fibration $\mathcal{S}^{-1}\mathcal{S} \longrightarrow \mathcal{S}^{-1}\mathcal{X} \longrightarrow \langle \mathcal{S}, \mathcal{X} \rangle$

We finish this section with a result which identifies the homotopy-fibre of the functor

$$p: \mathcal{S}^{-1}\mathcal{X} = \langle \mathcal{S}, \mathcal{S} \times \mathcal{X} \rangle \longrightarrow \langle \mathcal{S}, \mathcal{X} \rangle$$

induced by $pr_2: \mathcal{S} \times \mathcal{X} \longrightarrow \mathcal{X}$. Given an object Y of $\mathcal{X}$, we obtain a functor

$$i_Y : \mathcal{S} \longrightarrow \mathcal{X} \quad , \quad S \longmapsto S \# Y$$

which is ~~compatible~~ map of cats with $\mathcal{S}$-action. Hence it yields a comm. square

$$( ) \qquad \begin{tikzcd} \mathcal{S}^{-1}\mathcal{S} \arrow[r, "\mathcal{S}^{-1}(i_Y)"] \arrow[d] & \mathcal{S}^{-1}\mathcal{X} \arrow[d, "p"] \\ \langle \mathcal{S}, \mathcal{S} \rangle \arrow[r] & \langle \mathcal{S}, \mathcal{X} \rangle . \end{tikzcd}$$

Prop: Assume
i) every morphism in $\mathcal{X}$ is a mono
ii) $\forall X \in \mathcal{X}$, the functor $\mathcal{S} \to \mathcal{X}$, $S \mapsto S \# X$ is faithful.
Then the above square is homotopy-cartesian. Hence the homotopy-fibre of p over any object of $\langle \mathcal{S}, \mathcal{X} \rangle$ is homotopy equivalent to $\mathcal{S}^{-1}\mathcal{S}$.

Proof: The last assertion follows from the first one because the category $\langle \mathcal{S}, \mathcal{S} \rangle$ ~~has an initial object, hence~~ is contractible (ref.).

~~Since $\langle \mathcal{S}, \mathcal{S} \rangle$ has initial object 0, there is a natural transf. to the functor $p\, \mathcal{S}^{-1}(i_Y)$ from the constant functor with value Y. Thus we obtain~~

~~Let $\mathcal{S}$ act on $\mathcal{X}$~~

Hyp. i) + ii) $\Longrightarrow$ any arrow $x : X' \longrightarrow X$ in $\langle \mathcal{S}, \mathcal{X} \rangle$ has a ~~unique~~ rep.

$$a_x : T_x \# X' \longrightarrow X$$

where (T_x, a_x) is determined up to unique isomorphism. In particular, given

$$X'' \xrightarrow{x'} X' \xrightarrow{x} X$$

the composition xx' is represented by

$$(T_x \perp T_{x'}) \# X'' = T_x \# (T_{x'} \# X'') \xrightarrow{id \# a_{x'}} T_x \# X' \xrightarrow{a_x} X$$

hence there is a canonical isom.

$$T_x \perp T_{x'} \simeq T_{xx'}$$

satisfying the evident transitivity conditions.

So now let $\mathcal{S}$ act on itself (to the right) and let $\mathcal{F}$ denote the fibred cat over $\langle \mathcal{S}, \mathcal{X} \rangle$ with $\mathcal{F}_X = \mathcal{S}$ for all X and $\ni$ for $x : X' \longrightarrow X$

$$x^* = (? \perp T_x) : \mathcal{S} \longrightarrow \mathcal{S}$$

(thus $\quad (xx')^* \cong (? \perp T_x \perp T_{x'}) \cong x'^* x^*$)

Specifically an object of $\mathcal{F}$ is a pair (S, X); a map $(S, X) \longrightarrow (S', X')$ is an iso class of triples $(T, S \simeq S' \perp T, T \# X \longrightarrow X')$; obvious composition

Have functor

$$(*) \qquad \mathcal{F} \longrightarrow \mathcal{X} \qquad \cancel{(S,X) \longmapsto S\# X}$$

sending $\quad (S,X) \longmapsto S\# X$

$$(T,\ S \simeq S' \perp T,\ T\# X \to X') \longmapsto \left(\begin{array}{l} S\# X \simeq (S'\perp T)\# X \\ \qquad\qquad \| \\ \qquad S'\#(T\# X) \to S'\# X' \end{array} \right)$$

Claim $(*)$ is a heq. Define

$$(**) \qquad \mathcal{X} \longrightarrow \mathcal{F} \qquad X \longmapsto (0,X)$$

$$(X \to X') \longmapsto (0,\ 0 \simeq 0\perp 0,\ 0\# X = X \to X')$$

Clearly $(*)(**) \simeq \mathrm{id}$; on other hand $\exists\ (**)(*) \leftarrow \mathrm{id}$

$$\begin{tikzcd} (S,X) \arrow[r, mapsto] \arrow[d, mapsto] & (S\# X) \arrow[r, mapsto] & (0, S\# X) \\ (S,X) \arrow[urr, "{(S,\ S \simeq 0\perp S,\ S\# X \xrightarrow{id} S\# X)}"'] & & \end{tikzcd}$$

Now define $\mathcal{S}$-action on $\mathcal{F}$ by

$$T\# (S,X) = (T\perp S, X)$$

Then this action is cartesian relative to $\langle \mathcal{S}, \mathcal{X}\rangle$ so we know $\mathcal{S}^{-1}\mathcal{F}$ is fibred over $\langle \mathcal{S}, \mathcal{X}\rangle$ with fibres $\mathcal{S}^{-1}\mathcal{F}_X = \mathcal{S}^{-1}\mathcal{S}$ and base change functors:

$$x^* : (S_1, S_2) \longmapsto (S_1, S_2 \perp T_x)$$

By commutativity of $\mathcal{S}$, we know these base change functors are heq's $\Longrightarrow$ $\mathcal{S}^{-1}\mathcal{F}_X$ = h-fibre of $\mathcal{S}^{-1}\mathcal{F} \longrightarrow \langle \mathcal{S}, \mathcal{X} \rangle$ over X.

Also clear that $(*) : \mathcal{F} \to \mathcal{X}$ is compatible with $\mathcal{S}$-action, so it induces

$$\mathcal{S}^{-1}(*) : \mathcal{S}^{-1}\mathcal{F} \longrightarrow \mathcal{S}^{-1}\mathcal{X}$$

which is a heq.

$$\begin{tikzcd}
\mathcal{S}^{-1}\mathcal{X} \arrow[dr, dotted] & \mathcal{S}^{-1}\mathcal{F} \arrow[l, "\mathcal{S}^{-1}(*)"', "heq"] \arrow[d] \\
 & \langle \mathcal{S}, \mathcal{X} \rangle
\end{tikzcd}$$

To show Δ h-comm.

$$\begin{tikzcd}
(s_1, s_2, X) \arrow[r, mapsto, "\mathcal{S}^{-1}(*)"] \arrow[d, mapsto] & (s_1, s_2 \# X) \arrow[r, mapsto, "p"] & (s_2 \# X) \\
X \arrow[urr, dashed] & &
\end{tikzcd}$$

and it is clear this gives a nat. transf.

Finally one has

$$\underset{(s_1,s_2)}{\mathcal{S}^{-1}\mathcal{S}} \longrightarrow \underset{(s_1,s_2,X)}{\mathcal{S}^{-1}\mathcal{F}} \longrightarrow \underset{(s_1,s_2X)}{\mathcal{S}^{-1}\mathcal{X}}$$

is our friend $\mathcal{S}^{-1}(i_X)$, so its all working now!

Higher algebraic K-theory, II

to be written with the idea of clearing off the rest of your ideas on the subject.

K-theory of a ring: $BGL(A)^+$, K_iA.

stable splitting and comparison theorems.

The localization thm. and fundamental thm.? NOT this time

Stability and finite generation

Adams operations, products, delooping Q, semi-simp approach

Remark: One advantage of the proof with $\mathcal{F}$ is that one doesn't need to know $\mathcal{S}$ is commutative, only that

$$(S_1, S_2) \longmapsto (S_1, S_2 + T)$$

is a heq of $\mathcal{S}^{-1}\mathcal{S}$, $\forall\, T \in \mathcal{S}$.

Other proof:

$Y \backslash p$ consists of $((S, X), x : Y \to X)$

and a map $(S, X, x) \to (S', X', x')$ rep by

$$TS = S'$$
$$TX \to X'$$

and an isom $TT_x \cong T_{x'}$ $\ni$

$$\begin{tikzcd} TT_x Y \arrow[r, "Ta_x"] \arrow[d, "\wr"] & TX \arrow[d] \\ T_{x'} Y \arrow[r, "a_{x'}"'] & X' \end{tikzcd}$$

commutes.

Thus can define a functor

(1) $$Y \backslash p \longrightarrow S^{-1} S$$

$$(S, X, x) \longmapsto (S, T_x)$$

$$(TS = S',\ TX \to X',\ TT_x = T_{x'}) \longmapsto (TS = S',\ TT_x = T_{x'})$$

$$(S', X', x') \longmapsto (S', T_{x'})$$

On the other hand can define functor

(2) $$S^{-1} S \longrightarrow Y \backslash p$$

$$(S_1, S_2) \longmapsto (S_1, S_2 Y, c_{S_2, Y} : Y \to S_2 Y)$$

$$c_{S_2, Y} \overset{d}{=} (S_2,\ id : S_2 Y \to S_2 Y)$$

~~Clearly~~

$$\begin{tikzcd}
(S_1, S_2) \arrow[r, mapsto] \arrow[d, "{(TS_1 \simeq S_1',\ TS_2 \simeq S_2')}"'] & (S_1, S_2Y, (S_2, id: S_2Y \Rightarrow S_2Y)) \arrow[d, "{TS_1 = S_1',\ TS_2Y \simeq S_2'Y,\ T^{S_2} = T^{S_2'}_{S_2}}"] \\
(S_1', S_2') \arrow[r, mapsto] & (S_1', S_2'Y, (S_2', id: S_2'Y \Rightarrow S_2'Y))
\end{tikzcd}$$

Clearly $(2)(1) \simeq id_{S^{-1}S}$

$$(S_1, S_2) \longmapsto (S_1, S_2Y, (S_2, id)) \longmapsto (S_1, S_2).$$

$$\begin{tikzcd}
(S, X, a_x: T_x \# Y \to X) \arrow[r, mapsto, "(1)"] \arrow[dr, "id"'] & (S, T_x) \arrow[r, mapsto, "(2)"] & (S, T_xY, (T_x, T_xY = T_xY)) \arrow[dl, "{(0S = S,\ 0T_xY = T_xY \xrightarrow{a_x} X,\ 0T_x = T_x)}"] \\
& (S, X, x) &
\end{tikzcd}$$

and one must check this is a natural transf from $(2)(1) \longrightarrow id$.

Finally given $Y' \xrightarrow{y} Y$ $\quad T_y \# Y \xrightarrow{a_y} Y'$

$$\begin{tikzcd}
Y/P \arrow[r] \arrow[d] & S^{-1}S \arrow[d, "T_y"] \\
Y'/P \arrow[r] & S^{-1}S
\end{tikzcd}$$

$$
\begin{array}{ccc}
(S, X, x) & \longmapsto & (S, T_x) \\
\big\downarrow & & \\
\multicolumn{3}{c}{(S, X, xy : (T_x T_y,\ T_x T_y Y' \to T_x Y \to X))} \\
& \longmapsto & (S, T_x T_y)
\end{array}
$$

So the commutativity of the square shows that $Y \backslash p \to Y' \backslash p$ heq $\forall\ Y' \to Y$ in $\langle S, X \rangle$. $\Rightarrow$ $Y \backslash p$ = h-fibre of p over Y.

$$\Longrightarrow \quad S^{-1}S \xrightarrow{(s_1, s_2) \mapsto (s_1, s_2 Y)} S^{-1}X \longrightarrow \langle S, X \rangle$$

h-fibration

original proof hypotheses i), ii) as before

Prop: $S^{-1}S \longrightarrow S^{-1}X \xrightarrow{\;p\;} \langle S, X\rangle$ h-fibration

$$(S_1, S_2) \longmapsto (S_1, S_2 X)$$
$$(S, X) \longmapsto X$$

Proof: hyp. i) + ii) $\Longrightarrow$ p cofibred with

$$p^{-1}(X) = S$$

$$x^* = (T_x \perp ?) : S \longrightarrow S.$$

Make S act on $S^{-1}X$ by

$$T\#(S, X) = (T+S, X)$$

action is fibre-wise and cocartesian rel to $\langle S, X\rangle$, and moreover we know the action is invertible (by earlier stuff using commutativity) Thus have $S^{-1}(S^{-1}X)$ cofibres over $\langle S, X\rangle$ with fibres $S^{-1}S$ and base change as above, $\therefore$ get

$(S_1, S_2) \mapsto (S_1, S_2, X_0)$, $(S_1, S_2, X) \mapsto X$

$$\begin{tikzcd}
S^{-1}S \arrow[r] & S^{-1}(S^{-1}X) \arrow[r, "\text{h-fibn}"] & \langle S, X\rangle \\
& S^{-1}X \arrow[u, "\text{heq}"'] \arrow[ur, "p"'] &
\end{tikzcd}$$

$(0, S, X) \leftarrow\!\shortmid (S, X)$

maps $(S_1, S_2, X) \longrightarrow (S_1', S_2', X')$ are

$$(T_1 + S_1 \simeq S_1',\ T_1 + S_2 + T_2 \simeq S_2',\ T_2 + X \to X')$$

hence we get

$$\begin{tikzcd}
 & S^{-1}\mathcal{D}_0 \arrow[r] \arrow[dl] \arrow[d] & S^{-1}\mathcal{E}_0 \arrow[d] & \\
Q(\mathcal{H}) & S^{-1}\mathcal{D} \arrow[l] \arrow[r] & S^{-1}\mathcal{C} \arrow[r] \arrow[d] & S^{-1}\bar{Q}(\mathcal{V}) \arrow[d] \\
 & & Q(\mathcal{P}) \arrow[r] & Q(\mathcal{V})
\end{tikzcd}$$

$$(S_1, S_2) \longmapsto (S_1, S_2, X_0) \qquad (0, S_2, S_1X_0)$$

$$\begin{tikzcd}
S^{-1}S \arrow[r] \arrow[dr] & S^{-1}(S^{-1}\mathcal{X}) & \\
 & S^{-1}\mathcal{X} \arrow[u] &
\end{tikzcd}$$

$$(S_2, S_1X_0) \qquad (0, S, X) \leftarrow (S, X)$$

and you have

$$(0, S_2, S_1X_0) \longrightarrow (S_1, S_1S_2, S_1X_0) \longleftarrow (S_1S_2, X_0)$$

showing that $S^{-1}S \longrightarrow S^{-1}(S^{-1}\mathcal{X}) \longleftarrow S^{-1}\mathcal{X}_0$ in the h-cat is the negative of $S^{-1}S$ followed by the map $S^{-1}S \to S^{-1}\mathcal{X}$

$$(S_1, S_2) \longmapsto (S_1, S_2X_0).$$

Let $\mathcal{S}$ act on $\mathcal{X}$ and consider the functor

$$\mathcal{S}^{-1}\mathcal{X} = \langle \mathcal{S}, \mathcal{S}\times\mathcal{X}\rangle \longrightarrow \langle \mathcal{S}, \mathcal{X}\rangle$$

induced by $pr_2 : \mathcal{S}\times\mathcal{X} \longrightarrow \mathcal{X}$. Under what conditions will this be cofibred. ~~Fibre over X consists of (S, X) with maps classes of~~

~~$T\#S \to S'$~~

~~$T\#X \to X'$ must be identity~~

~~$T \simeq$~~ Fix an arrow $(T, T\#X \to X')$ in $\langle \mathcal{S}, \mathcal{X}\rangle$. Assume

[In $\mathcal{X}$ every arrow is a mono
$\forall X \quad T \mapsto T\#X$ faithful from $\mathcal{S}$ to $\mathcal{X}$

Then if $(T, T\#X \xrightarrow{u} X')$ is isom to $(T', T'\#X \xrightarrow{u'} X')$ say by $v : T \xrightarrow{\sim} T'$ we see v is unique

$$\begin{tikzcd}
T\#X \arrow[dr, "u"] \arrow[d, shift right, "v_1\#id_X"'] \arrow[d, shift left, "v_2\#X"] & \\
T'\#X \arrow[r, "u'"'] & X'
\end{tikzcd}$$

since u' mono $\Rightarrow v_1\#id_X = v_2\#id_X \Rightarrow v_1 = v_2$. In particular T is determined up to unique isom. Now then any map $(S, X) \longrightarrow (S', X')$ lying over $T\#X \longrightarrow X'$ is same as a map $T\#S \longrightarrow S'$, so the above functor is cofibred with fibres $\simeq \mathcal{S}$, and cobase changes with $T\#X \to X'$ isom to $T\#?$ on $\mathcal{S}$.

Assume $\mathcal{S}$ commutative so we can make $\mathcal{S}$ act on $\mathcal{S}^{-1}\mathcal{X}$ over $\langle \mathcal{S}, \mathcal{X} \rangle$ by

$$T_0 \# (S, X) = (T_0 \# S, X)$$

(This is ~~an action~~ well-defined because we send

$$T_0 S \to S' \qquad T_0 X \to X'$$

to

$$\begin{array}{ccc} T_0(T_0 S) \to T_0 S' & \qquad & (T_0 X) \to T_0 X' \\ \| & & \# \\ T(T_0 S) & & \end{array}$$

and we argue that this is well defined ~~and~~ comp. with comp.

$$T'S' \to S'' \qquad T'X' \to X''$$

$$\begin{array}{ccccc} T_0 T' T S & \to & T_0 T' S' & \to & T_0 S'' \\ \| & & \| & & \\ T'(T_0 T S) & \to & T' T_0 S' & & \\ \| & & & & \\ T' T (T_0 S) & & & & \end{array}$$

~~So then I can form~~

~~$S^{-1}(S^{-1}X) \longrightarrow \langle S, X\rangle$~~

~~which,~~

So I have this S-action on $S^{-1}X$ (first comp) and it is cocartesian relative to $\langle S, X\rangle$, hence I know that

$$S^{-1}(S^{-1}X) \longrightarrow \langle S, X\rangle$$

is cofibred with fibres $\simeq S^{-1}S$ and with $(TX \to X')_* \simeq ((S,S') \mapsto (S, TS'))$. Also I know $S^{-1}(S^{-1}X)$ fibres over $S^{-1}(pt)$ with fibres $\simeq S^{-1}X$ and $(TS \to S')_* \simeq ((S,X) \mapsto (TS,X))$.

(To be clearer $S^{-1}(S^{-1}X)$ is isom the foll. cat. Objects (S_1, S_2, X) and a morph. $(S_1,S_2,X) \longrightarrow (S_1', S_2', X')$ consists of iso class of T_1, T_2, α

$$T_1 S_1 \xrightarrow{\sim} S_1' \qquad T_2 T_1 S_2 \longrightarrow S_2' \qquad T_2 X \to X')$$

Then have

$$\begin{aligned} S^{-1}(S^{-1}X) &\longrightarrow \langle S, X\rangle \\ (S_1, S_2, X) &\longmapsto X \end{aligned}$$

cofibred fibres $\cong S^{-1}S$, $(T_2X \to X')_* = ((S_1,S_2) \mapsto (S_1, T_2S_2))$

and have

$$\begin{aligned} S^{-1}(S^{-1}X) &\longrightarrow S^{-1}(pt) \\ (S_1, S_2, X) &\longmapsto S_1 \end{aligned}$$

cofibred with fibres $\cong S^{-1}X$, $(T_1S_1 \to S_1')_* = ((S_2, X) \mapsto (T_1S_2, X))$

But we know already that

$$(S_1, S_2) \longmapsto (S_1, T_2 S_2)$$

is invertible on $S^{-1}S$, and that

$$(S_2, X) \longmapsto (T_1 S_2, X)$$

is invertible on $S^{-1}X$. Thus conclude that we get hfibrations. Also $S^{-1}(pt)$ contractible $\Longrightarrow$

$$S^{-1}X \hookrightarrow S^{-1}(S^{-1}X) \qquad \text{heq}$$

$$(S_2, X) \longmapsto (0, S_2, X)$$

and also we get fibration

$$(S_1, S_2) \longmapsto (S_1, S_2, X_0)$$

$$\begin{tikzcd}
S^{-1}S \arrow[r] & S^{-1}(S^{-1}X) \arrow[r] & \langle S, X \rangle \\
 & S^{-1}X \arrow[u, "\cup\ \text{heq}"'] \arrow[ur, "\text{can}"'] &
\end{tikzcd}$$

$$(S_2, X) \longmapsto X$$

~~So we can say that~~

$$\begin{tikzcd}
S^{-1}S \arrow[r, "{(0,S,X_0)}"] & S^{-1}(S^{-1}X) \\
S \arrow[u, "{(0,S)}"] \arrow[r, "{(S,X_0)}"'] \arrow[ur, phantom, "\text{commutes}"] & S^{-1}X \arrow[u, "\text{heq}"']
\end{tikzcd}$$

So the only thing left is to ~~[illegible]~~ connect up with $S^{-1}S \to S^{-1}X$ $\quad (S_1, S_2) \longmapsto (S_1, S_2 X)$, but

$$\begin{tikzcd}
S^{-1}S \arrow[r] \arrow[dr] & S^{-1}(S^{-1}(X)) \\
 & S^{-1}X \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
(s_1, s_2) \arrow[r, mapsto] \arrow[d] & (s_1, s_2, x_0) \arrow[r] & (s_1, s_1 s_2, s_1 x_0) \\
\cancel{(s_1, s_2, x_0) \to (0, s_1, s_2, x_0)} & & \\
(s_2, s_1 x_0) \arrow[r, mapsto] & (0, s_2, s_1 x_0) \arrow[uur] &
\end{tikzcd}$$

Thus we see that we get the inverse map.

<u>Conclusion</u>: S a truc-groupoid ACU

$\phi \neq X \ni s \mapsto s\#X$ all arrows monos faithful from S to X.

Then have in homotopy category a fibration

$$S^{-1}S \longrightarrow S^{-1}X \longrightarrow \langle S, X \rangle$$

$$(s, x) \longmapsto x.$$

$$(s_1, s_2) \mapsto (s_1, s_2 x)$$

<u>Example</u>: Assume S truc-groupoid ACU & $\perp : S \times S \to S$ is <u>faithful</u>. Then get fibration

$$S^{-1}S \longrightarrow S^{-1}\langle S, S\rangle \longrightarrow \langle S \times S, S\rangle$$

so $S^{-1}S \sim \Omega\langle S \times S, S\rangle$.

1

relation with Bass-Milnor approach

A ring (assoc. with 1), GL_nA, E_nA, $GL(A) = \bigcup GL_nA$, $E(A) = \bigcup E_n(A)$. Whitehead lemma $\Rightarrow$ $E(A) = (GL(A), GL(A))$ and $E(A) = (E(A), E(A))$. $St(A)$ = Steinberg group = universal central extension of $E(A)$. Put $H_i(G)$ = group coh with $\mathbb{Z}$ coeffs.

Definition:

$$K_1 A = H_1(GL(A)) = GL(A)/E(A)$$

$$K_2 A = H_2(E(A)) = \operatorname{Ker}\{St(A) \to E(A)\}.$$

To extend to higher dimensions it is necessary to interpret these groups as homotopy groups of a space.

In rest of section we work only with connected spaces with basepoint which are of h type of a CW complex. Maps are basepoint-preserving and we identify homotopy maps in the usual way.

Let BG be a classifying space for G; it is a Eilenberg-Maclane space type $(G,1)$. Local coeff. systems on BG may be identified with G-modules, and we have canon. isos.

$$H_*(BG, M) = H_*(G, M)$$

where on right is singular cohomology of the space BM and on the left the group cohomology.

Recall that a space is simple if $\pi_1 X$ acts triv. on $\pi_* X$, and that an H-space is simple.

Prop 1: Let $f: Y \to X$ be a homology isom. Given a simple space Z and a map $g: Y \to Z$, there exists $h: X \to Z$, unique up to homotopy such

Defn. $K_iA = \pi_i BGL(A)^+$ for $i \geq 1$.
(K_0A = Groth. gp of $\mathcal{P}(A)$).

Prop. 1 $\Rightarrow$ the space $BGL(A)^+$ is determined up to heq. Moreover given $u : A \to A'$ ring homo it induces a diagram

$$\begin{tikzcd} BGL(A) \arrow[r] \arrow[d, "u_*"'] & BGL(A)^+ \arrow[d, dashed] \\ BGL(A') \arrow[r] & BGL(A')^+ \end{tikzcd}$$

where the dotted arrow is unique up to h. by Prop. 1. From this one sees that the groups K_iA are well-defined covariant functors of A.

Prop. 1 $\Rightarrow$ the couple $(BGL(A)^+, f)$ is unique determined up to homotopy equivalence, so the groups K_iA are well-defined. ~~We will now ~~state~~ show this defn. to is consistent ( ) when i = 1, 2~~

First because $BGL(A)^+$ is simple, its π_1 is abelian so

$$K_1A = \pi_1 BGL(A)^+ \xrightarrow{\sim} H_1(BGL(A)^+)$$

As f is a H_*-isom, this is isom. to

$$H_1(BGL(A)) = H_1(GL(A)) = GL(A)/E(A),$$

proving the consistency for $i = 1$.

that hf is homotopic to g. Thus, if X is simple, the map $f: Y \to X$ is a universal map of Y to a simple space, and hence the couple (X, f) is unique up to homotopy equivalence.

In effect the existence of h and its uniqueness up to homotopy result from obstruction theory. The obstructions lie in the groups $H^*(\mathrm{Cone}(f), \pi_* Z)$, (untwisted coefficients as Z is simple), and these groups are zero as f is a homology isom. The last assertion is clear.

~~Theorem 1: There exists a homotopy associative and commutative H-space $BGL(A)^+$ and a map $f: BGL(A) \to BGL(A)^+$ inducing isos. on homology.~~

~~From the preceding proposition~~

Theorem 1: There exists a simple space $BGL(A)^+$ and a map $f: BGL(A) \longrightarrow BGL(A)^+$ inducing isos. on homology. Moreover the space $BGL(A)^+$ is a h-assoc + comm. H-space in a natural way.

This will be proved later. There are several ways of constructing the space $BGL(A)^+$ (see Gersten article). [Segal] [Anderson] method shows that $BGL(A)^+$ is an infinite loop space.

~~Lemma: $f_n : Y_n \to X_n$ ...~~

<u>Prop</u>: Def. consistent with .... Furthermore
$K_3A = H_3(St(A))$.

<u>Proof</u>: $X = BGL(A)^+$, $Y = BGL(A)$, $\{X_n\}$ = Postnikov system of X ~~&~~ X_n $(n-1)$-conn + $\pi_q X_n = \pi_q X$ $q \geq n$.
$Y_n = f^{-1}(X_n)$, $f_n : Y_n \to X_n$ ~~nav.~~ image of f.

$$\begin{tikzcd}
Y_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "p_n'"] & X_{n+1} \arrow[r] \arrow[d, "p_n"] & L(\pi_n X, n) \arrow[d] \\
Y_n \arrow[r, "f_n"'] & X_n \arrow[r, "k_n"'] & K(\pi_n X, n)
\end{tikzcd}$$

k_n ! map ~~inducing identity on $\pi_n X$ if ...~~ compatible with $\pi_n X_n = \pi_n X = \pi_n(K(\pi_n X, n))$.

$X_0 = BGL(A)^+$ has abelian π_1
$Y_0 = BGL(A)$

From

$$\begin{tikzcd}
GL(A) = \pi_1 Y \arrow[r] \arrow[d] & \pi_1 X \arrow[d, "\wr"] \\
GL(A)/E(A) = H_1 Y \arrow[r, "\sim"] & H_1 X
\end{tikzcd}$$

(vertical iso because $\pi_1 X$ is abelian) we get $\pi_1 X = GL(A)/E(A)$.
showing consistency in dim 1.

$k_1 f : BGL(A) \to K(\pi_1 X, 1) = B(K_1 A)$ is clearly the map corresp. to the epim. $GL(A) \to K_1 A$. as

Y_2 = fibre of ${}^k_1 f_*$ $\Rightarrow$.

<u>Lemma</u>: $Y_2 = BE(A)$

<u>Lemm.</u> f_2 H_* isom.

Consider the map of ~~...~~ fibrations

$$\begin{tikzcd}
BE(A) \arrow[r, phantom, "="] \arrow[d] & Y_2 \arrow[r] \arrow[d] & X_2 \arrow[d] \\
BGL(A) \arrow[r, phantom, "="] \arrow[d] & Y \arrow[r] \arrow[d] & X \arrow[d] \\
B(K_1 A) \arrow[r, phantom, "="] & B\pi_1 X \arrow[r, phantom, "="] & B\pi_1 X
\end{tikzcd}$$

leads to

$$\begin{array}{ccc}
E^2_{pq} = H_p(\overset{\pi_1 X}{\cancel{K_1A}}, H_q(\overset{Y_2}{\cancel{EA}})) & \Rightarrow & H_n(Y) \\
\downarrow & & \downarrow \wr \\
{}'E^2_{pq} = H_p(\quad , \quad) & \Rightarrow & {}'H_n(X)
\end{array}$$

as before. Insert next two pages

<u>Lemma</u>: f_n H_* isom:

Have map of fibrations

$$\begin{tikzcd}
K(\pi_n X, n-1) \arrow[r] \arrow[d, equal] & Y_{n+1} \arrow[r] \arrow[d] & Y_n \arrow[d] \\
K(\pi_n X, n-1) \arrow[r] & X_{n+1} \arrow[r] & X_n
\end{tikzcd}$$

same fibres and local syst. of homology is constant as X_n is 1-conn. $n \geq 2$.
done by spec. sequence.

Insert

The space X_2 is the universal covering of X, and is obtained by pulling back the universal bundle $E(\pi_1 X) \longrightarrow B(\pi_1 X)$ via the canonical map $k_1 : X \longrightarrow B(\pi_1 X)$. Thus we get cartesian squares

$$\begin{tikzcd} Y_2 \arrow[r, "f_2"] \arrow[d] & X_2 \arrow[r] \arrow[d] & E(\pi_1 X) \sim pt \arrow[d] \\ Y \arrow[r, "f"'] & X \arrow[r, "k_1"'] & B(\pi_1 X) \end{tikzcd}$$

~~Since f maps $\pi_1 Y = GL(A)$ onto $\pi_1 X = K_1 A$ with kernel $E(A)$, Y_2 is the covering of Y with $\pi_1 Y_2 = E(A)$, hence Y_2~~ where the vertical arrows are principal coverings with group $\pi_1 X = K_1 A$. Hence we get a map of spectral sequences

$$( )\qquad \begin{array}{ccc} E^2_{pq} = H_p(\pi_1 X, H_q Y_2) & \Longrightarrow & H_n(Y) \\ \Big\downarrow (f_2)_* & & \Big\downarrow \\ {}'E^2_{pq} = H_p(\pi_1 X, H_q X_2) & \Longrightarrow & H_n(X) \end{array}$$

associated to the maps $k_1 f$ and f. Since X is an H-space, $\pi_1 X$ acts trivially on $H_q X_2$, so ${}'E^2_{0q} \cong H_q X_2$. If we show $\pi_1 X$ acts trivially on $H_* Y_2$, then $E^2_{0q} \cong H_q Y_2$, and we will be able to use the comparison thm for spectral sequence to show $H_q Y_2 \xrightarrow{\sim} H_q X_2$ for all q as desired.

But it is clear that $k_1 f : BGL(A) \longrightarrow B(\pi_1 X)$ is the map of classifying spaces induced by the surjection $GL(A) \twoheadrightarrow GL(A)/E(A)$ followed by the isom. $GL(A)/E(A) \cong \pi_1 X$ described above.

Insert

Hence the fibration $Y_2 \to Y \to B(\pi_1 X)$ is h-equivalent to the one obtained by applying B to the exact seq

$$1 \to E(A) \to GL(A) \to K_1 A \to 1$$

and so the ~~?~~ spectral sequence at the top of ( ) is the H-S spectral sequence in group homology assoc to this exact seq. Thus the triviality of the $\pi_1 X$ action on $H_* Y_2$ follows from:

<u>Lemma</u>: $K_1 A$ acts trivially on $H_*(E(A))$.

<u>Proof</u>. Let $y \in K_1 A$ and $x \in H_g(E(A))$. As $H_g(E(A)) = \varinjlim H_g(E_n(A))$, w.m.a. x comes from $x' \in H_g(E_n(A))$. We can represent y by a matrix of the form $I_n \oplus \alpha$ in $GL(A)$. As this matrix centralizes $E_n(A)$, it follows that y acts trivially on x as claimed.

Now using Hurewicz thm, we have

$$K_2 A = \pi_2 X$$
$$= \pi_2 (X_2)$$
$$= H_2 (X_2)$$

Using the lemma we get

$$K_2 A = H_2(Y_2) = H_2(E(A))$$

showing consistency of Def — with — in dim. 2.

Lemma: $Y_3 = BSt(A)$.

Have fibration

$$(*) \qquad B\pi_2 X \longrightarrow Y_3 \longrightarrow BE(A)$$

so $Y_3 = BG$, $G = \pi_1 Y_3$. Since $(*)$ is induced from universal fibration over $K(\pi_2 X, 2)$, it follows the extension

$$1 \longrightarrow K_2 A \longrightarrow G \longrightarrow E(A) \longrightarrow 1$$

obtained by applying π_1 to $(*)$ is central. But $H_q(G) = H_q(Y_3) = H_q(X_3) = 0$, $q = 1, 2$, hence G is perfect + has no non-trivial central extension $\Rightarrow$ G must be univ. ext. of $E(A)$. $\therefore G = St(A)$.

$$\therefore \quad K_3 A = \pi_3 X = \pi_3 X_3 = H_3 X_3 = H_3 Y_3 = H_3(St(A))$$

~~**Remark:** Y_n has trivial hom. in degrees $< n$, so from~~

$$\cancel{[Y_n, K(H_n Y_n, n)] = H^n(Y_n, H_n(Y_n)) = Hom(H_n Y_n, H_n Y_n)}$$

~~we get a canonical map $Y_n \to$~~

~~The map $k_n f_n : Y_n \to K(\pi_n X, n)$ is the unique map such that compat. with isos~~

$$\cancel{H_n Y_n \longrightarrow H_n X_n \simeq \pi_n X_n = \pi_n X}$$

~~Remark: The tower Y_n may be constructed inductively directly from $Y_1 = BGL(A)$ as follows. (independently of $BGL(A)^+$). Assuming as part of the induction that $H_i(Y_n) = 0$ for $i < n$, let $k_n : Y_n \longrightarrow K(H_n Y_n, n)$ be the map corresp. to id under~~

~~$$[Y_n, K(H_n Y_n, n)] = H^n(Y_n, H_n Y_n) = \mathrm{Hom}(H_n Y_n, H_n Y_n).$$~~

~~Then Y_{n+1} = fibre of k_n. Here $H_n Y_n = K_n A$ which gives a definition of higher K-groups close to Bass-Milnor one, and ind. of $BGL(A)^+$~~

~~Remark: The tower Y_n appears in DROR~~

Remark: The tower $\{Y_n\}$ has been considered by DROR. Starting from $Y_2 = BE(A)$ (and more gen. any space with perf. π_1), he recursively const. a tower Y_n of spaces $\ni$ $H_i(Y_n) = 0$ $i < n$ by letting Y_{n+1} be the fibre of the map $k_n : Y_n \longrightarrow K(H_n Y_n, n)$ corresp to id under

$$[Y_n, K(H_n Y_n, n)] = \ldots\ldots$$

In terms of this tower one has

$$K_n A = H_n Y_n \qquad n \geq 2.$$

Moreover one can show that if $Y_\infty = \varprojlim Y_n$, then $BGL(A)^+$ is the cofibre of the map $Y_\infty \longrightarrow BGL(A)$, which gives another construction of the space $BGL(A)^+$. (see DROR, GERSTEN).

~~If k is a field, and X is a connected space fact~~

~~If X is an H-space, one has by MM thm that Hurewicz homo induces an isom~~

$$\cancel{\pi_q X \otimes \mathbb{Q} \xrightarrow{\sim} \operatorname{Prim} H_q(X, \mathbb{Q})}$$

~~for q>0, where Prim on the left is the space of primitive elements.~~

If X is a space with basepoint x_0, ~~let $\Delta, i', i'' : X \to X \times X$ be the maps~~ and M is an abelian group, we put

$$\operatorname{Prim} H_q(X, A) = \{\alpha \in H_q(X, A) \mid \Delta_* \alpha = i'_* \alpha + i''_* \alpha\}$$

where $\Delta, i', i'' : X \longrightarrow X \times X$ are the maps given by $\Delta(x) = (x, x)$, $i'(x) = (x, x_0)$, $i''(x) = (x_0, x)$. If X is an H-space, a thm of M-M asserts that the Hurewicz map induces an isom

$$\pi_q X \otimes \mathbb{Q} \xrightarrow{\sim} \operatorname{Prim} \{H_q(X, \mathbb{Q})\}$$

for $q > 0$. Thus applying this to $X = BGL(A)^+$ and using the fact that f induces an isom

$$\operatorname{Prim} H_q(BGL(A), \mathbb{Q}) \xrightarrow{\sim} \operatorname{Prim} H_q(BGL(A)^+; \mathbb{Q})$$

we get:

<u>Prop:</u> $K_q A \otimes \mathbb{Q} \simeq \operatorname{Prim}(H_q(GL(A); \mathbb{Q}))$.

Splitting thm.

$\mathcal{P}$ additive category essentially small

V fixed obj of $\mathcal{P}$,

~~$\mathcal{E}_V$ groupoid formed of objects $u: P \to V$ of $\mathcal{P}$ over V such that (i) Ker(u) is representable (ii) u has a section. morphisms in $\mathcal{E}_V$ are isom. of objects over V.~~

~~$\mathcal{E}_V$ is the full category~~

~~$\mathcal{E}_V$ is the following cat. An object is a pair (P, u) where $u: P \to V$ is a map in $\mathcal{P}$ such that (i) u has a section (ii) Ker(u) exists.~~

$\mathcal{E}_V$ is the full subcategory of the groupoid objects of $\mathcal{P}$ over V and their isomorphisms consisting of $u: P \to V$ which are isomorphic to $pr_2 : Q \oplus V \longrightarrow V$. Equivalently $u: P \to V$ is an object of $\mathcal{E}_V$ iff (i) u has a section (ii) the kernel of u exists in $\mathcal{P}$.

If $E = (u: P \to V)$ and $E' = (u': P' \to V)$ are two objects of $\mathcal{E}_V$ define

$$E \perp E' = (P \times_V P' \xrightarrow{u\,pr_1} V).$$

In this way get an operation

$$\perp : \mathcal{E}_V \times \mathcal{E}_V \longrightarrow \mathcal{E}_V$$

which is associative, commutative, and unitary up to canonical isomorphisms.

$\mathcal{E}_0 = \mathrm{Iso}(\mathcal{P})$ and $\perp = \oplus$. Have functors

$$k : \mathcal{E}_V \longrightarrow \mathcal{E}_0 \quad , \quad (P \xrightarrow{u} V) \longmapsto \mathrm{Ker}(u)$$

$$i : \mathcal{E}_0 \longrightarrow \mathcal{E}_V \quad , \quad Q \longmapsto (Q \oplus V \xrightarrow{pr_2} V).$$

compatible with $\perp$ operation.

Let $S = \pi_0(\mathcal{E}_0)$ = iso classes of P; it is a comm. monoid and ~~...~~ clearly $\pi_0(\mathcal{E}_V) \simeq S$. Let P_s rep. the class s. ~~...~~ Then have equivalences

$$\mathcal{E}_V \sim \coprod_{s \in S} \mathrm{Aut}(P_s) \tilde{\times} \mathrm{Hom}(V, P_s)$$

$$\mathcal{E}_0 \sim \coprod_{s \in S} \mathrm{Aut}(P_s)$$

Fix a field k and ~~...~~ for any ess. small cat. $\mathcal{X}$, let $H_*(\mathcal{X})$ be the homology of $\mathcal{X}$ with coefficients in k. Have Künneth isom

$$H_*(\mathcal{X} \times \mathcal{X}') \simeq H_*(\mathcal{X}) \otimes H_*(\mathcal{X}')$$

so get cogebra structure on $H_*(\mathcal{X})$

$$H_*(\mathcal{X}) \xrightarrow{\Delta} H_*(\mathcal{X}) \otimes H_*(\mathcal{X}).$$

Then ~~...~~ $H_*(\mathcal{E}_V)$ is a bicomm. Hopf algebra with product

$$H_*(\mathcal{E}_V) \otimes H_*(\mathcal{E}_V) \simeq H_*(\mathcal{E}_V \times \mathcal{E}_V) \xrightarrow{(\perp)_*} H_*(\mathcal{E}_V).$$

Because k, i commute with $\perp$, get Hopf alg. ~~...~~ maps

$$H_*(\mathcal{E}_V) \underset{i_*}{\overset{k_*}{\rightleftarrows}} H_*(\mathcal{E}_0)$$

$\ni$ $k_* i_* = \mathrm{id}$. In degree zero $H_0(\mathcal{E}_V) = H_0(\mathcal{E}_0) = k[S]$.

Thm: $S^{-1} H_*(\mathcal{E}_V) \simeq S^{-1} H_*(\mathcal{E}_0)$.

Proof. Let $\varphi = \rho\, i_* k_* : H_*(\mathcal{E}_W) \longrightarrow H_*(\mathcal{E}_W) \xrightarrow{\rho} S^{-1}H_*(\mathcal{E}_W)$
To show $\rho = \varphi$. Now ~~the identity~~ for any $E : (P \xrightarrow{\pi} V)$
we have canonical isom

$$E \perp E \simeq E \perp ik E \qquad \underset{V}{P \times P} \xrightarrow{} \underset{V}{P \times} (K \oplus V)$$

$$(x, y) \mapsto (x, (y-x) + px)$$

compatible with $\mathrm{Aut}(E)$. This shows that

$$H_*(\mathcal{E}_W) \xrightarrow{\Delta} H_*(\mathcal{E}_W)^{\otimes 2} \underset{id \otimes i_* k_*}{\overset{id \otimes id}{\rightrightarrows}} H_*(\mathcal{E}_W)^{\otimes 2} \xrightarrow{(\perp)_*} H_*(\mathcal{E}_W)$$

commutes, which implies

$$H_*(\mathcal{E}_W) \xrightarrow{\Delta} H_*(\mathcal{E}_W)^{\otimes 2} \underset{\rho \otimes \varphi}{\overset{\rho \otimes \rho}{\rightrightarrows}} S^{-1}H_*(\mathcal{E}_W)^{\otimes 2} \longrightarrow S^{-1}H_*\mathcal{E}_W$$

commutes.

But ~~previously~~ given a graded ring $R = R_0 \oplus \cdots$
we have defined a ring structure on

$$\mathrm{Hom}^{(0)}_{\mathrm{mod\,gr}(k)}(H_*(X), R) = \prod_{i \geq 0} \mathrm{Hom}(H_i X, R_i)$$

$$\simeq \prod_{i \geq 0} H^i(X, R_i)$$

by defining the product of u, v to be

$$H_*(X) \xrightarrow{\Delta} H_*(X) \otimes H_*(X) \xrightarrow{u \otimes v} R \otimes R \xrightarrow{\mu} R$$

i.e. $\quad (u * v)(x) = \sum u(x_i') v(x_i'') \quad$ if $\quad \Delta x = \sum x_i' \otimes x_i''$

The assertion to prove is:

<u>Proposition:</u> Let $R = R_0 \oplus \cdots\cdots$ ~~[illegible]~~ be a graded ring, $\mathcal{X}$ a category (ess. small). ~~[illegible]~~ Put

$$H^0(\mathcal{X}, R) = \operatorname{Hom}_{\text{modgr}(k)}(H_*(\mathcal{X}), R)$$
$$= \prod_{n\geq 0} \operatorname{Hom}_k(H_n(\mathcal{X}), R_n)$$
$$\cong \prod_{n\geq 0} H^n(\mathcal{X}, R_n)$$

and define the product $u*v$ ~~[illegible]~~ of two elts $u, v \in H^0(\mathcal{X}, R)$ to be the composition

$$H_*(\mathcal{X}) \xrightarrow{\Delta} H_*(\mathcal{X})^{\otimes 2} \xrightarrow{u \otimes v} R^{\otimes 2} \xrightarrow{\mu} R$$

μ being the product in R. (hence

$$(u*v)(x) = \sum u(x_i')v(x_i'') \quad \text{if} \quad \Delta x = \sum x_i' \otimes x_i'')$$

Then $H^0(\mathcal{X}, R)$ is a ring. Further u is an invertible element of $H^0(\mathcal{X}, R)$ iff $u_0 : H_0(\mathcal{X}) \to R_0$ carries the ~~[illegible]~~ set $\pi_0 \mathcal{X}$ of generators of $H_0(\mathcal{X})$ into invertible elements of R_0.

Proof: well-known. You might identify:

$$H^0(\mathcal{X}, R) = \prod_{n\geq 0} H^n(\mathcal{X}, R_n)$$

so that if $u = (u_n \in H^n(\mathcal{X}, R_n) = \operatorname{Hom}_k(H_n\mathcal{X}, R_n))$ then the product is

$$u*v = \left(n \longmapsto \sum_{i+j=n} u_i \cdot v_j\right)$$

where $u_i \cdot v_j$ is $\cup : H^i(X, R_i) \otimes H^j(X, R_j) \to H^{i+j}(X, R_{i+j})$.
Thus ~~[illegible]~~ u is invertible iff $u_0 \in H^0(X, R_0)$ is.

Have to put in fact it works for arb. coefficients.

Take Trans(S). From localizing in the graded ring

$$H_*(\mathcal{E}_V) = \coprod_{s \in S} H_*(\underbrace{Aut(P_s) \tilde{\times} Hom(V_s, P_s)}_{= Aut(E_s)})$$

we get

$$S^{-1}H_*(\mathcal{E}_V) = k[\bar{S}] \otimes \varinjlim_{Trans(S)} (s \mapsto H_*(Aut(E_s)))$$

<u>Cor. 1</u>: $\varinjlim_{s} H_*(Aut(P_s)) \xrightarrow{\sim} \varinjlim_{s} H_*(Aut(P_s) \tilde{\times} Hom(V, P_s))$

Now take a ring A, whence have cofinal functor $\mathbb{N} \longrightarrow Trans(S)$, $n \mapsto [A^n]$. and we get taking $V = A^r$

<u>Cor. 2</u>: Consider the inclusion of subgroups of $GL_{r+n}(A)$ indicated:

$$\begin{pmatrix} I_r & 0 \\ 0 & GL_n A \end{pmatrix} \subset \begin{pmatrix} I_r & M_{r,n}(A) \\ 0 & GL_n A \end{pmatrix} = \Gamma_{rn}$$

This inclusion induces isom of homology in the limit as $n \to \infty$.

$$\varinjlim_{n \to \infty} H_* \begin{pmatrix} I_r & 0 \\ 0 & GL_n A \end{pmatrix} \xrightarrow{\sim} \varinjlim_{n \to \infty} H_* \begin{pmatrix} I_r & M_{r,n}(A) \\ 0 & GL_n(A) \end{pmatrix}$$

For $0 \le r \le \infty$, we have

<u>Cor. 3.</u> $H_*\begin{pmatrix} GL_r A & 0 \\ 0 & GL_\infty A \end{pmatrix} \xrightarrow{\sim} H_*\begin{pmatrix} GL_r A & M_{r,\infty}(A) \\ 0 & GL_\infty A \end{pmatrix}$

<u>Proof:</u> Consider exact sequence

$$\begin{pmatrix} I_r & M_{r,n}A \\ & GL_n A \end{pmatrix} \to \begin{pmatrix} GL_n A & M_{r n} A \\ 0 & GL_r A \end{pmatrix} \longrightarrow GL_r A$$

and corresp. one for subgroup. Now compare two spectral sequences.

<u>Lemma:</u> If a map of exact sequences

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d, "\wr"] & 1 \\ 1 \arrow[r] & N' \arrow[r] & G' \arrow[r] & Q' \arrow[r] & 1. \end{tikzcd}$$

If $H_*(N) \xrightarrow{\sim} H_*(N')$ and $Q \simeq Q'$, then $H_*(G) \xrightarrow{\sim} H_*(G')$.

Immediate from the spectral sequence
$$E^2_{pq} = H_p(Q, H_q N) \Rightarrow H_{p+q}(G).$$

<u>Cor. 4:</u> Let $\theta \in H^*(GL(A), M)$. Given a rep $E : G \to \mathrm{Aut}(P)$ of G over A, can pull θ back

$$G \to \mathrm{Aut}(P) \xrightarrow[\text{unique up to inner autos}]{} GL(A)$$

$\theta(E) \in H^*(G, M)$. Then given an exact sequence have $\theta(E) = \theta(E' \oplus E'')$.

$$\begin{pmatrix} GL_\infty A & \\ & GL_\infty A \end{pmatrix} \longrightarrow \begin{matrix} GL & M \\ & GL \end{matrix} \rightrightarrows GL$$

Cor. 5: $\mathbb{F}_q$ finite field ~~elem~~ $q=p^d$ elements.
Then $H_i(GL(\mathbb{F}_q), \mathbb{F}_p) = 0$ for $i > 0$.

Proof: ~~suffices to show $H_i(GL_n \mathbb{F}_q) \to H_i(GL(\mathbb{F}_q)) = M$ is zero. Let P be the sylow p-subgroup; then $H_i(P) \to H_i(GL_n(\mathbb{F}_q))$ is surj.~~

Enough to show ~~for any~~ $GL_n(\mathbb{F}_q) \to GL(\mathbb{F}_q)$ induces zero map on H_i, hence enough to show for any rep E of a finite group G over $\mathbb{F}_q$, and $\forall\ \theta \in H^i(GL(\mathbb{F}_q), \mathbb{F}_p)$, $i>0$, that $\theta(E) = 0$ in $H^i(G, \mathbb{F}_p)$. But if P is the sylow p-subgroup, $H^i(G, \mathbb{F}_p)$ injects into $H^i(P, \mathbb{F}_p)$, whence w.m.s. G is a p-group. In this case E has a filtration by subrepresentation E_i $\ni$ E_i/E_{i-1} is a trivial repn. Applying preceding cor:

$$\theta(E) = \theta(\amalg E_i/E_{i-1})$$

hence can suppose E is trivial, whence it is clear since ~~then the rep is pull back via~~ $H^i(e, \mathbb{F}_p) = 0$, e = trivial gp.

Unstable thm:

I want now to consider carefully the the localization theorem again.

<u>Notation</u>: A, S, $S^{-1}A$. $\mathcal{T} = \mathcal{H}^1_S(A) = \{M \in \mathcal{P}_1(A) \mid S^{-1}M = 0\}$

Basic construction: Form over $Q(\mathcal{T}) \times Q(\mathcal{P}_A)$ the fibre cat. with fibre over (T, P) = the groupoid of

$$E \twoheadrightarrow T \times P \qquad E \in \mathcal{P}(A)$$

and their isomorphisms.

Unstable splitting:

$$H_*(X) = H_*(X, k) \qquad k \text{ field}$$

$$\begin{array}{ccccc} \mathrm{Hom}(V,P) & \longrightarrow & \mathrm{Aut}(P) \tilde{\times} \mathrm{Hom}(V,P) & \longrightarrow & \mathrm{Aut}(P) \\ \| & & \| & & \| \\ N & & G = Q \tilde{\times} N & & Q \end{array}$$

P is $\mathbb{Z}[\ell^{-1}]$-linear, ℓ prime $\ni$ $\ell \mid \mathrm{char}(k)$.

Claim then that

$$H_*(\mathrm{Aut}(P) \tilde{\times} \mathrm{Hom}(V,P)) \xrightarrow{\sim} H_*(\mathrm{Aut}(P)).$$

Proof. trivial if $\mathrm{char}(k) = \ell$ for then $H_*(\mathrm{Hom}(V,P)) = H_*(pt) = k$.

Suppose $\mathrm{char}(k) = 0$. Then one knows

$$H_i(N, k) = \Lambda^i(N \otimes_{\mathbb{Z}} k)$$

Homological

3) Splitting of exact sequences

stable splitting thm.

<u>Cor</u>: $S^{-1}\mathcal{E}_0 \longrightarrow S^{-1}\mathcal{E}_V \longrightarrow S^{-1}\mathcal{E}_0$ hegs.

~~Applications to the homology of~~

finite field application

unstable results

D

Here's what remains:

The comparison thm.

filtered rings

schemes

$\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \xleftarrow[\sim]{\Delta} \mathbb{Z}$

$\boxed{X \times X}$

2-days trying to get comparison thm. in shape

$A \longrightarrow B$ map of rings

$P(A) \longrightarrow P(B)$

want fibre, roughly $\langle P(A), P(B) \rangle$ if $A \hookrightarrow B$

But if $A \not\hookrightarrow B$ then I ought to make $P(A) \times P(A)$ act on $P(A) \times P(B)$.

$$\begin{array}{ccc} \langle P(A) \times P(A), P(A) \times P(B) \rangle \leftarrow & & \\ \uparrow & & \\ (P(A) \times P(B))^{+} & & (P \oplus Q, fQ) \\ \uparrow & & \uparrow \\ (P(A) \times P(A))^{+} & & (P, Q) \end{array}$$

Results of second part

(1.) **Comparison thm.** $\mathcal{M}$ exact cat (ess. small) $\ni$ every exact sequence splits $\Longrightarrow$ $\Omega Q(\mathcal{M}) \sim (\mathrm{Iso}\,\mathcal{M})^{-1}(\mathrm{Iso}\,\mathcal{M})$.

application: shows the consistency of definitions of the K-theory of a ring, and it implies

$$H_*(\Omega Q(P_A)_0) = H_*(BGL(A)).$$

Unsolved problem to compute $H_*(\Omega Q(\mathcal{M}))$ in general, even for vector bundles over a complete curve

(2.) **Resolution (revisited)** Two versions

(3.) **Localization.**

Thm: exact sequence

description of d.

Variants $\begin{cases} \text{abelian case} & S^{-1}A \text{ semi-simple} \\ \text{schemes.} \end{cases}$

(4.) Fund. thm,

(5.) Applications of fundamental thm

Reg. coherent rings

Rings of char. p

Second part of paper: logical structure

1. Comparison thm

Recall $\mathcal{M}$ exact category, admissible monos $\rightarrowtail$, epis $\twoheadrightarrow$, filtrations.

Recall $Q(\mathcal{M})$; i^*, $i_!$, $j^!$; functoriality for exact functors.

Definition. $\mathcal{E}_M$, $\phi^*: \mathcal{E}_M \to \mathcal{E}_{M'}$ if $\phi: M' \twoheadleftarrow M_0 \rightarrowtail M$

$\overline{Q}(\mathcal{M})$, $q: \overline{Q}(\mathcal{M}) \to Q(\mathcal{M})$ fibred

Given $(L \twoheadrightarrow M)$, $(L' \twoheadrightarrow M')$ notion of when a map $L \to L'$ induces a map $M \to M'$ in $Q(\mathcal{M})$.

Lemma 1: $\overline{Q}(\mathcal{M})$ equivalent to subdivision of cat of admiss monos so $\overline{Q}(\mathcal{M})$ is contractible.

2. ~~Resolution [illegible]~~ Resolution revisited. (outline)

State two versions of res. thm.

Thm.

Thm'.

Sketch reduction of Thm. to Thm'

Proof of Thm':

Construction of fibred cat. $\mathcal{F}$ over $Q(M)\times Q(P)$ using (i)

Functors:

$$Q(M) \xleftarrow{f} \mathcal{F} \xrightarrow{g} Q(P).$$

L1: f heq. (ref. to equiv $\overline{Q}(M) \to \mathrm{Sub}(\mathcal{J})$) L1 of §1

f fibred; $f^{-1}(M)$ equiv $\mathrm{Sub}(R_M)$. R_M conically contractible

L2: g heq

analogous proof

Recall $Q(M)$ connected h-assoc. comm. H-space, so $[X, Q(M)]$ is an abelian group. ~~[illegible]~~

~~[illegible]~~

Let $i: Q(P) \to Q(M)$ be induced by inclusion of $P \subset M$.

L3: $ig, f: \mathcal{F} \to Q(M)$ are negatives of each other for the H-space structure on $Q(M)$.

---

need notation $P_n(A)$, $P_\infty(A)$ for 3. Perhaps should put this in as corollary?

NO.

3. Localization

references required

$V \mapsto S^{-1}\mathcal{E}_V \ni$ ~~[illegible]~~ φ^* heq [S not nec. equal to $I_0(\mathcal{V}) - \mathcal{E}_0$

S acts h-invertibly on $C \Rightarrow C \hookrightarrow S^{-1}C$ heq

after comparison thm. or after S^{-1}
calculation should discuss $\mathcal{V} \subset \mathcal{P}$
full & cofinal.

?
4. Variants of localization thm.
discussion of d
~~[illegible]~~ abelian case, when $S^{-1}A$ field, or
more gen. semi-simple.
schemes — Cartier divisor affine complement.

3. Localization.

A, $S \subset$ Center A, ~~[illegible]~~ $S^{-1}A$, $\gamma: A \longrightarrow S^{-1}A$ canonical hom.
$\gamma^*: K_0A \longrightarrow K_0(S^{-1}A)$ induced map

$P_n(A)$ = full subcat of $\mathrm{Mod}(A)$ cons. of M having ~~[illegible]~~ $P(A)$-resolutions of length $\leq n$

$$P_\infty(A) = \bigcup P_n(A)$$

By res. thm. have

(1) $$K_*A = K_* P_1(A) = \cdots = K_* P_\infty(A)$$

Let $\mathcal{H}_S(A)$ = full subcat of $P_\infty(A)$ cons. of $M \ni S^{-1}M = 0$.

(2) $$c: K_* \mathcal{H}_S(A) \longrightarrow K_* A$$

the homo induced by inclusion $\mathcal{H}_S(A) \subset P_\infty(A)$ tog. with ( ).

Thm: If S consists of non-zero divisors, then have long exact sequence

where d is a canonical homom to be defined in the course of the proof.

Remark 1: I take "canonical" ~~[illegible]~~ as implying "functorial". Thus the sequence above is functorial in the pair (A, S) in the evident sense.

Remark 2: The sequence stops at $K_0(S^{-1}A)$ and $\gamma^*: K_0A \longrightarrow K_0(S^{-1}A)$ is not onto in general.

Proof to occupy rest of section.

define $\mathcal{H}_{S,n}(A)$

Lemma 1: $K_*(\mathcal{H}_{S,1}(A)) = K_*(\mathcal{H}_{S,2}(A)) = \cdots \cong K_*(\mathcal{H}_S(A))$.

Put $\mathcal{H} = \mathcal{H}_{S,1}(A)$ and identify c with the homo-

(3) $$c : K_q(\mathcal{H}) \longrightarrow K_q A$$

induced by the inclusion $\mathcal{H} \subset \mathcal{P}_1(A)$ followed by (1).

Define $\mathcal{V} \subset \mathcal{P}(S^{-1}A)$

Lemma 2:

$$K_q \mathcal{V} \xrightarrow{\sim} \begin{cases} \operatorname{Im}\{K_0 A \longrightarrow K_0(S^{-1}A)\} & q=0 \\ K_q(S^{-1}A) & q>0 \end{cases}$$

(application of comp. thm, and might go in there. You want to know

$$S^{-1}\,\mathrm{Iso}(\mathcal{V}) \sim \Omega Q(\mathcal{V}) \ ]$$

where $S = \mathrm{Iso}(\mathcal{P}(A))$ (or finite sets + isos.).

$$\pi_0\left[S^{-1}\,\mathrm{Iso}(\mathcal{V})\right] = (\pi_0 S)^{-1}\, \pi_0(\mathrm{Iso}\,\mathcal{V})$$

$$H_*\left(S^{-1}\,\mathrm{Iso}(\mathcal{V})\right) = (\pi_0 S)^{-1}\, H_*(\mathrm{Iso}\,\mathcal{V})$$
$$= \mathbb{Z}[K_0\mathcal{V}] \otimes H_*(GL(S^{-1}A)).$$

I think we should put this lemma as a corollary of the comp. thm.) IMPORTANT later to know that $S^{-1}\bar{Q}(\mathcal{V}) \longrightarrow Q(\mathcal{V})$ has all basechanges heq's, where $S = \mathrm{Iso}(\mathcal{P}(A))$.

$\mathcal{P} = \mathcal{P}(A)$, $\mu : \underline{Q(\mathcal{P})} \longrightarrow Q(\mathcal{V})$, $P \mapsto S^{-1}P$.

Outline proof of thm. Will construct a diagram

$$\begin{tikzcd}
Q(\mathcal{H}) & \mathcal{D} \arrow[l, "f"'] \arrow[r, "h"] \arrow[dr, "g"'] & \mathcal{C} \arrow[r, "\bar{\mu}"] \arrow[d, "\gamma'"] & \bar{Q}(\mathcal{V}) \arrow[d, "\gamma"] \\
 & & Q(\mathcal{P}) \arrow[r, "\mu"'] & Q(\mathcal{V})
\end{tikzcd}$$

$\mathcal{C}$ to be defined so that square is cart.
Will show using comp. thm. proof that square is
? h-cartesian, whence $\mathcal{C}$ is the h-fibre of γ, and we
get a long exact seq

$$\longrightarrow \pi_{g+1}\mathcal{C} \longrightarrow \pi_{g+1} Q(\mathcal{P}) \xrightarrow{\mu_*} \pi_{g+1} Q(\mathcal{V}) \longrightarrow$$

By lemma μ_* essentially γ^*.

Definition of $\mathcal{D}$ analogous to $\mathcal{F}$ in res. thm
(In fact restriction of $\mathcal{F}$ to $Q(\mathcal{H}) \times Q(\mathcal{P}) \subset Q(\mathcal{M}) \times Q(\mathcal{P})$
$\mathcal{M} = \mathcal{P}_1(A)$.)

<u>L3</u>: f heq

<u>L4</u>: homo $K_g\mathcal{H} = \pi_{g+1} Q(\mathcal{H}) \xrightarrow[\sim]{(f_*)^{-1}} \pi_{g+1}\mathcal{D} \xrightarrow{g_*} \pi_{g+1}^{Q}(\mathcal{P}) = K_g A$
is negative of homom. c ( ).

Def of $\mathcal{C}$ fibred over $Q(\mathcal{P})$ fibre $\mathcal{E}_{S^{-1}P}$
Def of $h : \mathcal{D} \to \mathcal{C}$

<u>L5</u>: h heq.
Ref to that it suffices to show $h_P : g^{-1}(P) \to \mathcal{E}_{S^{-1}P}$

~~L7: Square in (*) is h-cartesian~~
~~Make $\mathcal{S} = \mathrm{Iso}(\mathcal{P})$~~

Define $\mathcal{S} = \mathrm{Iso}(\mathcal{P})$ action on $\mathcal{D}, \mathcal{C}, \bar{Q}(\mathcal{V})$.

L6: $\forall T \in \mathcal{S}$, $T\# ? : \mathcal{C} \to \mathcal{C}, \mathcal{D} \to \mathcal{D}$ homotopic to the identity.

L7: Square

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \bar{Q}(\mathcal{V}) \arrow[d] \\ Q(\mathcal{P}) \arrow[r] & Q(\mathcal{V}) \end{tikzcd}$$

is h-cartesian.

Proof. The ~~square~~

$$\begin{tikzcd} \mathcal{S}^{-1}\mathcal{C} \arrow[r] \arrow[d] & \mathcal{S}^{-1}\bar{Q}(\mathcal{V}) \arrow[d] \\ Q(\mathcal{P}) \arrow[r] & Q(\mathcal{V}) \end{tikzcd}$$

~~is cartesian square~~ has vertical maps fibred with same fibres & base changes are heq's, so h-fibre = fibre $\Rightarrow$ square is h-cart. But L6 + (ref) $\Rightarrow$ $\mathcal{C} \hookrightarrow \mathcal{S}^{-1}\mathcal{C}$, $\bar{Q}(\mathcal{V}) \hookrightarrow \mathcal{S}^{-1}(\bar{Q}(\mathcal{V}))$ are heq's, so done.

Description. $d: K_{q+1}(S^{-1}A) \to K_q(\mathcal{H}_S(A))$ is the map of ~~Homotopy~~ induced by

$$\begin{tikzcd} & \mathcal{S}^{-1}g^{-1}(0) \arrow[r, "heq"] \arrow[dl] \arrow[d] & \mathcal{S}^{-1}\mathcal{E}_0 \arrow[d] \\ Q(\mathcal{H}) & \mathcal{S}^{-1}\mathcal{D} \arrow[l, "heq"'] \arrow[r, "heq"] & \mathcal{S}^{-1}\mathcal{C} \end{tikzcd}$$

so can be described at least on representations by ~~lattices~~ ~~of~~ 1.

Resolution (new proof).

Consider the following two results:

Thm: Let $\mathcal{P}, \mathcal{M}$ be essentially small full subcategories of an abelian category $\mathcal{A}$ which are both closed under extensions in $\mathcal{A}$. Assume $\mathcal{P} \subset \mathcal{M}$ and

i) If $0 \to M' \to P \to M'' \to 0$ is exact with M', M'' in $\mathcal{M}$ and $P \in \mathcal{P}$, then $M' \in \mathcal{P}$.

ii) For every M in $\mathcal{M}$ there exists an exact sequence

(*) $\quad 0 \to P_n \to \cdots \to P_0 \to M \to 0$

such that $P_i \in \mathcal{P}$ and $\mathrm{Im}\{P_{i+1} \to P_i\} \in \mathcal{M}$ for each i.

Then the inclusion of $\mathcal{P}$ in $\mathcal{M}$ induces isos. $K_* \mathcal{P} \xrightarrow{\sim} K_* \mathcal{M}$.

Thm': Same as Thm. but with hypothesis ii) replaced by

ii)' For every M in $\mathcal{M}$ there is an ex. seq

$$0 \to P' \to P \to M \to 0$$

with P' and P in $\mathcal{P}$.

Evidently Thm' is a special case of Thm. On the other hand Thm reduces to Thm' by the following method. Let $\mathcal{P}_n$ = full subcat of $\mathcal{M}$ cons. of $M \ni \exists$ (*) of length $\leq n$. Then Thm' applies to the inclusion $\mathcal{P}_n \subset \mathcal{P}_{n+1}$, so assuming this result we get

$$K_* \mathcal{P} = K_* \mathcal{P}_0 \simeq K_* \mathcal{P}_1 \simeq \cdots \simeq K_* \mathcal{P}_n \simeq \cdots$$

yielding $K_* \mathcal{P} \simeq K_* \mathcal{M}$ by passing to the limit over n. For the details see (   ).

In the rest of this section we will give a proof of Thm' which is different from the one appearing in (   ). Several of the arguments will ~~occur in the~~ be used again in the localization thm. of the next section.

Resolution thm (new proof)

~~In this section we give a different proof of the resolution thm.~~

Resolution thm. $\mathcal{P}, \mathcal{M}$ essentially small full subcategories of an abelian category $\mathcal{A}$ which are both closed under extensions in $\mathcal{A}$ ~~and have a set of iso. classes.~~
Assume $\mathcal{P} \subset \mathcal{M}$ and

1) If $0 \to M' \to P \to M'' \to 0$ exact in $\mathcal{M}$, $P \in \mathcal{P} \Longrightarrow M' \in \mathcal{P}$

2) $\forall\, M \in \mathcal{M}$, $\exists$ exact sequence

$$(*) \qquad 0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

such that $P_i \in \mathcal{P}$ and ~~which is exact in $\mathcal{M}$ in the sense that it is exact in $\mathcal{A}$ and each~~ $\mathrm{Im}(P_{i+1} \to P_i) \in \mathcal{M}$ for each i.

Then the inclusion of $\mathcal{P}$ in $\mathcal{M}$ induces isos.
$K_*(\mathcal{P}) \simeq K_*(\mathcal{M})$.

As in ~~Col.~~ paper one lets $\mathcal{P}_n$ be the full subcat of $\mathcal{M}$ consist of $M \ni \exists$ a resolution $(*)$ of length $\leq n$. Then one has $\mathcal{M} = \bigcup \mathcal{P}_n$

Recall the reduction ~~to~~ :

Resolution thm:

Theorem: Let $\mathcal{P}, \mathcal{M}$ be full subcategories of an abelian cat $\mathcal{A}$ which are closed under extensions and have a set of iso. classes. Assume $\mathcal{P} \subset \mathcal{M}$ and

1) $0 \to M' \to P \to M'' \to 0$ exact in $\mathcal{M}$, $P \in \mathcal{P}$
$\Longrightarrow M' \in \mathcal{P}$

2) $\forall M \in \mathcal{M}$ $\exists$ exact sequence
$$0 \to P' \to P \to M \to 0$$
with $P, P' \in \mathcal{P}$.

Then the ~~functor $Q(\mathcal{P}) \to Q(\mathcal{M})$ induced by the~~ inclusion of $\mathcal{P}$ in $\mathcal{M}$ ~~is a heq.~~ induces iso $K_*\mathcal{P} \xrightarrow{\sim} K_*\mathcal{M}$

Proof: (Steps:

construction of $\mathcal{F}$ fibred over $Q(\mathcal{M}) \times Q(\mathcal{P})$

L1: $\mathcal{F} \xrightarrow{f} Q(\mathcal{M})$ heq

Show $f^{-1}(M)$ equiv to $\mathrm{Sub}(R_M)$.
R_M conically contractible. ~~by prev. crit.~~

L2: $\mathcal{F} \xrightarrow{g} Q(\mathcal{P})$ heq

Similar proof

L3: The composition
$$K_q\mathcal{M} = \pi_{q+1} Q(\mathcal{M}) \xrightarrow[\sim]{(f_*)^{-1}} \pi_{q+1}\mathcal{F} \xrightarrow[\sim]{g_*} \pi_{q+1} Q(\mathcal{P}) = K_q\mathcal{P}$$
is the negative of the map induced by the inclusion of $\mathcal{P}$ in $\mathcal{M}$.)

Proof: ~~Suppose~~ We begin by constructing a fibred category $\mathcal{F}$ over $Q(\mathcal{M}) \times Q(\mathcal{P})$. Given objects M in $Q(\mathcal{M})$, P in $Q(\mathcal{P})$, let $\mathcal{F}_{M,P}$ denote the following groupoid. An object of $\mathcal{F}_{M,P}$ is an $\mathcal{M}$-admissible epimorphism $L \to M \times P$ such that $L \in \mathcal{P}$. A map from $(L \to M \times P)$ to $(L' \to M \times P)$ is an isom. $L \simeq L'$ over $M \times P$.

Suppose given morphisms $\phi : M' \to M$, $\psi : P' \to P$ in $Q(\mathcal{M})$, $Q(\mathcal{P})$ respectively, represented by

$$M' \twoheadleftarrow M_0 \rightarrowtail M,$$
$$P' \twoheadleftarrow P_0 \rightarrowtail P.$$

If $(L \to M \times P)$ is an ob. of $\mathcal{F}_{M,P}$, then one sees easily using the hyp. (1) that the composite arrow

$$(M_0 \times P_0) \times_{M \times P} L \xrightarrow{pr_1} M_0 \times P_0 \longrightarrow M' \times P'$$

is an object of $\mathcal{F}_{M',P'}$ which we will denote $(\phi,\psi)^*(L \to M \times P)$. In this way we obtain a functor

$$(\phi,\psi)^* : \mathcal{F}_{M,P} \longrightarrow \mathcal{F}_{M',P'}$$

associated to each map in $Q(\mathcal{M}) \times Q(\mathcal{P})$.

The category $\mathcal{F}$ will be the fibred cat over $Q(\mathcal{M}) \times Q(\mathcal{P})$ having fibre $\mathcal{F}_{M,P}$ over (M,P) and the above base-change functors. Thus an object of $\mathcal{F}$ is an $\mathcal{M}$-admiss epi $(L \to M \times P)$ with $M \in \mathcal{M}$ and $L, P \in \mathcal{P}$, and a map from $(L' \to M' \times P')$ to $(L \to M \times P)$ in $\mathcal{F}$ is a diagram