ is a succ. ext. of deg -1 and $\mathcal{O}$'s. By exactness reduce to $E/L = k(P)$. Have

$$\begin{tikzcd}
0 \arrow[d] & & \\
V \otimes \mathrm{Hom}(\mathcal{O}, E/L) \arrow[r] \arrow[d] & E/L & \\
V \otimes \mathrm{Hom}(V, E/L) \arrow[r] \arrow[d] & E/L \arrow[r] & 0 \\
V \otimes \mathrm{Hom}(\mathcal{O}, E/L) \arrow[d] & & \\
0 & &
\end{tikzcd}$$

and so it is clear.

9) March 73

Serre's theorem. Let E be a vector bundle over a noetherian scheme S, say affine so that E has enough sections. To prove that E has an everywhere non-vanishing sections, provided the rank of E is larger than the dimension of the space $X = \max(S)$.

Example 1. Suppose X is of dimension 0, which means that X is semi-local. Here one knows that if x_i are the finitely many points of X, and if we give e_i in $E(x_i)$ for each i, then can find a section s such that $s(x_i) = e_i$ all i.

Example 2. Suppose X is of dimension 1, e.g. a curve. Then the first thing one does is to look at the generic points, and choose a section s non-vanishing at each of these. Then one looks at the dependency locus of s which is a finite set of closed points. One selects a section t which is non-zero at each of the points where s vanishes and which is generically independent of s. Then one takes $s + gt$ where g is chosen to be zero at the points where s and t become dependent, and $= 1$ at the points where s vanishes.

Very interesting combinatorial argument.

X has a canonical filtration according to the dimension of the support. Possibly one wants to use induction on the dimension of X.

Over each open set we have sections, easily approximated by global sections without changing independence.

Example 3. Suppose k is a field and E is a vector space over k of dimension n.

Why is the usual K-theory a cohomological functor. Thus why can you work with stable vector bundles and get a satisfactory theory. One reason is that every bundle is a direct summand of a trivial bundle. Recall the argument. Given a vector bundle over a top. space X, we wish to show that it is a direct summand of a trivial bundle, which signified that there exists sections. How to produce sections: Easy. at each point x of X you choose a ~~section which is~~ local frame s_i. Then you extend.

Now suppose k is a discrete field, and let X be a finite complex, and let E be a k-vector bundle over X of ~~dim~~ rank greater than the dimensions of X. Then over X, I get a simplicial complex whose vertices are ~~vertices~~ non-zero vector in the fibres over vertices ~~on~~ of X. ~~and whose~~ The point is that over each simplex of X we get a ~~sheaf~~ stalk and we consider the ~~inside~~ unimodular complex associated. If E is obtained from a representation, then the space $U(E)$ over X is ~~also~~ an induced bundle. Then there has to exist a section, which means I think that by passing to a subdivision, we can construct over ~~the~~ each vertex a non-zero vector in the fibre, and over each p-simplex a $(p+1)$-frame in the fibre.

~~Suppose then I~~ The significance of a $(p+1)$-frame in the ~~fir~~ fibre is clear. You need this in order to ~~make~~ know that the ~~barycentric~~ section ~~is~~ constructed from this frame and the barycentric coordinates on the simplex is non-vanishing. What sort of gadget is it that you get when you divide out by ~~the~~ this section? Thus in the usual situation, you would get a quotient bundle of one less dimension. Thus we get what?

We get ~~a sh~~ on a simplex x of dimension p we get a ~~bundle~~ vector space ~~W~~ F_x ~~spa~~ of dimension $n-p$, quotient of E_x, and when x is a face of y, F_y is a ~~quotient~~ quotient of F_x ~~by a frame vector space~~ with kernel essentially generated by the vertices of y not in x. Therefore if we consider the ~~space~~ system ~~provided by the~~ x goes to the k-vector space of ~~k-vector space of~~ functions on the vertices ~~which~~ we get a $\pm$ system.

March 22, 1973

On the homotopy type of small categories:

1. Sheaves over a category

Let $\mathcal{C}$ be a small category. By a sheaf of sets on $\mathcal{C}$ we will mean a covariant functor from C to Sets. We choose this variance for sheaves because of the following example. Let C be the ordered set of simplices of a simplicial complex K. If F is a sheaf over K which is locally constant over each open simplex of K, let $F(\sigma)$ be the stalk of F over any interior point of σ. If τ is a face of σ, then any element of $F(\tau)$ extends to a section of a nbd of the center of τ, and the nbd contains an interior point of σ, so we get a map $F(\tau) \longrightarrow F(\sigma)$. Thus F determines a functor from $\mathcal{C}$ to Sets.

Fact: Sheaves over K constant over each open simplex are the same as functors from the ordered set of simplices of K to Sets.

Thus we have associated to C the topos Funct(C, Sets). Hence we have the cohomological apparatus: Abelian sheaves, Complexes, and the derived category which we denote $D(\mathcal{C})$. Also fundamental groupoid, higher homotopy groups, and the homotopy category of simplicial sheaves over $\mathcal{C}$.

If $f : \mathcal{C} \longrightarrow \mathcal{C}'$ is a functor, then it determines adjoint functors

$$\begin{tikzcd} \mathrm{Funct}(\mathcal{C}, \mathrm{Sets}) \arrow[r, shift left=2.5ex, "f_!^{\mathrm{sets}}"] \arrow[r, shift right=2.5ex, "f_*"'] & \mathrm{Funct}(\mathcal{C}', \mathrm{Sets}) \arrow[l, "f^*"'] \end{tikzcd}$$

$$(f^* G)(X) = G(fX)$$

$$(f_!^{\mathrm{sets}} F)(Y) = \varinjlim_{(X,u) \in f/Y}{}^{\mathrm{sets}} F(X) \qquad (X, u : fX \to Y)$$

$$(f_* F)(Y) = \varprojlim_{(X,v) \in Y \backslash f} F(X) \qquad (X, v : Y \to fX)$$

2. Homotopy invariance, $D_{lc}(\mathcal{C})$.

One knows that only locally constant sheaves and their cohomology enjoy the property of homotopy invariance. More generally we introduce the subcat $D_{lc}(\mathcal{C})$ of $D(\mathcal{C})$ consisting of complexes with locally constant homology sheaves. Given a functor $f : \mathcal{C} \to \mathcal{C}'$ it induces a functor

$$(1) \qquad f^* : D_{lc}(\mathcal{C}') \longrightarrow D_{lc}(\mathcal{C})$$

and the basic fact is that this is an equivalence of categories when f is a homotopy equivalence, and conversely.

Suppose f such that for any locally constant L on $\mathcal{C}$, $Lf_!(L) \in D_{lc}(\mathcal{C}')$. Since

$$L_q f_!(L)(Y) = H_q(f/Y, L)$$

this will be the case provided $\forall (Y \to Y')$ in $\mathrm{Ar}\,\mathcal{C}'$ we have that $f/Y \longrightarrow f/Y'$ is a heq. Then we know that $Lf_! : D^-(\mathcal{C}) \longrightarrow D^-(\mathcal{C}')$ induces a functor

$$(2) \qquad Lf_! : D^-_{lc}(\mathcal{C}) \longrightarrow D^-_{lc}(\mathcal{C}')$$

which is necessarily left adjoint to (1) (restricted to the "$-$" subcategories).

Suppose now that f is arbitrary and that there is a factorization

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "i"] \arrow[dr, "f"'] & & \mathcal{E} \arrow[dl, "p"] \\ & \mathcal{C}' & \end{tikzcd}$$

where i is a heq, and p is such that the [redacted] homotopy type of p/Y is locally constant in Y. Then one sees (because $i^*: D_{lc}(\mathcal{E}) \simeq D_{lc}(\mathcal{C})$) that [redacted] (1) has a left adjoint (defined on D^-), namely $Lp_! \circ (i^*)^{-1}$. This suggests (2) should always exist.

3. <u>Problem</u>: Show the inclusion $D^-_{lc}(\mathcal{C}) \longrightarrow D^-(\mathcal{C})$ has a left adjoint.

Thus given a complex K in $D^-(\mathcal{C})$ we wish to produce a universal arrow $K \to E$ with E in $D^-_{lc}(\mathcal{C})$. Universal ~~map~~ means

$$Hom(E, E') \xrightarrow{\sim} Hom(K, E')$$

for all E' in $D^-_{lc}(\mathcal{C})$. Taking the Postnikov decomp. of E', one sees it is enough to have

$$Hom(E, L[q]) \xrightarrow{\sim} Hom(K, L[q])$$

for all $L \in \mathcal{L} =$ (locally constant sheaves) and all q. (Recall $L[q]$ is the q-th susp. of L, it lives in degree $-q$ with cohomological indexing.)

The construction is inductive. Suppose we have found $K \longrightarrow E(n)$ such that for all L

$$Hom(E(n), L[q]) \longrightarrow Hom(K, L[q]) \qquad \begin{array}{l} \text{iso} \quad q < n \\ \text{inj.} \quad q = n \end{array}$$

Let $C(n) = \text{Cofibre}\ (K \to E(n))$, whence this is equivalent to

$$Hom(C(n), L[q]) = 0 \qquad \forall q \leq n \qquad \forall L.$$

It follows that

$$L \longmapsto Hom(C(n), L[n+1]) \qquad \mathcal{L} \longrightarrow Ab$$

is left exact, and since it commutes with products for essentially trivial reasons, it must be representable, say

$$(*) \qquad Hom(C(n), L[n+1]) = Hom_{\mathcal{L}}(Q_{n+1}, L)$$

Thus we have a canonical map

$$C(n) \longrightarrow Q_{n+1}[n+1]$$

such that

$$(\ast\ast) \qquad \mathrm{Hom}(C(n), L[q]) \longleftarrow \mathrm{Hom}(Q_{n+1}[n+1], L[q])$$

is an isom for $\forall L$ and $q \leq n+1$. In addition since

$$\mathrm{Hom}(Q_{n+1}[n+1], L[n+1]) = \mathrm{Hom}(Q_{n+1}, L[1])$$
$$\cong \mathrm{Ext}^1_{\mathcal{L}}(Q_{n+1}, L)$$

one knows that $(\ast\ast)$ is injective for $q = n+2$. (The point is that both sides are δ-functors and $\mathrm{Ext}^1_{\mathcal{L}}(Q_{n+1}, L)$ is effaceable.)

So now define $E(n+1)$ as the fibre:

$$E(n+1) \longrightarrow E(n) \longrightarrow Q_{n+1}[n+1].$$

Thus we have

$$\begin{tikzcd}
K \arrow[r] \arrow[d, equal] & E(n+1) \arrow[r] \arrow[d] & E(n+1) \arrow[d] \\
K \arrow[r] \arrow[d] & E(n) \arrow[r] \arrow[d] & C(n) \arrow[d] \\
0 \arrow[r] & Q_{n+1}(n+1) \arrow[r, "="] & Q_{n+1}(n+1)
\end{tikzcd}$$

and $(\ast\ast)$ implies $\mathrm{Hom}(C(n+1), L[q]) \simeq 0$ $\quad q \leq n+1$. Completing the induction.

So now one puts $E = \varprojlim E(n)$ and checks that it works.

(This was the approach used June, 1972.)

New approach:

Take $\Delta : \mathcal{C} \longrightarrow \mathcal{C}\times\mathcal{C}$ and consider its standard factorization

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "i"] \arrow[dr, "\Delta"'] & & \mathcal{M}_\Delta \arrow[dl, "p"] \\ & \mathcal{C}\times\mathcal{C} & \end{tikzcd}$$

where $\mathcal{M}_\Delta$ consists of triples $\big(Z, (X,Y), \Delta Z \to (X,Y)\big)$ with the evident notion of map. Thus ~~cons~~ $\mathcal{M}_\Delta$ consists of diagrams of the form

$$X \longleftarrow Z \longrightarrow Y$$

or equivalently

$$\mathcal{M}_\Delta = \mathrm{Funct}\big(Sd(0<1), \mathcal{C}\big)$$

because $\quad Sd(0<1) = (0,0) < (0,1) > (1,1).$

Now p is cofibred, and i is a heq. $\quad i(Z) = (Z, \Delta Z, id_{\Delta Z})$

We have a diagram

$$\begin{tikzcd} \mathcal{C} \arrow[dr, "i"] \arrow[drr, bend left, "id"] \arrow[ddr, bend right, "id"'] & & \\ & \mathcal{M}_\Delta \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathcal{C} \\ & \mathcal{C} & \end{tikzcd}$$

with p_1, p_2 cofibred. Given a sheaf F on $\mathcal{C}$ let

$$\boxed{\Phi(F) = p_{1!}\, p_2^* F}$$

Thus

$$\Phi(F)(X) = \varinjlim_{X \leftarrow Z \to Y} F(Y)$$

and

$$p_{1!}\, p_2^* = pr_{1!}\, p_!\, p^*\, pr_2^*$$

$$(p_!\, p^*(G))(X,Y) = \varinjlim_{p^{-1}(X,Y)} p^*G \qquad \text{as } p \text{ is cofibred}$$

$$= \varinjlim_{\Delta Z \to (X,Y)} G(X,Y) = (\Delta_!\mathbb{Z} \otimes G)(X,Y)$$

(In general $p_!\, p^*(G) = p_!\mathbb{Z} \otimes G$ for f cofibred, and here we have $i_!\mathbb{Z} = \mathbb{Z}$ because i/W are contractibles)

Thus

$$\boxed{\Phi(F) = p_{1!}\, p_2^* F = pr_{1!}(\Delta_!\mathbb{Z} \otimes pr_2^* F)}$$

Note that there is a canonical map

$$F \longrightarrow \Phi(F)$$

defined by either by adjunction

$$id = p_{1!}\, i_!\, i^* p_2^* \longrightarrow p_{1!}\, p_2^*$$

or by the evident map

$$F(X) \longrightarrow \varinjlim_{X \leftarrow Z \to Y} F(Y)$$

associated to the object $X = X = X$ of $p_1^{-1}(X)$.

<u>Lemma</u>: If F is a sheaf on $\mathcal{C}$ the following are equiv.

a) $F \xrightarrow{\sim} \Phi(F)$

b) F is locally constant.

<u>Proof</u>: b) $\Rightarrow$ a) obvious because ~~[illegible]~~ the category $p_1^{-1}(X)$ consisting of $X \leftarrow Z \rightarrow Y$ has an obvious contraction "along the path".

a) $\Rightarrow$ b). Suppose we are given $X \xrightarrow{\alpha} X'$. Then consider the diagram

$$\begin{tikzcd}
F(X) \arrow[rr, "\varepsilon_{X=X=X}"] \arrow[d, "F(\alpha)"'] & & \varinjlim\limits_{X \leftarrow Z \rightarrow Y} F(Y) \arrow[d, "\beta"] \\
F(X') \arrow[urr, dashed, "\varepsilon_{X=X\xrightarrow{\alpha}X'}"] \arrow[rr, "\varepsilon_{X'=X'=X'}"'] & & \varinjlim\limits_{X' \leftarrow Z' \rightarrow Y'} F(Y')
\end{tikzcd}$$

(where $\varepsilon_i : X_i \rightarrow \varinjlim X_i$ is the canonical map). The map β is defined by $\beta\, \varepsilon_{X \xleftarrow{u} Z \rightarrow Y} = \varepsilon_{X' \xleftarrow{\alpha u} Z \rightarrow Y}$. Thus

$$\beta\, \varepsilon_{X=X\xrightarrow{\alpha}X'} = \varepsilon_{X' \xleftarrow{\alpha} X \xrightarrow{\alpha} X'}$$

and as there is an evident arrow

$$\rho : \begin{tikzcd}
X' \arrow[d, equal] & X \arrow[l] \arrow[r] \arrow[d] & X' \arrow[d, equal] \\
X' \arrow[r, equal] & X' \arrow[r, equal] & X'
\end{tikzcd}$$

one knows by definition of the inductive limit that

$$\begin{tikzcd}
F(X' \leftarrow X \rightarrow X') \arrow[rr, "\varepsilon_{X' \leftarrow X \rightarrow X'}"] \arrow[d, "F(\gamma)"] & & \varinjlim F(X') \\
F(X' = X' = X') \arrow[urr, bend right, "\varepsilon_{X'=X'=X'}"] & &
\end{tikzcd}$$

commutes; since $F(\gamma)$ is the identity, we see that the lower right triangle commutes. Similarly we have a map

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, equal] \arrow[d, equal] & X \arrow[d, "\alpha"] \\
X \arrow[r, equal] & X \arrow[r] & X'
\end{tikzcd}$$

which leads to the commutativity of the upper left Δ.

Now by assumption a), the horizontal arrows $\varepsilon_{X=X=X}$, $\varepsilon_{X'=X'=X'}$ are isomorphisms, so it follows that $F(\alpha)$ is an isom. QED.

Given an arbitrary sheaf F put

$$\Phi^\infty(F) = \varinjlim \Phi^n(F)$$

NO $\exists$ at least two possibilities for $\Phi(F) \to \Phi^2(F)$

Since Φ commutes with $\varinjlim$'s we see that

$$\Phi^\infty(F) \xrightarrow{\sim} \Phi(\Phi^\infty(F))$$

hence $\Phi^\infty(F)$ is locally constant. In fact, the arrow

$$F \longrightarrow \Phi^\infty(F)$$

is clearly a universal arrow from F to a locally const. sheaf.

Now define $\Phi : D^{-}(\mathcal{C}) \longrightarrow D^{-}(\mathcal{C})$ by

$$\Phi(F) = Lp_{1!}(p_2^{*}F)$$

so that again we have a natural transf.

$$F \longrightarrow \Phi(F)$$

<u>Lemma:</u> $F \in D^{-}_{lc}(\mathcal{C}) \iff F \xrightarrow{\sim} \Phi(F)$ in $D^{-}(F)$.

~~Proof: (⇒) Because we have heqs.~~ $\cancel{\mathcal{C} \underset{p_1}{\overset{i}{\rightleftarrows}} M_{\Delta}}$
~~and that p_1 is cofibred with contractible fibres and hence $Lp_{1!}$ preserves D_{lc}, one knows that~~

Proof: $(\Rightarrow)$ p_1 is cofibred with contractible fibres hence $Lp_{1!}$ is the homotopy $p_{1!}$ (notation $L^{\pi}p_{1!}$?), hence $Lp_{1!}$ is adjoint to p_1^{*}, hence inverse to p_1^{*} as p_1^{*} is a heq. But i^{*} is also inverse to p_1^{*}, so for F in D^{-}_{lc}

$$Lp_{1!}(p_2^{*}F) = i^{*}p_2^{*}F = F.$$

Can also use the formula

$$F = L^{\pi}p_{1!}\, L^{\pi}i_{!}(i^{*}p_2^{*}(F)) \longrightarrow L^{\pi}p_{1!}(p_2^{*}F)$$

and the fact that $L^{\pi}i_{!} \circ i^{*} \to \mathrm{id}$ since i is an heq.

$(\Leftarrow)$ Have to show $H_q(F) \xrightarrow{\sim} H_q(\Phi(F))$.
Consider least $q \ni H_q(F) \neq 0$. Then

$$H_q(\Phi(F)) = \Phi(H_q(F))$$

($\exists$ spec. seq.

$$E^2_{pq} = L_p p_{1!}(p_2^{*}H_q(F)) \Rightarrow H_n(\Phi(F))\,)$$

so it follows that $H_q(F)$ is locally constant. Now remove it from F and use the fact that ($\Longrightarrow$) has been proved for $\mathcal{H}_q(F)[q]$. Then one finishes by induction.

To compute Φ^2:

$$\begin{tikzcd}
\mathcal{M}_\Delta \times_{\mathcal{C}} \mathcal{M}_\Delta \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & \mathcal{M}_\Delta \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathcal{C} \\
\mathcal{M}_\Delta \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathcal{C} & \\
\mathcal{C} & &
\end{tikzcd}$$

$$\begin{aligned}
\Phi^2(F) &= p_{1!}\, p_2^*\, p_{1!}\, p_2^* \\
&= p_{1!}\, pr_{1!}\, pr_2^*\, p_2^*
\end{aligned}$$

The reason
$$p_2^*\, p_{1!} = pr_{1!}\, pr_2^*$$
is because p_1 is cofibred. But $\mathcal{M}_\Delta \times_{\mathcal{C}} \mathcal{M}_\Delta$ has for its objects diagrams
$$X \leftarrow Z \rightarrow Y' \leftarrow Z' \rightarrow Y$$
and the evident morphisms. Thus it appears that
$$\mathcal{M}_\Delta = \mathrm{Funct}\left(\mathrm{Sd}(0<1), \mathcal{C}\right)$$
$$\mathcal{M}_\Delta \times_{\mathcal{C}} \mathcal{M}_\Delta = \mathrm{Funct}\left(\mathrm{Sd}^2(0<1), \mathcal{C}\right)$$

Thus what you seem to be constructing is a category of paths in $\mathcal{C}$. Thus

$$\mathcal{M}_\Delta^n = \mathcal{M}_\Delta \times_{\mathcal{C}} \cdots \times_{\mathcal{C}} \mathcal{M}_\Delta \qquad n\text{-times}$$

is the category of diagrams in $\mathcal{C}$ of the form

$$X_0 \leftarrow Z_{01} \rightarrow X_1 \leftarrow Z_{12} \rightarrow X_2 \cdots \rightarrow X_n .$$

and $\mathcal{M}_\Delta^n$ leads to the operation Φ^n on sheaves, and on complexes of sheaves. Now I have to understand how to take the limit. There are many choices, for example, if $\alpha_F : F \to \Phi(F)$ is the canonical map, then we have two possibilities

$$\Phi(F) \underset{\Phi(\alpha_F)}{\overset{\alpha_{\Phi(F)}}{\rightrightarrows}} \Phi^2(F)$$

corresponding to the two maps

$$\begin{tikzcd}
 & X \leftarrow Z \rightarrow Y \arrow[dl, mapsto] \arrow[dr, mapsto] \arrow[ddd, mapsto] & \\
X \leftarrow Z \rightarrow Y = Y = Y & & X = X = X \leftarrow Z \rightarrow Y \\
 & & \\
 & X \leftarrow Z = Z = Z \rightarrow Y &
\end{tikzcd}$$

and we also have a third $\rightarrow$

Perhaps a reasonable way to proceed is as follows. Given a subdivision $0 < t_1 < \cdots < t_k < 1$ of the unit interval it defines a category of simplices

$$0 \longrightarrow (0, t_1) \longleftarrow t_1 \longrightarrow \cdots$$

and so it leads to one of the $\mathfrak{M}^n_\Delta$ which we should perhaps denote $\mathfrak{M}^\tau(\mathcal{C})$, τ denoting the subdivision of I. Then if τ' refines τ we have a functor

$$\mathrm{cat}(\tau') \longrightarrow \mathrm{cat}(\tau)$$

hence a functor

$$\mathfrak{M}^{\tau'}(\mathcal{C}) \longleftarrow \mathfrak{M}^\tau(\mathcal{C}).$$

And it is fairly certain that in this way we get an inductive system of categories.

Another simpler way to get an inductive system is to consider the system

$$n \longmapsto \mathrm{Funct}(\mathrm{Sd}^n(0<1), \mathcal{C})$$

(Special case of preceding corresponding to $1/2^n$ divisions)

March 24, 1973:

Consider the category $P(k)$ of f.d.v.s. over a field k to fix the ideas, and form the category of pairs (V^+, V^-) of the same dimension, call it $\mathcal{D}$ for difference. Now the idea is that in $\mathcal{D}$ we can complete squares:

$$\begin{tikzcd}[row sep=small] & \bullet \arrow[dr, dashed] & \\ \bullet \arrow[ur] \arrow[dr] & & \bullet \\ & \bullet \arrow[ur, dashed] & \end{tikzcd}$$

in a functorial way. Check this

$$\begin{tikzcd}[row sep=small] & \vec{V} \oplus \Delta A \arrow[dr] & \\ \vec{V} \arrow[ur] \arrow[dr] & & \vec{V} \oplus \Delta A \oplus \Delta B \\ & \vec{V} \oplus \Delta B \arrow[ur] & \end{tikzcd}$$

where $\vec{V} = (V^+, V^-)$ and $\Delta A = (A, A)$. To be a bit more convincing I should perhaps take the injection-proj. model

$$\begin{tikzcd}[row sep=small] & \vec{V}_1 \arrow[dl, shift left] \arrow[dr, shift left] & \\ \vec{V} \arrow[ur, shift left] \arrow[dr, shift left] & & \vec{V}_1 \times_V \vec{V}_2 \arrow[ul, shift left] \arrow[dl] \\ & \vec{V}_2 \arrow[ul, shift left] & \end{tikzcd}$$

but it's pretty clear.

functorality is no good; see p. 5

The thing to do now is to suppose we are given a category $\mathcal{C}$ such that we have such a completion property.

Suppose now I form the category of diagrams

$$X \longrightarrow Z \longleftarrow Y$$

in $\mathcal{C}$, call this $\mathcal{F} = \underline{\mathrm{Hom}}(\mathrm{Sd}(0<1), \mathcal{C})$. Then $\mathcal{F}$ arises by factoring $\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ into a heq followed by a fibred functor, so $\mathcal{F}$ is fibred over $\mathcal{C}^2$ and cofibred over $\mathcal{C}$, in fact $\mathcal{F}$ is is the fibred cat over $\mathcal{C}^2$ with fibres $(X,Y) \mapsto (X,Y)\backslash\Delta$.

To show that $\mathcal{F} \to \mathcal{C}^2$ is a h-fibration. Thus to show that $\forall\ (X,Y) \to (X',Y')$, we have $(X,Y)\backslash\Delta \leftarrow (X',Y')\backslash\Delta$ is a heq. Enough to suppose $Y = Y'$. Thus have

$$\Big(X \xrightarrow{u\alpha} \underset{\text{variable}}{Z} \xleftarrow{v} Y\Big) \longleftarrow \Big(X' \xrightarrow{u} \underset{\text{var.}}{Z} \xleftarrow{v} Y\Big)$$

where $\alpha : X \to X'$, and I want to prove this is an heq. But define a map backwards as follows

$$(X \to Z \leftarrow Y) \longmapsto (X' \longrightarrow X' \overset{X}{+} Z \leftarrow Y)$$

This is ~~indeed~~ a functor. Now compute the composites.

$$(X \to Z \leftarrow Y) \longmapsto (X' \to X' \overset{X}{+} Z \leftarrow Y)$$

$$(X \to X' \to X' \overset{X}{+} Z \leftarrow Y)$$

seems OKAY.

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & X' \arrow[r, "u"] \arrow[d, "in_2"] & Z \arrow[d, "in_2"] & Y \arrow[l, "v"'] \arrow[dl] \\
X' \arrow[r, "in_1"'] & X' \overset{X}{+} X' \arrow[r, "id+u"'] & X' \overset{X}{+} Z &
\end{tikzcd}$$

Thus there is a problem for what we end up with is

$$X' \xrightarrow{in_1} X' \overset{X}{+} Z \xleftarrow{in_2 v} Y$$

and we start with → $\quad X' \xrightarrow{u} Z \xleftarrow{v} Y$

and there is no obvious map from one to the other. (YES see p. 5)

Probably it will be so that what we are trying to do won't work for the pair category. Thus take $Y = 0$ whence an object

$$X \longrightarrow Z \longleftarrow 0$$

consists of a stable isomorphism

$$X^+ \oplus P \cong X^- \oplus P$$

and morphisms arise by adding on Q. And the point is that two different stable isos. with the same P never become equal. Thus I have not succeeded in killing elementary autos.

But notice that in general, when we can complete, the space of paths

$$X \longrightarrow \bullet \longleftarrow X'$$

is a retract of the space of paths

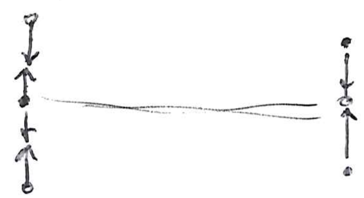
$$X \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \cdot \longleftarrow X'$$

which is a retract of the space of paths

$$X \to \leftarrow \to \leftarrow \bullet \to \leftarrow \cdot \to \leftarrow X'$$

for the embeddings associated to the maps

$$Sd(0<1)^3 \longrightarrow Sd(0<1)^2 \longrightarrow Sd(0<1)$$



etc.

So this example seems to show that the ~~[illegible]~~ conjecture saying that

$$\begin{pmatrix}\text{path} \\ \text{space} \\ \text{from } X \text{ to } X'\end{pmatrix} \underset{\substack{\text{up to} \\ \text{homotopy}}}{\cong} \varinjlim_n \underset{\substack{0 \mapsto X \\ 1 \mapsto X'}}{\mathrm{Funct}}\left(Sd(0<1)^n, \mathcal{C}\right)$$

is wrong. NOT CLEAR

?

The reason one has problems for the pair category is that the square-completion is not functorial. Thus the map

$$\begin{tikzcd}
Z & Z \arrow[l] \arrow[r, "="] & Z \\
Z \arrow[u, "\|"] & 0 \arrow[l] \arrow[r] \arrow[u] & Z \arrow[u, "\|"']
\end{tikzcd}$$

leads to

$$\begin{tikzcd}
Z \arrow[r, "="'] & Z & Z \arrow[l, "="'] \\
Z \arrow[u, "\|"] \arrow[r, "in_1"] & Z \oplus Z \arrow[u, "+"'] & Z \arrow[l, "in_2"'] \arrow[u, "\|"']
\end{tikzcd}$$

and $+$ is not injective.

But on page 3 we have the following map

$$\begin{tikzcd}
X' \arrow[r, "in_1"] \arrow[d, equal] & X' +^{X} Z \arrow[d] & Y \arrow[l, "in_2 v"'] \arrow[d, equal] \\
X' \arrow[r, "in_1"] \arrow[dr, "u"'] & X' +^{\cancel{X}X'} Z \arrow[d, equal] & Y \arrow[l, "in_2 v"'] \arrow[dl, "v"] \\
& Z &
\end{tikzcd}$$

So what we were doing there actually works.

March 25, 1973

To understand stability:

Suppose k is a field to fix the ideas. Let $\mathcal{E}$ be the fibred cat over $Q = Q(P(k))$ with fibre over V the groupoid of extensions

$$0 \longrightarrow K \longrightarrow E \longrightarrow V \longrightarrow 0$$

Then $\mathcal{E}$ is contractible, because the functor $\left(\begin{smallmatrix} E \\ \downarrow \\ V \end{smallmatrix}\right) \longmapsto E$

from $\mathcal{E}$ to injections in $P(k)$ is cofibred and the fibres are contractible.

The subcategory of $E \twoheadrightarrow V$ such that $\mathrm{Ker}(E \twoheadrightarrow V)$ has dim $\geq N$ is also contractible. In effect, it cofibres over the cat of injections of v.s. of dim $\geq N$, and the fibres have initial objects. Call this cat $\mathcal{E}_N$

Let $\overline{\mathcal{E}}_N$ be the fibred category over Q whose fibre over V is the set of $n \geq N$, with $(V, n) \longrightarrow (V', n')$ be a map $V \xrightarrow{u} V'$ such that $n = n' + \dim(\mathrm{Ker}\, u)$. Then $(\overline{\mathcal{E}}_N)_V$ = iso classes of $(\mathcal{E}_N)_V$ and there is an evident functor

$$(1) \qquad \mathcal{E}_N \longrightarrow \overline{\mathcal{E}}_N .$$

On the other hand, we know that the universal covering $\tilde{Q}$ of Q is the fibred cat of all pairs (V, n) with $n \in \mathbb{Z}$. Thus $\overline{\mathcal{E}}_N$ is the sub-fibred cat. of $\tilde{Q}$ consisting of pairs (V, n) with $n \geq N$. Claim

$$(2) \qquad \overline{\mathcal{E}}_N \longrightarrow \tilde{Q}$$

is a h.eq. But $\tilde{Q} = \lim\left(\overline{\mathcal{E}}_N \xrightarrow{+1} \overline{\mathcal{E}}_N \longrightarrow \cdots\right)$

?

Now we use the exact sequence

$$0 \leftarrow \mathbb{Z} \leftarrow C_0(X) \leftarrow C_1(X) \leftarrow C_2(X) \leftarrow M \leftarrow 0$$

$M = H_2(X)$ free abelian group on which G acts. This leads to a spectral sequence with E^1_{*q}:

$$0 \leftarrow H_q(G) \leftarrow H_q(B) \xleftarrow{d_1} H_q(N, \mathbb{Z}^{sg}) \xleftarrow{d_1} H_q(\Sigma_3 \times k^*, \mathbb{Z}^{sg}) \leftarrow H_q(G, M) \leftarrow 0$$

and trivial abutment

First to understand $q = 0$.

$$H_0(N, \mathbb{Z}^{sg}) = \mathbb{Z}/2$$
$$H_0(\Sigma_3 \times k^*, \mathbb{Z}^{sg}) = \mathbb{Z}/2$$

and the map will be induced by taking a 2-simplex (x, y, z) into its boundary $(y, z) - (x, z) + (x, y) \sim$ one 1-simplex; thus d_1 will be an isomorphism between these groups, so

$$E^2_{p0} = 0 \qquad p = 0, 1, 2, 3$$

Next to do $q = 1$:

$$1 \to T \to N \to W \to 1$$

$$H_p(W, H_q(T, \mathbb{Z}^{sg})) \Rightarrow H_n(N, \mathbb{Z}^{sg}).$$

$$0 \to \underset{\substack{\wr\wr \\ k^*}}{(H_1(T) \otimes \mathbb{Z}^{sg})_W} \to H_1(N, \mathbb{Z}^{sg}) \rightleftarrows \underset{\substack{\| \\ \mathbb{Z}/2}}{H_1(W, \mathbb{Z}^{sg})} \to 0$$

$$0 \to \underset{\substack{\| \\ k^*/(k^*)^2}}{(H_1(k^*) \otimes \mathbb{Z}^{sg})_{\Sigma_3}} \to H_1(\Sigma_3 \times k^*, \mathbb{Z}^{sg}) \rightleftarrows \underset{\substack{\| \\ \mathbb{Z}/2}}{H_1(\Sigma_3, \mathbb{Z}^{sg})} \to 0$$

March 27, 1973

On $GL_2(k)$, k a field.

Let X be the simplicial complex whose simplices are the finite non empty subsets of $\mathbb{P}^1_k$. Then X is the full simplex with vertices $\mathbb{P}^1_k$, hence X is contractible, and so we have an exact sequence

$$0 \leftarrow \mathbb{Z} \leftarrow C_0(X) \leftarrow C_1(X) \leftarrow \cdots$$

of $G = GL_2(k)$ modules. Here $C_q(X)$ will denote the group of singular q-chains of X, i.e. $C_{p-1}(X)$ is the free abelian group generated by $(\ell_1, \ldots, \ell_p)$ $\ell_i \in \mathbb{P}^1_k$.

$$C_0(X) = \mathbb{Z}[G/B] \qquad B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} = \text{stabilizer of line } \begin{pmatrix} * \\ 0 \end{pmatrix}$$

$$C_1(X) = \coprod_{\substack{x, y \in \mathbb{P}^1_k \\ x \neq y}} \mathbb{Z}(x,y) \oplus \coprod_{x \in \mathbb{P}^1_k} \mathbb{Z}(x,x)$$

$$= \mathbb{Z}[G/T] \oplus \mathbb{Z}[G/B]$$

where $T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ stabilizer of $\begin{pmatrix} * \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ * \end{pmatrix}$

$$C_2(X) = \coprod_{\substack{(x,y,z) \\ \text{dist.}}} \mathbb{Z}(x,y,z) \oplus \coprod_{\substack{(x,x,y) \\ x \neq y}} \mathbb{Z}(x,x,y) \oplus \coprod_{\substack{(x,y,y) \\ x \neq y}} \mathbb{Z}(x,y,y)$$

$$\oplus \coprod_{\substack{(x,y,x) \\ x \neq y}} \mathbb{Z}(x,y,x) \oplus \coprod_{(x,x,x)} \mathbb{Z}(x,x,x)$$

$$= \mathbb{Z}[G/k^*] \oplus \mathbb{Z}[G/T] \oplus \mathbb{Z}[G/T] \oplus \mathbb{Z}[G/T] \oplus \mathbb{Z}[G/B]$$

Recall that there is a unique proj. transf carrying 3 pts to any other 3 points.

So the exact sequence

$$0 \leftarrow \mathbb{Z} \leftarrow C_0(X) \leftarrow C_1(X) \leftarrow C_2(X) \leftarrow Z \leftarrow 0$$

(where $Z = Z_2(X)$) leads to a spectral sequence with E^1:

$$H_*(G) \leftarrow H_*(B) \leftarrow \begin{matrix} H_*(B) \\ \oplus \\ H_*(T) \end{matrix} \longleftarrow \begin{matrix} H_*(B) \\ \oplus \\ H_*(T)^3 \\ \oplus \\ H_*(k^*) \end{matrix} \leftarrow H_*(G, Z)$$

and with trivial abutment.

Perhaps we shouldn't introduce Z. The point is that

$$E^1_{pq} = H_q(G \text{ acting on } (\mathbb{P}^1_k)^{p+1})$$

is a simplicial abelian group. Now I have to determine the normalized complex, which means you want to take the quotient by the degenerate stuff.

Thus let us take the orbit of $(x_1, \ldots, x_p)$. This is non-degenerate iff $x_1 \neq x_2 \neq x_3 \neq \cdots \neq x_p$, but it can very well happen that $x_1 = x_3$. Non-degenerate quotient complex:

$$H_*(G) \leftarrow H_*(B) \leftarrow H_*(T) \longleftarrow \begin{matrix} H_*(T) \\ \oplus \\ H_*(k^*) \end{matrix} \longleftarrow \begin{matrix} H_*(T) \\ \oplus \\ H_*(k^*)^3 \\ \oplus \\ H_*(k^*) \otimes \mathbb{Z}[k^* - 1] \end{matrix}$$

Possible (x_1, x_2, x_3, x_4): Among the non-degenerate ones we have

$$\begin{array}{lcccccl}
 & x & y & x & y & & x \neq y \\
3 & x & y & x & z & & x, y, z \text{ distinct} \\
 & x & y & z & x & & \text{"} \\
 & x & y & z & y & & \text{"} \\
4 & x & y & z & w & & x, y, z, w \text{ distinct.}
\end{array}$$

But one knows the orbits of GL_2 acting on 4 distinct points are classified by the cross-ratio, which is an element of k^* <u>not equal to 1.</u>

All it remains to do is to ~~stratify~~ determine d_1.

Start with $H_*(B) \longleftarrow H_*(T)$. We have the basic lines $\infty = \binom{*}{0}$ and $0 = \binom{0}{*}$ in $\mathbb{P}^1_k$, and T is their stabilizer. Then

$$\mathbb{Z}[G/T] = \coprod_{x \neq y} \mathbb{Z}(x,y) \underset{d_1}{\overset{d_0}{\rightrightarrows}} \coprod_{x} \mathbb{Z}(x) = \mathbb{Z}[G/B].$$

$$d_0(x,y) = y$$
$$d_1(x,y) = x$$

$$\begin{tikzcd}
(0,\infty) \arrow[r] \arrow[dr, "d_0"'] & 0 \\
 & \infty
\end{tikzcd}$$

$$T \xhookrightarrow{d_0} B \qquad d_0 \text{ is inclusion}$$

$$G/T \longrightarrow G/B \qquad \omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$(0,\infty) \longmapsto 0 = \omega\infty$$

It would seem that d_1 is then induced by conjugating using w.

$(\infty, 0, \infty)$ has stabilizer T

has faces $d_0 = (0, \infty)$ st. T

$d_1 = (\infty, \infty)$ degenerate simplex

$d_2 = (\infty, 0)$ st T

It would seem then that the map would be $id + w^*$.

$(\infty, 1, 0)$ has stab. k^*

has faces

$d_0 = (1, 0)$ st

$d_1 = (\infty, 0)$ st T

$d_2 = (\infty, 1)$ st

but because k^* is central, there is no action so all the d_i coincide. So what we have so far is:

$$H_*(G) \xleftarrow{in} H_*(B) \xleftarrow{in(id - w^*)} H_*(T) \xleftarrow{id + w^*} H_*(T) \oplus H_*(k^*)$$

with $H_*(T) \xleftarrow{in} H_*(k^*)$.

Example: Rational cohomology of $GL_2(k)$.

Consider now the first attempt: Use the simplicial complex of finite subsets of P^1_k and the resolution

$$0 \leftarrow \mathbb{Q} \leftarrow C_0(X,\mathbb{Q}) \leftarrow C_1(X,\mathbb{Q}) \leftarrow \cdots$$

where now we use cellular chains.

$$C_{p-1}(X,\mathbb{Q}) = \coprod_{\substack{(x_1,\dots,x_p) \\ \text{distinct}}} \mathbb{Q}(x_1,\dots,x_p) / \text{action of } \Sigma_p$$

What this means is I have to consider

$$(P^1_k)^p_{reg} / \Sigma_p \times G$$

But one knows that

$$(P^1_k)^p_{reg} = P^p_k - (\text{discriminant}=0) \qquad k = \bar{k}$$

It won't work. What you wanted to show was that every G-orbit has a transposition, so the sign representation would be non-trivial. But it is clearly possible to arrange $(0,1,\infty,z)$ so that no permutation has the same cross-ratio.

March 28, 1973. (Stability continued)

Suppose X is a simplicial complex on which a group G operates. Assume ~~___~~ connectivity of X begins in dim m, and that G operate transitively on the ~~simplices~~ on the ordered q-simplices for each $q = 0, \ldots, m$. Better, I should suppose m such that $\tilde{H}_i(X) = 0$ $i \leq m$, and such that G acts transitively on the ordered q-simplices for $0 \leq q \leq m$.

Now consider singular chains of X.

$$\cdots \longrightarrow C_1(X) \longrightarrow C_0(X)$$

as a complex of G-modules. $C_p(X)$ has a $\mathbb{Z}$-basis consisting of the $(p+1)$-tuples $(v_0, \ldots, v_p)$ of vertices of X which are simplices of X. Thus if we fix a standard m-simplex $(e_0, \ldots, e_m)$, and let G_p = stabilizer of $(e_0, e_1, \ldots, e_p)$

$$G \supset G_0 \supset G_1 \supset \cdots \supset G_p$$

then $H_*(G, C_p(X))$ can be described in terms of the homology of the subgroups G_i for $0 \leq i \leq p$ as follows

First I have to ~~___~~ describe carefully a certain (semi-) simplicial set.

Let S be a finite set with n elements. Then

$$p \longmapsto S^{p+1}$$

is a standard contractible ~~___~~ simplicial set for $S \neq \phi$. It is the singular simplicial set ~~___~~ associated to the simplex with S as vertices. Now ~~_~~ $\mathrm{Aut}(S)$ ~~___~~

acts in an evident way on $[p] \mapsto S^{p+1}$, so we get a quotient simplicial set

$$[p] \longmapsto S^{p+1}/\mathrm{Aut}(S).$$

which we might denote $J(S)$. Observe that $J(S)$ depends only on the cardinality of S. Moreover if we have an <u>injection</u> $\theta : S \hookrightarrow T$, then we get an induced map

$$(*) \qquad S^{p+1}/\mathrm{Aut}(S) \longrightarrow T^{p+1}/\mathrm{Aut}(T)$$

since any auto. of S extends to an auto. of T. (send $(s_0, \dots, s_p) \longmapsto (\theta s_0, \dots, \theta s_p)$; $(\tau s_0, \dots, \tau s_p) \mapsto (\theta\tau s_0, \dots, \theta\tau s_p)$ $= (\tau'\theta s_0, \dots, \tau'\theta s_p) \sim (\theta s_0, \dots, \theta s_p)$ if τ' extends τ.).

Moreover $(*)$ is injective and an isomorphism for $p+1 \leq \mathrm{card}(S)$. Proof. Let $\phi \in \mathrm{Aut}(T)$ be such that

$$\phi(\theta s_0, \dots, \theta s_p) = (\theta s_0', \dots, \theta s_p')$$

To show that we may assume $\phi(\theta S) = \theta S$. But ϕ is given as a bijection from a subset of θS to another subset of θS, so we can choose a bijection of the complements, and so this is clear. Thus $(*)$ is injective, and surjectivity is evident because any element of T^{p+1} can be moved by an isomorphism into S^{p+1} provided $p+1 \leq \mathrm{card}(S)$.

~~We therefore obtain an increasing sequence of simplicial sets~~

$$\cancel{pt = J(1) \subset J(2) \subset J(3) \subset \cdots}$$

~~and we have~~ $\cancel{J(n) = J(n+1)}$ ~~in degrees~~ $\cancel{\leq n-1}$.

We therefore get

$$pt = J(\underline{0}) \subset J(\underline{1}) \subset J(\underline{2}) \subset \cdots$$

where $\underline{p}$ has $(p+1)$-elements, and $J(\underline{n}) = J(\underline{n+1})$ in degrees $\leq n$.

Lemma: The inclusion $J(\underline{n}) \subset J(\underline{n+1})$ is null-homotopic.

Proof: Recall that one shows $(p \mapsto S^{p+1}) \longrightarrow (p \mapsto (S \cup e)^{p+1})$ is null-homotopic by "cone" construction. Thus one defines

$$s_{-1}(z_0, \ldots, z_p) = (e, z_0, \ldots, z_p)$$

and clearly

$$d_0 s_{-1} = id$$
$$d_i s_{-1} = s_{-1} d_{i-1}$$

so that

$$s_{-1}\Big(\sum_{i=0}^{p-1} (-1)^i d_i\Big) + \Big(\sum_{i=0}^{p} (-1)^i d_i\Big) s_{-1} = id$$

But now observe that since e is different from any of the z_i, s_{-1} induces a map on the quotients by $Aut(S)$ and $Aut(S \cup e)$. In effect $(e, z_0, \ldots, z_p)$ and $(e, \phi z_0, \ldots, \phi z_p)$ are conjugate under $Aut(S \cup e)$ for any ϕ in $Aut(S)$. Thus the inclusion $J(S) \subset J(S \cup e)$ is null-homotopic as claimed.

Corollary: $\tilde{H}_i(J(\underline{n})) = 0 \qquad i < n$

In fact one has in dimension n the simplex $(0, 1, \ldots, n)$ of $J(\underline{n})$ whose cone does not exist in $J(\underline{n})$, which suggests that $\tilde{H}_n(J(\underline{n}))$ is cyclic. Note

$$d(0,1,\dots,n) = \sum_{i=0}^{n}(-1)^i(0,1,\dots,n-1)$$

$$= \begin{cases} 0 & n \text{ odd} \\ (0,\dots,n-1) & n \text{ even} \end{cases}$$

Thus if n is even we can define

$$h: C_q(J(\underline{n})) \longrightarrow C_{q+1}(J(\underline{n}))$$

~~to~~ be s_{-1} for $q < n$, and $= s_{-1}$ on n-simplices $\neq (0,1,\dots,n)$ and 0 on $(0,1,\dots,n)$. Because $s_{-1}(0,\dots,n) = (0,\dots,n+1)$ even here.

so $H_n(J(\underline{n})) = 0$ n even.

$$J(\underline{n}) \longrightarrow J(\underline{n+1}) \longrightarrow J(\underline{n+1})/J(\underline{n})$$

1 simplex in degree $n+1$

$$\to H_{n+1}(J(\underline{n+1})) \longrightarrow \tilde{H}_{n+1}(J(\underline{n+1})/J(\underline{n})) \xrightarrow{\partial} H_n(\underline{J}(n)) \xrightarrow{0} H_n(\underline{J}(n+1))$$

$$\tilde{H}_{n+1}(J(\underline{n+1})/J(\underline{n})) = \mathbb{Z}/?$$

$J(\underline{n+1})/\underline{J(n)}$ in deg. n has $(0,1,\dots,n+1)$

$n+1$ symbols $0,1,\dots,n+1$ one is duplicated.

e.g. $(0,1,0,2,\dots,n+1)$

typical case $(0,1,\dots,p,p+1,\dots,p+1,p,p+2,\dots,n+1)$

duplicated vertex

The boundary of this: If a face does not delete p, then it lies in $J(\underline{n})$, so only two faces count, so it's clear that we get [redacted] $\overset{o}{2}$ $(0, 1, \dots, n+1)$.

$$\tilde{H}_{n+1}\left(J(\underline{n+1})/J(\underline{n})\right) = \mathbb{Z}/2. \qquad n \geq 1.$$

Formulas:

$$\boxed{\begin{array}{lll} \tilde{H}_i\left(J(\underline{n})\right) = & 0 & i < n \\ H_n\left(J(\underline{n})\right) = & \begin{cases} \mathbb{Z}/2 & n \text{ odd} \\ 0 & n \text{ even.} \end{cases} & \end{array}}$$

<u>Lemma:</u> $\tilde{H}_*(J(\underline{n}), \mathbb{Q}) = 0.$

<u>Proof:</u>

$$J(\underline{n})_p = (0, \dots, n)^{p+1} / \Sigma_{n+1}$$

$$J(S) = S^{\bullet+1} / Aut(S)$$

so

$$\mathbb{Z}[J(S)] = \mathbb{Z}[S^{\bullet+1}] / Aut(S)$$

And over characteristic zero one would have

$$H_*(J(S), \mathbb{Q}) = H_*(S^{\bullet+1})_{Aut(S)} = \mathbb{Q}.$$

Now return to the situation at the beginning. X is a simp comp on which G acts, m integer, and I suppose that G acts transitively on the set of ordered p simplices for each $p \le m$. I want to explicitly compute the complex

$$\cdots \longrightarrow H_*(G, C_p(X)) \longrightarrow H_*(G, C_{p-1}(X)) \longrightarrow \cdots$$

in terms of the homology of the stabilizers. Here $C_p(X)$ is the free abelian group generated by sequences $(x_0, \ldots, x_p)$ of vertices which lie in the same simplex; call this set $S_p(X)$. Then

$$C_p(X) = \mathbb{Z}[S_p(X)].$$

Further there is a canonical map of simplicial sets

$$(*) \qquad S_\bullet(X)/G \longrightarrow J(S_0(X))$$

which associates to each singular simplex in X the corresponding simplex in the full simplex with the same vertices as X. The assumption that G acts transitively on ordered simp. of degrees $\le m \Longrightarrow (*)$ is an isom. in degrees $\le m$. The point is that any sing. simp. $(x_0, \ldots, x_p)$ determines a repetition pattern — meaning this: Consider those j such that $x_j \notin \{x_0, \ldots, x_{j-1}\}$. Get a subset $0 = j_0 < j_1 < \cdots < j_t$ and in between these changes, we get certain repetitions. Canonical form is therefore a sequence

$$\begin{array}{ccccccc} j_0 & & j_1 & & j_2 & \quad j_t & & p \\ 0 & \underbrace{\quad}_{\substack{\text{any}\\ \text{seq of}\\ 0\text{'s}}} & 1 & \underbrace{\quad}_{\substack{\text{any}\\ \text{seq of}\\ 0,1}} & 2 & \quad t & \underbrace{\qquad\qquad}_{\substack{\text{any}\\ \text{seq of}\\ 0,\dots,t}} & \end{array}$$

Clearly this pattern is independent of the orbit of G on $S_p(X)$, and because G is transitive on the ordered simplices of degree $\leq m$, two elts of $S_p(X)$ with the same repetition pattern are conjugate.

$$p \longmapsto (S_p(X), G) \qquad \text{is a simplicial } G\text{-set}$$

in particular a simplicial groupoid. And what I am constructing now is a functor

$$(S(X), G) \longrightarrow J(\infty)$$

of cofibred categories over Δ°, and I think this is cofibred. The transitivity hypothesis guarantees that the fibre over a pattern is G_t where t is the size of the pattern.

So we have

$$H_*(G, S_p(X)) = \coprod_\alpha H_*(G_{|\alpha|})$$

where α runs over all patterns of length p and $|\alpha|$ is the size of α.

Now I want to compute the face operators. If α is a pattern of size t and $d_j\alpha$ is of size $t-1$, I would like to show that $H_*(G_{|\alpha|}) \longrightarrow H_*(G_{|d_j\alpha|})$ is independent of d_j.

In other words, I would like to know that the local system of homology on $\mathcal{J}(\infty)$ factors through the size functor $\alpha \mapsto |\alpha|$ from $\mathcal{J}(\infty)$ to $\mathbb{N}_+$. So what I have to show is this:

$$\begin{pmatrix}\text{category of}\\ \text{ordered } p\text{-simplices}\end{pmatrix} \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \\ \vdots \\ \xrightarrow[d_p]{} \end{array} \begin{pmatrix}\text{cat. of ordered}\\ (p-1)\text{-simplices}\end{pmatrix}$$

To show these different face functors induce the same map on homology, for example that they are isomorphic functors. Take case $p = 1$. Then pick standard 1-simplex (e_0, e_1), $G_0 =$ stab. of e_0, $G_1 =$ stab. of (e_0, e_1), and let $\omega \in G$ reverse e_0 and e_1. Note that given another ω', then $\omega' \in \omega G_1$. Thus we have two maps

$$\begin{array}{ccc} G_1 & \xrightarrow{\text{inclusion}} & G_0 \\ & \longrightarrow & \\ g & \longmapsto & \omega g \omega^{-1} \end{array}$$

so in fact unless conjugation by ω on G_0 is an inner auto, the two functors will not be isomorphic. So this won't work.

But all I really need however is that $\alpha \mapsto H_*(G_{|\alpha|})$ be locally constant. Thus if I know that the inclusion

$$G_p \hookrightarrow G_{p-1}$$

induces isomorphisms on homology, the same will be true for any of the other face operators. Thus Σ_p acts on G_p mod inner autos, as it permutes the faces around.

List assumptions: 1) G act transitively on ordered p-simplices for $p \leq m$. It follows that

$$H_*(G, S_p(X)) = \coprod_{\alpha \in J(\infty)_p} H_*(G_{|\alpha|}) \qquad p \leq m$$

2) Assume $H_i(G_p) \xrightarrow{\sim} H_i(G_0)$ some range

Now consider

$$E^1_{pq} = H_q(G, S_p(X)) = \coprod_{\alpha \in J_p} H_q(G_{|\alpha|}) \qquad \text{if } p \leq m$$

p-chains on J with coefficients in the local coeff. system.

$$\begin{array}{ccccc} E^1_{0n} \leftarrow E^1_{1n} & & & & \\ & E^1_{1,n-1} & & & \\ & & & E^1_{n+1,1} & \\ & & E^1_{n0} & E^1_{n+1,0} & E^1_{n+2,0} \end{array}$$

I trying to show that $H_{\leq n}(G_0) \xrightarrow{\sim} H_{\leq n}(G)$. Thus I want to know

$\alpha)$ $E^2_{p,q} = 0$ $\qquad p+q \leq n+1$ $\qquad q < n$

so that therefore it seems that we need $n+2 \leq m$ and that $H_q(G_p) \xrightarrow{\sim} H_q(G_0)$ for $q < n$, $p+q \leq n+2$

$\beta)$ $\tilde{H}_p(X) = 0$ for $p \leq n+2$.

Lemma: α) G acts transitively on ordered p-simplices for $p \leq n+2$

β) $\tilde{H}_p(X) = 0 \qquad p \leq n+2$

γ) $H_q(G_p) \xrightarrow{\sim} H_q(G_{p-1}) \xrightarrow{\sim} \cdots \xrightarrow{\sim} H_q(G_0)$ for $\begin{matrix} q < n \\ p+q \leq n+2 \end{matrix}$

Under these conditions I can conclude that there is an exact sequence

$$H_n(G_1) \xrightarrow{id - w} H_n(G_0) \longrightarrow H_n(G) \longrightarrow 0$$

March 30, 1973 (stability continued)

Suppose now $G^N = GL_N$ with N very large (in a nbd. of infinity). Let G^N act on the complex of unimodular vectors in N-space with standard basis $(e_1, e_2, \cdots, e_N)$. Then G_i^N is the stabilizer of $(e_1, \cdots, e_{i+1})$

$$G_i^N = \left(\begin{array}{c|c} I_{i+1} & * \\ \hline 0 & GL_{N-i-1} \end{array}\right)$$

(I need a better notation, eventually). The w operators on G_i^N are the permutation matrices

$$\Sigma_{i+1}^N = \left(\begin{array}{c|c} \Sigma_{i+1} & 0 \\ \hline 0 & I_{N-i-1} \end{array}\right)$$

I am interested in n-dimensional homology. First I should worry about what happens in the split case, i.e. coh. mod l where $l^{-1} \in A$. In this case

$$H_*(G_i^N) \cong H_*(GL_{N-i-1})$$

and now if we are working by induction, we have already established that

$$H_q(GL_N) = H_q(GL_{N-1})$$

for $0 \leq q < n$ and N suff. large. We are assuming that the unimodular complex is highly-connected and that GL_N acts transitively on the ordered simplices of dimensions $\leq n+2$. So by the lemma I get an exact sequence

$$H_n\begin{pmatrix} 1 & & * \\ & 1 & \\ & & GL_{N-2}\end{pmatrix} \xrightarrow{1-w} H_n\left(\begin{array}{c|c} 1 & * \\ \hline & GL_{N-1}\end{array}\right) \to H_n(GL_N) \to 0$$

But because I am assuming that $*$ can be removed it follows that w acts trivially on the former group. $\therefore$ get $H_n(GL_{N-1}) \xrightarrow{\sim} H_n(GL_N)$.

Now try to do the non-split case. Let $\Gamma = \left(\begin{array}{c|c} 1 & * \\ \hline & GL_{N-1}\end{array}\right) = G_0^N$ and I want it to $\overset{\text{right}}{\wedge}$act on the complex of vectors $(1 * * \cdots *)$. Another way of saying this is that we consider the homo

$$\Gamma \xrightarrow{\alpha \mapsto \alpha^{-1}} \begin{pmatrix} 1 & \\ * & GL_{N-1}\end{pmatrix}$$

& the natural action of the latter on the affine space $\begin{pmatrix} 1 \\ * \\ \vdots \\ * \end{pmatrix}$. Again the action is transitive on simplices. Let

$$\Gamma_i = \text{stabilizer of } {}_{i+1}\left\{\begin{array}{l} (1\ 0 \cdots 0) \\ (1\ 1\ 0 \cdots 0) \\ \\ (1\ 0 \cdots 1) \end{array}\right.$$

Thus

$$\Gamma_0 = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & GL_{N-1}\end{array}\right) \qquad \Gamma_1 = \begin{pmatrix} 1 & & \\ & 1 & 0 \\ & * & GL_{N-2}\end{pmatrix}$$

$$\Gamma_j = \left(\begin{array}{c|c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & 0 \\ \hline * & GL_{N-j-1}\end{array}\right) \updownarrow j$$

We are assuming by induction that for $q < n$

$$H_q(GL_N) \xleftarrow{\sim} H_q\begin{pmatrix} 1 & * \\ & GL_{N-1} \end{pmatrix} \xleftarrow{\sim} H_q(GL_{N-1}) \qquad N \text{ suff. large}$$

This is equivalent (applying the transpose conjugate automorphism) to

$$H_q(GL_N) \xleftarrow{\sim} H_q\begin{pmatrix} 1 & 0 \\ * & GL_{N-1} \end{pmatrix} \xleftarrow{\sim} H_q(GL_{N-1})$$

Now consider $\Gamma_j \hookrightarrow \Gamma_{j-1}$. The point:

$$\Gamma_{j-1} = \begin{pmatrix} 1 & & & \\ & I_{j-1} & & \\ & * & \boxed{\begin{matrix} * & * \\ * & * \end{matrix}} \\ & * & & \end{pmatrix} \supset \begin{pmatrix} 1 & & \\ & I_j & \\ & * & GL_{N-j-1} \end{pmatrix} = \Gamma_j$$

$$\bigcup_{\substack{\text{coh. iso} \\ \deg < n}} \qquad\qquad \bigcup$$

here you need $H_q\begin{pmatrix} I_r & * \\ & GL_{N-r} \end{pmatrix} = H_q(GL_{N-r})$

$$\begin{pmatrix} 1 & & \\ & I_{j-1} & \\ & & GL_{N-j} \end{pmatrix} \supset \begin{pmatrix} 1 & & & \\ & I_{j-1} & & \\ & & 1 & \\ & & & GL_{N-j-1} \end{pmatrix}$$

Therefore by the lemma we get an exact sequence

$$H_n(\Gamma_1) \xrightarrow{1-w} H_n(\Gamma_0) \longrightarrow H_n(\Gamma) \longrightarrow 0$$

Computation of w. Recall Γ_1 stabilizes $\begin{matrix} 1 & 0 & \cdots \\ 1 & 1 & \cdots \end{matrix}$ and $w \in \Gamma$ interchanges these. Thus

$$w = \left(\begin{array}{cc|c} 1 & 1 & \\ & -1 & \\ \hline & & I_{N-2} \end{array}\right) \in \Gamma .$$

Now must compute conjugation by w on Γ_1

$$\begin{pmatrix} 1 & 1 & \\ & -1 & \\ & & I \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \\ & \beta & \alpha \end{pmatrix}\begin{pmatrix} 1 & 1 & \\ & -1 & \\ & & I \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & \beta & \alpha \end{pmatrix}\begin{pmatrix} 1 & 1 & \\ & -1 & \\ & & I \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\beta & \alpha \end{pmatrix}.$$

However observe that this auto of Γ_1 is induced by conjugation within Γ_0:

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & -I \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \\ & \beta & \alpha \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -I \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & \\ & 1 & \\ & -\beta & -\alpha \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -I \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & -\beta & \alpha \end{pmatrix}$$

So therefore $\quad 1 - w : H_n(\Gamma_1) \longrightarrow H_n(\Gamma_0)$ is zero, and

so we get

$$H_n(\Gamma_0) \xrightarrow{\sim} H_n(\Gamma)$$

i.e.

$$H_n(GL_{N-1}) \xrightarrow{\sim} H_n\begin{pmatrix} 1 & * \\ & GL_{N-1} \end{pmatrix}$$

for N large.

Now using the fact that everything works for dimensions $< n$, we can derive as before an exact sequence

$$H_n\left(\begin{array}{cc|c} 1 & & * \\ & 1 & \\ \hline 0 & & GL_{N-2}\end{array}\right) \xrightarrow{1-w} H_n\left(\begin{array}{c|c} 1 & * \\ \hline 0 & GL_{N-1}\end{array}\right) \longrightarrow H_n(GL_N) \to 0$$

However

$$\begin{pmatrix} 1 & & \\ & 1 & * \\ & & * \end{pmatrix} \overset{\text{no}}{\triangleleft} \begin{pmatrix} 1 & \cdot & * \\ & 1 & * \\ & & * \end{pmatrix}$$

$$\cup \qquad\qquad\qquad \cup$$

$$\begin{pmatrix} 1 & & \\ & 1 & 0 \\ & & * \end{pmatrix} \overset{\text{no}}{\triangleleft} \begin{pmatrix} 1 & & * \\ & 1 & 0 \\ & & * \end{pmatrix} \quad ?$$

Still doesn't work

Must strengthen the induction

Assume for $q < n$ that

$$H_n(GL_{N-r}) \xrightarrow{\sim} H_n\left(\begin{array}{c|c} I_r & * \\ \hline 0 & GL_{N-r}\end{array}\right)$$

<u>Idea:</u> Show homology onto to get trivial w-action.

Let's go back to the idea of inverting. The idea has to do with Hopf alg.

Key point: A connected H-space has a homotopy inverse.

$$\begin{tikzcd} \overset{a\ \ b}{M \times M} \arrow[rr] \arrow[dr, "pr_1"'] & & \overset{a\ \ ab}{M * M} \arrow[dl, "pr_1"] \\ & M & \end{tikzcd}$$

map of filtrations heq over $a=0$

inverse [illegible]

$$\bar{H}_*(M) \otimes H_*(M) \longrightarrow H_*(M) \otimes H_*(M)$$

$$\alpha \otimes \beta \longmapsto \sum_i \alpha_i' \otimes \alpha_i'' \beta.$$

should be an isomorphism. Indeed we filter both sides by setting

$$F_p\left(\bar{H}_*(M) \otimes H_*(M)\right) = \bigoplus_{i \le p} H_i(M) \otimes H_*(M)$$

$$\alpha \in F_p H(M)$$

$$\sum_i \alpha_i' \otimes \alpha_i'' \beta \in F_p(H(M) \otimes H(M))$$

has leading term $\alpha \otimes \beta$.

Thus map is an isom. on gr's hence is an isomorphism.