# Tate's letter

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Lichtenbaum's conjecture relating the behavior of $\zeta_F(s)$ at negative integers to the higher K groups of the ring of integers A of F

Let F be a number field of finite degree, $n$, over $\mathbb{Q}$.
Let $F \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{C}^{r_2} \times \mathbb{R}^{r_1}$; $n = 2r_2 + r_1$, the map being given by

(1) $$\alpha \otimes 1 \longmapsto (\varphi_1(\alpha), \dots, \varphi_{r_2}(\alpha); \varphi_{r_2+1}(\alpha), \dots, \varphi_{r_2+r_1}(\alpha))$$

Let A be the ring of integers in F; its image is a lattice in $\mathbb{C}^{r_2} \times \mathbb{R}$.
Let $\zeta_F(s)$ be the zeta function of F. For each integer $m \geq 0$, let $d_m$ be the order of the zero of $\zeta_F(s)$ at $s = -m$, and let $C_m > 0$ be the positive constant such that

(2) $$\zeta_F(s) \sim \pm C_m (s+m)^{d_m}, \quad \text{as } s \to -m.$$

The functional equation is (modulo mistakes)

(3) $$\zeta_F(s) = \frac{2^{ns-r_2}\, \pi^{ns-n}}{|d_F|^{s-1/2}} (\sin \pi s)^{r_2} \left(\sin \pi \tfrac{s}{2}\right)^{r_1} \left(\Gamma(1-s)\right)^n \zeta_F(1-s),$$

where $d_F$ is the discriminant of F. Since $\zeta_F$ is non-zero for $s > 1$ and has a pole of order 1 at $s = 1$ we obtain from (3)

(4) $$d_m = \begin{cases} r_2, & \text{if } m \text{ odd} \\ r_2 + r_1, & \text{if } m \text{ even} > 0 \\ r_2 + r_1 - 1, & \text{if } m = 0. \end{cases} \qquad C_0 = \frac{|d_F|^{1/2}}{2^{r_1+r_2}\, \pi^{r_2}} \operatorname{res}_{s=0} \zeta_F(s)$$

$$C_m = \frac{|d_F|^{m+1/2} (m!)^n}{2^{mn+d_m}\, \pi^{mn+n-d_m}} \zeta_F(1+m)$$

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Associated with A are abelian groups $K_i A$, $i \geq 0$, namely $K_0 A = \operatorname{Pic} A$, $K_1 A = A^\circ$, $K_2 A = \dots$ (cf. Milnor, Quillen, et.al.). Presumably they are finitely generated. If so, then

(5) $K_{2m} A$ is finite, and $K_{2m+1}$ is of rank $d_m$,

because Borel (after Garland's beginning) has shown $(K_{2m} A) \otimes \mathbb{Q} = 0$, and $\dim_{\mathbb{Q}} ((K_{2m+1} A) \otimes \mathbb{Q}) = d_m$.

Presumably Borel does this via Quillen's result that

(6) $$(K_i A)\otimes \mathbb{Q} \approx [H_i(GL(A),\mathbb{Q})]_{\text{primitive}},$$

(This has nothing to do with arithmetic — it holds for any ring $A$). Using (6), one (eg. Bott, Borel) can define canonical homomorphisms

$$e_{2m+1} : K_{2m+1}(\mathbb{C}) \longrightarrow \mathbb{R}$$

For example, $e_1$ is induced by $X \mapsto \log|\det X|$, for $X\in GL_n(\mathbb{C})$. Composing these $e$-maps with the maps $K_{2m+1}A \to K_{2m+1}\mathbb{C}$ induced by the imbeddings $\varphi$ of (1), Lichtenbaum defines "higher regulators" $R_m$ as follows:

Let $\alpha_1,\dots,\alpha_{d_m}$ be a base for $K_{2m+1}A$ mod torsion

Let $\lambda_{ij} = \begin{cases} 2\,e_{2m+1}(\varphi_i(\alpha_j)) & ,\text{if } 1\le i\le r_2 \\ e_{2m+1}(\varphi_i(\alpha_j)) & ,\text{if } r_2+1\le i\le r_2+r_1\end{cases}$

Then, by definition,

(7) $$R_m = \text{abs.val.} \det_{1\le i,j\le d_m}(\lambda_{ij})$$

Lichtenbaum then conjectures:

(8) $$c_m = \frac{|K_{2m}A|\,R_m}{|(K_{2m+1}A)_{\text{tors.}}|}$$

<u>Example 1</u>: <u>The case $m=0$</u>. For $m=0$ the conjecture reads $c_0 = \frac{hR}{w}$, where $h$ is the class number, $R$ the usual regulator, and $w$ the number of roots of 1 in $F$. By (4) this is equivalent to the classical formula

(9) $$\operatorname{res}_{s=1}\zeta_F(s) = \frac{2^{r_1+r_2}\pi^{r_2}\,hR}{|d_F|^{1/2}\,w}$$

Hence the conjecture is true and well known for $m=0$.

Example 2: The case $d_m = 0$, i.e. $m$ odd and $F$ totally real. In this case $R_m = 1$ and the conjecture reads

(11) $$\zeta_F(-m) = \frac{|K_{2m}A|}{|K_{2m+1}A|}, \quad \text{for } m \text{ odd, } F \text{ tot. real.}$$

In particular, for $m=1$,

(12) $$\zeta_F(-1) = \frac{|K_2 A|}{|K_3 A|}, \quad \text{for } F \text{ tot. real.}$$

This last fits with the conjecture of Birch, because the map of $K_2 A$ onto the "tame kernel" in $K_2 F$ is presumably an isomorphism, and because Quillen conjectures $|(K_3 A)_{tors}| = w_2(F)$. More precisely, Quillen's (and Lichtenbaum's ?) conjectural cohomological description of $K_2 A$ implies that

(13) $$(K_{2m+1}A)_{tors} \text{ is cyclic of order } w_{m+1}(F),$$

where $w_r(F)$ is defined, for each integer $r \geq 1$, as the largest integer $k$ such that $Gal(F(\zeta_k)/F)$ is killed by $r$, $\zeta_k$ being a primitive $k$-th root of unity.

For example, $w_1(F)$ is the number of roots of unity in $F$, for any $F$. For $F = \mathbb{Q}$ and any $r$, the number $w_r(\mathbb{Q})$ is the largest integer $k$ such that $x^r \equiv 1 \pmod{k}$ for all integers $x$ prime to $k$. From this it is easy to see that

(14) $$w_r(\mathbb{Q}) = \begin{cases} 2, & \text{if } r \text{ is odd} \\ 2 \cdot \prod_{\substack{p \text{ prime} \\ p-1 \text{ divides } r}} p^{1+\mathrm{ord}_p r}, & \text{if } r \text{ is even}. \end{cases}$$

For the Riemann zeta function $\zeta = \zeta_{\mathbb{Q}}$ we have

(15) $$\zeta(1-m) = \pm \frac{B_{m/2}}{m}, \quad \text{for even } m > 0,$$

where the $B$ is a Bernoulli number:

(16) $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, $B_4 = \frac{1}{30}$, $B_5 = \frac{5}{66}$, $B_6 = \frac{691}{2730}$, $B_7 = \frac{7}{6}$
$B_8 = \frac{3617}{510}$, $B_9 =$

Using (13), (14), and (15) in (11) (with $A = \mathbb{Z}$ and $m$ replaced by $m-1$) we get for even $m > 0$

$$(17) \qquad B_{\frac{m}{2}} = \frac{m\,|K_{2m-2}\mathbb{Z}|}{w_m} = \frac{|K_{2m-2}\mathbb{Z}|\ \prod_{p\notin S_m} p^{\operatorname{ord}_p m}}{2 \cdot \prod_{p\in S_m} p},$$

where $S_m$ is the set of primes $p$ such that $p-1$ divides $m$. By The Clausen–Von Staudt theorem, the denominator of $B_{m/2}$ is $\prod_{p\in S_m} p$. Hence (17) is equivalent to:

$$(18) \qquad \text{numerator of } B_{\frac{m}{2}} = \frac{|K_{2m-2}\mathbb{Z}|}{2} \cdot \prod_{p\notin S_m} p^{\operatorname{ord}_p m} \qquad (m \text{ even})$$

Thus, conjecturally, the order of $K_{4r-2}\mathbb{Z}$ is twice the "essential part" of the numerator of $B_r$", e.g. $|K_N\mathbb{Z}| = 2$ for $N = 2, 6, 10, 14, 18, 2$[illegible] $|K_{22}\mathbb{Z}| = 2 \times 691$, and $K_{30}\mathbb{Z} = 2 \times 3617$. (Presumably it is well known that $\prod_{p\notin S_m} p^{\operatorname{ord}_p m}$ divides $B_{\frac{m}{2}}$, although I don't recall having seen such a result. What is a reference for it?) (Also, what is the "j-homomorphism"?)

found one – see p5.

Example 3: $F = \mathbb{Q}$, $d_m = 1$: Suppose $m$ even, $> 0$. Combining (4), (8) with $A = \mathbb{Z}$, (13) and (14) we get

$$\zeta(1+m) = \frac{(2\pi)^m}{m!}\,|K_{2m}\mathbb{Z}|\ e_{2m+1}(\varphi(\alpha)),$$

where $\alpha = \alpha_{2m+1}$ is a generator for $K_{2m+1}\mathbb{Z}$ (mod torsion) and where $\varphi : K\mathbb{Z} \to K\mathbb{C}$ is induced by the inclusion $\mathbb{Z} \subset \mathbb{C}$. For example, taking $m = 2$ we should have

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = 2\pi^2 |K_4\mathbb{Z}|\ e_5(\alpha_5).$$

Is it possible to test this experimentally? If we could compute $e_5(\beta) \neq$ for some $\beta \in K_5\mathbb{Z}$ or in $[H_5(GL(\mathbb{Z}); \mathbb{Q})]_{prim}$ (cf. (6)) then it would be fun to find that $\dfrac{\zeta(3)}{2\pi^2 e_5(\beta)}$ was approximately the ratio of two small integers

Example 4: F imaginary quadratic. In this case we should have, for all $m > 0$,

$$\frac{|K_{2m}A|}{W_{m+1}(F)}\, e_{2m+1}(\alpha_{2m+1}) = (m!)^2 \left(\frac{|d_F|^{1/2}}{2\pi}\right)^{2m+1} \zeta_F(1+m).$$

In particular,

$$\frac{|K_2 A|}{W_2(F)}\, e_3(\alpha_3) = \frac{|d_F|^{3/2}}{8\pi^3}\, \zeta_F(2).$$

Here again we could make some interesting experiments if we could compute $e_3(\alpha_3)$. Is it, up to a small rational factor, the volume of the fundamental domain of $SL_2 A$ operating as usual on $\mathbb{C} \times \mathbb{R}^+_{>0}$ ??

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Reference for divisibility (cf p.4) of $B_r$ by primes dividing $r$ not dividing denom of $B$: Uspensky + Heaslet, Elementary No. Theory, McGraw Hill 1939, p 261. They prove what we want via "Voronoi's Theorem" which states, if $B_r = P_r/Q_r$ in lowest terms, that for any $N > 1$ and any $a$ prime to $N$,

$$(a^{2r}-1)P_r \equiv (-1)^{r-1}\, 2r\, a^{2r-1} Q_r \sum_{\nu=1}^{N-1} \nu^{2r-1}\left[\frac{\nu a}{N}\right] \pmod N$$

Taking $N = 2r$ we get

$$(a^{2r}-1)P_r \equiv 0 \pmod{2r}$$

Taking for $a$ a primitive root of a prime $p \mid r$, we see that if $p-1$ does not divide $2r$, then $a^{2r}-1$ is prime to $p$, and so the power of $p$ occurring in $2r$ also occurs in $P_r$.

$E$ vector space over $\mathbb{C}$ of dimension $n$.
$\mathcal{O}$ ring of integers in a quadratic no. field $\mathbb{Q}[\sqrt{-d}]$.
I want to let $X$ be the set of lattices in $E$ which are $\mathcal{O}$-modules. Thus if we choose one of these $L$ we have that

$$X \xleftarrow{\sim} \mathrm{Aut}_{\mathbb{C}}(E)/\mathrm{Aut}_{\mathcal{O}}(L).$$

no.

$$X = \left\{ L \subset E \;\middle|\; \begin{array}{l} L \text{ lattice i.e. free abelian grp of rank } 2n \\ \text{which spans } E \\ L \text{ an } \mathcal{O}\text{-module, i.e. } \sqrt{-d}\cdot L \subset L. \end{array} \right\}$$

Then if $\theta \in \mathrm{Aut}_{\mathbb{C}}(E)$ and $L \in X$, then $\theta L \in X$.

Does $\mathrm{Aut}_{\mathbb{C}}(E) = G$ act transitively on $X$.
If $L$ and $L'$ $\ni$ $\theta L = L'$ then

$$\theta : L \xrightarrow{\sim} L'$$

is an $\mathcal{O}$-module isomorphism. So ~~the~~ if $L$ and $L'$ are not isomorphic as $\mathcal{O}$-module then they are not in the same $G$-orbit. Thus there are finitely many $G$-orbits, because the ranks of $L, L'$ are same so only other invariant is the determinant $\Lambda^n_{\mathcal{O}} L \in \mathrm{Pic}(\mathcal{O})$. which is finite. Gives a direct proof patterned on the finiteness of class number.

Recall that proof. One starts with a lattice $\mathfrak{a} \subset \mathcal{O} \subset \mathbb{C}$ and applies Minkowski to find an element $z \in \mathfrak{a}$ with small abs. value

Let $P$ be the projective space of lines in $E$. It is a compact complex manifold on which $\Gamma$ operates. ~~We consider the category of sheaves on $P$ endowed with a compatible action of $\Gamma$.~~

In general, given a space $X$ endowed with an action of a discrete group $\Gamma$, one can consider the category $\mathrm{Top}(X,\Gamma)$ of $\Gamma$-sheaves over $X$, that is, sheaves of sets over $X$ on which $\Gamma$ acts in a way compatible with the action on $X$. The $\Gamma$-sheaves of abelian groups form an abelian category with enough injectives (ref. to Tohoku) and one defines the equivariant cohomology of $X$ with coefficients in a $\Gamma$-sheaf $F$, denoted $H^i(X,\Gamma;F)$, as the derived functors of $F \longmapsto H^0(\Gamma, H^0(X,F))$, the latter being the group of invariant global sections.

In general, given a topological space on which a discrete group $\Gamma$ operates, one can consider $\Gamma$-sheaves over $X$, that is, sheaves over $X$ endowed with a $\Gamma$-action compatible with the action on $X$. The equivariant cohomology of $X$ with coefficients in a $\Gamma$-sheaf $F$ (of abelian groups), denoted $H^i(X,\Gamma;F)$ is defined as the derived functors of the functor from the category of $\Gamma$-sheaves of abelian groups to the category of abelian groups which sends $F$ to the group $H^0(\Gamma, H^0(X,F))$ of invariant global sections.

Let $\Gamma$ be a group and let $E$ be a complex vector space of finite dimension endowed with a linear action of $\Gamma$. To the representation $E$ belongs Chern classes

$$c_m(E) \in H^{2m}(B\Gamma, \mathbb{Z}) \qquad 1 \leq m \leq \dim(E)$$

where $B\Gamma$ is the classifying space of $\Gamma$. These are the Chern classes of the complex vector bundle over $B\Gamma$ associated to $E$.

~~One knows that $c_m(E)$ is a torsion class hence that its image in $H^{2m}(B\Gamma, \mathbb{C})$~~

To the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}^* \longrightarrow 0$$
$$z \longmapsto \exp(2\pi i z)$$

is associated a long exact sequence

$$\cdots \; H^{2m-1}(B\Gamma, \mathbb{C}^*) \xrightarrow{\;\delta\;} H^{2m}(B\Gamma, \mathbb{Z}) \longrightarrow H^{2m}(B\Gamma, \mathbb{C}) \cdots$$

It is known that the image of $c_m(E)$ in the complex cohomology is zero, and in fact zero for "canonical" reasons. This suggests the possibility of defining characteristic classes

$$e_m(E) \in H^{2m-1}(B\Gamma, \mathbb{C}^*)$$

such that $\delta e_m(E) = c_m(E)$.

For $m=1$ this may be done as follows. When $E$ is one-dimensional, it is ~~determined~~ classified up to isomorphism

Introduction: Let $\Gamma$ be a discrete group and let $E$ be a complex vector space of finite dimension endowed with a linear action of $\Gamma$. To the representation $E$ belong Chern classes

$$c_m(E) \in H^{2m}(E, \mathbb{Z}) \qquad 1 \leq m \leq \dim(E)$$

which may be thought of as the Chern classes of the complex vector bundle over the classifying space $B\Gamma$ associated to $E$. Let

$$\cdots H^{i-1}(\Gamma, \mathbb{C}^*) \xrightarrow{\delta} H^i(\Gamma, \mathbb{Z}) \longrightarrow H^i(\Gamma, \mathbb{C}) \cdots$$

be the long exact sequence in cohomology associated to the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}^* \longrightarrow 0$$
$$z \longmapsto \exp(2\pi i z) \quad .$$

~~It is known that the image of $c_m(E)$ in $H^{2m}(\Gamma, \mathbb{C})$ is zero, hence it lies in the image of $\delta$. In fact, as we shall show, the image of $c_m(E)$ in complex cohomology is zero for a "canonical" reason, hence there are classes~~

~~$$e_m(E) \in H$$~~

It is known that the image of $c_m(E)$ in $H^{2m}(\Gamma, \mathbb{C})$ is zero, and hence $c_m(E)$ lies in the image of $\delta$. We are going to produce characteristic classes

$$e_m(E) \in H^{2m-1}(\Gamma, \mathbb{C}^*) \qquad 1 \leq m \leq \dim(E)$$

such that $\delta e_m(E) = c_m(E)$. For example, with respect to the canonical isom

$$H^1(\Gamma, \mathbb{C}^*) = \mathrm{Hom}(\Gamma, \mathbb{C}^*)$$

$e_1(E)$ will correspond to the homomorphism $\gamma \longmapsto \det(\gamma_E)$.

~~In order to define $e_m(E)$ it will be necessary to review the definition of $c_m(E)$ and to show that it vanishes in complex cohomology for a~~

Before defining the classes $e_m(E)$, we review the definition of $c_m(E)$ in a suitable form and prove that it vanishes in complex cohomology. The reason for the vanishing will lead to the classes $e_m(E)$:

Let $P$ be the projective space of lines in $E$. It is a compact complex manifold with a natural action of $\Gamma$. In general, given a topological space $X$ on which a discrete group $\Gamma$ operates, one can consider $\Gamma$-sheaves over $X$, that is, sheaves over $X$ endowed with a $\Gamma$-action compatible with the action on $X$. The equivariant cohomology of $X$ with coefficients in the $\Gamma$-sheaf $F$ (of abelian groups), denoted $H^i(X, \Gamma; F)$, is defined to be the right derived functors of the functor ~~[illegible]~~ which associates to a $\Gamma$-sheaf of abelian groups the group of its invariant global sections.

Dear Raoul,

I thought a bit about the characteristic classes with real coefficients you have defined using curvature for bundles stratified with respect to a foliation. It seems that these classes may be viewed as Chern classes in a style of cohomology[1] intimately connected with the complex tori defined by Griffiths.
In addition, it appears that characteristic classes with coefficients in $\mathbb{C}^*$ can be defined which yield real classes under the homom. $z \longmapsto \log|z|$.

Let $X$ be a $C^\infty$-manifold endowed with a subbundle $S$ of the complexified tangent bundle such that both $S$ and $S+\bar{S}$ are integrable. Such an $S$ I will call a _quasi-foliation_ of $X$. Particular cases are: i) foliated manifolds, ~~ii) complex manifolds ($S \cap \bar{S} = 0$, $S+\bar{S} = T\otimes\mathbb{C}$). In general~~ where $S$ is the subbundle of complex vectors tangent to the leaves, ii) complex manifolds, where $S$ is the subbundle of anti-holom. vectors. In general, the quasi-foliation defines locally on $X$ two submersions

(1) $$X \xrightarrow{f} X/S\cap\bar{S} \xrightarrow{g} X/S+\bar{S}$$

where the fibres of $g$ are complex manifolds. (Newlander–Nirenberg thm)

1) I learned about this cohomology while at Bures.

Let $\Omega_X^m$ be the sheaf of complex-valued $C^\infty$ $m$-forms on $X$ which are flat with respect to the quasi-foliation ($i(v)\omega = i(v)\,d\omega = 0$ if $v \in S$). In terms of the local picture (1), these are the inverse images of forms on $X/S\cap\bar{S}$ which are holomorphic on the fibres of $q$. The De Rham complex

$$\Omega_X^{\cdot}: \qquad \Omega_X^0 \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \cdots\cdots$$

is a resolution of $\mathbb{C}$. We put $\mathcal{O}_X = \Omega_X^0$ (after Serre).

Let $E$ be a complex vector bundle on $X$ stratified with respect to the quasi-foliation, that is, endowed with an $S$-connection $\delta: E \to E \otimes S^*$ such that $\delta^2 = 0$. By the Frobenius-Nirenberg theorem, $E$ is spanned by the subsheaf of flat sections. From this it follows that the category of stratified bundles is equivalent to the category of locally free $\mathcal{O}_X$-modules of finite rank. In particular, the group of isomorphism classes of stratified line bundles is isom. to $H^1(X, \mathcal{O}_X^*)$.

~~I propose to define Chern classes for stratified vector bundles~~ $\cancel{E}$

$$\cancel{c_i(E) \in H^{2i}(X, W_X^{(i)})}$$

~~where~~ $\cancel{W_X^{(i)}}$ ~~is a~~

Given a stratified vector bundles $E$, I shall define Chern classes

~~Our goal is to define~~

$$c_m(E) \in H^{2m}(X, W^{(m)}_X)$$

with values in the hypercohomology of a certain complex $W^{(m)}_X$. In order to define this complex I need the analogue for complexes of a well-known construction in homotopy theory.

Let $f : A \longrightarrow C$ and $g : B \longrightarrow C$ be morphisms of complexes in an additive category. By the homotopy-fibred product of $f$ and $g$, I mean the complex defined by

$$(A \overset{2}{\times}_C B)^n = A^n \oplus B^n \oplus C^{n-1}$$

$$d(a, b, c) = (da, db, fa - gc - db).$$

For any complex $K$, the morphisms $K \longrightarrow A \overset{2}{\times}_C B$ may be identified with triples $(u, v, h)$ where $u : K \longrightarrow A$, $v : K \longrightarrow B$ are morphisms and $h : K \longrightarrow C$ is a homotopy:

$$[d, h] = fu - gv.$$

Here $[d, \alpha] = d\alpha - (-1)^p \alpha d$ if $\alpha : K \longrightarrow C$ is of degree $p$.

For the moment $X$ will be fixed and we abbreviate $\Omega_X$ to $\Omega$,

Let $m$ be an integer and let $F_m \Omega^\bullet_X$ be the subcomplex of $\Omega^\bullet_X$ which is the same in degrees $\geq m$ and $0$ in degrees $< 0$, and let $g : F_m \Omega^\bullet_X \longrightarrow \Omega^\bullet_X$ be the inclusion. Let $(2\pi i)^m \mathbb{Z} \subset \Omega^0_X$

denote the evident constant subsheaf, ~~[illegible]~~ regard it as a complex concentrated in degree zero, and let $f: (2\pi i)^m\mathbb{Z} \longrightarrow \Omega^{\cdot}_X$ denote the evident inclusion. ~~Define $W^{(m)}_X$ as the homotopy fibred product of $f$ and $g$~~ Set

$$W^{(m)}_X = (2\pi i)^m\mathbb{Z} \overset{2}{\underset{\Omega^{\cdot}_X}{\times}} F_m\Omega^{\cdot}_X$$

so that there is a homotopy-cartesian square

$$\begin{tikzcd}
W^{(m)} \arrow[r] \arrow[d] & F_m\Omega^{\cdot} \arrow[d] \\
(2\pi i)^m\mathbb{Z} \arrow[r, hook, "f"] & \Omega^{\cdot}
\end{tikzcd}.$$

Then $W^{(m)}_X$ is the complex

$$\begin{tikzcd}
(2\pi i)^m\mathbb{Z} \arrow[r, "f"] & \Omega^0_X \arrow[r, "-d"] & \cdots \arrow[r] & \Omega^{m-2}_X \arrow[r, "-d"] & \Omega^{m-1}_X \arrow[r, "-d"] & \Omega^{m+1}_X \arrow[r, "-d"] & {} \\
& \oplus & & \oplus & \oplus & & \\
& 0 & & 0 & \Omega^m_X \arrow[r, "d"'] \arrow[uur, "-1"] & \Omega^{m+1}_X \arrow[r, "d"'] \arrow[uur, "-1"'] & {}
\end{tikzcd}$$

so it is quasi-isomorphic to the complex $\tilde{W}^{(m)}_X$

$$0 \longrightarrow (2\pi i)^m\mathbb{Z} \xrightarrow{f} \Omega^0_X \xrightarrow{-d} \cdots\cdots \Omega^{m-1}_X \longrightarrow 0 \longrightarrow$$

with $\Omega^j_X$ in degree $j+1$. ~~Thus $W^{(0)} = \mathbb{Z}$ and because of the exact sequence~~

~~$$0 \longrightarrow 2\pi i\mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}^*_X \longrightarrow 0$$~~

~~$W^{(0)}$ is quasi-isomorphic to $\mathcal{O}^*_X$~~

~~we have that~~

~~$W$~~

Thus

$$W^{(0)} = \mathbb{Z}$$

$$W^{(1)} \sim \mathcal{O}_X^*[-1]$$

and for $m \geq 2$

$$\mathcal{H}^q(W^{(m)}) = \begin{cases} \operatorname{Ker}\{\Omega_X^m \xrightarrow{d} \Omega_X^{m-1}\} & q = m \\ \mathbb{C}/(2\pi i)^m\mathbb{Z} \simeq \mathbb{C}^* & q = 1 \\ 0 & q \neq 1, m. \end{cases}$$

~~Using~~ From the exact sequence

$$0 \to \Omega_X^{\cdot}/F_m\Omega_X^{\cdot}[-1] \longrightarrow W_X^{(m)} \longrightarrow (2\pi i)^m\mathbb{Z} \longrightarrow 0$$

one obtains a long exact sequence

$$H^{2m-1}(X, (2\pi i)^m\mathbb{Z}) \to H^{2m-1}(X, \Omega_X^{\cdot}/F_m\Omega_X^{\cdot}) \to H^{2m}(X, W_X^{(m)})$$

$$\longrightarrow H^{2m}(X, (2\pi i)^m\mathbb{Z}) \longrightarrow H^{2m}(X, \Omega_X^{\cdot}/F_m\Omega_X^{\cdot}).$$

~~In the case of a Kahler manifold this seq~~ This shows in the case of a Kahler manifold that $H^{2m}(X, W_X^{(m)})$ is an extension of the integral classes of type $(m,m)$ by the Griffiths torus of degree $m$.

There are pairings

$$W^{(m)} \otimes_{\mathbb{Z}} W^{(m')} \longrightarrow W^{(m+m')}$$

defined as follows. It suffices to functorially associate to a pair of morphisms ~~maps~~ $t: K \to W^{(m)}$ and $t': K' \to W^{(m')}$ of complexes a morphism $K \otimes_{\mathbb{Z}} K' \longrightarrow W^{(m+m')}$. Let $t$ ~~Suppose the~~ ~~forms~~ be represented by the triple $(u, v, h)$ where $u: K \to (2\pi i)^m \mathbb{Z}$, $v: K \to F_m \Omega^{\cdot}$ are morphisms and where $h: K \to \Omega^{\cdot}$ is a homotopy joining $u$ to $v$ in $\Omega^{\cdot}$, that is, $[d, h] = u - v$. Let $t'$ be represented by $(u', v', h')$. Using the exterior products of forms, we obtain maps

$$K \otimes_{\mathbb{Z}} K' \xrightarrow{v \otimes v'} F_m \Omega^{\cdot} \otimes_{\mathbb{Z}} F_{m'} \Omega^{\cdot} \longrightarrow F_{m+m'} \Omega^{\cdot}$$

which will be denoted $vv'$, as well as similar maps $uu'$, $hu'$, etc. Since

$$\begin{aligned} uu' - vv' &= (u-v)u' + v(u'-v') \\ &= [d,h]u' + v[d,h'] \\ &= [d, hu' + vh'] \end{aligned}$$

the triple ~~[illegible]~~ $(uu', vv', hu' + vh')$ represents the desired morphism from $K \otimes_{\mathbb{Z}} K'$ to $W^{(m+m')}$.

It is easy to check that these pairings are associative, and that they are homotopy commutative. Therefore

$$\bigoplus_{j,m} H^j(X, W_X^{(m)})$$

is a bigraded ring, anti-commutative with respect to the degree $j$.

We can now define Chern classes for stratified bundles in the W-cohomology. For a line bundle $L$, let $c_1(L)$ be the image of the isom. class of $L$ under the canon. isom.

$$H^1(X, \mathcal{O}_X^*) \simeq H^2(X, W_X^{(1)}).$$

For a stratified bundle $E$ of dimension $n$, form the projective bundle $\pi: \mathbb{P}E \longrightarrow X$ of lines in $E$, and let $\mathcal{O}(1)$ be the canonical line bundle. Note that $\mathbb{P}E$ has a canonical quasi-foliation induced by that of $X$ and the holomorphic structures on the fibres of $\pi$.

<u>Theorem:</u> Let $\xi = c_1(\mathcal{O}(1)) \in H^2(\mathbb{P}E, W_{\mathbb{P}E}^{(1)})$. Then $H^*(\mathbb{P}E, W_{\mathbb{P}E}^{(*)})$ is a free module over $H^*(X, W_X^{(*)})$ with basis $1, \ldots, \xi^{n-1}$. In other words, $(a_g) \mapsto \sum a_g \xi^g$ yields an isomorphism

~~Proof: We must show that the map is an isomorphism~~

$$\bigoplus_{0 \leq g < n} H^{j-2g}(X, W_X^{(m-g)}) \xrightarrow{\sim} H^j(\mathbb{P}E, W_{\mathbb{P}E}^{(m)}).$$

$$(a_g) \longmapsto \sum a_g \xi^g$$

is an isomorphism, where $W_X^{(r)} = \mathbb{Z}$ for $r \leq 0$. To the square ( ) is associated a Mayer-Vietoris sequence

$$\cdots \to H^j(\mathbb{P}E, W_{\mathbb{P}E}^{(m)}) \longrightarrow H^j(\mathbb{P}E, \mathbb{Z}) \oplus H^j(\mathbb{P}E, \mathbb{C}) \longrightarrow H^j(\mathbb{P}E, F_m \Omega^{\cdot}_{\mathbb{P}E}) \to \cdots$$

Using the similar exact sequences on $X$ and the five lemma, we see it suffices to prove that the maps

Proof: To the square ( ) is associated a Mayer-Vietoris style sequence

$$\to H^j(\mathbb{P}E, W^{(m)}_{\mathbb{P}E}) \longrightarrow H^j(\mathbb{P}E, \mathbb{Z}) \oplus H^j(\mathbb{P}E, F_m\Omega^{\cdot}_X) \longrightarrow H^j(\mathbb{P}E, \mathbb{C}) \longrightarrow \cdots\cdots$$

~~Comparing this with the analogous exact sequences over X, one sees that the theorem results from the projective bundle theorems for cohomology with coefficients in $\mathbb{Z}$, $\mathbb{C}$~~

~~Using the analogous exact sequences over X and the 5 lemma, it suffices to show the maps~~

$$\bigoplus_{q=0}^{n-1} H^{j-2q}\left(X, \begin{matrix}\mathbb{Z}\\ \mathbb{C}\end{matrix}\right) \xrightarrow{\sim} H^j\left(\mathbb{P}E, \begin{matrix}\mathbb{Z}\\ \mathbb{C}\end{matrix}\right)$$

$$\bigoplus_{q=0}^{n-1} H^{j-2q}(X, F_{m-q}\Omega^{\cdot}_X) \xrightarrow{\sim} H^j(\mathbb{P}E, F_m\Omega^{\cdot}_{\mathbb{P}E})$$

sending $(a_q) \longmapsto \sum a_q \xi^q$. ~~are isomorphisms~~

Comparing this sequence with a direct sums of analogous sequences over X, it suffices to ~~show the maps~~ establish isomorphisms

The former follows from the classical projective bundle theorems in ordinary cohomology, because the canonical hom. $W^{(m)}_{\mathbb{P}E} \longrightarrow \mathbb{Z}$ carries $\xi$ to the ordinary first Chern class of $\mathcal{O}(1)$ in $H^2(\mathbb{P}E, \mathbb{Z})$.

To establish the latter ism, we can use the the exact sequences

$$0 \longrightarrow F_{m+1}\Omega^{\cdot} \longrightarrow F_m\Omega^{\cdot} \longrightarrow \Omega^m[-m] \longrightarrow 0$$

both on $\mathbb{P}E$ and $X$ to reduce to proving isomorphisms

$$\bigoplus_{q=0}^{n-1} H^{j-q}(X, \Omega_X^{m-q}) \xrightarrow{\sim} H^j(\mathbb{P}E, \Omega_{\mathbb{P}E}^m).$$

Using the Leray spectral sequence for $\pi$, it suffices to prove

$$(*) \qquad \bigoplus_{q=0}^{n-1} \Omega_X^{m-q} \xrightarrow{\sim} R^q\pi_*(\Omega_{\mathbb{P}E}^m).$$

Let $\Omega^1_{\mathbb{P}E/X}$ be defined by the exact sequence

$$0 \longrightarrow \pi^*\Omega_X^1 \longrightarrow \Omega^1_{\mathbb{P}E} \longrightarrow \Omega^1_{\mathbb{P}E/X} \longrightarrow 0.$$

~~One knows that~~

$$\cancel{R^q\pi_*(\Omega^m_{\mathbb{P}E/X}) = \begin{cases} 0 & q \neq m \\ \mathcal{O}_X & q = m \end{cases}}$$

Filtering $\Omega^m_{\mathbb{P}E}$ by $F_p\Omega^m_{\mathbb{P}E} = \pi^*\Omega^p_X \cdot \Omega^{m-p}_{\mathbb{P}E}$, we have

$$gr_p\,\Omega^m_{\mathbb{P}E} = \pi^*\Omega^p_X \otimes \Omega^{m-p}_{\mathbb{P}E/X}.$$

On the other hand, one knows by classical projective space computations that

$$R^q\pi_*(\Omega^{m-p}_{\mathbb{P}E/X}) = \begin{cases} 0 & q \neq m-p \\ \mathcal{O}_X & q = m-p. \end{cases}$$

The formula $(*)$ results easily. Q.E.D.

want $\Gamma$ equiv. maps $\bar{C}_{et} \longrightarrow BU$
so I can perhaps identify this with $\Gamma$-fixpoints
on Map$(\bar{C}_{et}, BU)$

---

Thus the homotopy analysis of a curve over a finite field $k$ would amounts to statement that there is an exact sequence

$$K_*(C) \longrightarrow K_*(C \otimes_k \bar{k}) \xrightarrow{\text{Frob} - 1}$$

The K-groups of $\bar{C}$ are clear, namely

$$K_{2i} = \text{Pic}(\bar{C}) \otimes T^{\otimes i}$$

$$K_{2i-1} = 2 \text{ copies of } K_{2i-1}(k).$$

Still not clean!

---

The situation: I would like to get completely cleaned up the lower bound side of the [illegible] homology of $GL(A)$.

(1) $A$ is a local field $K$ containing $\mu_\ell$. Then

$$H_*(BGL(K)) = S[\tilde{H}_*(K^*)]$$

except possibly for the prime $\ell = 2$.

(2) $A$ is the ring of S-integers in a number field $K$, $A \supset \mu_\ell, \ell^{-1}$. Then [illegible] yet have convincing conjectures. The point is to [illegible] the homotopy type of the answer.

In the case of [illegible] $C$ over a finite field $k$ what happens is [illegible] the homotopy type of a surface of genus $g$ [illegible] Betti nos. 1, 2g, 1 and there is a Frobenius action.

$$\begin{array}{c} \bar{C}_{et} \\ \downarrow \Gamma \\ C_{et} \end{array}$$

Thus $C_{et}$ is a sort of three manifold fibred over a circle with fibre $\bar{C}_{et}$.

$$\begin{array}{c} BU \\ \downarrow \\ \bar{C}_{et} \times^{\Gamma} BU \\ \downarrow \\ C_{et} \end{array}$$

now we can try to decompose $C_{et}$ skeleta-wise!

$$\dim H^1(\mathcal{O}[\ell^{-1}],\mu_\ell) = r_2 + \operatorname{card} S_\ell + \dim_\ell \operatorname{Pic}\mathcal{O}$$

$$\|$$

$$1 + \dim \operatorname{Hom}_\Gamma(H_1(\bar{\mathcal{O}}[\ell^{-1}]),\mu_\ell)$$

$$\|$$

$$1 + r_2 + \dim (J \otimes_\Gamma \mathbb{Z}/\ell\mathbb{Z}).$$

so $$\boxed{\dim(J\otimes_\Gamma \mathbb{Z}/\ell\mathbb{Z}) = \operatorname{card} S_\ell + \dim_\ell \operatorname{Pic}\mathcal{O} - 1}$$

it would seem then that

$$(J\otimes \mathbb{Z}/\ell\mathbb{Z})^\Gamma \quad \text{and} \quad (J\otimes \mathbb{Z}/\ell\mathbb{Z})_\Gamma$$

have the same dimensions $\longrightarrow$ CLEAR.

---

$$\overbrace{\qquad\qquad}^{(J\otimes\mathbb{Z}/\ell\mathbb{Z})_\Gamma + r_2 \text{ things}}$$

$$0 \longrightarrow \operatorname{Hom}_\Gamma(H_1(\bar{\mathcal{O}}[\ell^{-1}]),\mu_\ell) \longrightarrow \operatorname{Hom}(H_1\bar{A},\mu_\ell) \longrightarrow \bigoplus_{p\in S_f - S_\ell} \mathbb{Z}/p\mathbb{Z}$$

$$\hookrightarrow \underbrace{\operatorname{Ext}^1_\Gamma(H_1\bar{\mathcal{O}}[\ell^{-1}],\mu_\ell)}_{\operatorname{Hom}((J\otimes\mathbb{Z}/\ell\mathbb{Z})^\Gamma,\mu_\ell)} \longrightarrow \operatorname{Ext}^1_\Gamma(H_1\bar{A},\mu_\ell) \to \bigoplus_{p\in S_f - S_\ell} \mathbb{Z}/p\mathbb{Z} \to 0$$

so the exact sequence doesn't split.

local field case

Case 1: K field (of char. 0) complete with respect to a d.v.
residue field k alg closure of $\mathbb{F}_p$

Then

$$\begin{array}{ccccccccc} & & 1+m & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \to & \mathcal{O}^* & \to & K^* & \to & \mathbb{Z} & \to & 0 \\ & & \downarrow & & & & & & \\ & & k^* & & & & & & \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array}$$

the point is that $k^*$ is divisible, hence the $\cong \bigoplus_{\ell \neq p} \mathbb{Q}_\ell/\mathbb{Z}_\ell$
mod $\ell$ cohomology is that of the circle group.

So $H^*(K^*) = H^*(\mathbb{Z}) \times H^*(k^*) = \Lambda[d^\circ 1] \otimes P[d^\circ 2]$.

In this situation, what we have proved about elem. symm. functions for a finite field should apply so we find

$$S[\tilde{H}_*(K^*)] \hookrightarrow H_*(GL(K))$$

---

case 2: k alg. closed field $\ell$ prime no. $\neq$ char$(k)$.
Then $H_*(BGL(k))$ is at least a polynomial ring.

---

In the local field case with finite residue field then $H_*(K^*) = \Lambda[d^\circ 1] \otimes \Gamma[d^\circ 2]$, and I still want to understand if ~~why~~ the Chern classes separate the ~~___~~ ~~___~~ monomials in the basis

$$\to \coprod_{\substack{p\in S\\ p\nmid \ell}} \Gamma \times_{G_p} T_\ell \longrightarrow H_1(\bar A) \longrightarrow H_1(\bar{\mathcal O}[\ell^{-1}]) \longrightarrow 0$$

thing of abelian unramified $\ell$-extensions of $\bar K = K[\mu_{\ell^\infty}]$ (only the $0\to$ is unclear)

so

$$0 \to \underset{r_2+s}{A^*/(A^*)^\ell} \longrightarrow H^1(A_{et},\mu_\ell) \longrightarrow {}_\ell(\mathrm{Pic}\,A) \longrightarrow 0$$

$$0 \longrightarrow \mu_\ell \longrightarrow H^1(A_{et},\mu_\ell) \longrightarrow \mathrm{Hom}_\Gamma(H_1\bar A,\mu_\ell) \longrightarrow 0$$

$$\dim H^1(A_{et},\mu_\ell) = r_2 + \mathrm{card}\, S_f + \dim {}_\ell\mathrm{Pic}(A).$$

$$\dim H^1(A_{et},\mu_\ell) = 1 + \mathrm{card}\, S_f - \mathrm{card}\, S_\ell + \dim \underbrace{H_1(\bar{\mathcal O}[\ell^{-1}]) \otimes_\Gamma \mathbb Z/\ell\mathbb Z}_{r_2 + \dim J\otimes_\Gamma \mathbb Z/\ell\mathbb Z}.$$

Thus it would seem that

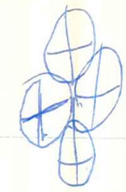
$$\dim J \otimes_\Gamma \mathbb Z/\ell\mathbb Z = \dim {}_\ell\mathrm{Pic}(A) + \mathrm{card}\, S_\ell - 1$$

$$0 \longrightarrow \overset{r_2+\mathrm{card}\,S_f+\dim_\ell \mathrm{Pic}\,A}{H^1(A,\mu_\ell)} \longrightarrow \overset{r_2+\mathrm{card}\,S_\ell+\dim_\ell \mathrm{Pic}\,\mathcal O[\ell^{-1}]}{H^1(\mathcal O[\ell^{-1}],\mu_\ell)} \longrightarrow \overset{\mathrm{card}\,S_f-\mathrm{card}\,S_\ell}{\bigoplus_{p\in S_f-S_\ell} \mathbb Z/\ell\mathbb Z}$$

$$\longrightarrow H^2(A,\mu_\ell) \longrightarrow H^2(\mathcal O[\ell^{-1}],\mu_\ell) \longrightarrow \bigoplus_{p\in S_f-S_\ell} H^1(\ ,\mathbb Z/\ell\mathbb Z) \longrightarrow 0$$

$${}_\ell(\mathbb Q/\mathbb Z)^{\mathrm{card}\,S_f-1} \longrightarrow {}_\ell(\mathbb Q/\mathbb Z)^{\mathrm{card}\,S_\ell-1} \qquad \mathbb Z/\ell\mathbb Z$$

Thus it seems that $\dim {}_\ell\mathrm{Pic}\,A = \dim {}_\ell\mathrm{Pic}\,\mathcal O[\ell^{-1}]$ which is nonsense



$r_2 +$ card $S_\ell + \dim_\ell \mathrm{Pic}\,\mathcal{O}$ $\qquad$ $r_2 +$ card $S_f + \dim_\ell \mathrm{Pic}\,A$

$$0 \to H^1(\mathcal{O}[\ell^{-1}], \mu_\ell) \to H^1(A, \mu_\ell) \to \bigoplus_{p \in S_f - S_\ell} \mathbb{Z}/\ell\mathbb{Z}$$

$$\hookrightarrow H^2(\mathcal{O}[\ell^{-1}], \mu_\ell) \to H^2(A, \mu_\ell) \to \bigoplus_{p \in S_f - S_\ell} \mathbb{Z}/\ell\mathbb{Z} \to 0$$

$\dim_\ell \mathrm{Pic}\,\mathcal{O} +$ card $S_\ell - 1$ $\qquad$ card $S_f - 1 + \dim_\ell \mathrm{Pic}\,A$

$$0 \to H^1(\mathcal{O}[\ell^{-1}], T_\ell^{\otimes i}) \to H^1(A, T_\ell^{\otimes i}) \to \bigoplus_{p \in S_f - S_\ell} T_\ell^{\otimes(i-1)}/\mathrm{Gal}_p)$$

$$\mapsto 0$$

$$\hookrightarrow \underbrace{H^2(\mathcal{O}[\ell^{-1}], T_\ell^{\otimes i})} \to H^2(A[\ell^{-1}], T_\ell^{\otimes i}) \to$$

$$\mathrm{Hom}\left(H_1(\bar{\mathcal{O}}[\ell^{-1}]), T_\ell^{\otimes i}\right)_\Gamma$$

$$\| $$

$$\mathrm{Hom}(J, T_\ell^{\otimes i})_\Gamma$$

$$H^2(\mathcal{O}[\ell^{-1}], \mu_\ell) = \mathrm{Hom}(H_1(\bar{\mathcal{O}}[\ell^{-1}]), \mu_\ell)_\Gamma$$

$\dim_\ell \mathrm{Pic}\,\mathcal{O} + \mathrm{card}(S_\ell) - 1$

$$\mathrm{Hom}(J, \mu_\ell)_\Gamma$$

$J \otimes \mathbb{Z}/\ell\mathbb{Z}$

$$\mathrm{Hom}(J \otimes \mathbb{Z}/\ell\mathbb{Z}, \mu_\ell)_\Gamma$$

Now I start adjoining

Suppose $_\ell \mathrm{Pic}\, \mathcal{O}[\ell^{-1}] = {}_\ell \mathrm{Pic}\, \mathcal{O}$ has $\ell$-rank $\lambda$

then by ~~adding~~ removing $\lambda$ points (equidist?)

should be able to reach an $A$ with $_\ell(\mathrm{Pic}\, A) = 0$.

so $s = \lambda + s_0$ here, so

$$\dim H^1(A_{et}, \mu_\ell) = \lambda + 1 + r_2$$

$$\dim H^1(A_{et}, \mu_\ell) = 1 + r_2 + \lambda + \dim J \otimes_\Lambda \mathbb{Z}/\ell\mathbb{Z}$$

so conclude that $\dim J \otimes_\Lambda \mathbb{Z}/\ell\mathbb{Z} = 0$ contradiction of sorts.

---

$\mathcal{O} = \mathbb{Z}[\mu_\ell]$

$\dim {}_\ell \mathrm{Pic}(\mathcal{O}) = \dim {}_\ell \mathrm{Pic}(\mathcal{O}[\ell^{-1}]) = \lambda$.

now ~~adjoin~~ remove $\lambda$ nice points and call the result. $A$. Should be so that $_\ell \mathrm{Pic}(A) = 0$, if the $\lambda$ points can be chosen to generate $_\ell \mathrm{Pic}(\mathcal{O})/\ell\, \mathrm{Pic}(\mathcal{O})$.

$$\dim H^1(A_{et}, \mu_\ell) = 1 + \lambda + r_2$$

$$H^1(A_{et}, \mu_\ell) = 1 + r_2 + \lambda + \dim J \otimes_\Lambda \mathbb{Z}/\ell\mathbb{Z}$$

So there's a mistake. In any case starting with $A = \mathcal{O}[\ell^{-1}]$ we get

$$\mathrm{rank}\, H^1(A, \mu_\ell) = r_2 + \{\text{no. of } v \mid \ell\} + \mathrm{rank}\{_\ell \mathrm{Pic}(\mathcal{O})\}$$

$$= 1 + r_2 + \text{no. of } \not{\!\,}\infty, \ell + \mathrm{rank}\, J^{\infty}_\Lambda .$$

~~The point perhaps is that if k is a field~~

Now I know that

$$K_{2i-3}(A/\mathfrak{p})\otimes \mathbb{Z}_\ell \simeq T_\ell^{\otimes(i-1)} \Big/ \text{effect of } x\to x^{q}$$

Thus it would appear that corresponding to the prime $\mathfrak{p}\in S$ $\mathfrak{p}\neq\infty, \ell$ we expect

$$\mathrm{Ind}_{gal(\overset{A/\mathfrak{p}[\mu_{\ell^\infty}]}{\overline{A/\mathfrak{p}}}/A/\mathfrak{p})\hookrightarrow\Gamma}\left(T_\ell^*\right)$$

in $X$?

$$\cancel{\Gamma\times_{gal}(T_\ell)\longrightarrow X}$$

$$\Gamma\times_{gal}(T_\ell)$$

$$\cancel{\mathrm{Map}_{gal}(\Gamma, T_\ell)}$$

$$\boxed{H^1(\Gamma, \mathrm{Map}_{gal}(\Gamma, T_\ell))}$$

$$H^1(\Gamma, \mathrm{Hom}(\Gamma\times_{gal} T_\ell, T_\ell^{\otimes i}))$$
$$\shortparallel$$
$$\mathrm{Map}_{gal}(\Gamma, T_\ell^{\otimes(i-1)})$$

---

Thus the conjecture is

$$X = \Lambda^{\Gamma_2}\oplus\bigoplus_{\substack{\mathfrak{p}\in S\\ \mathfrak{p}\neq\infty,\ell}}\underbrace{\mathrm{Ind}_{gal(A/\mathfrak{p})\hookrightarrow\Gamma}(T_\ell)}_{=\ \Gamma\times_{gal(A/\mathfrak{p})}T_\ell}\oplus J$$

where $J$ is the interesting part.

~~Definition of~~

$$\to \mu_\ell \longrightarrow H^1(A_{et}, \mu_\ell) \longrightarrow \mathrm{Hom}_\Gamma(X, \mu_\ell) \to 0$$

$$(\mathbb{Z}/\ell\mathbb{Z})^{r_2} \oplus \begin{pmatrix}\text{no. of } v \in S \\ \text{of } \infty, \ell\end{pmatrix} \oplus J \otimes_\Lambda \mathbb{Z}/\ell\mathbb{Z}$$

$$0 \longrightarrow A^*/(A^*)^\ell \longrightarrow H^1(A_{et}, \mu_\ell) \longrightarrow {}_\ell\mathrm{Pic}A \longrightarrow 0$$

rank $s + r_2$ rank $\lambda$

$$0 \to \mathcal{O}^* \to A^* \longrightarrow \coprod_{\substack{v \in S \\ v \nmid \infty}} \mathbb{Z} \longrightarrow \mathrm{Pic}(\mathcal{O}) \longrightarrow \mathrm{Pic}(A) \longrightarrow 0$$

$$S = \underbrace{\{v|\infty\}}_{r_2} + \overbrace{\{v|\ell\} + \{v \nmid \infty, \ell\}}^{s}$$

If $A = \mathbb{Z}[\ell^{-1}, \mu_\ell]$, then $s = 1$

$$\text{rank } H^1(A, \mu_\ell) = 1 + r_2 + \lambda$$
$$= 1 + r_2 + 0 + \text{rank}(J \otimes_\Lambda \mathbb{Z}/\ell\mathbb{Z})$$

$\therefore$ rand $(J \otimes_\Lambda \mathbb{Z}/\ell\mathbb{Z}) = \lambda$ here.

If $A = \mathcal{O}[\ell^{-1}]$. Then

$$\text{rank } H^1(A, \mu_\ell) = 1 + r_2 + \text{rank}(J \otimes_\Lambda \mathbb{Z}/\ell\mathbb{Z})$$
$$= s + r_2 + \lambda$$

$$\text{rank}(J \otimes_\Lambda \mathbb{Z}/\ell\mathbb{Z}) = \text{rank } {}_\ell\mathrm{Pic}A + \text{no of } v|\ell - 1$$

interesting point is possibility of killing $\mathrm{Pic}(A)$.

rank ${}_\ell\mathrm{Pic}(\mathcal{O}) = \lambda$ must remove $\lambda$ primes to make trivial.

$$X = H_1(\bar{A}_{et}, \mathbb{Z}_\ell) = \pi_1(\bar{A})_{ab,\ell}$$

$$\begin{array}{ccccccc} 0 \to (T_\ell^{\otimes i})_\Gamma & \longrightarrow & H^1(A_{et}, T_\ell^{\otimes i}) & \longrightarrow & \mathrm{Hom}_\Gamma(X, T_\ell^{\otimes i}) & \longrightarrow & 0 \\ \text{cyclic order } w_i & & \wr\ \text{conj.} & & \wr\ \text{conj} & & \\ & & (K_{2i-1}A) \otimes \mathbb{Z}_\ell & & \mathbb{Z}_\ell^{\Gamma_2} & & \end{array}$$

$$\begin{array}{ccc} \mathrm{Hom}(X, T_\ell^{\otimes i})_\Gamma & \xrightarrow{\sim} & H^2(A_{et}, T_\ell^{\otimes i}) \\ \underline{\text{finite}} & & \wr\ \text{conj.} \\ & & (K_{2i-2}A) \otimes \mathbb{Z}_\ell \end{array}$$

I want to express $X$ in three parts, one factor $\Lambda^{\Gamma_2}$, another from $v \nmid \ell$, and then the interesting part. Now let $\mathfrak{p}$ be a prime in $A$ not dividing $\ell$, and suppose $q = N\mathfrak{p} = \mathrm{card}(A/\mathfrak{p})$. Then $\ell^a | q-1$, let $b \doteq v_\ell(q-1)$. Now

$$\bar{A}/\bar{A}\mathfrak{p} \simeq (A/\mathfrak{p}[\mu_{\ell^\infty}])^{\ell^{b-a}}.$$

In effect

$$A/\mathfrak{p} \otimes_A \bar{A} \longrightarrow A/\mathfrak{p}$$

is a principal covering with group $\Gamma$, hence corresponding to the $\mathrm{Gal}(\overline{A/\mathfrak{p}}/A/\mathfrak{p})$ set $\Gamma$ with gen. acting via $q$. But $q$ generates $1 + \ell^b\mathbb{Z}_\ell \subset \Gamma = 1 + \ell^a\mathbb{Z}_\ell$, so its clear. Thus there are $\ell^{b-a}$ primes in $\bar{A}$ over $\mathfrak{p}$. Now

$$0 \to K_{2i-2}(\mathcal{O}[\ell^{-1}]) \to K_{2i-2}(A) \to \bigoplus_{\substack{v \in S \\ v \nmid \infty, \ell}} K_{2i-3}(k_v) \to 0.$$

$\ell$ odd prime
$K$ number field containing $\mu_\ell$
$S$ set of primes of $K$ including $\infty$ ones and those $|\ell$.
$A = \mathcal{O}_S$ ring of $S$-integers.

Observe $K$ totally imag. i.e. $[K:\mathbb{Q}] = 2r_2$

Dirichlet $\Longrightarrow$ $A^* \simeq \mu_K \times \mathbb{Z}^{s+r_2-1}$

where $s = \operatorname{card} S - \{v|\infty\}$.

From Kummer theory get

$$0 \longrightarrow A^*/(A^*)^\ell \longrightarrow H^1(A_{et}, \mu_\ell) \longrightarrow {}_\ell\operatorname{Pic}(A) \longrightarrow 0$$

$$A^*/(A^*)^\ell \cong (\mathbb{Z}/\ell\mathbb{Z})^{s+r_2}$$

let $\lambda = \dim {}_\ell\operatorname{Pic}(A)$.

$$0 \to \operatorname{Pic}(A)\otimes\mathbb{Z}/\ell \longrightarrow H^2(A_{et}, \mu_\ell) \longrightarrow {}_\ell Br(A) \longrightarrow 0.$$

$$\operatorname{Pic}(A)\otimes\mathbb{Z}/\ell \cong (\mathbb{Z}/\ell)^\lambda$$

It is known that

$$0 \longrightarrow Br(A) \longrightarrow Br(K) \longrightarrow \bigoplus_{v\notin S} \underset{\mathbb{Q}/\mathbb{Z}}{\underset{\shortparallel}{Br(K_v)}}$$

and last map onto since $s \geq 1$. In general

$$0 \longrightarrow Br(K) \longrightarrow \bigoplus_{\substack{\text{all } v\\ \text{for } K}} Br(K_v) \xrightarrow{\text{sum}} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

so we conclude

$$0 \longrightarrow Br(A) \longrightarrow \bigoplus_{v\in S} \mathbb{Q}/\mathbb{Z} \xrightarrow{\text{sum}} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

The problem is this:

$\pi_1(A)_{ab}$ is "understood" by class field theory it is abelian extensions of $K$ unramified outside of $S$. Hence must be described in terms of

$\bar{A}^*$ is not $l$-divisible
otherwise there would

$$H^1(\bar{A}_{et}, \mathbb{Z}/l) \quad \text{classifies cyclic ext. deg. } l.$$

$$H^1(\bar{A}_{et}, \mathbb{Z}/l) \qquad \text{forget this}$$

~~In any case, suppose that $\mu_l$ is finite.~~

$$H^1(\bar{A}_{et}, \mu_l) = \mathrm{Hom}(\pi_1 \bar{A}_{ab}, \mu_l)$$

so we set $\boxed{\pi_1 \bar{A}_{ab} = X}$

in which case

$$H^1(\bar{A}_{et}, T_l^{\otimes i}) = \mathrm{Hom}(X, T_l^{\otimes i})$$

in the function field case

$$H^1(\bar{C}, T_l) = T_l(J)$$
$$\| $$
$$\mathrm{Hom}(X, T_l^{\otimes})$$

hence $X = T_l(J) \otimes T_l(\mathbb{G}_m)$

in fact you have $T_l(J) \times J_l(J) \to T_l$ so can identify $T_l(J)$ and $X$ by auto-duality of the Jacobian.

The fundamental problem is to detect these classes. What's missing is the analogue of the elementary symm. functions. $\boxed{\text{YES}}$

---

Suppose $A$ is ~~a part of a no. field~~ $S$-integers in a number field $K$. Assume $\ell^{-1}, \mu_\ell \subset A$. $\ell$ odd. Then

$s = \operatorname{card} S - \{\infty\}$ ~~card of S-inf. primes~~

$$0 \to A^*/(A^*)^\ell \to H^1_{et}(A, \mu_\ell) \longrightarrow {}_\ell\operatorname{Pic}(A) \to 0$$

$$A^*/(A^*)^\ell = (\mathbb{Z}/\ell\mathbb{Z})^{s+r_2-1+1} \qquad {}_\ell\operatorname{Pic}(A) = (\mathbb{Z}/\ell\mathbb{Z})^\lambda$$

$$0 \to \operatorname{Pic} A/\ell \operatorname{Pic} A \longrightarrow H^2_{et}(A, \mu_\ell) \longrightarrow {}_\ell\operatorname{Br}(A) \to 0$$

$$\operatorname{Pic} A/\ell \operatorname{Pic} A \cong (\mathbb{Z}/\ell\mathbb{Z})^\lambda$$

$$0 \longrightarrow \operatorname{Br}(A) \longrightarrow \operatorname{Br}(K) \longrightarrow \bigoplus_{\sigma \notin S - \{\infty \text{ primes}\}} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

$$\Rightarrow \operatorname{Br}(A) \simeq \operatorname{Ker}\Big( \bigoplus_{\sigma \in S - \{\infty \text{ primes}\}} \mathbb{Q}/\mathbb{Z} \xrightarrow{+} \mathbb{Q}/\mathbb{Z} \Big)$$

Thus

$${}_\ell\operatorname{Br}(A) = (\mathbb{Z}/\ell\mathbb{Z})^{s-1}$$

---

Thus

$$H^0(A) = \mathbb{Z}/\ell\mathbb{Z}$$

$$H^1(A) = (\mathbb{Z}/\ell\mathbb{Z})^{s+r_2+\lambda}$$

$$H^2(A) = (\mathbb{Z}/\ell\mathbb{Z})^{s+\lambda-1}$$

$r_2 =$ no. of $\infty$ primes $= \frac{1}{2}[K:\mathbb{Q}]$

$s =$ no. of finite primes in $S$.

goal — to understand conjecturally the K-groups for a number field $K$

$\boxed{\text{Notation:}}$ $\mathcal{O}$ ring of integers in $K$.

$S$ finite set of places

$\mathcal{O}_S$ $S$-integers

$\ell$ prime number (odd to simplify)

suppose that

$v \mid \ell \Rightarrow v \in S$

$\mu_\ell \subset K$

principal conjecture is that

$$\boxed{K_{2i-1}(\mathcal{O}_S)_\ell}$$

---

let $\mathcal{O}$ be the ring of integers. The idea is that the K-groups are understood except for the field itself.

$$0 \longrightarrow K_{2i}(\mathcal{O}) \longrightarrow K_{2i}(\mathcal{O}_S) \longrightarrow \bigoplus_{v \in S_f} K_{2i-1}(\mathcal{O}/m_v) \longrightarrow 0$$

so better way to write this is

$$0 \longrightarrow K_{2i}(\mathcal{O}_S) \longrightarrow K_{2i}(K) \longrightarrow \bigoplus_{v \notin S} K_{2i-1}(\mathcal{O}/m_v) \xrightarrow{\delta} K_{2i-1}(\mathcal{O}_S) \longrightarrow K_{2i-1}(K) \longrightarrow 0$$

the idea is that only globally ($S = \infty$ primes) might the boundary be $\neq 0$.

$A$ Dedekind domain with quotient field $K : \mathbb{Q} < \infty$

assume $A \hookleftarrow \mathbb{Z}[\mu_\ell, \ell^{-1}]$. ~~Then adjoin~~

$$\bar{A} = A \otimes_{\mathbb{Z}[\mu_\ell]} \mathbb{Z}[\mu_{\ell^\infty}]$$

$\tilde{X} = H_1(\tilde{A})$. module d'Iwasawa a $\tilde{\Gamma} = Gal(\tilde{A}/A)$-module

$\bar{A}$ isn't connected.

The point is to choose a conn. component

$$\bar{A} = A[\mu_{\ell^\infty}] \subset \bar{\mathbb{Q}}. \text{ fixed alg. closure.}$$

then $\Gamma = Gal(\bar{A}/A)$ multiples of fixed $\gamma$.

of form $\gamma(\zeta) = \zeta^{\ell^a}$ where $a$ least $\ni$ $\mu_{\ell^a} \subset A$.

$X = H_1(\bar{A})$ and if one has $\mathfrak{p} \subset A$ with $\mathrm{card}(A/\mathfrak{p}) = q$ then

$$\ell^a \mid q-1.$$

and in $\bar{A}$ $\exists$ finitely many primes over $\mathfrak{p}$ in number $\frac{\ell^b}{\ell^a}$ where

$v_\ell(q-1) = b$.

My conjectures about K-groups

$$0 \to H^{j-1}(\bar{A}_{et}, T_\ell^{\otimes i})_\Gamma \to H^j(A_{et}, T_\ell^{\otimes i}) \longrightarrow H^j(\bar{A}_{et}, T_\ell^{\otimes i})^\Gamma \to 0$$

$\cancel{H^j(\bar{A}_{et},}$

But $H^1(\bar{A}_{et}, T_\ell^{\otimes i}) = \mathrm{Hom}(X, T_\ell^{\otimes i})$

$$\bar{A}^*/(\bar{A}^*)^{\ell^n} \longrightarrow H^1(\bar{A}_{et}, \mu_{\ell^n}) \longrightarrow {}_{\ell^n}\mathrm{Pic}(\bar{A})$$

$$\begin{tikzcd}
\mu_{\ell^n} \arrow[r] \arrow[d, "\ell"] & G_m \arrow[r, "\ell^n"] \arrow[d, "\ell"] & G_m \arrow[d] \\
\mu_{\ell^{n-1}} \arrow[r] & G_m \arrow[r, "\ell^{n-1}"] & G_m
\end{tikzcd}$$

~~But~~ $\bar{A}$ want analogues of $T_\ell(J)$

$$\mathcal{O}_{\bar{C}}^*/\ell^n \to H^1(\bar{C}, \mu_{\ell^n}) \longrightarrow {}_{\ell^n}J \to 0$$

Thus it appears that

$$\Gamma(\bar{C}, \bar{\mathcal{O}}_C^*)/ \to H^1(\bar{C}, \mu_{\ell^n}) \to {}_{\ell^n}J \to 0$$

$\| \quad 0$

. Thus

$$\varprojlim_n H^1(\bar{C}, \mu_{\ell^n}) = \varprojlim {}_{\ell^n}J = T_\ell(J)$$

$$\bigoplus_{\mathfrak{p}\in S} K_{2i-1}(A/\mathfrak{p}) \longrightarrow K_{2i-1}(A) \longrightarrow K_{2i-1}(B)$$

$$0 \to K_{2i}(A) \to K_{2i}(B) \to \bigoplus_{\mathfrak{p}\in S} K_{2i-1}(A/\mathfrak{p}) \to 0$$

$$\begin{array}{ccc} & & \| \\ H^1(\Gamma, H^1(\bar{A}_{et}, T_\ell^{\otimes i})) & = & H^2(A_{et}, T_\ell^{\otimes i-1}) \\ \| & & \\ \mathrm{Hom}(X, T_\ell^{\otimes i-1})_\Gamma & & \end{array}$$

so in addition to having

$$\mathrm{Hom}(W, T_\ell^{\otimes i})_\Gamma$$

so ~~X~~ should have a copy of

$$T_\ell^{-1}$$

for every ~~copy of~~ $\mathfrak{p}$ in $S$ not dividing $\ell$.

---

$s = (\mathrm{card}\, S) - r_2$

$$X = \underset{\text{complex places}}{\Lambda^{r_2}} + (T_\ell^{-1})^{s-1} + W$$

$$H_1(X)$$

$$0 \to (T_\ell^{\otimes i})_\Gamma \longrightarrow H^1_{et}(A, T_\ell^{\otimes i}) \to \mathrm{Hom}_\Gamma(X, T_\ell^{\otimes i}) \to 0$$

$K_{2i-1}$

$\cong \mathbb{Z}_\ell^{r_2}$

2 things wrong $i=1$
Dirichlet condition ~~product formula~~
deleting points means more units.

---

anyway for $i \geq 2$ $K_3$ etc.

$$0 \longrightarrow (T_\ell^{\otimes i})_\Gamma \longrightarrow H^1(A_{et}, T_\ell^{\otimes i}) \longrightarrow \mathrm{Hom}_\Gamma(X, T_\ell^{\otimes i}) \longrightarrow 0$$

$H^1(A_{et}, T_\ell^{\otimes i}) = K_{2i-1}$

$\cong \mathbb{Z}_\ell^{r_2}$ assuming that ~~roots~~ integers ~~$\mathbb{Z}_\ell^\times$~~ are not eigenvalues.

---

$$X = (\pi_1 \bar{A})_{ab} = \Lambda^{r_2} + \prod \mathbb{Z}_\ell + W$$

infinity

for each finite prime left out

Tate of Jac. of complete curve.

$(\pi_1 \bar{A})_{ab}$

---

$$0 \to H^0(\bar{A}, T_\ell^{\otimes i})_\Gamma \to H^1_{et}(A, T_\ell^{\otimes i}) \longrightarrow \mathrm{Hom}_\Gamma(X, T_\ell^{\otimes i}) \to 0$$

$H^1_{et}(A, T_\ell^{\otimes i}) = (K_{2i-1}A)_\ell$

point is the finite part.
affects the even K groups

~~for example~~ the situation is this. On one hand the base ~~[illegible]~~ is like a highly non-orientable surface, yet the subgroups we consider basic ~~are~~ is $K^\times$ which is a torus $(S^1)^{[K:\mathbb{Q}_p]+1}$ product with BC. **Unclear!!**

---

Point will be roughly that cohomologically we have a tensor product

$$H_*(BU) \otimes H_*(\Omega BU)^{\otimes 2g} \otimes H_*(\overbrace{\Omega^2 BU}^{\text{take conn. component}}).$$

which will be a <u>free</u> anti-commutative algebra.

~~[illegible]~~ generators of

| | | |
|---|---|---|
| poly generators | degrees | $2, 4, 6, 8, \ldots\ldots$ |
| ext gen. | $2g = 2 + [K:\mathbb{Q}_p]$ | $1, 3, 5, 7, 9, \ldots\ldots$ |
| poly. gen. | | $(0)\ 2, 4, 6, 8$ |

?

now from $K^*$ I get the following list:

$1 = \xi_0$ by stability. $\xi_1, \xi_2, \ldots\ldots$

$\sigma_i \xi_0, \sigma_i \xi_1, \ldots\ldots$ $\quad i = 1, 2$

$\sigma_1\sigma_2 \xi_0,$ ~~$\xi_0$~~ $\sigma_1\sigma_2 \xi_1$

$H^*(\pi, \mathbb{F}_\ell)$ $\pi = \mathrm{Gal}(\bar{K}/K)$ ~~$\mathbb{F}_\ell$~~ — cohomology ring of a surface of genus ~~$[K:\mathbb{Q}_p]$~~

1 for $\ell \neq p$

$[K:\mathbb{Q}_p]+1$ for $\ell = p$.

symmetric products of surfaces are manifolds.

$X$ surface cohomology torsion-free

$X^n/\Sigma_n$ manifold of dim $2n$.

$$H^*(X^n/\Sigma_n) \longrightarrow H^*(X)^{\Sigma_n}$$ is this an isom??

$X = \mathbb{C}P^n$.

$$H^*(X) = \mathbb{Z} + \mathbb{Z}\varepsilon \qquad \text{degree } \varepsilon = 2$$

$$H^*(X^n) = \cancel{\Lambda[\varepsilon_1, \ldots, \varepsilon_n]} \quad P[\varepsilon_1, \ldots, \varepsilon_n]/(\varepsilon_i^2)$$

and in each degree their is one ~~invariant~~ basic invariant namely

$$\sum_{i_1 < \cdots < i_q} \varepsilon_{i_1} \cdots \varepsilon_{i_q}.$$

---

$$\left(\sum \varepsilon_i\right)^2 = \sum_{i,j} \varepsilon_i \varepsilon_j$$

$$= 2\sum_{i<j} \varepsilon_i \varepsilon_j$$

Thus for example: on the top cell.

$$H^{2n}(X^n/\Sigma_n) \longrightarrow H^{2n}(X^n)^{\Sigma_n}$$

The degree is $n!$ = order of symmetric group.

$G$ group and we have a point $S \to G$.
Then we have $S^n \longrightarrow G^n \longrightarrow G$, hence
a map $SP_n(S) \longrightarrow G$. In the case of a line $S$
if we have

$$t \mapsto f(t) \qquad D \to G$$

then have

$$D^n \longrightarrow G$$

$$t_1 \cdots t_n \qquad \sum f(t_i)$$

and we identify $D^n / \Sigma_n$ with $D^n$ via
elementary symmetric functions. But the point
should be to eliminate this theorem - prove it
directly.

$$x_1 \cdots\cdots x_n \qquad \mathbb{Q}_p \quad \begin{matrix} K \otimes \mathbb{Q}_p \\ \| \\ \oplus K_v \\ v \mid p \end{matrix} \qquad \mathbb{Q} \to K$$

$$\boxed{\mathbb{Q} \xrightarrow[p-1]{} \mathbb{Q}(\mu_p)}$$

Geometrically, it amounts to this.

$K$ local field ~~...~~ $[K : \mathbb{Q}_p] < \infty$ $\mu_p \subset K$.
Then ~~...~~ known that

$$H^2(\pi, \mu_p) = \mathbb{Z}/p\mathbb{Z}$$

only one prime over $p$. $\left[\mathbb{Q}_p(\mu_p) : \mathbb{Q}_p\right] = p-1$ even

since $H^2(\pi, \bar{K}^*) = \mathbb{Q}/\mathbb{Z}$. & $H^1(\pi, \bar{K}^*) = 0$.

~~and~~

$$H^1(\pi, \mathbb{Z}/p\mathbb{Z}) \otimes H^1(\pi, \mu_p) \longrightarrow H^2(\pi, \mu_p) = \mathbb{Z}/p\mathbb{Z}$$

good duality. Thus

$$H^1(\pi, \mathbb{Z}/p\mathbb{Z}) \otimes H^1(\pi, \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^2(\pi, \mathbb{Z}/p\mathbb{Z}) = \mu_p^\vee \underset{\zeta}{\updownarrow} \mathbb{Z}/p\mathbb{Z}$$

is a good duality. But $p$ odd $\Rightarrow$ this cup product
is skew-symmetric $\to$ dim $H^1(\pi, \mathbb{Z}/p\mathbb{Z})$ even. but

$$\dim H^1(\pi, \mathbb{Z}/p\mathbb{Z}) = \dim K^*/(K^*)^\ell = 1 + 1 + [K : \mathbb{Q}_p]$$ ~~since~~

no contradiction: since $[\mathbb{Q}_p : \mathbb{Q}_p(\mu_p)] = \frac{?}{p-1}$

<u>Problem:</u> I have a twisted form of $BU_L$ over a space $S$ and I want the cohomology of the space of sections.

$V$ graded vector space over $k$.

$$S(\Gamma(V)) \longrightarrow \Gamma(S(V)).$$

$$\Gamma(V) \longrightarrow \Gamma(S(V)) \quad \text{cogebra map.}$$

<u>But anyway</u>

theorem on elementary symmetric functions

Let $x_1, \dots, x_n$

Let's prove the thm. on elem. sym. function for all $n$ at once. Thus I consider

$$\bigoplus_{n \geq 0} A[x_1, \dots, x_n]^{\Sigma_n} = \bigoplus_n \left(A[x]^{\otimes n}\right)^{\Sigma_n}$$

$$= \bigoplus \Gamma(A[x]).$$

~~And I have an inclusion~~

$$P[x_1, \dots, x_n]^{\Sigma_n} =$$

idea if we have a curve $f(t)$ in $G$ then we have the sum curve $\sum_G f(t_i)$.

$A$ coalgebra

$\Rightarrow$ have a map $\quad A \longrightarrow P[x] \quad$ ring homo

$A \to \Gamma_n(A)$

$$A \longrightarrow P[x]^{\otimes n}$$

all $n$.

$a \mapsto \Delta^{(n)}(a)$

$\Delta^{(n+m)}(a) = \Delta^{(n)}(a) \otimes \Delta^{(m)}(a)$

so $\quad Br(A) \simeq (\mathbb{Z}/\ell\mathbb{Z})^{s-1}$ .

---

$$0 \longrightarrow A^* \longrightarrow K^* \longrightarrow \coprod_{v \notin S} \mathbb{Z} \longrightarrow Pic(A) \longrightarrow 0 \longrightarrow \coprod$$

?

?

$$Br(A) \longrightarrow Br(K) \longrightarrow \coprod_{v\notin S} Br(K_v) \xrightarrow{\text{sum}} \mathbb{Q}/\mathbb{Z} \to 0$$

---

$\bar{A} = A[\mu_{\ell^\infty}]$ adjoined. $\quad \Gamma = Gal(\bar{A}/A)$

$$\Gamma \hookrightarrow 1+\ell\mathbb{Z}_\ell$$

say $\quad \Gamma = 1+\ell^a\mathbb{Z}_\ell$ . $\quad a =$ largest $\ni$ $\mu_{\ell^a} \subset K$.

$$X = H_1(\bar{A}_{et})_\ell = \pi_1(\bar{A}_{et})_{ab,\ell}$$

Then one wants to understand $X$. $\quad \exists$ part of

$$\bar{A}_{et}$$

$(\pi_1 \bar{A})_{ab} = X$ = Galois group of the maximal unramified abelian extension of the $\Gamma$extension $\bar{A}$.

---

$$H^0(\bar{A}, \mu_\ell)_\Gamma \longrightarrow H^1(A_{et}, \mu_\ell) \longrightarrow H^1(\bar{A}, \mu_\ell)^\Gamma \longrightarrow 0$$

so

$$H^1(A_{et}, \mu_\ell) \simeq \mu_\ell + \mathrm{Hom}_\Gamma(X, \mu_\ell)$$

dual of $X/\Gamma, \ell$

conjectured structure for X says its $\Lambda^{r_2} + W$. in case. $\mathbb{Z}_\ell$

where $\Lambda = \mathbb{Z}_\ell[\Gamma]$ and $W \simeq \mathbb{Z}_\ell^\lambda$.

~~therefore~~ so $X/(\Gamma\ell) = (\mathbb{Z}/\ell\mathbb{Z})^{r_2} + (\mathbb{Z}/\ell\mathbb{Z})^\lambda_\Gamma$

accounted for

---

$$H^1(A_{et}, \mu_\ell) = \mu_\ell \oplus \mathrm{Hom}_\Gamma(X, \mu_\ell)$$

$$\mathrm{Hom}\left((X \otimes_\Gamma \mathbb{Z}/\ell\mathbb{Z})^*, \mu_\ell\right)$$

$X = \underline{\Lambda}^{r_2} \oplus W$

$$X = \Lambda^{\text{card } S} + W$$

(Tate of Jacobian of the complete curve).

---

$$\overset{\circ}{\overrightarrow{\text{~~…~~}}}\ T_{\ell\,\Gamma} \longrightarrow H^1_{et}(A, T_\ell) \xrightarrow{w_1} \mathrm{Hom}_\Gamma(X, T_\ell) \longrightarrow 0$$

$X$ has rank $\qquad\qquad T_\ell^{r_2}$

anyhow I do have

$$0 \longrightarrow \bar{A}^*/(\bar{A}^*)^{\ell^n} \longrightarrow H^1(\bar{A}_{et}, \mu_{\ell^n}) \longrightarrow {}_{\ell^n}\mathrm{Pic}(\bar{A}) \to 0$$

now unlike the case of a complete curve I see no reason why $\bar{A}^*$ should be $\ell$-divisible. ~~[illegible]~~ In fact thinking of

$$\bar{A} = \varinjlim_n A[\mu_{\ell^n}]$$

then $\bar{A}^* = \varinjlim_n A[\mu_{\ell^n}]^*$

$$0 \longrightarrow \mu_{\ell^\infty} \longrightarrow \bar{A}^* \longrightarrow \text{big group of } \infty \text{ rank} \to 0$$

unfortunately it is clear that $\bar{A}^*$ is not $\ell$-divisible in effect if so that no $\ell$-extension i.e.

$$\pi_1(A) \longrightarrow \mathrm{Gal}(\bar{A}/A)$$

is the maximal $\ell$-extension, which is non-sense.

The point is that

$\pi_1(A) \longrightarrow$

${}_{\ell^n}\mathrm{Pic}(A$

~~suppose that $A$ is a strictly local ring of res. char. $p$. Want to prove periodicity thm. which says~~ ~~$H_*(BGL(A), \mathbb{F}_\ell) = H_*(BU, \mathbb{F}_\ell)$.~~

~~The idea is to establish Kummer sequence~~

~~Chern classes!~~

my idea is simply to understand once and for all what can be proved by mod $\ell$ cohomology methods

The main point which seems to be accessible is to establish a lower bound, in fact a bigebra summand.

~~torsion free~~

If $X$ a space, then $H^*(SP_n X)$ is determined by $H^*(X)$ as a complex.

2-sphere $\qquad BU(1)$

$\infty$ symm. product of $S^2$.

maybe $B(K^*)$ is simply $\qquad B(S^1) \times B\,\mathbb{Z}_p(C)$ $\quad$ • $[K:\mathbb{Q}_p] \neq 1$

is an EM $\qquad Sp^\infty(X)$

where $X$ is our surface?

so don't give up hopes. We have this coh. ring.

$$H^*(X)$$

and our Chern classes take their values over it.

Thus

$$\begin{tikzcd} BU \arrow[d] \\ X \end{tikzcd}$$

and I want the space of sections $\Gamma$

no problem when $k^\times$ $\ell$-divisible.

roughly $X$ should decompose into

$$BU \times U^{2g} \times \Omega^2 BU$$

precisely, we have the following

$$\begin{tikzcd} \Omega^2 BU \arrow[d] & \Omega^1(BU) \arrow[d] & BU \\ \Gamma(X, \mathcal{B}) \arrow[r] & \Gamma(X_{(1)}, \mathcal{B}) \arrow[r] & \Gamma(X_{(0)}, \mathcal{B}) \end{tikzcd}$$

and these fibrations must be translated into cohomology, the point being that they are all totally non-homologous ~~to~~ zero.

---

Observe that it's all beautiful!!!

Thus the point somehow is the fact that the various Kunneth components of the arithmetic Chern classes give the classes needed to prove totally non-homol. to zero.

On the other hand a map

$$\underline{\Gamma(S(V))} \longrightarrow A$$

$$\begin{tikzcd}
\Gamma(A) \otimes \Gamma(A) \arrow[r, "\exists!\ \text{coalgebra}"] \arrow[d] & \Gamma(A) \arrow[d] \\
A \otimes A \arrow[r] & A
\end{tikzcd}$$

$\Gamma(S(V)) \oplus A$

$$\Delta(e^a) = e^a \otimes e^a$$

$$\begin{tikzcd}
e^a \otimes e^b \arrow[d] & e^{ab} \\
a \otimes b & ab
\end{tikzcd}$$

$\Delta(e^{f(x)})$

usual alg str.

$$\begin{tikzcd}
\Gamma(A) \otimes \Gamma(A) \arrow[r] \arrow[d, "e^a \quad e^b"] & \Gamma(A) \arrow[d, "e^{a+b}"] \\
A \oplus A \arrow[r, "+"] & A
\end{tikzcd}$$

$$e^a \otimes e^b \Downarrow a \otimes b$$

$\underline{\mathrm{Hom}}(\hat{W}, \mathbb{G}_m)(A)$ = natural homos. from $\hat{W}$ to $\mathbb{G}_m$ on the category of A-algebras

---

Basic pairing runs as follows
uses product in W?

---

$\boxed{\text{Cartier duality}}$

---

$\hat{W}$ sort of universal thing generated by a curve. Thus given $G$ and a curve $\gamma : \hat{G}_a \to G$

one

$$\begin{tikzcd} a & \hat{G}_a \arrow[r] \arrow[d] & G \\ 1+aT & \hat{W} \arrow[ur, "\Theta"'] & \end{tikzcd}$$

---

$$\begin{tikzcd} \hat{G}_a \arrow[r] \arrow[d] & G \\ \hat{W} \arrow[ur] & \end{tikzcd}$$

$$W(A) = (1 + A[[T]]^+) \qquad T = (T_1, \dots, T_n)$$

With $n \geq 1$, $\exists$ no way of factoring these series at all.

If $V$ is a vector space in characteristic zero, then $SV^\wedge$ has exponential

$$e^v = \sum \frac{v^n}{n!}$$

satisfying $\Delta e^v = e^{\Delta v} = e^v \otimes e^v$

In general consider $S(V)^\wedge$ and

$1+$

$$S(\Gamma(V)) \longrightarrow A$$

$$t_1 e_1 + \cdots + t_m e_m \longmapsto \sum_{\alpha \geq 0} a_\alpha t^\alpha$$

group-like ie

$$\Delta a_\alpha = \sum a_\beta \otimes a_\gamma$$

---

Therefore a homomorphism

---

In my situation $V$ is 3-dimensional generated by $\sigma$, $\tau$, $\xi$ where $\sigma$ and $\tau$ are of degree 1, and $\xi$ of degree 2.

To give a Hopf alg map

$$\begin{array}{c} S(\Gamma(V)) \longrightarrow A \\ \| \\ P[\xi_j, \sigma\xi_j, \tau\xi_j, \sigma\tau\xi_j]_{j \geq 0} \longrightarrow A \end{array}$$

is the same thing as a poly map

$$X\xi + S\sigma + T\tau \longmapsto \sum_{\substack{0 \leq \alpha \\ 0 \leq \beta, \gamma \leq 1}} a_{\alpha\beta\gamma} X^\alpha S^\beta T^\gamma$$

which is group-like

$$\begin{tikzcd} S(\Gamma(V)) \arrow[r] \arrow[d] & \Gamma(S(V)) \\ A & \end{tikzcd}$$

Hopf algebra map $S(\Gamma(V)) \longrightarrow A$

same as a coalg map $\Gamma(V) \longrightarrow A$

which means for each $v$ we give elements

$$\theta_n(v) \in A$$

such that $\sum \theta_n$ commutes with $\Delta$.

hence we give a map of $V$ into ~~?~~ $G(A)$

---

polyn. $V \longrightarrow G(A)$

e.g. $V$ one dimensional

$$\begin{array}{ccc} S(\Gamma(V)) & \longrightarrow & A \\ \parallel & & \mid \\ \mathbb{Z}[b_1, \cdots\cdots] & \longrightarrow & A \\ b_i & \longmapsto & a_i \end{array}$$

$$\Delta a_n = \sum_{i+j=n} a_i \otimes a_j$$

thus map $V \longrightarrow G(A)$

$$t \qquad \sum_{n\geq 0} a_n t^n \quad \text{group-like}$$

## Witt vectors

$A$ variable ring

$W(A) = 1 + A[[T]]T$ under $\times$

Thus $W(A) \doteq \mathrm{Hom}(\mathbb{Z}[c_1, \dots], A)$

$$\Delta c_n = \sum_{i+j=n} c_i \otimes c_j \qquad c_0 = 1.$$

also we have

$\hat{W}(A) =$ subgroup of $W(A)$ consisting of series $1 + \sum a_n T^n$ with $a_i$ nilp and $0$ for $i$ large.

$$\hat{W}(A) = \varinjlim_N \mathrm{Hom}(\mathbb{Z}[c_1, \dots]/I_N, A)$$

where $I_N$ gen by monomials $c^\alpha$ of weight $\sum i\alpha_i \geq N$.

According to Cartier, there is a basic duality

$$W \times \hat{W} \longrightarrow \mathbb{G}_m$$

precisely $W \xrightarrow{\sim} \underline{\mathrm{Hom}}(\hat{W}, \mathbb{G}_m)$

same Poincaré series

$$S(\Gamma(V)) \longrightarrow \Gamma(S(V))$$

$A$ Hopf algebra

Witt vectors

$S(\Gamma(V)) \longrightarrow A$

same as coalgebra map $\Gamma(V) \longrightarrow A$

same as a polynomial mapping $\varphi : V \longrightarrow \cancel{B}\, G(A)$

of Hopf algebras

a map $\Gamma(S(V)) \longrightarrow A$

$S(V) \longrightarrow A$

is basically a map

$$\begin{tikzcd}
\Gamma(S(V)) \arrow[r] & A \\
S(V) \arrow[u, "\gamma"] \arrow[ur] &
\end{tikzcd}$$

---

$$\begin{tikzcd}
C \arrow[r] \arrow[dr, "\exists !"'] & V \\
& \Gamma(V) \arrow[u]
\end{tikzcd}$$

$\Gamma(A)$ is a Hopf algebra

1

~~Let M be a simplicial monoid, let $h_*$ be a generalized homology theory and let $\pi$ be the group-completion of the monoid $\pi_0 M$, i.e. $\pi$ is a group equipped with a homomorphism $\pi_0 M \to \pi$ which is universal homomorphism $\pi_0 M \longrightarrow \pi$ from $\pi_0 M$ to a group~~

Let M be a simplicial monoid and let $\pi_0 M \longrightarrow \pi$ the group-completion of the monoid $\pi_0 M$, i.e. this map is universal among homomorphisms from $\pi_0 M$ to a group. If A is an abelian group on which $\pi_0 M$ acts to the right, we put

$$A[\pi_0 M^{-1}] = A \otimes_{\mathbb{Z}[\pi_0 M]} \mathbb{Z}[\pi].$$

and call it the localization of A with respect to $\pi_0 M$. We will assume that the functor $A \mapsto A[\pi_0 M^{-1}]$ is exact, or equivalently that $\mathbb{Z}[\pi]$ is a flat left $\mathbb{Z}[\pi_0 M]$-module. This is the case if $\pi_0 M$ is abelian, for then $A[\pi_0 M^{-1}]$ is the localization, in the sense of commutative algebra, of the module A with respect to the multiplicative system $\pi_0 M$ in the ring $\mathbb{Z}[\pi_0 M]$.

②

Let $h_*$ be a generalized homology theory on simplicial sets. Right multiplication by $M$ on itself induces a right action of $\pi_0 M$ on $h_i(M)$ with respect to which we can form the localization $h_i(M)[\pi_0 M^{-1}]$. ~~On the other hand left multiplication induces a left action of $\pi_0 M$ $\pi_0 M$ also acts to the left on $h_i(M)$ and the localization. We will assume the left action is~~

On the other hand, $\pi_0 M$ also acts to the left on the localization. We will assume the left action on $h_i(M)[\pi_0 M^{-1}]$ is invertible for all $i$, that is to say, the endomorphism produced by left multiplying by an element of $\pi_0 M$ is an automorphism. This will be the case if left and right multiplication by an element of $\pi_0 M$ on $h_i(M)$ coincide, for example, if left and right multiplication by $m_0 \in M_0$ on $M$ are homotopic.

By a *simplicial M-set* we mean a simplicial set E endowed with a right action of M. We say M acts *freely* on E if for each q, $E_q$ is isomorphic to $(E_q/M_q) \times M_q$ as a right $M_q$-set.

*Proposition:* ~~Under the above assumptions~~ Let E be a simplicial M-set on which M acts freely, and put $X = E/M$. Under the above assumptions there is a spectral sequence

$$E^2_{pq} = H_p(X, \mathcal{L}_q) \Longrightarrow h_{p+q}(E)[\pi_0 M^{-1}]$$

where $\mathcal{L}_q$ is a local coefficient system on X with fibre $h_q(M)[\pi_0 M^{-1}]$.

This will be proved in the following sections. We now deduce its consequences.

---

~~Cor. [illegible] If $X' \to X$ [illegible]~~

*Proposition:* Let $E' \to E$ be a map of simplicial M-sets

We ~~say~~ will call a map $f: X' \to X$ of simplicial sets a homotopy equivalence if ~~it becomes an isomorphism in the homotopy category,~~ the induced map of geometric realizations is a homotopy equivalence. One has to be careful that this does not in general imply that there is a map $X \to X'$ which is a homotopy-inverse for $f$, unless $X$ and $X'$ are Kan complexes.

Proposition: Let $E' \longrightarrow E$ be a map of simplicial $M$-sets on which $M$ acts freely such that $E'/M \longrightarrow E/M$ is a homotopy equivalence. Then, under ~~with~~ the above assumptions, we have an isomorphism

$$h_i(E')[\pi_0 M^{-1}] \xrightarrow{\sim} h_i(E)[\pi_0 M^{-1}]$$

This will be proved in the following section. ~~We now deduce some consequences.~~ We now show why it implies the group-completion theorem

## The idea now:

Put $B = BM$, $E = EM$, and let $P \longrightarrow B$ be ~~a fibration with $P$ contractible s~~ Choose $\phi: E \to P$ over $B$; $\phi': M \longrightarrow \Omega$ canon. map — ind. of choice of $\phi$.

Fits into

$$\begin{tikzcd} M \arrow[r, "\phi'"] \arrow[d, "j"'] & \Omega \arrow[d, "i"] \\ P\times_B E \arrow[r, "\phi''"'] & P\times_B P \end{tikzcd}$$

Proposition $\Longrightarrow$ $h_*(M)[\pi_0 M^{-1}] \xrightarrow{\sim} h_*(P\times_B E)[\pi_0 M^{-1}]$.

$$\cancel{h_*(P\times_B E) \xrightarrow{\sim} h_*(P\times_B P) \xleftarrow{\sim} h_*(\Omega)}$$

~~The diagram ......... shows that $\phi'$ is an H-map, and that $\phi''$~~

On the other hand, right multiplication by $m \in M_0$ on $P\times_B E$ is a homotopy equivalence since

Now I want to obtain the group-completion thm.
Define $B = BM = \mathrm{diag}\,(p \mapsto M^p)$, $E = EM =$

Let $P \to B$ be a fibration, $P$ contractible, $\Omega$ = fibre, so $\Omega \simeq \Omega BM$. Choose $\phi : E \to P$ over $B$. Then have basic square

$$\begin{tikzcd} M \arrow[r, "\phi'"] \arrow[d, "j"'] & \Omega \arrow[d, "i"] \\ P \times_B E \arrow[r, "\phi''"'] & P \times_B P \end{tikzcd}$$

$j(m) = (p_0, e_0 m)$
$i(\omega) = (p_0, \omega)$
$\phi'(m) = \phi(e_0 m)$
$\phi''(p, e) = (p, \phi e)$.

$i$ inclusion of the fibre for the fibration $pr_1 : P \times_B P \to P$ which has contractible base $\Longrightarrow$ $i$ is a heq.
$\phi''$ heq because it's the pull-back of $\phi$ by the fibration $P \to B$. $\phi'$ independent up to homotopy of choice of $\phi$. $\phi'$ is the canonical map $M \to \Omega BM$.

---

Better: Let $B$ be a space with basepoint $b_0$, $P \to B$ a fibration, $P$ contractible with basepoint $p_0$ lying over $b_0$. The fibre $\Omega$ of $P \to B$ is canon. heq to $\Omega B$. Its H-space structure obtained as follows. The map $pr_1 : P \times_B P \to P$ is a fibr with cont. base, hence the inclusion of the fibre $i : \Omega \to P \times_B P$, $i(\omega) = (p_0, \omega)$ is a heq.

$$\begin{tikzcd} \Omega \times \Omega \arrow[d, "i \times i"] & & \\ (P \times_B P) \times (P \times_B P) & P \times_B P \times_B P \arrow[l, "(pr_{12}, pr_{23})"'] \arrow[r, "pr_{23}"] & {} \end{tikzcd}$$

basic diagram

$$\begin{tikzcd}
(P\times_B E)\times M \arrow[rr] \arrow[dd] & & (P\times_B P)\times(P\times_B P) \\
& \Omega\times M \arrow[ul] \arrow[r] \arrow[dl] & P\times_B P\times_B P \arrow[u, "(pr_{12},\, pr_{23})"'] \arrow[d, "pr_{13}"] \\
P\times_B E \arrow[rr, "id\times z"'] & & P\times_B P
\end{tikzcd}$$

$(p,e,m) \mapsto (p, z(e)) \times (p_0, z(e_0 m))$

$(p,e,m) \mapsto (p, em)$

$(\omega, e_0, m) \mapsto (\omega, p_0), (p_0, z(e_0 m))$

commutes

$(\omega, m) \mapsto (\omega, p_0, z(e_0 m))$

$(\omega, m) \mapsto (\omega, e_0 m)$

$(\omega, e_0 m) \mapsto (\omega, z(e_0 m))$

$(p, e) \mapsto (p, z(e))$.

---

$$\begin{tikzcd}
M\times M \arrow[r] \arrow[dd] & (P\times_B E)\times M \arrow[rr] \arrow[dd] & & (P\times_B P)\times(P\times_B P) & \Omega\times\Omega \arrow[l, "i\times i", "\sim"'] \arrow[dd] \\
& & \Omega\times M \arrow[ul, hook] \arrow[r] \arrow[dl] & P\times_B P\times_B P \arrow[u, "(pr_{12},\, pr_{23})"'] \arrow[d, "pr_{13}"] & \\
M \arrow[r] & P\times_B E \arrow[rr] & & P\times_B P & \Omega \arrow[l, "i", "\sim"']
\end{tikzcd}$$

$(\omega, m) \mapsto (\omega, e_0, m)$

$(\omega, m) \mapsto (\omega, p_0, z(e_0 m))$

$(\omega, m) \mapsto (\omega, e_0 m)$

The following diagram shows that $\phi'$ is an H-map, and also that $\phi''$ is compatible with the $M$ action on $P\times_B E$ and

It follows that $(\phi')_*$ factors

$$h(M) \longrightarrow h(P\times_B E) \xrightarrow[\sim]{(i_*)^{-1}(\phi'_*)} h(\Omega)$$

$\Omega \to P \longrightarrow B$

$$i : \Omega \longrightarrow P \times_B P \qquad i(\omega) = (p_0, \omega)$$

is a homotopy equivalence. The

$$\cancel{\Omega \times \Omega \xrightarrow{i \times i} (P \times_B P) \times (P \times_B P)}$$

$$\Omega^2 \xrightarrow{i^2} (P \times_B P)^2 \xleftarrow{(pr_{12}, pr_{23})} P \times_B P \times_B P \xrightarrow{pr_{13}}$$

E contractible

$$\begin{tikzcd} & P \arrow[d, "\text{fibn}"] \\ E \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

$$\begin{tikzcd} \Omega \arrow[r, "\text{heq}"] \arrow[d] & E \times_B P \arrow[r] \arrow[d] & P \arrow[d] \\ pt \arrow[r, "\text{heq}"] & E \arrow[r] & B \\ & (E\times M)\times_B P \arrow[uu, shift left] \arrow[uu, shift right] \arrow[d] & \\ & E\times M \arrow[uu, shift left] \arrow[uu, shift right] & \end{tikzcd}$$

~~m ∈ M fixed.~~

Using $z : E \to P$ over $B$ we construct a map from $M$ to $\Omega$ namely $m \mapsto z(e_0 m)$

This is clearly an h-map as it comes from

$$E \times M^\nu \longrightarrow (P/B)^{\nu+1}$$

$$e_0, m_1, \dots, m_\nu \longmapsto (z(e_0), z(e_0 m_1), \dots, z(e_0 m_1 \cdots m_\nu)).$$

~~as~~ a map of simplicial spaces.

But the next point would be to relate this to

$$M \longrightarrow P \times_B E \longleftarrow \Omega$$

$$m \qquad (p_0, e_0 m)$$

$$(\omega, e_0) \longmapsfrom \omega$$

and the point is that

~~$M \to P \times_B E$~~

$$\Omega \longrightarrow P \times_B E \longrightarrow E$$

$$\equiv \; (P \times_B E) \times M \rightrightarrows (P \times_B E)$$

$$\begin{tikzcd}
M \arrow[r] & E \times M \arrow[r] & P \times_B P & \Omega \arrow[l] \\
M^2 \arrow[u] \arrow[r] & E \times M^2 \arrow[u] \arrow[r] & (P/B)^3 \arrow[u] & \Omega^2 \arrow[u] \arrow[l]
\end{tikzcd}$$

$e, m \mapsto (\varphi(e), p(em))$

This is my h-map.

$$\begin{tikzcd}
M \arrow[r, "m \,\mapsto\, (\varphi(*),\, p(*m))"] \arrow[d, "(*,\, *m)"'] & P \times_B P & \Omega \arrow[l, "heq"'] \arrow[dl, "heq"] \\
& P \times_B E \arrow[u] &
\end{tikzcd}$$

---

Recall how canonical map $M \longrightarrow \Omega$ defined: one chooses $\varphi : E \longrightarrow P$ over $B$, then

~~$M \longrightarrow E \times M \longrightarrow E \times E$~~

$$m \longmapsto e_0 m \longmapsto \varphi(e_0 m)$$

$$\begin{tikzcd}
M \arrow[r] \arrow[rrr, bend right, "canon"'] & E \arrow[r, "\varphi"] & P & \Omega \arrow[l]
\end{tikzcd}$$

$m \longmapsto \varphi(e_0 m)$

why homotopic to

$$\begin{tikzcd}
M \arrow[r, "m \,\mapsto\, p_0,\, e_0 m"] & P \times_B E & \Omega \arrow[l]
\end{tikzcd}$$

$(\omega, e_0) \qquad \omega$

# group-completion theorem.

$M$ simplicial monoid

$EM = |M^{P} \times M|$, $BM = |M^{P}|$

$P \to BM$ fibration with $P$ contractible, $\Omega =$ the fibre over basepoint; $\Omega$ has the homotopy type of $\Omega BM$. Because $pr_2 : P \times_{BM} EM \to EM$ is a fibration with fibre $\Omega$ and contractible base:

$$H_i(\Omega) \xrightarrow{\sim} H_i(P \times_{BM} EM).$$

~~Right multiplication by $m \in M_0$ on $P \times_{BM} EM$ is a heq~~

Make $M$ act on $P \times_{BM} EM$ by $(p,e)m = (p, em)$; as mult by $m$ on $EM$ is a heq, so also is mult. by $m$ on $P \times_{BM} EM$. Thus $\pi_0 M$ acts invertibly on $H_i(P \times_{BM} EM)$.

~~$H_i(P \times_{BM} EM) \cong H_i(P \times_{BM} EM)[\pi_0 M^{-1}]$~~

~~Hypothesis 1: $(\pi_0 M, \pi_0 M)$ is filtering $\Rightarrow A \mapsto A[\pi_0 M^{-1}]$ exact~~

## Localization:

$\pi_0 M \longrightarrow \gamma$ group completion

If $A$ is a right $\pi_0 M$-module, we set

$$A[\pi_0 M^{-1}] = A \otimes_{\mathbb{Z}[\pi_0 M]} \mathbb{Z}[\gamma]$$

~~localization~~ .. Hypothesis 1: $\mathbb{Z}[\gamma]$ flat left $\mathbb{Z}[\pi_0 M]$-mod.

Hypothesis 2: $\pi_0 M$ acts invertibly on $H_*(M)[\pi_0 M^{-1}]$

So we have a canonical map of right $\pi_0 M$-modules

$$H_i(M) \to H_i(P \times_{BM} EM) \cong H_i(\Omega)$$

hence a canonical map

(*) $$H_i(M)[\pi_0 M^{-1}] \longrightarrow H_i(\Omega BM).$$

Group-completion thm. says Assume hyp. 1 and 2; then (*) is an isomorphism.

$C: \quad \Delta(n) \times M \to P$

1) $E^2_{pq} = H_p(C, (\Delta(n)\times M \to P) \mapsto H_q(M)) \Rightarrow H_{p+q}(P)$.

2) Hyp on $\pi_0 M$-action on $H_*(M)$
Hyp on $(P, M) \sim X$. Then define a local coefficient system $\mathcal{L}_q$ on $X$ and have a spectral sequence
$$E^2_{pq} = H_p(X, \mathcal{L}_q) \Rightarrow H_{p+q}(P)[\pi_0 M^{-1}].$$

3) group completion thm. says
$$H_*(M)[\pi_0 M^{-1}] \xrightarrow{\sim} H_*(\Omega BM).$$

~~$M \to \Omega BM$~~

$$\Omega \to P \to B$$

~~$\Omega \to P \to B$ yes.~~

$$P \times \Omega^2 \rightrightarrows P \times \Omega \rightrightarrows P$$
$$\wr \qquad\qquad \wr$$
$$(P/B)^3 \rightrightarrows (P/B)^2 \rightrightarrows P$$

$$E \times_B (P/B)^3 \rightrightarrows \cancel{E} \times_B P \times_B P \times_B E = \cancel{E} \times_B P \times_B E$$

$$(\cancel{P \times_B E}) \times M = P \times_B E$$

$$\rightrightarrows (P \times_B E) \times M \rightrightarrows P \times_B E$$

---

$M$ monoid, so have $M \times M \to M$ which induces $H_*(M) \otimes H_*(M) \to H_*(M)$.
Similarly for

---

The map $M \to \Omega BM$

$$\begin{tikzcd} M \arrow[r] \arrow[d] & EM \arrow[r] \arrow[d] & BM \\ \Omega BM \arrow[r] & X \arrow[r] & BM \end{tikzcd}$$

$$\begin{tikzcd} \Omega BM \arrow[r] \arrow[d, equal] & X \times_{BM} EM \arrow[r] \arrow[d] & EM \arrow[d] \arrow[dl, dotted] \\ \Omega BM \arrow[r] & X \arrow[r] & BM \end{tikzcd}$$

~~equivalent~~ $\quad \Omega BM \xrightarrow{heq} X \times_B P \xleftarrow{\text{act on basept.}} M$

one gets also ~~a map of M~~ this way a h class of maps $\Omega BM$ assoc. to $\alpha \in \pi_0 M$.

---

But also there is a product $\Omega Y \times \Omega Y \to \Omega Y$

$(P/Y)^3 \equiv P \times_Y P \quad = \quad P \to Y$, $\quad P \downarrow pt$

similarly one has
$$P \times_Y P \times_Y P = (P \times_Y P) \times_P (P \times_Y P)$$
$$\uparrow heq$$
$$\Omega Y \times \Omega Y$$

$$\begin{tikzcd} \Omega Y \arrow[r, "heq"] & P \times_Y P \arrow[r] \arrow[d] & P \arrow[d] \\ & P \arrow[r] & Y \end{tikzcd}$$

---

~~EM~~

$$\begin{tikzcd} & & P \times_B E \arrow[r] \arrow[d, shift left] \arrow[d] \arrow[d, shift right] & EM \arrow[d] \\ (P/BM)^3 \arrow[r, shift left] \arrow[r] \arrow[r, shift right] & (P/BM)^2 \arrow[r, shift left] \arrow[r, shift right] & P \arrow[r] & BM \end{tikzcd}$$

① Suppose $P \to B$ fibration with $P$ contractible. $\Omega$ = fibre over basepoint $b_0$. Then we have a fibration

$$\begin{array}{ccccc} \Omega & \xrightarrow{i} & P \times_B P & \xrightarrow{pr_1} & P \\ \omega & \longmapsto & (p_0, \omega) & & \end{array}$$

$$\begin{tikzcd} \Omega \arrow[r] \arrow[d] & P \times_B P \arrow[d, "pr_1"] \\ pt \arrow[r, "heq"] & P \end{tikzcd}$$

with contractible base, showing $i$ is a heq. Also we have a cartesian square

$$\begin{tikzcd} (P/B)^3 \arrow[r, "(pr_{12}, pr_{23})"] \arrow[d, "pr_2"] & (P/B)^2 \times (P/B)^2 \arrow[d, "pr_2 \times pr_1"] \\ P \arrow[r] & P \times P \end{tikzcd}$$

where the vertical maps are fibrations, and the bottom is a heq.; thus the top arrow is a heq. Now we have

$$\begin{tikzcd} \Omega \times \Omega \arrow[r, "i \times i"] & (P/B)^2 \times (P/B)^2 & (P/B)^3 \arrow[l] \arrow[d, "pr_{13}"] \\ & \Omega \arrow[r, "i"] & (P/B)^2 \end{tikzcd}$$

where the horizontal arrows are heq's. This defines the h-structure on $\Omega$; i.e. a map $\Omega \times \Omega \to \Omega$ in the homotopy category.

---

Note that one uses only the fact that fibrations are flat for h-base change, so could have assumed $P \to B$ is a quasi-fibration.

General assertion is that if $P \to B$ is a q-fibn with $P$ contractible, then

$$\nu \longmapsto (P/B)^{\nu+1}$$

is a special simplicial space which is invertible.

---

$M$ = simp mon.

$B = BM$, $E = EM = \mathrm{diag}(\mapsto M^\nu \times M)$.

$P \longrightarrow B$ fibration with fibre $\Omega$, $P$ contractible

Def of the canonical h-map $M \longrightarrow \Omega$: Choose $z : E \longrightarrow P$ over $B$; then $m \longmapsto z(e_0 m)$ maps $M$ to $\Omega$.

Why this is an h-map:

$$\begin{tikzcd}
M^2 \arrow[r, "{(m_1,m_2) \mapsto (z(e_0),\, z(e_0 m_1),\, z(e_0 m_1 m_2))}"] \arrow[d, "\mathrm{prod}"] & P \times_B P \times_B P \arrow[r, phantom, "\sim"] \arrow[d, "pr_{13}"] & \Omega \times \Omega \arrow[d, dotted, "\text{h-structure map}"] \\
M \arrow[r, "{m \mapsto (z(e_0),\, z(e_0 m))}"] & P \times_B P & \Omega \arrow[l, "{z(e_0 m) \mapsfrom z(e_0 m)}"']
\end{tikzcd}$$

$m \longmapsto (z(e_0), z(e_0 m)) \longleftrightarrow z(e_0 m)$.

Observe we have a comm.

$$\begin{tikzcd}
M \arrow[r] \arrow[dr] & P \times_B E \arrow[d, "id \times z"] & \Omega \arrow[l] \arrow[dl, "(*)"] \\
 & P \times_B P &
\end{tikzcd}$$

$m \longmapsto (p_0, e_0 m)$, $\quad (p_0, e_0) \mapsfrom \omega \mapsto (\omega, p_0)$

$m \longmapsto (p_0, z(e_0 m))$

We note that the arrow $(*)$ is $\omega \longmapsto (\omega, p_0)$ which is h-inverse to the old heq.

$$\begin{tikzcd}
\Omega \arrow[r, "i"] \arrow[rr, bend right, "\text{inverse}"'] & P \times_B P & \Omega \arrow[l]
\end{tikzcd}$$

(5)

We now deduce the group-completion thm. It will be necessary to recall the canonical H-map $M \to \Omega BM$.

~~Let $B$ be a simplicial set with basepoint $b_0$. Then its "loop space" $\Omega B$ is the simplicial set, which is unique up to canonical homotopy equivalence, obtained by taking the fibre over $b_0$ of a fibration $P \to B$, where $P$ is contractible with basepoint $p_0$ lying over $b_0$.~~

Let $B$ be a simp. set with basepoint $b_0$. Its "loop "space", denoted $\Omega B$, may be defined as the fibre over $b_0$ of a (Kan) fibration $P \to B$, where $P$ is contractible with basepoint $p_0$ lying over $b_0$. One defines an H-space map $\Omega B \times \Omega B \to \Omega B$ as follows. Note first that $pr_1 : P \times_B P \to P$ is a fibration with contractible base, hence the inclusion of the fibre $i : \Omega B \to P \times_B P$, $i(\omega) = (p_0, \omega)$ is a h. eq. In the diagram

$$\begin{tikzcd}
\Omega B \times \Omega B \arrow[r, "i \times i", "heq"'] & (P \times_B P) \times (P \times_B P) \arrow[r, "(pr_2, pr_1)"] & P \times P \\
 & P \times_B P \times_B P \arrow[u, "(pr_{12}, pr_{23})"', "heq"] \arrow[r, "pr_2"] \arrow[d, "pr_{13}"] & P \arrow[u, "\Delta"', "heq"] \\
\Omega B \arrow[r, "i", "heq"'] & P \times_B P &
\end{tikzcd}$$

the square is cartesian, so $(pr_{12}, pr_{23})$ is a homotopy equivalence (one uses here the basic fact that the base change of a homotopy equivalence by a fibration is a homotopy equivalence.) From the diagram, we get a well-defined map $\Omega B \times \Omega B \to \Omega B$ in the homotopy category.

~~If $C$ is a simplicial category, its nerve $NC$ is by definition the bisimplicial set $N_p C_q$, where $N_p C_q$ is the set of diagrams $X_0 \leftarrow \cdots \leftarrow X_p$ in the category $C_q$. Its classifying "space" $BC$ is defined to be the diagonal simplicial set $p \mapsto N_p C_p$ of the nerve. will be useful to think of $NC$ as a simplicial object $p \mapsto N_p C$ in~~

~~In the same way one thinks of a simplicial category as corresponding to a topological category, it is useful to think of $NC$ as a simplicial object~~

If $\mathcal{C}$ is a simplicial category, its <u>nerve</u> $N\mathcal{C}$ is by definition the bisimplicial set $N_p\mathcal{C}_q$, where $N_p\mathcal{C}_q$ is the set of diagrams $X_0 \leftarrow \cdots \leftarrow X_q$ in the category $\mathcal{C}_q$. Its classifying "space" $B\mathcal{C}$ is defined to be the diagonal simplicial set $p \mapsto N_p\mathcal{C}_p$ of the nerve. In particular, regarding the simp. monoid $M$ as a simplicial category, we have its classifying "space

$$BM = \operatorname{diag}((M_q)^p).$$

Put

$$EM = B\overline{M} = \operatorname{diag}((M_q)^p \times M_q)$$

where $\overline{M}_q$ is the category having elements of $M_q$ as its objects, and havings as morphism from $x$ to $y$ those $m$ in $M_q$ such that ~~...~~ $y = mx$. It is clear that $M$ acts freely on the right of $EM$ and that $EM/M \cong BM$. Furthermore, the simplicial set $p \mapsto N_p\overline{M}_q$ is contractible for each $q$, because the category $\overline{M}_q$ has an initial object; thus $EM$, ~~...~~ being the diagonal of a bisimplicial set which is contractible in each ~~...~~ vertical degree, is itself contractible (see below).

$$\begin{tikzcd}
m \arrow[r, mapsto] & (e_0, m) \arrow[r, mapsto] & (p_0, \varphi(e_0 m)) & \varphi(e_0 m) \arrow[l, mapsto] \\
M \arrow[r] \arrow[dr, "m \mapsto (p_0, e_0 m)"'] \arrow[rrr, bend right, "\text{canonical map}"] & E \times M \arrow[r] & P \times_B P & \Omega \arrow[l, "(p_0,\omega) \leftarrow\!\shortmid\, \omega"'] \arrow[dl, "(\omega, e_0) \leftarrow\!\shortmid\, \omega"] \\
& & P \times_B E \arrow[u, "(\mathrm{id} \times \varphi)"] &
\end{tikzcd}$$

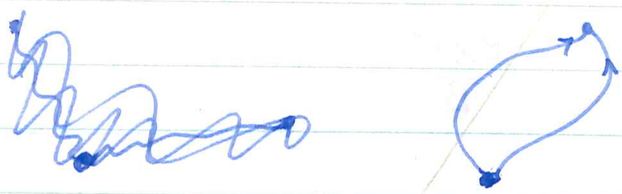
commutes

reduces us to showing

$$\begin{tikzcd}
\omega \arrow[r, mapsto] & (\omega, e_0) \\
\Omega \arrow[r] \arrow[dr, "\omega \mapsto (p_0, \omega)"'] & P \times_B E \arrow[d, "(\omega, p_0)"] \\
& P \times_B P
\end{tikzcd}$$


looks opposite in sign.

$P \times_B P$



$b_0$

---

Thus our basic map is

$$\begin{array}{ccccccc}
M & \longleftarrow & E \times M & \longrightarrow & (P/B)^2 & \longleftarrow & \Omega \\
m & & (e_0, m) & & (\cancel{z(e_0)}\, p_0, z(e_0 m)) & & z(e_0 m)
\end{array}$$

which is what it should be. This map induces

$$h(M)[\pi_0 M^{-1}] \longrightarrow h(\Omega)$$

on the other hand, we have the ~~isomorphism~~ map

$$M \longrightarrow P \times_B E \longleftarrow \Omega$$

how the h-structure on $\Omega$ is defined:

$$\begin{array}{ccc} (P/B)^3 & & \\ \big\downarrow{\scriptstyle pr_{13}} & & \\ (P/B)^2 & \xleftarrow[\text{heq}]{} & \Omega \\ (p_0, \omega) & & \omega \end{array}$$

Claim $\exists$ heq. $(P/B)^3 = P \times_B P \times_B P$ with $\Omega^2$ which if $P$ were a principal $\omega$ bundle would be

$$(\omega_1, \omega_2) \longmapsto (p_0, p_0\omega_1, p_0\omega_1\omega_2)$$

$$\Omega^2 \longrightarrow (P/B)^3$$

But the equivalence is ~~the~~ follows

$$(P/B)^3 \xrightarrow[(d_2 = pr_{12},\ d_0 = pr_{23})]{} (P/B)^2 \times (P/B)^2 \leftarrow \Omega \times \Omega \qquad \leftarrow\!\!-\!\!-\leftarrow$$

$$M \times M \ \big\downarrow {\scriptstyle d_0 = pr_2,\ d_2 = pr_1}$$

should be a heq.

$$(p_0, p_0\omega_1, p_0\omega_1\omega_2) \longmapsto (p_0, p_0\omega_1), (\underline{p_0}\omega_1, \underline{p_0}\omega_1\omega_2)$$

$$\wr$$

$$(p_0, p_0\omega_1), (p_0, p_0\omega_2). \text{ OKAY.}$$

and it will be a heq - why

$$\begin{array}{ccc} (P/B)^3 & \longrightarrow & (P/B)^2 \times (P/B)^2 \\ \big\downarrow & & \big\downarrow {\scriptstyle pr_2 \times pr_1} \\ P & \longrightarrow & P \times P \end{array}$$

I will work in the simplicial setup, replacing the category of spaces by the ~~sub~~category of simplicial sets (which can be viewed as a subcategory ~~of~~ by means of the geometric realization functor.) ~~So from now on spaces are~~

Let $p \mapsto X_p$ be a simplicial object in the cat of simplicial sets, i.e. a bisimplicial set $(p,q) \mapsto X_{pq}$.

Prop: The realization of the simplicial space $p \mapsto |X_p|$ is homeomorphism to the realization of the diagonal simplicial set $p \mapsto X_{pp}$:

$$|p \mapsto |q \mapsto X_{pq}|| \xrightarrow{\simeq} |p \mapsto X_{pp}|$$

This proposition shows that the analogue of the ~~geometric~~ realization of a simplicial space in the simplicial setup is the diagonal simplicial set of a bisimplicial set. If $X_{pq}$ is a bisimplicial set we sometimes write

$$\mathrm{diag}(p \mapsto X_{p*})$$

for this diagonal.

Prop ~~Corollary~~: Let $X_{pq} \to Y_{pq}$ be a map of bisimplicial sets such that $X_{p*} \to Y_{p*}$ is a heq for each $p$. Then

$$\mathrm{diag}(X_{p*}) \longrightarrow \mathrm{diag}(Y_{p*})$$

is a heq.

Prop. Let $h_*$ be a generalized homology theory, and X a bisimplicial set. Then have Segal spectral sequence.

---

Comments: 1). Do not formally write out these propositions

2) must find a way of describing bisimplicial things.

The spectral sequence.

Suppose $E$ is a simplicial set on which the simplicial monoid $M$ acts to the right. Introduce the category $\mathcal{C}$ whose objects are maps $[n] \times M \longrightarrow E$ of simplicial $M$-sets and whose morphisms are commutative triangles

$$\begin{tikzcd} {[n] \times M} \arrow[rr] \arrow[dr] & & {[n'] \times M} \arrow[dl] \\ & E & \end{tikzcd}$$

We then have an evident functor, which we denote $F$, from $\mathcal{C}$ to the category of simplicial $M$-sets over $E$ and we can form the simplicial object of chains with coeff. in this functor

$$C_p(\mathcal{C}, F) = \coprod_{[n_0]\times M \to \cdots \to [n_p]\times M \to E} [n_0] \times M$$

which has an evident augmentation $C_\cdot(\mathcal{C}, F) \longrightarrow E$.

<u>Lemma 1:</u> $|C(\mathcal{C}, F)| \longrightarrow E$ is a heq.

<u>Proof.</u> suffices to show the horizontal simp. obj. $p \mapsto C_p(\mathcal{C}, F)_q$ for each $q$ contracts to $E_q$. But

<u>Lemma 1:</u>

$$N_p\mathcal{C} \times_{N_0\mathcal{C}} F$$

$M$ simplicial monoid

$BM = \operatorname{diag}(p \mapsto M^p)$

$EM = \operatorname{diag}(p \mapsto M^p \times M)$ contractible  $M$ acts by right mult.

~~$P \to BM$ a fibration with $P$ contractible~~

$\Omega BM =$ fibre ~~[illegible]~~ of a fibration $P \to BM$, $P$ cont.

$$\begin{tikzcd}
\Omega BM \arrow[r] \arrow[d] & P \times_{BM} EM \arrow[r] \arrow[d] & EM \arrow[d] \\
\Omega BM \arrow[r] & P \arrow[r] & BM
\end{tikzcd}$$

Group-completion theorem: Assume $\pi_0 M$ is abelian and that left and right multiplication by an element of $\pi_0 M$ on $h_q(M)$ coincide. Then $h_q(M)[\pi_0 M^{-1}] \xrightarrow{\sim} h_q(\Omega BM)$.

$\boxed{\text{Localization:}}$ $\pi_0 M \longrightarrow G$ group-completion

suppose $\mathbb{Z}[G]$ flat as a left $\mathbb{Z}[\pi_0 M]$ module

$$A[\pi_0 M^{-1}] = A \otimes_{\mathbb{Z}[\pi_0 M]} \mathbb{Z}[G]$$

Then $A \longmapsto A[\pi_0 M^{-1}]$ is exact, so can localize

$$E^2_{pq} = H_p(N\mathcal{C}, \quad )$$

On other hand

Proposition 1: $E$ simplicial $M$-set, $\mathcal{C}$ = category of s.$M$-sets over $E$ of the form $\Delta(n)\times M \to E$, $K_*$ homology theory. Then $\exists$ spectral sequence

$$E^2_{pq} = H_p(N\mathcal{C}, \mathcal{K}_q) \Longrightarrow F_{p+q}(E) \qquad F \text{ no good.}$$

where $\mathcal{K}_q$ denotes: $\Delta(n)\times M \to E \longmapsto F_q(\Delta(n)\times M) \simeq F_q(M)$.

Proof: $N_p\mathcal{C} \times_\theta F_r \qquad F_r = \coprod_{\Delta(n)\times M \to E} (\Delta(n)\times M)_r$

idea is to consider the simplicial object

$p \longmapsto N_p\mathcal{C}\times_\theta F \qquad F = \coprod_\theta \Delta(n)\times M$

~~and the Segal spectral sequence~~

$$\cancel{E^1_{pq} = K_q(N_p\mathcal{C}\times_\theta F) \Longrightarrow}$$
$$\cancel{\cong \coprod K_q}$$

and the associated Segal spectral sequence

$$E^1_{pq} = K_q(N_p\mathcal{C}\times_\theta F)$$

right $E_2$

$$= C_p(N\mathcal{C}, \mathcal{K}_q)$$

$$\boxed{E^2_{pq} = H_p(N\mathcal{C}, \mathcal{K}_q).}$$

Lemma: Fix $r$ and consider $N_p\mathcal{C}\times_\theta F_r \longrightarrow E_r$

this is an heq

right abutment.

Proposition 2: $E$ simplicial $M$-set such that $M_r$ acts freely on $E_r$ for each $r$ (whence $E_r \simeq E_r/M_r \times M_r$). Then $N\mathcal{C}$ is heq to $E/M$ in a canonical way.

~~We consider the~~ Consider the diag. of trisimplicial sets

$$\begin{tikzcd}
N_p\mathcal{C} \times_{\mathcal{O}} F_r \times M_r^q \arrow[r, "(1)"] \arrow[d, "(2)"] & E_r \times M_r^q \arrow[d, "(3)"] \\
N_p\mathcal{C} \times_{\mathcal{O}} F_r/M_r \arrow[r] \arrow[d, "(4)"] & E_r/M_r \\
N_p\mathcal{C} &
\end{tikzcd}$$

Previous lemma $\Rightarrow$ (1) induces a heq on realizations (or diagonal simplicial sets) because this is true with $q\,r$ fixed. (Logic:

$$\int_p N_p\mathcal{C}\times_{\mathcal{O}} F_r\times M_r^q \longrightarrow E_r\times M_r^q \qquad \text{heq}$$

$$\Rightarrow \int_r\int_p \longrightarrow \int_r \qquad \text{heq}$$

$$\Rightarrow \int_q\int_r\int_p \longrightarrow \int_{qr} \qquad \text{heq}$$ )

(2) and (3) induce heq's on realizations because $M_r$ acts freely on $F_r$ and $E_r$ respectively

(4) induces an heq on realizations because ~~the fibre~~

$$F_r/M_r = \coprod_{q} \Delta(n)$$

---

we have a (weak) homotopy equivalence

$$\mathrm{diag}(p \mapsto E \times M^p) \longrightarrow X .$$

Consider now the bisimplicial object in the category of simplicial $M$-sets

$$(p,q) \longmapsto N_p\mathcal{C} \times_{\mathrm{Ob}\,\mathcal{C}} F \times M^q$$

This has a "vertical" augmentation to $N\mathcal{C}$ with fibres the simplicial objects $q \mapsto \Delta(n) \times M \times M^q$, which are contractible. It also has a "horizontal" augmentation to $q \mapsto E \times M^q$ which

~~commut~~

iso class$\{M^{(q)}\} = I_q$    simplicial abelian monoid.

$I_0 = pt$

$I_1$ = iso classes of bundle

$I_2$    $M_1 \subset M_2 \; + \; M_2 \subset M_3$

in the spectral sequence $E^2_{11} =$ ~~Bass~~ $K_1$

so one has also

$$E^2_{30} \xrightarrow{d_2} E^2_{11} \longrightarrow K_1 \longrightarrow E^2_{10} \longrightarrow 0$$

$$\hat{I}_3 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; \hat{I}_2 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \hat{I}_1 \rightrightarrows pt$$

---

over NC have a simplicial M-set

$$\begin{tikzcd} & PM \arrow[d] \\ X \arrow[r] & BM \end{tikzcd}$$

$F \times M \rightrightarrows F$

---

bisimplicial object

$$X_{pq} = \coprod_{[n_0]\times M \to \cdots \to [n_p]\times M \to E} [n_0]\times M \times M^q = \coprod_{x_0 \to \cdots \to x_p} F(x_0) \times M^q$$

$$X_{pq} = Ar_p C \times_{Ob\, C} F \times M^q$$

$$= N_p C \times_{N_0 C} F \times M^q$$

$N_p C \times_{N_0 C} F \times M^q$    $F = \coprod_{x \in N_0 C} F(x)$

~~$\not{E} \neq X \not{\,}$~~

$E \times M$

$$\coprod_{\mathrm{ar}\,\mathcal{C}} \Delta(n) \times M \rightrightarrows \coprod_{\mathrm{Ob}\,\mathcal{C}} \Delta(n) \times M \to E$$

$$\begin{array}{ccc} X_{pq} = \coprod\limits_{[n_0]\times M \to \cdots \to [n_p]\times M} [n_0] \times M \times M^q & \xrightarrow[\text{aug}]{\text{horiz}} & E \times M^q \\ \Big\downarrow {\scriptstyle \text{vertical aug}} & & \\ (N\mathcal{C})_p & & \end{array}$$

horizontal augmentation is ~~~~ a heq in ~~~~ for each $q$.
vertical augment

$$\begin{array}{ccccc} |X_{\cdot\cdot}| & \xrightarrow{\sim} & |E \times M^q| & \xrightarrow{\sim} & X. \\ \downarrow{\scriptstyle S} & & \downarrow & & \\ \Big| \coprod\limits_{[n_0]\times M \to \cdots \to [n_p]\times M} [n_0] \Big| & \longrightarrow & X & & \\ \downarrow{\scriptstyle S} & & & & \\ N\mathcal{C} & & & & \end{array}$$

64      70-71
65      71-72
66      72-
67
68
69

~~Theorem: Let $E \xrightarrow{f} B$ be a map of s. sets on which $M$ acts on the right, the action being trivial on $B$, and assume $E_q \simeq B_q \times M_q$ for each $q$.~~

~~Then ∃ spectral sequence~~

$$E^2_{pq} = H_p(B, L_q) \Longrightarrow H_{p+q}(E)[\pi_0 M^{-1}]$$

~~where $L_q$ is a local coeff. system to be understood.~~

~~Set $E = P \times_{BM} EM$, $B = P$~~

~~Proof: Let $E \to B$ be a map~~

Proof. Let $E = P \times_{BM} EM$. This is a simp. $M$-set.

$\mathcal{C}$ = category of $\Delta(n) \times M \longrightarrow E$; get a bisimple

$$\coprod_{x_0 \to \cdots \to x_n} \Delta \times M$$

$$C_q(\mathcal{C}, F) = \coprod_{\substack{M(n_0) \to \cdots \to M(n_n) \to E \\ [n_0]\times M \to \cdots \to [n_q]\times M \to P}} [n_0] \times M$$

so now consider the two spectral sequences

so you get a spectral sequence:

$$E^2_{pq} = H_p(\mathcal{C}, \quad ) \Longrightarrow H_q(E)$$

now localize

Problem: Define H-map $M \to \Omega BM$.

Choose $\begin{tikzcd} E \arrow[rr] \arrow[dr] & & P \arrow[dl] \\ & B & \end{tikzcd}$ then you get $\begin{tikzcd} E\times M \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & P\times_B P \arrow[d, shift left] \arrow[d, shift right] \\ E \arrow[r] \arrow[d] & P \arrow[d] \\ B \arrow[r, equal] & B \end{tikzcd}$

In dimension 1 + 2 get

$$\begin{tikzcd} M & E\times M \arrow[l, "heq"'] \arrow[r] & P\times_B P & \Omega B \arrow[l, "heq"'] \\ M^2 \arrow[u, "\mu"] & E\times M^2 \arrow[l] \arrow[r] \arrow[u, "id\times\mu"] & P\times_B P\times_B P \arrow[u, "pr_{13}"] & \Omega B\times\Omega B \arrow[l, "heq"'] \arrow[u, dotted, "\text{loop composition}"'] \end{tikzcd}$$

---

~~Conclude that~~ $\cancel{M \to \boxed{\Omega BM}}$

~~B, paths~~ Suppose $P \to B$ fibration with $P$ contractible; set $\Omega$ = fibre over basepoint of $B$. Then $\exists$ $E \to P$ over $B$, and the space of these maps is contractible. So we get maps

$$\begin{tikzcd} M & E\times M \arrow[l] \arrow[r] & P\times_B P & \Omega \arrow[l, "h.eq."'] \end{tikzcd}$$

$m$ $\quad(*, m)$

$(e, m) \mapsto (\varphi(e), \varphi(em)),\ (*, z) \longleftarrow\!\!| \ z$

~~whose composition is the~~ $\therefore$ Get a well-defined ~~unique~~ homotopy class of maps from $M$ to $\Omega$

---

$B = BM$ = realiz. of nerve of $M$ acting on pt

$E = EM$ = " " " " $M$ left acting on $M$; $M$ right acts on $E$

$P \to B$ fibration with $P$ contractible

$\Omega$ = fibre over basepoint.

Because $E$ contractible $\exists$ map $E \xrightarrow{\varphi} P$ over $B$ + space of such maps is contractible. Then have

$$M \longrightarrow P\times_B P \longleftarrow \Omega$$

$$(*, z) \longleftarrow\!\!| \ z$$

$$m \longmapsto (*, \varphi(*\cdot m))$$

with second map a heq. Thus get a well-defined homotopy class of maps $M \to \Omega$. Following diagram

$$\begin{tikzcd}
M^2 \arrow[r] \arrow[d] & \overset{(z_1, \cancel{z_1}, z_2)}{P\times_B P\times_B P} \arrow[d, "pr_{13}"] & \overset{(z_1, z_2)}{\Omega\times\Omega} \arrow[l] \arrow[d] \\
M \arrow[r] & P\times_B P & \Omega \arrow[l, "h.eq"']
\end{tikzcd}$$

$$(m_1, m_2) \longmapsto *, \varphi(*m_1), \varphi(*m_1m_2)$$

shows this is an $H$-map.

Using the spectral sequence I am going to show that the inclusion of the fibre over the basepoint induces an isom

$$H_i(M)[\pi_0 M^{-1}] \xrightarrow{\sim} H_i(P\times_B E)[\pi_0 M^{-1}]$$

On other hand, I have that $\pi_0 M$ acts invertibly on $P\times_B E$:

$$\begin{tikzcd}
P\times_B E \arrow[r, "\cdot m"] \arrow[d] & P\times_B E \arrow[d] \\
P \arrow[r, "\cdot m"] & P
\end{tikzcd}$$

and the map on the base is a h.eq. so

$$H_i(P\times_B E) \xrightarrow{\sim} H_i(P\times_B E)[\pi_0 M^{-1}].$$

Finally have that $P\times_B E$ fibres over $P$ with fibre $\Omega$, so

$$H_i(\Omega) \xrightarrow{\sim} H_i(P\times_B E).$$

Putting these isoms. together get

$$H_i(M)[\pi_0 M^{-1}] \xrightarrow{\sim} H_i(\Omega)$$

# E' class, E'' class.

group-completion. M, P,

standard resolution

$\rightrightarrows \coprod \Delta(\ )\times M \rightrightarrows \coprod \Delta(n)\times M \to P$

assertion: it is a simplicial object in the category of right M-sets. Formula:

dim n: $\coprod_{x_0\to\cdots\to x_n} Q(x_0)$ and given $\varphi:[n]\to[m]$

with $\varphi(0)=i$, then $\varphi_* : \coprod$

Introduce category $S(M,P)$ consisting of $\Delta(n)\times M\to P$

pair $(n, \Delta(n)\times M\to P)$ and a map

objects consists of $n\geq 0$ and $x\in P_n$; an arrow $(n,x)\to(n',x')$ consists of $\varphi: n\to n'$ and $m\in M_n$ with $\varphi^*(x')^m = x$. Put another way, $S(M,P)$ is isom. to the full subcategory of s. M-sets over P consisting of the objects $\Delta(n)\times M \to P$.

$$\begin{tikzcd} \Delta(n)\times M \arrow[rr, "(\varphi{,}\ m)"] \arrow[dr] & & \Delta(n')\times M \arrow[dl, "\varphi^* x'\cdot m"] \\ & P & \end{tikzcd}$$

Then we have the functor $(n,x)\mapsto \Delta(n)\times M$ from $S(M,P)$ to s.M sets/P, so can form the simplicial object

C category: $F: C \to$ a category with direct sums can form simplicial object

$C_n(C,F) \quad \coprod_{x_0\to\cdots\to x_n} F(x_0)$

chains of Nerve of C coeff. in A. This gives rise to homology of C coeff

$H_*(C,F)$.

If F locally constant, meaning morphism-inverting and A abelian, then have

C = cat of s. M-sets/P of the form $\Delta(n)\times M\to P$

Claim that

$C_*(\ )$

Assertion:

$\coprod \Delta(n)\times M$

$(n_0)$

Logic: have a simp. object

$\rightrightarrows \coprod \Delta(n)\times M$

+ an augmentation to P

Assertion: $\forall_g$, $(\coprod \Delta(n)\times M)_g \to$ is acyclic.

Prop 1: $\exists$ a spectral sequence

$H_*(P) \Longleftarrow H_*(C, x\mapsto H_*(x))$.

Proof: Construct a bisimplicial set and look at the two spectral sequences.

$$\begin{tikzcd} & M \arrow[d] & \\ \Omega BM \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & PM \arrow[d] \\ \Omega BM \arrow[r] & X \arrow[r] & BM \end{tikzcd}$$

then $H_*(\Omega BM) \xrightarrow{\sim} H_*(P)$

$\Omega BM\times M$

The problem is that I need the map $M\to\Omega BM$ and the fact it is an H-map. The point

M

CLAIM: $\exists$ canonical map $M\to \Omega BM$ which is an H-map.

November 5, 1972: To understand $H_*(\Omega Q(\mathcal{M}))$.

~~vvg~~ over a finite field; $\mathbb{Q}$ coefficients:

$$\coprod_{\substack{\text{iso}\\ \text{classes}\\ \text{of extensions}}} \mathbb{Q} \qquad \coprod_{\substack{\text{iso}\\ \text{classes}}} \mathbb{Q} \qquad \overset{\circ}{\mathbb{Q}}$$

useless

$$E^{(2)} \qquad I = E^{(1)} \qquad\qquad M$$

question: Is

$$E^{(3)} \xrightarrow{\sim} E^{(2)} \times_{E^{(1)}} E^{(2)}.$$

~~clear~~ categorically. $\qquad 0$

---

$$0 \subset F_1M \subset F_2M \subset M$$

$$\{F_1M \subset F_2M \subset M\} \longmapsto (\{F_1M \subset F_2M\}, \{F_2M \subset M\}).$$

suppose then that $\exists$ isom $F_1M' \subset F_2M' \simeq F_1M \subset F_2M$.

& $\exists$ isom $F_2M' \subset M' \simeq F_2M \subset M$.

then there is no reason why these fit.

---

to start with the K-theory of ~~~~

---

bicartesian squares?

$$\begin{array}{ccccccccc}
C & \leftarrow & X & \leftarrow & X' & \leftarrow & K & \leftarrow & 0 \\
\| & & \downarrow & & \downarrow & & \| & & \\
C & \leftarrow & Y & \leftarrow & Y' & \leftarrow & K & \leftarrow & 0
\end{array}$$

determines an element of $\mathrm{Ext}^2(C, K)$ which is clearly an invariant.

another idea: Instead of ~~[illegible]~~ starting with iso classes. Regard a vector bundle as a K module + extra structure.

vector bundle = a K-vector space V together with a point of the building $X(V)$ of lattices. The category of these is obtained by letting $GL(V)$ act. The category of vector bundles is now somewhat reasonable. Next we must add in the ~~[illegible]~~ ordinary building

an extension is a ~~[illegible]~~ K-vector space together with two point in the

This is interesting combinatorial geometry which one ought to be able to understand in dim. 1 + 2.

$H_2(Q(\mathcal{M}))$.

CLUE: an $E' \subset E$ consists of an $E'$ class, an $E''$ class, and an orbit of ~~[illegible]~~ $Ext^1(E'', E')$ under $Aut(E') \times Aut(E'')$.

$$\coprod_{E' \hookrightarrow E} BAut(E' \hookrightarrow E) \rightrightarrows \coprod_{E} BAut(E) \rightrightarrows pt$$

$$\bigoplus_{E' \subset E} \qquad \bigoplus_{E} H_q(Aut(E)) \rightrightarrows 0$$

Problem: $M$ simplicial monoid

$$\begin{tikzcd}
\Omega BM \arrow[r] \arrow[d, equal] & P\times_{BM} EM \arrow[r] \arrow[d] & EM \arrow[d] \\
\Omega BM \arrow[r] & P \arrow[r] & BM
\end{tikzcd}$$

$M \longrightarrow P\times_{BM} EM$ defined by choosing a basepoint of $P\times_{BM} EM$.

$\Omega BM \xrightarrow{heq} P\times_{BM} EM$ fibre over basepoint of $EM$.

Then why is

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & \Omega BM \arrow[d] \\
EM \arrow[r, "\varphi"] \arrow[d] & P \arrow[dl] \\
BM &
\end{tikzcd}$$

① why is

$$\begin{tikzcd}
M \arrow[r] \arrow[dr] & \Omega BM \arrow[d] \\
& P\times_{BM} EM
\end{tikzcd}$$

homotopy commutative?

---

② why does it commute with the section of ~~$\Omega BM$~~ $M$?

$$\begin{tikzcd}
& M \arrow[r, "\sim"] \arrow[d] & M \arrow[d] \\
\Omega BM \arrow[r] \arrow[d, "="] & P\times_{BM} EM \arrow[l, bend right] \arrow[r, shift left] \arrow[d] & EM \arrow[l, dashed, shift left] \arrow[d] \\
\Omega BM \arrow[r] & P \arrow[r] & BM
\end{tikzcd}$$

Statement:

Let $M$ ($p \mapsto M_p$) be a simplicial monoid, and $X$ (resp. $Y$) a simplicial set on which $M$ acts to the right (resp. left). We can form a simplicial object $p \mapsto X \times M^p \times Y$ in the cat. of simplicial sets in which the $i$-th face operator multiplies the $i$-th and $(i+1)$-th components together. Viewing this simplicial object as a bisimplicial set ~~$p,q \mapsto X_q \times M_q^p \times Y_q$~~, we can form the diagonal simplicial set $p \mapsto X_p \times M_p^p \times Y_p$, denoted $\mathrm{diag}(X \times M^p \times Y)$.

Taking $X = Y = pt$, where $pt$ consists of a one-element set in each dimension, we obtain the classifying "space"

$$BM = \mathrm{diag}(p \mapsto M^p)$$

and taking

$\boxed{\text{Realization}}$

The <u>group-completion</u> of a topological monoid $M$ is by definition $\Omega BM$, the loop space of its classifying space. The group-completion theorem enables one to compute $h_*(\Omega BM)$ in terms of $h_*(M)$ under suitable assumptions, for any generalized homology theory $h_*$.

basepoint $P$, $EM$. get a map $M \to P \times_{BM} EM$ to which proposition can be applied yielding

(1) $$h_i(M)[\pi_0 M^{-1}] \xrightarrow{\sim} h_i(P \times_{BM} EM)[\pi_0 M^{-1}].$$

On the other hand right

$$\begin{tikzcd} P \times_{BM} EM \arrow[r, "\cdot m"] \arrow[d] & P \times_{BM} EM \arrow[d] \\ EM \arrow[r, "\cdot m"] & EM \end{tikzcd}$$

Right multiplication by $m \in M_0$ is a heq on $P \times_{BM} EM$

(2) $$h_i(P \times_{BM} EM) \xrightarrow{\sim} h_i(P \times_{BM} EM)[\pi_0 M^{-1}].$$

Hence inclusion of the fibre ~~...~~ gives an isomorphism

(3) $$h_i(\Omega BM) \xrightarrow{\sim} h_i(P \times_{BM} EM).$$

Somewhere have to note that the isom. ~~...~~ compatible with a map $M \to \Omega BM$ produced by a map $EM \to P$ over $BM$.

~~Theorem~~

Group-completion Theorem: ~~...~~. Assume that $\mathbb{Z}[\pi]$ is flat as a left $\mathbb{Z}[\pi_0 M]$-module, and that the ~~$\pi_0 M$ acts invertible~~ left action of $\pi_0 M$ on $h_*(M)[\pi_0 M^{-1}]$ is invertible. ~~...~~ (In part. if $\pi_0 M$ is abelian and if the left and right action of $\pi_0 M$ on $h_*(M)$ coincide). Then

$$h_*(M)[\pi_0 M^{-1}] \xrightarrow{\sim} h_*(\Omega BM).$$

finish section by discussing the translation with ~~...~~ topological monoids.

$E$ vector space over $\mathbb{C}$ of dimension $n$.
$\mathfrak{o}$ ring of integers in a quadratic no. field $\mathbb{Q}[\sqrt{-d}]$.
I want to let $X$ be the set of lattices in $E$ which are $\mathfrak{o}$-modules. Thus if we choose one of these $L$ we have that

$$X \xleftarrow{\sim} \mathrm{Aut}_{\mathbb{C}}(E)/\mathrm{Aut}_{\mathfrak{o}}(L).$$

no.

$$X = \left\{ L \subset E \;\middle|\; \begin{array}{l} L \text{ lattice i.e. free abelian grp of rank } 2n \text{ which spans } E \\ L \text{ an } \mathfrak{o}\text{-module, i.e. } \sqrt{-d}\cdot L \subset L. \end{array} \right\}$$

Then if $\theta \in \mathrm{Aut}_{\mathbb{C}}(E)$. and $L \in X$, then
$\theta L \in X$.

Does $\mathrm{Aut}_{\mathbb{C}}(E) = G$ act transitively on $X$.

If $L$ and $L'$ $\ni$ $\theta L = L'$ then

$$\theta : L \xrightarrow{\sim} L'$$

is an $\mathcal{O}$-module isomorphism. So if $L$ and $L'$ are not isomorphic as $\mathcal{O}$-module then they are not in the same $G$-orbit. Thus there are finitely many $G$-orbits, because the ranks of $L, L'$ are same so only other invariant is the determinant $\Lambda^n_{\mathfrak{o}} L \in \mathrm{Pic}(\mathfrak{o})$. which is finite. Gives a direct proof patterned on the finiteness of class number.

Recall that proof. One starts with a lattice $\mathfrak{a} \subset \mathfrak{o} \subset \mathbb{C}$ and applies Minkowski to find an element $z \in \mathfrak{a}$ with small abs. value