October 1, 1972. Segal's proof of the group-completion thm.

Suppose M is a top monoid, $a \in M$, and left and right multiplication by a, $M \longrightarrow M$, are cofibrations (*). Suppose also that the image of a in $\pi_0 M$ is in the center of $H_*(M, k)$, (k = field, to simplify). Then Segal computes the homology of the monoid $M[a^{-1}]$ obtained by adjoining a^{-1} to M.

(*) M free seems necessary

He filters $M[a^{-1}]$ using the number of occurrences of a^{-1}.

$$M \subset \bigcup_k M a^{-k} M \subset \bigcup_k M a^{-k} M a^{-k} M \subset \cdots$$

Consider Filt_1 consisting of products $x = m_1 a^{-k} m_2$. It should be true that if $x \notin M$ and k is minimal, then m_1 and m_2 are uniquely determined $m_1 \in M - Ma$, $m_2 \in M - aM$. Thus set-theoretically

$$\bigcup_k M a^{-k} M = M \amalg \coprod_{k>0} (M - Ma) \times (M - aM)$$

?

Better, set $a^{-\infty} M = \bigcup a^{-k} M$; it is the limit of the sequence of cofibrations

$$M \xrightarrow{a\cdot} M \xrightarrow{a\cdot} M \xrightarrow{a\cdot} \cdots$$

and hence has the homology we are after, namely $H_*(M)[a^{-1}]$. Now the next step is to consider

$$\bigcup_k a^{-\infty} M a^{-k} M$$

We have cocart squares

$$\begin{tikzcd}
{[Ma \times a^{-k}M] \cup [M \times a^{-k+1}M]} \arrow[r, two heads] \arrow[d, hook] & Ma^{-k+1}M \arrow[d, hook] \\
M \times a^{-k}M \arrow[r, two heads] & Ma^{-k}M
\end{tikzcd}$$

(if $m_1 a^{-k} m_2 \in Ma^{-k+1}M$, then either $m_1 \in Ma$ or $m_2 \in aM$, and moreover if $m_1 a^{-k} m_2 \notin Ma^{-k+1}M$, then $m_1 \in M - Ma$ and $m_2 \in M - aM$ are uniquely determined.) Also

$$\begin{tikzcd}
Ma \times a^{-k+1}M \arrow[r, hook] \arrow[d, hook] & M \times a^{-k+1}M \arrow[d, hook] \\
Ma \times a^{-k}M \arrow[r, hook] & {[Ma \times a^{-k}M] \cup [M \times a^{-k+1}M]}
\end{tikzcd}$$

is cocartesian. Now multiply on the left by a^m and take the limit as $m \to \infty$. Since $a^{-\infty}Ma \longrightarrow a^{-\infty}M$ is a homology isomorphism, it follows from the second square that

$$a^{-\infty}Ma \times a^{-k}M \hookrightarrow [a^{-\infty}Ma \times a^{-k}M] \cup [a^{-\infty}M \times a^{-k+1}M]$$

is a homology isomorphism, and hence from the first square that

$$a^{-\infty}Ma^{-k+1}M \hookrightarrow a^{-\infty}Ma^{-k}M$$

is a homology isomorphism $\forall k$. Conclude

$$a^{-\infty}M \longrightarrow a^{-\infty}Ma^{-\infty}M$$

is a homology isomorphism.

Segal claims something similar works for the remaining maps

$$a^{-\infty}Ma^{-\infty}M \hookrightarrow a^{-\infty}Ma^{-\infty}Ma^{-\infty}M \hookrightarrow \cdots$$

I think it desirable to understand if the spaces $Ma^{-\infty}M$ or $a^{-\infty}Ma^{-\infty}M$ are nerves of suitable categories in the case where

$$M = \coprod_n BGL_nA$$

For example

$$Ma^{-\infty} = \mathbb{Z} \times BGL(A)$$

since

$$Ma^{-\infty} = \lim\ M \xrightarrow{\cdot a} M \xrightarrow{\cdot a} M \longrightarrow \cdots\cdots$$

and right mult. by a = basept of BGL_1A is the map

$$BGL_nA \longrightarrow BGL_{n+1}A$$
$$\theta \longmapsto \theta \oplus \mathrm{id}.$$

Similarly

$$a^{-\infty}Ma^{-\infty} = \mathbb{Z} \times B\left(\begin{matrix}\text{doubly infinite}\\ \text{matrices}\end{matrix}\right).$$

The point of the argument on page 3 is

$$Ma^{-k}M / Ma^{-k+1}M = M/Ma \wedge M/aM,$$

i.e. any product $m_1 a^{-k} m_2$ not in $Ma^{k+1}M$ has a uniquely determined $m_1 \in M - Ma$, $m_2 \in M - aM$. Now taking the limit we have

$$a^{-\infty}Ma^{-k}M / a^{-\infty}Ma^{-k+1}M = a^{-\infty}M / a^{-\infty}Ma \wedge M/aM$$

Finally $a^{-\infty}M/a^{-\infty}Ma$ has no homology by the commutativity up to homotopy.

<u>October 3, 1972</u>: Segal's proof of the group-completion theorem

M free top (simplicial if you prefer) monoid, $a \in M$ such that $cl(a) \in \pi_0 M$ is central in $H_*(M)$. Segal proposes to show that $M[a^{-1}]$ has homology $H_*(M)[cl(a)^{-1}]$ by using the filtration

$$a^{-\infty} M a^{-\infty} \subset a^{-\infty} M a^{-\infty} M a^{-\infty} \subset \cdots$$

I only understand the first step at the moment which is based upon the cocartesian square

$$\begin{tikzcd}
\bigcup_k a^{-\infty} M a^{-k} \times a^{k} M a^{-\infty} \arrow[r, hook] \arrow[d] & (a^{-\infty} M a^{-\infty})^2 \arrow[d] \\
a^{-\infty} M a^{-\infty} \arrow[r, hook] & a^{-\infty} M a^{-\infty} M a^{-\infty}.
\end{tikzcd}$$

which results by passing to the limit with respect to left + right mult. by a in the cocart. square

$$\begin{tikzcd}
\bigcup_k M a^{-k} \times a^{k} M \arrow[r, hook] \arrow[d, two heads] & M a^{-\infty} \times a^{-\infty} M \arrow[d, two heads] \\
M \arrow[r, hook] & M a^{-\infty} M
\end{tikzcd}$$

Now I would like to understand the space $M a^{-\infty} M$ in the case where $M = \coprod_n BG_n$ where $G_n = GL_n$ of some ring, or perhaps, Σ_n. I have to start by understanding the left vertical arrow and the union in the square immediately above.

Consider [redacted] the ordered set J of integers pairs (p,q) with $p+q \leq 0$ with $(p',q') \leq (p,q) \iff p' \leq p$ and $q' \leq q$. Then we have a functor

$$(p,q) \longmapsto Ma^{-p} \times a^{-q}M$$

and it would be nice to know that

$$\underrightarrow{holim}\left((p,q) \mapsto Ma^{-p} \times a^{-q}M\right) = \bigcup_k Ma^{-k} \times a^k M \,.$$

($\underrightarrow{holim}$ = telescope). Leaving this aside, [redacted] we work out the map of multiplication

$$Ma^{-p} \times [redacted] a^{-q}M \longrightarrow M.$$

$$(Ma^{-p})_k = BG_{p+k}$$

$$(a^{-q}M)_\ell = [redacted] = BG_{q+\ell}$$

The [redacted] mult takes $(Ma^{-p})_k \times ([redacted] a^{-q}M)_\ell$ to $M_{k+\ell}$ $= BG_{k+\ell}$ and it is clearly the map

$$BG_{p+k} \times BG_{q+\ell} \longrightarrow BG_{k+\ell}$$

$$(A \;,\; B) \longmapsto A \oplus \varepsilon^{-(p+q)} \oplus B$$

(recall $p+q \leq 0$).

<u>The hope:</u> The space $Ma^{-\infty}M$ might provide a useful description for the group completion. I recall [redacted] the filtration

$$M \subset Ma^{-1}M \subset Ma^{-2}M \subset \cdots\cdots\cdots$$

and the isos.

$$Ma^{-k}M / Ma^{-k+1}M \xleftarrow[\simeq]{\cdot a^{-k}\cdot} (M/Ma) \wedge (M/aM).$$

Now when $M = \coprod BG_n$ we have

$$\begin{array}{ccc}
\left(Ma^{-k}M / Ma^{-k+1}M\right)_n & \xleftarrow{\cong} & \left(M/Ma \wedge M/aM\right)_{n+k} \\
 & & \wr\| \\
 & & \bigvee_{i+j=n+k} (M/Ma)_i \wedge (M/aM)_j \\
 & & \| \\
 & & \bigvee_{i+j=n+k} BG_i \times BG_j / BG_{i-1} \times BG_j \cup BG_i \times BG_{j-1} \\
 & & \| \\
 & & \left(BG_i/BG_{i-1}\right) \wedge \left(BG_j/BG_{j-1}\right)
\end{array}$$

This is nice, because a stability result for BG_i/BG_{i-1} with i large implies stability for the filtration $Ma^{-k}M$. The converse has certain possibilities.

It seems desirable, therefore, to understand the inclusion $M \subset Ma^{-1}M$.

$$\begin{tikzcd}
Ma^{-1}\times aM \cup M\times aM \arrow[r, hook] \arrow[d] & Ma^{-1}\times M \arrow[d] \\
M \arrow[r, hook] & Ma^{-1}M
\end{tikzcd}$$

In degree n, this square looks as follows:

$$\begin{tikzcd}
& \coprod\limits_{i+j=n} BG_i\times BG_j \arrow[dr, "{(A,B)\mapsto(A,\varepsilon\oplus B)}"] & & \\
\coprod\limits_{i+j=n-1} BG_i\times BG_j \arrow[ur, "{(A,B)\mapsto(A+\varepsilon,B)}"] \arrow[dr, "{(A,B)\mapsto(A,\varepsilon\oplus B)}"'] & & \begin{pmatrix} Ma^{-1}\times aM \\ \cup M\times M \end{pmatrix}_n & \subset \coprod\limits_{i+j=n+1} BG_i\times BG_j \\
& \coprod\limits_{i+j=n} BG_i\times BG_j \arrow[ur, "{(A,B)\mapsto(A\oplus\varepsilon,B)}"'] \arrow[d, shift right, "\oplus"'] \arrow[d, shift left, "\oplus"] & & \\
& BG_n & &
\end{tikzcd}$$

What seems relevant: Inside of G_n we consider the family of subgroups

$$G_p \times \varepsilon^{n-p-q} \times G_q \subset G_n \qquad p+q<n$$

which are closed under intersection. [redacted] Form the union

$$\bigcup_{p+q<n} BG_p \times \varepsilon^{n-p-q} \times BG_q \subset BG_n .$$

Now the union can be mapped to BG_{n-1}, so we obtain a square

$$
\begin{tikzcd}
\bigcup\limits_{p+q<n} BG_p \times \varepsilon^{n-p-q} \times BG_q \arrow[r, hook] \arrow[d] & BG_n \arrow[d] \\
BG_{n-1} \arrow[r, hook] & ?
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & BG_{i-1} \times BG_{m-i} \arrow[d, "+"] \arrow[dr, "(A,B) \mapsto (A+\varepsilon, B)"] & \\
BG_{i-1} \times BG_{m-i-1} \arrow[ur, "(A,B) \mapsto (A,\varepsilon+B)"] \arrow[dr, "(A,B) \mapsto (A+\varepsilon, B)"'] & BG_{m-1} & BG_i \times BG_{m-i} \\
 & BG_i \times BG_{m-i-1} \arrow[u, "+"'] \arrow[ur, "(A,B) \mapsto (A,\varepsilon+B)"'] &
\end{tikzcd}
$$

Here $m = n+1$. Thus we get finally this cocartesian square

$$
\begin{tikzcd}
\coprod\limits_{i=0}^{n+1} \left[(BG_{i-1} \times \varepsilon) \times BG_{n+1-i}\right] \cup \left[BG_i \times \varepsilon \times BG_{n-i}\right] \arrow[r, phantom, "\subset"] \arrow[d] & \coprod\limits_{i=0}^{n+1} BG_i \times BG_{n+1-i} \arrow[d] \\
BG_n \arrow[r, phantom, "\subset"] & (Ma^{-1}M)_n
\end{tikzcd}
$$

<u>Conjecture:</u> $\exists$ cocartesian square

$$\begin{tikzcd}
\bigcup\limits_{p+q\leq n} BG_p \times \varepsilon^{n+1-p-q} \times BG_q \arrow[d] & \subset & BG_{n+1} \quad \boxed{\cancel{\bigcup\limits_{\substack{p+q=n+1 \\ p,q>0}} BG_p \times BG_q}} \arrow[d] \\
BG_n \arrow[rr] & & (Ma^{-1}M)_n
\end{tikzcd}$$

where the union is taken inside of BG_{n+1}. ?

The problem preventing a clear understanding is that it is difficult to interpret the union

$$\bigcup BH_\alpha \subset BG,$$

, where H_α is a family of subgroups of G, in a clear categorical way.

October 5, 1972

BG theory – spherical fibrations. This is the K-theory that arises from the category with objects ptd spaces of the homotopy type of spheres, with maps equal to ptd-homotopy equivalences, and with operation = smash product. Thus there is essentially one object S^n for each $n \geq 0$, and the space of the category is essentially

$$\coprod_{n \geq 0} B(\Omega^n S^n)_{\pm 1}$$

where $(\Omega^n S^n)_{\pm 1}$ is regarded as a monoid under composition, denoted G_n. The associated invertible H-space is

$$\mathbb{Z} \times BG \qquad\qquad G = \varinjlim G_n$$

(this is a consequence of the group-completion theorem and the fact that BG is a simple space.) The situation is exactly the same as for $\coprod_{n \geq 0} BU_n$.

$BG - q^{-1}$ theory: Here one replaces $(\Omega^n S^n)_{\pm 1}$ by $\coprod_k (\Omega^n S^n)_{\pm q^k}$. Again the objects are ptd spheres but the arrows are q^{-1} homotopy-equivalences. The associated invertible H-space should be

$$\mathbb{Z} \times B(\mathbb{Z}/2 \times \mathbb{Z}) \times BSG[\tfrac{1}{q}] .$$

~~Obs~~ Observation. In this model we have the maps $S^n \longrightarrow S^n$ breaking into the components of degree $\pm q^k$. The smash of maps of degrees m, n is of degree mn, and the same is true of the composition.

Look what one gets from the algebraic geometry: For each V of dim n, one gets a Frobenius map $V^{(q)} \longrightarrow V$ of degree q^n. Thus the only maps we would get would be map $S^{2n} \longrightarrow S^{2n}$ of degree q^n. This suggests examining the top. monoid

$$\coprod_{n \geq 0} \left(\Omega^{2n} S^{2n} \right)_{q^n}$$

the monoid operation coming from the smash product.

October 5, 1972

difference for a top. monoid:

Suppose M is a connected top. monoid. Consider the map

$$\begin{tikzcd} M\times M \arrow[rr, "(pr_1,\mu)"] \arrow[dr, "pr_1"'] & & M\times M \arrow[dl, "pr_1"] \\ & M & \end{tikzcd}$$

where $\mu: M\times M \longrightarrow M$ is the product. The vertical maps are fibrations, and over $1\in M$, the map of fibres is the identity. Conclude á la Dold that (pr_1,μ) is a fibre homotopy equivalence. If (pr_1, g) is an inverse we get

$$g: M\times M \longrightarrow M$$

such that

$$g(m_1, \mu(m_1, m_2)) \sim m_2$$

$$\mu(m_1, g(m_1, m_2)) \sim m_2 \quad .$$

Same argument works if $\pi_0 M$ is a group.

October 6, 1972

Segal's exploding process.

Let C be a small category (topological, eventually). By a homotopy commutative diagram of space indexed by C, one means

$\forall\ i \in C$ a space X_i

$\forall\ i \to j$ a map $X_i \longrightarrow X_j$

$\forall\ i \to j \to k$ a path joining $X_i \longrightarrow X_j \longrightarrow X_k$ and $X_i \longrightarrow X_k$

etc.

The way to describe this is to introduce for each pair (i,j) of $\mathrm{Ob}\,C$ the category whose objects are chains of arrows joining i to j, that is, functors

$$[n] \xrightarrow{\varphi} C \qquad 0 \mapsto i,\ n \mapsto j$$

in which an arrow $(\varphi : [n] \to C) \longrightarrow (\varphi' : [n'] \longrightarrow C)$ is a monotone map $n \xrightarrow{\psi} n'$ preserving endpoints such that $\varphi'\psi = \varphi$. Then we can form a top. category $\tilde{C}$ with $\mathrm{Ob}\,C = \mathrm{Ob}\,\tilde{C}$ but where the space of maps from $i \to j$ in C is the realization of this category of chains. The point is that a homotopy commutative diagram of spaces indexed by C is then identifiable with a functor

$$\tilde{C} \longrightarrow \text{Spaces}$$

Variant: Suppose to simplify that C has a single object, in fact, suppose C is a top. monoid M. Then $\tilde{C}$ is a free top. monoid which is made up of

$$
\begin{tikzcd}
M & \subset & M^2 & \subset & M^3 & \subset & M^4 \\
& M^2 \times I \arrow[ul] \arrow[ur] & & & & &
\end{tikzcd}
$$

Now I want to consider a category with exact sequences:

$$\mathcal{M}^{(3)} \;\begin{smallmatrix}\to\\\to\\\to\\\to\end{smallmatrix}\; \mathcal{M}^{(2)} \;\begin{smallmatrix}\to\\\to\\\to\end{smallmatrix}\; \mathcal{M} \;\rightrightarrows\; pt$$

The thing to try is to form the free monoid generated by $\coprod_{n \geq 0} \mathcal{M}^{(n)}$

and to introduce relations when one goes from a filtered object to its associated graded object.

<u>Precisely</u>: Consider the category $\mathcal{F}$ whose objects are sequences $(M_1, M_2, \ldots, M_p)$ of objects of $\mathcal{M}$, and in which a morphism

$$(M'_1, \ldots, M'_{p'}) \longrightarrow (M_1, \ldots, M_p)$$

consists of a surjective monotone map $\varphi : \{1, \ldots, p'\} \to \{1, \ldots, p\}$, and filtrations on M_j

$$0 \subset F_{i_0} M_j \subset \cdots\cdots \subset F_{i_1} M_j = M_j \qquad \varphi^{-1}\{j\} = \{i_0, i_1\}$$

and isom

$$M'_i \xrightarrow{\sim} F_i M_j / F_{i-1} M_j \qquad j = \varphi(i)$$

In other words a map from $(M_1', \ldots, M_p')$ to $(M_1, \ldots, M_p)$ consists of a filtration of the latter and an isomorphism of the associated graded object with the former. This category $\mathcal{F}$ clearly has a monoid structure. ∎

<u>Conjecture:</u> $Q(\mathcal{M})$ is the classifying space of $\mathcal{F}$.

October 7, 1972 Products in exact sequence K-theory

Notation: Given a full subcategory $\mathcal{M}$ of an abelian cat $\mathcal{A}$ closed under extensions, let $\mathcal{M}_n$ denote the groupoid of n-filtered objects of $\mathcal{M}$ and their isomorphisms. Thus an object of $\mathcal{M}_n$ consists of a functor

$$0 \leq i \leq j \leq n \longmapsto M_{ij}$$

[redacted] $$(i,j) \leq (i',j') \longmapsto (M_{ij} \to M_{i'j'})$$

such [redacted] that [redacted]

$$M_{ii} = 0$$

$$i \leq j \leq k \implies \quad 0 \to M_{ij} \to M_{ik} \to M_{jk} \to 0 \quad \text{exact}$$

Let $\mathcal{M}_{p,q}$ be the groupoid of (p,q)-filtered objects, i.e. functors

$$0 \leq (i_0,i_1) \leq (j_0,j_1) \leq (p,q) \mapsto M_{ij}$$

such that we get exact sequences horizontally + vertically.

$$0 \to M_{(i,a),(j,a)} \to M_{(i,a),(k,a)} \to M_{(j,a)(k,a)} \to 0$$

$$0 \to M_{(a,i)(a,j)} \to M_{(a\,i)(a\,k)} \to M_{(a\,j)(a\,k)} \to 0.$$

Similarly we can define $\mathcal{M}_{p_1 p_2 \cdots p_m}$ the groupoid of $(p_1,\ldots,p_m)$-filtered objects.

The program now is to show that the m-fold s. category $p_1 \cdots p_m \longmapsto \mathcal{M}_{p_1 \ldots p_m}$ is the m-th space in the spectrum associated to $\mathcal{M}$.

<u>Example:</u> Suppose C is a category with a nat. associative unitary operation $\oplus$. Let C_* be the fibred category over Δ with fibre C^n over $[n]$. To show that $\Omega C_* = C$, when $\pi_0 C$ is a group.

Form over C_* the fibred category $\mathcal{E}_*$ over Δ with

$$\mathcal{E}_n = C^{n+1} = C_n \times C$$

and with simplicial operations according to the scheme

$$\begin{tikzcd}
C^3 \arrow[r, shift left=2, "pr_{23}"] \arrow[r, "+\times id"] \arrow[r, shift right=2, "id\times +"'] \arrow[d, "pr_{12}"'] & C^2 \arrow[r, shift left, "pr_2"] \arrow[r, shift right, "+"'] \arrow[d, "pr_1"] & C \arrow[d] \\
C^2 \arrow[r, shift left=2, "pr_2"] \arrow[r, "+"] \arrow[r, shift right=2, "pr_1"'] & C \arrow[r, shift left] \arrow[r, shift right] & pt
\end{tikzcd}$$

Then $\mathcal{E}_*$ is contractible and $\mathcal{E}_* \longrightarrow C_*$ is fibred with fibre C over each object X of C_*. The base change functors are either the identity or left multiplication by an object of C. Since $\pi_0 C$ is a group these are all homotopy equivalences, so we can conclude that C is heq to the h-fibre of $\mathcal{E}_* \longrightarrow C_*$, hence to ΩC_*.

<u>Example:</u> Let M be a simplicial monoid. I can regard it as a fibred category over Δ. I can then form the simplicial category

$$\Rrightarrow M \times_{\Delta} M \Rrightarrow M \rightrightarrows \Delta$$

and when $\pi_0 M$ is a group I can conclude that M is heq to the loop space of the above. Observe that $M \times_{\Delta} M \longrightarrow M \times M$

is a heq by the Eilenberg-Zilber theorem.

<u>October 8, 1972</u> New proof of the group-completion theorem.

<u>The idea:</u> Let M be a top. monoid, BM the classifying space for M, and PM the M-bundle over BM.

Then on the category of spaces X over BM we have a functor

$$X \longmapsto H_*(X \times_{BM} PM)$$

$$(X, Y) \longmapsto H_*(X \times_{BM} PM, Y \times_{BM} PM)$$

satisfying exactness. Since M acts to the right on PM, we have a natural right action of the monoid $\pi_0 M$ on this theory. Supposing that the group-completion of $\pi_0 M$ admits calculation by right fractions (i.e. the category I obtained by letting $\pi_0 M$ act on itself to the right is filtering), we can then localize the above theory:

$$H_*(X \times_{BM} PM)[\pi_0 M^{-1}] = \varinjlim_{I} H_*(X \times_{BM} PM),$$

and still have exactness. ($\mathbb{M}$ a right $\pi_0 M$-module, it gives a functor $I \to \mathcal{Ab}$, $s \mapsto \mathbb{M}$, $(s \xrightarrow{t} st) \mapsto (\mathbb{M} \xrightarrow{t} \mathbb{M})$, whose inductive limit is the localization $\mathbb{M}[\pi_0 M^{-1}]$.)

Set

$$F_i(X, Y) = H_i(X \times_{BM} PM, Y \times_{BM} PM)[\pi_0 M^{-1}]$$

The point will be to prove this theory satisfies the

<u>homotopy axiom.</u> Equivalent conditions:

a) $\forall \quad X \times I \xrightarrow{h} B$ have $F(X, hi_0) \xrightarrow{\sim} F(X \times I, h)$

b) $\forall \quad X \xrightarrow{f} Y \xrightarrow{v} B$ such that f is a heq, we have
$$F(X, vf) \xrightarrow{\sim} F(\cancel{X}, v).$$

Clearly b) $\Rightarrow$ a) so assume a) holds and we are given $X \xrightarrow{f} Y \xrightarrow{v} B$ as in b). Let $g: Y \to X$ be a homotopy-inverse for f, and h = homotopy joining fg to id_Y.

$$\begin{tikzcd}
Y \arrow[r, shift left, "i_0"] \arrow[r, shift right, "i_1"'] \arrow[rr, bend left, "fg"] \arrow[rr, bend right, "id"'] \arrow[rrr, bend left=50, "vfg"] \arrow[rrr, bend right=50, "v"'] & Y \times I \arrow[r, "h"] & Y \arrow[r, "v"] & B
\end{tikzcd}$$

$$\begin{tikzcd}
h(Y, vfg) \arrow[r, "\cong", "(i_0)_*"'] \arrow[rr, bend left, "(fg)_*"] & h(Y \times I, vh) \arrow[r, "h_*"] & h(Y, v) \\
h(Y, v) \arrow[ur, "(i_1)_*"] \arrow[urr, bend right, "id"'] & &
\end{tikzcd}$$

Note $(i_1)_*$ is an isom. (use a) but reflect in I). $\therefore h_*$ and $(fg)_*$ are isom. Thus get

$$\begin{tikzcd}
& h(Y, vfg) \arrow[rr, "\cong"] \arrow[dr, "g_*"] & & h(Y, v) \\
h(X, vfgf) \arrow[ur, "f_*"] \arrow[rr, "\cong"'] & & h(X, vf) \arrow[ur, "f_*"'] &
\end{tikzcd}$$

where bottom isom will come from a similar argument using a homotopy : gf to id_X. Diagram shows $f_* : h(X,V) \to h(Y,V)$ is an isomorphism, proving a).

Assuming the homotopy axiom holds, let E be the space of paths in BM starting at the basepoint. Then ~~as~~ $pt \to E$ is a heq we have

$$H_*(pt \times_{BM} PM)[\pi_0 M^{-1}] \xrightarrow{\sim} H_*(E \times_{BM} PM)[\pi_0 M^{-1}]$$

The former is simply $H_*(M)[\pi_0 M^{-1}]$. On the other hand we have a fibration

$$\Omega BM \longrightarrow E \times_{BM} PM \longrightarrow PM$$

with contractible base, so

~~$H_*(E \times_{BM} PM)$~~

$$H_*(\Omega BM) \xrightarrow{\sim} H_*(E \times_{BM} PM)$$

and moreover $\pi_0 M$ acts invertibly. Thus we get the group-completion theorem

$$H_*(M)[\pi_0 M^{-1}] \xrightarrow{\sim} H_*(\Omega BM).$$

~~Perhaps one should proceed this way. We have cartesian squares~~

$$\begin{array}{ccccc} M & \longrightarrow & E \times_{BM} PM & \longrightarrow & PM \\ \downarrow & & \downarrow & & \downarrow \\ pt & \xrightarrow{heq} & E & \longrightarrow & BM \end{array}$$

~~on which M acts on the upper spaces to the right~~

Perhaps the argument should go as follows: There is a canonical map

$$H_*(M) \longrightarrow H_*(\Omega BM)$$

obtained as the composite

$$H_*(M) \longrightarrow H_*(E \times_{BM} PM) \xleftarrow{\ \cong\ } H_*(\Omega BM)$$

induced by the inclusions of the fibres of $E \times_B PM$ over the basepoints of E and BM respectively. ~~On the other hand $\pi_0 M$ acts invertibly on $H_*(E \times_{BM} PM)$, so we have a natural map~~

$$\cancel{H_*(M)[\pi_0 M^{-1}] \longrightarrow H_*(E \times_{BM} PM)}$$

The last isomorphism comes from the fibration

$$\Omega BM \longrightarrow E \times_{BM} PM \longrightarrow PM$$

with contractible base, which shows also that right multiplying by an element of M on $E \times_{BM} PM$ is a heq. Thus we have canonical maps

$$H_*(M)[\pi_0 M^{-1}] \longrightarrow H_*(EM \times_{BM} PM)[\pi_0 M^{-1}] \xleftarrow{\ \cong\ } H_*(EM \times_{BM} PM)$$
$$\uparrow \cong$$
$$H_*(\Omega BM).$$

The first map is an isomorphism when the homotopy axiom for the localized theory holds.

Now let us work on the homotopy axiom. The idea will be to show that it is a local property over B. So assume that B has a covering $\mathcal{U}$ such that given $X \xrightarrow{f} Y \xrightarrow{v} U \in \mathcal{U}$ with f a heq, then $F(X, vf) \xrightarrow{\sim} F(Y, v)$. Given now any map $X \times I \xrightarrow{h} B$ consider the set S of open subsets V of X such that $\exists$ ~~an open covering $\mathcal{W}$ of I such that~~ a sequence $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $V \times [t_{i-1}, t_i]$ is contained in a member of $h^{-1}\mathcal{U}$ for each i. Standard argument shows that S is a covering of X. Note that

$$V \times [0, t_{i-1}] \underset{V \times t_{i-1}}{\cup} V \times [t_{i-1}, t_i] = V \times [0, t_i]$$

so by exactness, etc. one gets

$$F(V, hi_0) \xrightarrow{\sim} F(V \times I, h)$$

for any V in S. Now by Mayer–Vietoris, etc. one builds up unions. So done with the local character of the homotopy axiom.

So now we use the specific model for BM described by Segal. To give a map $X \longrightarrow BM$ amounts to giving a partition $\sum_{i=0}^{\infty} \rho_i = 1$ of unity and maps $\forall i \leq j$

$$m_{ij} : V_{ij} \longrightarrow M \qquad V_i = \rho_i^{-1}(0, 1]$$

such that $m_{ij} m_{jk} = m_{ik}$ on V_{ijk}

A point of BM is thus a point (t_i) in the infinite simplex together with for each pair $i<j \ni t_i, t_j \neq 0$ an element m_{ij}, subject to transitivity. Description:

$$\left(t_{i_0}^{m_{i_0 i_1}}, t_{i_1}^{m_{i_1 i_2}} \cdots, t_{i_q}\right).$$

~~To form $X \times_{BM} PM$, one glues $V_i \times M$ together according to the maps~~

$$\begin{array}{ccc} V_{ij}\times M & \xrightarrow{(\mathrm{inpr},\ \mu(m_{ij}, id))} & V_j \times M \\ \big\downarrow {\scriptstyle in \times id} & & \\ V_i \times M & & \end{array} \qquad \begin{array}{c} (x, m) \longmapsto (x, m_{ij}(x) m) \\ \big\downarrow \\ (x, m) \end{array}$$

~~In fact PM is made up of points described so:~~

$$t_{i_0}^{m_{i_0 i_0}, m_{i_0 i_1}}, t_{i_1}^{m_{i_1 i_1}, m_{i_1 i_2}}, \ldots\ldots$$

~~so therefore, we find~~

A point of $X \times_{BM} PM$ over x should consist of an element $m_i \in M$ for each $i \ni \rho_i(x) > 0$ such that $m_i = m_{ij}(x) m_j$ for $\forall\, i<j$ such that $\rho_i(x), \rho_j(x) > 0$.

?

October 9, 1972

Use the model for BM such that a map $f: X \longrightarrow BM$ is the same as a partition of unity on X

$$\sum_{i=0}^{\infty} \rho_i = 1 \qquad \begin{array}{l} \rho_i : X \longrightarrow [0,1] \text{ continuous} \\ \text{finite sum} \quad \forall x \end{array}$$

together with maps $m_{ij} : V_i \cap V_j \longrightarrow M \quad \forall i \leq j$ satisfying transitivity, where $V_i = \rho_i^{-1}(0,1]$.

Let

$$P_f = \varinjlim_{\sigma} V_\sigma \times M$$

where σ runs over the finite subsets of $\mathbb{N}$

$$V_\sigma = \bigcap_{i \in \sigma} V_i$$

and if $\sigma \subset \tau$ then

$$V_\tau \times M \longrightarrow V_\sigma \times M$$

$$(x, m) \longmapsto (x, m_{ij}(x)\, m) \qquad \begin{array}{l} i = \text{last vertex } \sigma \\ j = \text{——— } \tau \end{array}$$

Then P_f is a space over X on which M acts on the right.

It seems clear that by assigning to (X, f) the homology $H_*(P_f)[\pi_0 M^{-1}]$ we should have the homotopy axiom. Unfortunately, it is no longer clear that we have contractibility of the universal bundle over BM.

Actually there are problems ~~I~~ don't understand very well.

Question: Let M be a ~~[illegible]~~ simplicial set, and let P be a right M-torsor over a simp. set X. Recall this means that M acts to the right of P over X, and for each point $x \in X_n$, the category formed ~~of~~ by M_n acting on the fibre $P_n(x)$ is ~~a~~ filtering, e.g. if $P_n(x)$ ~~[illegible]~~ $\simeq M_n$ as right M_n-sets. Then for every $x' \to x$ in Δ/X, is it true that

$$H_*(P_{x'}) \longrightarrow H_*(P_x)$$

induces an isomorphism of localizations? (assuming hypotheses of the group completion theorem).

Special case: $X = \Delta(0)$, and where P is a simplicial M-set such that $P_n \simeq M_n$ for each n. Is it true that a map $M \to P$ inducing an isomorphism in degree 0 induces an isom.

$$H_*(M)[\pi_0 M^{-1}] \xrightarrow{\sim} H_*(P)[\pi_0 M^{-1}] \ ?$$

Consider the category $\mathcal{E}$ whose objects are maps of simplicial M-sets over P $\Delta(n) \times M \longrightarrow P$; $\mathcal{E}$ = category of simp. M-sets of the form $\Delta(n) \times M \longrightarrow P$. We then have a functor

$$\mathcal{E} \xrightarrow{Q} \text{simp } M\text{-sets} \qquad Q(\Delta(n)\times M \longrightarrow P) = \Delta(n)\times M$$

and so I can form a bi-simplicial set

$$\cdots \coprod_{e' \to e} Q_{e'} \rightrightarrows \coprod_{e \in \mathrm{Ob}\,\mathcal{E}} Q_e$$

augmented vertically to the nerve of $\mathcal{E}$ and horizontally to P. I claim the horizontal augmentation is h-equivalence; this is standard - $\{\Delta(n)\times M\}$ are projective generators for the category of simplicial M-sets.

Now consider the resulting spectral sequence

$$E^2_{pq} = \varinjlim_{\mathcal{E}}{}_p \left(e \mapsto H_q(Q_e) \right) \Longrightarrow H_q(P)$$

and localize it with respect to $\pi_0 M$. Then it will be the case that $e \mapsto H_*(Q_e)[\pi_0 M^{-1}]$ will be a local system on $\mathcal{E}$. Note that $\mathcal{E}$ is fibred over Δ/X with fibre over $\Delta(n) \xrightarrow{x} X$, the category $(P_n(x), M_n)$ which we have assumed to be contractible. Thus $\mathcal{E}$ is homotopy equivalent to Δ/X, so the local system must descend to Δ/X and we get a spectral sequence

$$E^2_{pq} = H_p\left(X, x \mapsto H_q(M)[\pi_0 M^{-1}]\right) \Longrightarrow H_{p+q}(P)[\pi_0 M^{-1}]$$

which is what we want.

October 10, 1972 : Group-completion thm.

M simplicial monoid $\Rightarrow$ the localization of $H_* M$ w.r.t. $\pi_0 M$ admits calc. by right fractions. $BM =$ diag of $\mathrm{Nerv}(M)$

$$\begin{tikzcd}
\Omega BM \arrow[r] \arrow[d, equal] & Ex_{BM} PM \arrow[r] \arrow[d] & PM \arrow[d] \\
\Omega BM \arrow[r] & E \arrow[r] & BM
\end{tikzcd}$$

$E =$ path space of BM. Assume can construct spec. seq.

$$(*) \qquad E^2_{pq} = H_p(E, \mathcal{L}_q) \Longrightarrow H_{p+q}(Ex_{BM}PM)[(\pi_0 M)^{-1}]$$

where $\mathcal{L}_q$ is a local coeff. system over E with stalks $\cong H_*(M)[\pi_0 M^{-1}]$. If so, then because E contractible

~~$H_*(M)[\pi_0 M^{-1}] \xrightarrow{\sim} H_*(Ex_{BM}PM)[(\pi_0 M)^{-1}]$ E contractible~~

~~$H_*(\Omega BM) \xrightarrow{\sim} H_*(\Omega BM)[(\pi_0 M)^{-1}]$ because $Ex_{BM}PM$ fibres over PM which is contract.~~

~~because $\pi_0 M$ acts invertibly~~

$$H_*(M)[\pi_0 M^{-1}] \xrightarrow{\sim} H_*(Ex_{BM}PM)[\pi_0 M^{-1}]$$

Now ~~[illegible]~~ M acts invertibly on $Ex_{BM}PM$ because it does so on PM, $\Longrightarrow$

$$H_*(Ex_{BM}PM) \xrightarrow{\sim} H_*(Ex_{BM}PM)[\pi_0 M^{-1}].$$

Finally PM contractible $\Rightarrow$ $H_*(\Omega BM) \xrightarrow{\sim} H_*(Ex_{BM}PM)$ so we get group-completion thm.

$$H_*(M)[\pi_0 M^{-1}] \xrightarrow{\sim} H_*(\Omega BM).$$

Construction of spec. seq. ~~[illegible]~~ If P a ~~a~~ M-set, let

Δ_M/P be cat. of s. M-sets over P of form $\Delta(n)\times M \to P$.
Then have standard resolution of P

$$\begin{smallmatrix}\to\\\to\\\to\end{smallmatrix} \coprod_{\substack{\Delta(n')\times M \to \Delta(n)\times M \\ \searrow \quad \swarrow \\ P}} \Delta(n')\times M \rightrightarrows \coprod_{\Delta(n)\times M \to P} \Delta(n)\times M \longrightarrow P$$

Spectral sequence of this bis. set gives

$$E^2_{pq} = H_p\big(\Delta_M/P, (\Delta(n)\times M \to P) \mapsto H_q(M)\big) \Longrightarrow H_{p+q}(P)$$

Localize:

$$E^2_{pq} = H_p\big(\Delta_M/P, \underbrace{(\Delta(n)\times M \to P) \mapsto H_q(M)[\pi_0 M^{-1}]}_{\text{local coeff system on } \Delta_M/P.}\big) \Longrightarrow H_{p+q}(P)[\pi_0 M^{-1}]$$

Now if $X = P/M$, then $\Delta_M/P \longrightarrow \Delta/X$ (passage to $/M$) is fibred and fibre over $x \in X_n$ is cat of $y \in P_n$ over x with maps $y' \to y$ if $y' = my$ $m \in M_n$. Thus when these fibres are contractible, e.g. if M_n acts freely on P_n, $\forall n$, have $\Delta_M/P \to \Delta/X$ is a heq, and so we get the desired spectral sequence

$$E^2_{pq} = H_p(X, \mathcal{L}_q) \Longrightarrow H_{p+q}(P)[\pi_0 M^{-1}] .$$

October 10, 1972:

Definition: A map of simplicial sets $f: X \to Y$ is a quasi-fibration if $\forall\, y' \to y$ in Δ/Y, the map $X_{y'} \to X_y$ is a homotopy equivalence

Proposition: Let $f: X \to Y$ be a quasi-fibration, and let $S \to T$ be a map of simplicial sets over Y which is a heq. Then $X_S \to X_T$ is a heq.

Proof. Let L be any local coefficient system of abelian groups on X_T, and consider the map of spectral sequences

$$\begin{array}{ccc} E^2_{pq} = H_p(S, s \mapsto H_q(X_{S,s}, L)) & \Longrightarrow & H_{p+q}(X_S, L) \\ \downarrow & & \downarrow \\ {}'E^2_{pq} = H_p(T, t \mapsto H_q(X_{T,t}, L)) & \Longrightarrow & H_{p+q}(X_T, L) \end{array}$$

Note quite generally that the system $s \mapsto H_q(X_{S,s}, L)$ is the inverse image of the system $t \mapsto H_q(X_{T,t}, L)$ via the map $S \to T$. But by hypothesis the system homotopy types $t \mapsto X_{T,t}$ is locally constant. Specifically, let $g: T \to Y$ be the structural map. Then for $t' \to t$ in Δ/T

$$\begin{array}{ccc} X_{T,t} & = & X_{g(t)} \\ \uparrow & & \uparrow \text{ heq. by hypothesis} \\ X_{T,t'} & = & X_{g(t')} \end{array}$$

So in the above spectral sequence the local coefficient systems are locally constant, and we therefore have because $S \to T$ is a heq that $E^2 \xrightarrow{\sim} {}'E^2$. Therefore

$$H_*(X_S, L) \xrightarrow{\sim} H_*(X_T, L)$$

for all local coeff systems of abelian groups on X_T.

<u>Continued (Oct. 15)</u>: One still has the problem of π_1. One reduces to showing that in

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d, "qf"] \\ Y' \arrow[r, "heq"'] & Y \end{tikzcd} \qquad \pi_1 Y = \pi_1 Y' = \pi_1 Y = 0$$

that $\pi_1 X'$ is zero. One knows $H_1 X' = H_1 X = 0$, so $\pi_1 X'$ is perfect. ~~[illegible]~~ By lemma below for $x \in X'_0$ we have ~~[illegible]~~ $\pi_1(X'_{y'}) = \pi_1(X_y)$ is abelian, and $\pi_1(X'_{y'}) \twoheadrightarrow \pi_1 X'$ so done.

<u>Lemma</u>: $X \to Y$ quasi-fibr ~~[illegible]~~ $x \in X_0$ with image y in Y_0. Assume $\pi_1 Y = 0 = \pi_0 X$. Then

a) X_y connected

b) $\pi_1 X_y \twoheadrightarrow \pi_1 X$

c) $\mathrm{Ker}\{\ \text{"}\ \} \subset \text{center } \pi_1 X_y$.

Proof of c): Let $G = \pi_1 X_y$, with center Z. Consider those principal G-bundles over the fibres of $X \to Y$ which are 1-connected. Each P over $X_{y'}$ defines $\pi_1 X_{y'} \to G$ unique up to inner autos as Y 1-conn, can restrict to those $P \ni$ this homo. compatible with the given isom. at basepoint. Then one sees that P/Z glues to give a covering of X, whence

$$\begin{tikzcd} \pi_1 X_y \arrow[rr] \arrow[dr] & & \pi_1 X_y / Z \\ & \pi_1 X \arrow[ur, "\exists"'] & \end{tikzcd}$$

<u>October 10, 1972.</u> Group-completion theorem

Simplicial version:

Let M be a simplicial monoid and consider the category of simplicial (right) M-sets. It admits the system of projective generators $\Delta(n)\times M$, $n\geq 0$; let Δ_M denote the full subcat. of simp. M-sets consisting of these. Then we have a functor

$$f: \Delta/P \longrightarrow \Delta_M/P$$

which associates to $\Delta(n)\longrightarrow P$ its extension to a map $\Delta(n)\times M \longrightarrow P$ of M-sets. Form the standard factorization of f

$$\Delta/P \xrightarrow{i} \mathcal{M}_f \xrightarrow{p} \Delta_M/P$$

where $\mathcal{M}_f$ consists of triples $(x:\Delta(m)\to P,\ y:\Delta(n)\times M \longrightarrow P,\ f(x)\to y)$. $\mathcal{M}_f$ may be identified with the category of diagrams

$$\Delta(m) \longrightarrow \Delta(n)\times M \longrightarrow P$$

where the second is a map of simplicial M-sets. Recall that $\mathcal{M}_f$ is fibred over Δ/P and the fibres have initial objects, and $\mathcal{M}_f$ is cofibred over Δ_M/P, the fibre over $\Delta(n)\times M \longrightarrow P$ being $\Delta/\Delta(n)\times M$. The homology spectral sequence of p is of the form

$$E^2_{pq} = H_p\left(\Delta_M/P,\ (\Delta(n)\times M \to P) \mapsto H_q(\Delta(n)\times M)\right)$$

$$\Longrightarrow H_{p+q}(\mathcal{M}_f) \xrightarrow{\sim} H_{p+q}^{(P)}$$

Notice that M_0 acts to the

right on the fibres of p, i.e. right multiplication by m_0 induces a functor $\Delta/\Delta(n)\times M \longrightarrow \Delta/\Delta(n)\times M$ compatible with the similar functor on Δ/P. This induces an action of $\pi_0 M$ on the spectral sequence, whose ~~Suppose now that the localization~~ effect on the E^2-term (abutment) is the map induced by right multiplication by $\pi_0 M$ on $H_q(\Delta(n)\times M)$ (resp. $H_*(P)$).

Suppose now that the localization of $H_*(M)$ with respect to $\pi_0 M$ admits calculation by right fractions. Recall this means that (i) the category I formed by $\pi_0 M$ right acting on itself is filtering (ii) left multiplication by an element of $\pi_0 M$ on

$$H_*(M)[\pi_0 M]^{-1} \overset{def}{=} \varinjlim_{I} \left(s \mapsto H_* M,\ (s \xrightarrow{t} st) \mapsto \text{right mult by } t\right)$$

is invertible. Using (i) we can localize the spectral sequence with respect to $\pi_0 M$ obtaining a spectral sequence

$$E^2_{pq} = H_p\left(\Delta_M/P,\ (\Delta(n)\times M \to P) \longmapsto H_q(M)[\pi_0 M]^{-1}\right)$$
$$\Longrightarrow H_{p+q}(P)[\pi_0 M]^{-1}.$$

Assuming (ii) the functor $(\Delta(n)\times M \longrightarrow P) \longmapsto H_q(M)[\pi_0 M]^{-1}$ is invertible. In effect for the map $\Delta(k)\times M \longrightarrow \Delta(n)\times M$ given by $\varphi : \Delta(k) \longrightarrow \Delta(n)$, $m : \Delta(k) \longrightarrow M$ we have a comm. square

$$\begin{tikzcd}
H_*(\Delta(k)\times M) \arrow[r] & H_*(\Delta(n)\times M) \\
H_*(M) \arrow[u, "\cong"] \arrow[r] & H_*(M) \arrow[u, "\cong"']
\end{tikzcd}$$

where the bottom arrow is left multiplication by the element of $\pi_0 M$ determined by m. By hypothesis these left multiplication maps become isos. after localization. Thus the E^2 term is a homotopy invariant of Δ_M/P.

Finally, let $X = P/M$ and consider the functor of passing to the quotient by M:

$$g : \Delta_M/P \longrightarrow \Delta/X .$$

The fibre of g over $\Delta(n) \longrightarrow X$ sending $i_n \to x$ is the category formed of the elements of P_n over x acted on by M_n. Given $\varphi : \Delta(k) \longrightarrow \Delta(n)$ and $y' \in P_k$ over $\varphi^*(x)$ and $y \in P_n$ over x, then a map

$$\begin{tikzcd} \Delta(k) \times M \arrow[rr, dotted, "u"] \arrow[dr, "y'"'] & & \Delta(n) \times M \arrow[dl, "y"] \\ & P & \end{tikzcd}$$

lying over $\varphi : \varphi^*(x) \longrightarrow x$ is the same thing as $m \in M_k$ such that $y' = \varphi^*(y)\, m$. Thus

$$\mathrm{Hom}_{\Delta_M/P}\,(y', y)_{\varphi^*(x) \xrightarrow{\varphi} x} = \{ m \in M_k \mid \varphi^*(y) m = y' \}$$

$$= \mathrm{Hom}_{g^{-1}(\varphi^*(x))}\,(y', \varphi^*(y))$$

where to be precise we should have said above that a map $y' \longrightarrow y$ in $g^*(x)$ is an element m carrying y to y'. So it follows that g is fibred.

Now suppose that for every $x : \Delta(n) \longrightarrow X$ in Δ/X, the fibre $g^*(x)$ is contractible, for example if M_n acts freely on P_n for each n.

Then g is a homotopy equivalence, because its is fibred and its fibres are contractible. Thus the spectral sequence constructed above takes the form

$$E^2_{pq} = H_p(X, \mathcal{L}_q) \Longrightarrow H_{p+q}(P)[(\pi_0 M)^{-1}]$$

where $\mathcal{L}_q$ is the local coefficient system on X:

$$\mathcal{L}_q = g_!\big(((\Delta(n)\times M \to P) \mapsto H_q(M)[(\pi_0 M)^{-1}]\big)$$

Thus if we pick over $\Delta(n) \xrightarrow{x} X$ a lifting to $\Delta(n) \to P$ then we obtain an isomorphism

$$H_q(M)[\pi_0 M^{-1}] \xrightarrow{\sim} \mathcal{L}_q(x).$$

We have therefore proved:

<u>Proposition:</u> Assume M is a simplicial monoid such that the localization of $H_*(M)$ with respect to $\pi_0 M$ admits calculation by right fractions. Let P be a simplicial right M-set such that for each simplex $\Delta(n) \to X = P/M$ the category formed by M_n acting on the fibre of P_n over x is contractible. Then there is a spectral sequence

$$E^2_{pq} = H_p(X, \mathcal{L}_q) \Longrightarrow H_{p+q}(P)[(\pi_0 M)^{-1}]$$

where $\mathcal{L}_q$ is the local coefficient system defined above, whose stalks are $\simeq H_q(M)[(\pi_0 M)^{-1}]$.

Now apply this to get the group-completion theorem as follows. Assume P, X as in the theorem with P contractible, so that X is ~~[illegible]~~ homotopy equivalent to BM in an essentially canonical way. Let $X' \longrightarrow X$ be a fibration with X' contractible. Then we can apply the proposition to $X' \times_X P$ over X':

$$E^2_{pq} = H_p(X', \mathcal{L}_q) = \begin{cases} 0 & p > 0 \\ H_q(M)[\pi_0 M^{-1}] & p = 0. \end{cases}$$

so the spectral sequence degenerates. But we have the fibration

$$\Omega X \longrightarrow X' \times_X P \longrightarrow P$$

with contractible base, so

$$H_*(X' \times_X P) \xleftarrow{\sim} H_*(\Omega X).$$

Thus

$$H_*(M)[(\pi_0 M)^{-1}] \simeq \underset{\displaystyle H_*(\Omega X)}{\underset{\shortparallel}{H_*(\Omega X)[(\pi_0 M)^{-1}]}}$$

as desired.

October 14, 1972:

Let M be the monoid of C^∞ maps $f: S^1 \to S^1$ which are orientation-preserving submersions, hence étale. Then

$$M = \coprod_{n \geq 0} M_n$$

where M_n consists of the f of degree n. Thus $M_n M_{n'} \subset M_{nn'}$. Observe that we have a "functor"

$$(S^1, M) \longrightarrow \Gamma = \text{pseudo-group of orientation-preserving diffeos. of } \mathbb{R}.$$

Note that if f, g are in M_n, there is an $h \in M_1$

$$\begin{tikzcd} S^1 \arrow[rr, "h"] \arrow[dr, "f"'] & & S^1 \arrow[dl, "g"] \\ & S^1 & \end{tikzcd}$$

such that $gh = f$. If $g(z) = z^n$, then h is unique up to an element of $\mu_n = \{\zeta \mid \zeta^n = 1\}$. Thus

$$\mu_n \backslash M_1 \xrightarrow{\sim} M_n$$

$$\mu_n h \longmapsto (z \mapsto h(z)^n).$$

Improvement: Replace above M by those whose germ at $z = 1$ is the identity. Then M_n will be a right principal homogeneous space over M_1, so given a $\gamma_n \in M_n$ we can define a homomorphism $g \mapsto \underset{n}{\theta}(g)$ from M_1 to M_1 by the formula

$$g\,\gamma_n = \gamma_n \underset{n}{\theta}(g).$$

~~Then it is clear, that provided $\gamma_n \gamma_{n'} = \gamma_{nn'}$,~~

~~thus~~ Provided we choose γ_n so that $\gamma_n \gamma_{n'} = \gamma_{nn'}$,

$$g\,\gamma_n \gamma_{n'} = \gamma_n \underset{n}{\theta}(g)\,\gamma_{n'} = \gamma_n \gamma_{n'}\, \theta_{n'}(\theta_n g)$$

$$\therefore \boxed{\theta_{nn'}(g) = \theta_{n'}\,\theta_n(g)}.$$

Clearly M is the semi-direct product of the monoid $\mathbb{Z}^{\times}_{\geq 1}$ acting on $M_1 \cong$ diffeos of $(0,1)$ compact support.

θ_n is essentially the n-fold sum on BM_1.
Here $M_1 = G =$ diffeos of $\mathbb{R}$ with compact support.

October 15, 1972:

Definition: A map of simplicial sets $X \longrightarrow Y$ is a *quasi-fibration* if $\forall\, y' \to y$ in Δ/Y, the map $X_{y'} \longrightarrow X_y$ is a homotopy equivalence.

Proposition: Given a cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

with f a quasi-fibration. If g is a homotopy equivalences, so is g'.

Corollary 1: Let $X \longrightarrow Y$ be a quasi-fibration, and $U \to V$ a map of simplicial sets over Y. If $U \to V$ is a homotopy equiv., so is the induced map $X_U \to X_V$.

Apply the prop to the square

$$\begin{tikzcd} X_U \arrow[r] \arrow[d] & X_V \arrow[d] \\ U \arrow[r] & V \end{tikzcd}$$

and use the fact that quasi-fibrations are closed under base change.

Corollary 2: Quasi-fibrations are closed under composition.

Let $X \to Y$ and $Y \to Z$ be quasi-fibrations, and let $z' \to z$ be a morphism in Δ/Z. Then we have a cartesian square

$$\begin{tikzcd} X_{z'} \arrow[r] \arrow[d] & X_z \arrow[d] \\ Y_{z'} \arrow[r] & Y_z \end{tikzcd}$$

in which the bottom arrow is a heq and the vertical arrows are quasi-fibrations. Applying the proposition, the top arrow is a heq, so $X \to Z$ is a quasi-fibration.

<u>Corollary 3</u>: Given a triangle

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Z \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

with f and g quasi-fibrations and h a heq. Then for any simplicial set U over Y, the induced map $X_U \to Z_U$ is a heq.

Proof. Factor the arrow $U \to Y$ into $U \to V \to Y$ where $U \to V$ is a heq and $V \to Y$ is a fibration (this is always possible, see GZ). Then we have a square

$$\begin{tikzcd} X_U \arrow[r] \arrow[d] & Z_U \arrow[d] \\ X_V \arrow[r] & Z_V \end{tikzcd}$$

in which the vertical arrows and the bottom arrow are heq's by [redacted] Cor. 1. Thus the top arrow is also a heq.

<u>Remark</u>: Recall that if $f: X \to Y$ is a map of s. sets and y is a vertex of Y (or more generally a simplex of Y), then one defines the <u>homotopy-theoretic fibre</u> of f over y to be the fibre of $g: Z \to X$, where $f = gh$ is a factorization of f with g a fibration and h a homotopy equivalence. The preceding corollary shows that when f is a quasi-fibration, the homotopy-theoretic fibre Z_y is heq to the actual fibre X_y.

<u>Corollary 4</u>: Let $f: X \to Y$ be a quasi-fibration, let x be a vertex of X and $y = f(x)$. Then there is a long exact sequence

$$\pi_{i+1}(Y, y) \xrightarrow{\partial} \pi_i(X_y, x) \longrightarrow \pi_i(X, x) \longrightarrow \pi_i(Y, y) \xrightarrow{\partial} \cdots$$

October 15, 1972:

k finite field, $\bar{k}$ an algebraic closure of k.

Problem: To prove "directly" that $BGL(k)^+$ is the homotopy-fixpoint space of Frobenius on $BGL(\bar{k})^+$.

To begin with, consider the problem of the h-fibre of the map

$$Q(\mathrm{Modf}(k)) \longrightarrow Q(\mathrm{Modf}(\bar{k})).$$

Candidates:

1). The category C_1 whose objects are $\bar{k}$ ~~...~~ modules and in which a map $V' \to V$ is a complemented injection $V' \underset{\pi}{\leftrightarrows} V$ together with a reduction of the complement: $L \otimes \bar{k} \simeq \mathrm{Ker}\,\pi$.

2). The pseudo-simplicial category C_2 whoses fibre over $[n]$ is the groupoid of tuples $(V, L_1, \ldots, L_n)$ with V a $\bar{k}$-module and L_i a k-module. Faces are given by direct sum. Observe that the functor $C_2 \to C_1$ given by

$$(V, L_1, \ldots, L_n) \longmapsto V \oplus L_1 \oplus \cdots \oplus L_n$$

is a homotopy equivalences. Observe also that there is a spectral sequence

$$E^2 = \mathrm{Tor}^{H_*(M)}(H_*(\bar{M}), k) \implies H_*(C_2, k)$$

where

$$M = \coprod BGL_n k$$
$$\bar{M} = \coprod BGL_n \bar{k}.$$

This suggests that $\mathcal{C}_2$ has the correct homology as the h-fibre in question.

3) The pseudo-simp cat $\mathcal{C}_3$ whose fibre over $[n]$ is the groupoid consisting of filtered $\bar{k}$-module

$$V_0 \subset V_1 \subset \cdots \subset V_n$$

together with a reduction of V_n/V_0 to a filtered k-module

4) The category $\mathcal{C}_4$ whose objects are exact sequences of $\bar{k}$-modules

$$0 \longrightarrow V_0 \longrightarrow V_1 \longrightarrow L \otimes \bar{k} \longrightarrow 0.$$

For L fixed these exact sequences form a groupoid which depends contravariantly as L ranges over $Q(\mathrm{Modf}(k))$. Thus $\mathcal{C}_4$ is a pull-back

$$\begin{tikzcd}
\mathcal{C}_4 \arrow[r] \arrow[d] & \square \arrow[d] \\
Q(\mathrm{Modf}(k)) \arrow[r] & Q(\mathrm{Modf}(\bar{k}))
\end{tikzcd}$$

where $\square$ is the contractible category fibred over $Q(\mathrm{Modf}(\bar{k}))$ whose fibre over V is the groupoid of extensions of V.

(Oct. 21)

The moral is that all of these candidates are probably correct; each represents the result of letting k-modules act on $\bar{k}$-modules. The problem however is that we need the map $V \mapsto V - \sigma V$ in order to be able to identify this fibre with $BGL(\bar{k})^+$.

Possibility suggested by spherical fibration theory and algebraic geometry.

Observe by Lang's theorem we have a torsor over the alg. gp GL_n for the group $GL_n(k)$, whence a map

$$GL_n \longrightarrow BGL_n(k)$$

in some sense. This is compatible with Whitney sum, hence leads to a map of "monoids"

$$\coprod_n GL_n \longrightarrow \coprod_n BGL_n(k)$$

and then to a map of classifying "spaces"

$$B\{\coprod_n GL_n\} \longrightarrow B\{\coprod_n BGL_n(k)\}.$$

Now the homology of the former space can be computed via group-completion theorem. One knows that the ring $\bigoplus_n H_*(U_n)$ is commutative because the maps

$A \mapsto A \oplus \varepsilon,\ \varepsilon \oplus A$ from $\mathcal{U}_n$ to $\mathcal{U}_{n+1}$ are homotopic. Thus

$$\Omega B\{\amalg\, \mathcal{U}_n\} = \mathbb{Z} \times \mathcal{U},$$

so presumably (?)

$$B\{\amalg\, \mathcal{U}_n\} \simeq S^1 \times B\mathcal{U}.$$

October 10, 1972

Conjecture: If X is compact and $\mathcal{C}$ is a small category, then

$$\pi_0 \underline{\mathrm{Tors}}(X, \mathcal{C}) \simeq [X, B\mathcal{C}].$$

where

$$\underline{\mathrm{Tors}}(X, \mathcal{C}) = \underline{\mathrm{Homtop}}(\mathrm{Top}(X), \mathcal{C}^{\vee})$$

$$\mathcal{C}^{\vee} = \underline{\mathrm{Hom}}(\mathcal{C}, \mathrm{Sets}).$$

The ~~ca~~ case where $\mathcal{C}$ is an ordered set J is critical:

Make J into a topological space $\tilde{J}$ by calling $U \subset J$ open if $x \leq y,\ x \in U \Longrightarrow y \in U$. One then has a functor

$$\mathrm{Top}(\tilde{J}) \longrightarrow \underline{\mathrm{Hom}}(J, \mathrm{Sets})$$

$$F \longmapsto (j \mapsto F(U_j))$$

where $U_j = \{j' \geq j\}$. (Note $j \leq j' \Rightarrow U_j \supset U_{j'} \Rightarrow F(U_j) \to F(U_{j'})$. This functor is an equivalence of categories, the inverse functor being

$$(j \mapsto F_j) \longmapsto F(U) = \varprojlim_{j \in U} F_j.$$

Let $f: X \to \tilde{J}$ be continuous, and set

$$V_j = f^{-1} U_j$$

Then $j \leq j' \Longrightarrow V_j \supset V_{j'}$ and

$$x \in V_j \Longleftrightarrow f(x) \geq j$$

so that $f(x)$ is the largest j such that $x \in V_j$. Thus we see that

$$\coprod_{j \in J} V_j \longrightarrow X$$

is a J-torsor over X such that for each x, the functor $j \mapsto V_{j,x}$ is representable. One thus may identify "representable" J-torsors over X and maps $X \to \tilde{J}$.

The preceding conjecture then splits into parts:

$$[X, \tilde{J}] \cong [X, BJ] \quad ?$$

$$\pi_0 \underline{\mathrm{Hom}}(X, \tilde{J}) = [X, \tilde{J}] \quad ?$$

$$\pi_0 \underline{\mathrm{Hom}}(X, \tilde{J}) \simeq \pi_0 \mathrm{Homtop}(\mathrm{Top}(X), J^{\vee}) \ ?$$

where $\underline{\mathrm{Hom}}(X, J)$ is the ordered set of maps $X \to \tilde{J}$.

Example: Let K be an abstract simplicial complex and $\mathrm{Simp}(K)$ the ordered set of simplices in K. Then one has the quotient map

$$|K| \xrightarrow{p} \mathrm{Simp}(K)^{\sim}$$

collapsing each open simplex to a point. Observe

$$p^{-1}(U_\sigma) = \text{open star of } \sigma$$
$$= \bigcup_{\tau \geq \sigma} \tau ;$$

denote this U_σ. Then $\sigma \subset \tau \iff U_\sigma \supset U_\tau$. Sheaves on $\mathrm{Simp}(K)^\sim$ may be identified via p^* with sheaves on $|K|$ which are constant on each open simplex.

Now suppose we have a map $f: X \longrightarrow \mathrm{Simp}(K)^\sim$. Then for each vertex v we have $f^{-1}(U_v)$ and

$$f^{-1}(U_\sigma) = \bigcap_{v\in\sigma} f^{-1}(U_v)$$

since

$$U_\sigma = \{\tau \geq \sigma\} = \bigcap_{v\in\sigma}\{\tau \ni v\}.$$

If X is paracompact, we can choose a partition of 1 $\sum \rho_i = 1$, where $\mathrm{Supp}(\rho_i) \subset f^{-1}(U_{\alpha(i)})$ with $\alpha(i)$ a vertex of K. Then define

$$g: X \longrightarrow |K|$$
$$g(x) = \sum \rho_i\, \alpha(i).$$

since $\rho_i(x) \neq 0 \Rightarrow f(x) \in U_{\alpha(i)} \Longrightarrow \alpha(i) \in f(x)$

this is well-defined. Moreover the support of $g(x)$ is contained in $f(x)$ so we have

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "f"'] & {|K|} \arrow[d, "p"] \\ & \mathrm{Simp}(K)^\sim \end{tikzcd}$$

note $pg \leq f$
$\Longrightarrow pg \sim f$

Thus it seems, I can prove that the map

$$[X, |K|] \longrightarrow [X, \mathrm{Simp}(K)^\sim]$$

is surjective. Working relatively, it should not be much harder to prove it's an isomorphism.

Since $B\,\mathrm{Simp}(K) \cong |K|$ by the barycentric sub-division construction, we therefore have established the formula

$$[X, BJ] \simeq [X, \tilde{J}]$$

for ordered sets J of the form $\mathrm{Simp}(K)$. So what has to be done now is to show that

$$[X, (\mathrm{Ch}\,J)^{\sim}] \xrightarrow{\sim} [X, J^{\sim}]$$

where $\mathrm{Ch}\,J$ is the simplicial complex of chains in J.

<u>October 17, 1972.</u> Homology with coefficients in the Steinberg ~~[illegible]~~ representation.

$k = \overline{\mathbb{F}}_p$, l prime $\neq p$. Recall

$$\bigoplus_n H_*(GL_n k) = P[\xi_j]_{j\geq 0} \qquad \deg(\xi_j) = 2j$$

where the homology has coefficients in $\mathbb{F}_l$.

Let X_n be the (model) building of k^n, and $St(k^n) = \tilde{H}_{n-1}(X_n)$ the Steinberg module. I want to determine

$$H_*(GL_n k, St(k^n))$$

Letting $GL_n k = G_n$ act on the ~~[illegible]~~ chains of X_n

$$0 \to St(k^n) \to C_{n-1}X_n \to C_{n-2}X_n \to \cdots \to C_0 X_n \to \mathbb{F}_l \to 0$$

gives a spectral sequence

$$E^1_{st} = H_t(G_n, C_{s-1}) \Longrightarrow H_{s+t-n}(G_n, St(k^n))$$

$$\Big\|$$

$$\bigoplus_{\substack{i_1+\cdots+i_s=n \\ i_\nu>0}} H_t(G_{i_1}\times\cdots\times G_{i_s}).$$

Now the E^1 term is just the bar construction for computing Tor.

$$E^2_{s*} = \mathrm{Tor}_s^{\oplus H_* G_n}(\mathbb{F}_l, \mathbb{F}_l)_n$$

$\uparrow$ refers to grading of $\oplus H_* G_n$

Now this Tor is an exterior algebra on generators in $\hat{\xi}_j \in \mathrm{Tor}_1(\mathbb{F}_e, \mathbb{F}_e)_1$. corresponding to the ξ_j

Thus

$$E^2_{s*} = 0 \quad \text{for} \quad s \neq n.$$

and on the other hand, E^2_{n*} has basis $\hat{\xi}_{j_1} \cdots\cdots \hat{\xi}_{j_n}$ $j_1 < \cdots < j_n$.

October 23, 1972:

Problem: Consider the monoid $\amalg U_n$ under Whitney sum. By group-completion theorem, $\Omega B(\amalg U_n) \simeq \mathbb{Z} \times U$. Show that
$$B(\amalg U_n) \simeq S^1 \times BU.$$

Let I be the category of finite sets and injective maps. Suppose we are given a group G_S for each $S \in I$ and Whitney sum maps
$$G_S \times G_{S'} \longrightarrow G_{S''}$$
for any isomorphism $S \amalg S' \simeq S''$. These should be subject to various compatibility conditions to make the sequel work.

Then the cofibred category $I \backslash G$ inherits a product as follows. If (S,g), $(S',g') \in I\backslash G$ define
$$(S,g) + (S',g') = (S+S', g \oplus g') \qquad S+S' = S \amalg S'.$$
where $g \oplus g'$ denotes the image of (g,g') under the map $G_S \times G_{S'} \longrightarrow G_{S+S'}$. In fact $I \backslash G$ is a permutative category; the commutativity requires the square

$$\begin{tikzcd}
G_S \times G_{S'} \arrow[r] \arrow[d, "\text{interchange}"] & G_{S+S'} \arrow[d, "\text{isom. induces by interchange } S+S' \xrightarrow{\sim} S'+S"] \\
G_{S'} \times G_S \arrow[r] & G_{S'+S}
\end{tikzcd}$$

to commute.

Observe one has not used the group structure on G_S, and so G_S could be a set, or even a category or space. So we can also do the same for the cofibred category E over I with fibre the group G_S over S. We can think therefore of E as the category of pairs (S, x) with $x \in BG_S$.

Now the problem is to show that E is in some sense a delooping of $I \backslash G$. $B(I/G)$ is the simplicial category $[\nu] \mapsto (I/G)^\nu$ while E is heq to $I \backslash BG$, the cofibred category $S \mapsto BG_S$ where BG_S is the s. cat $\nu \mapsto G_S^\nu$.

$$(I \backslash G)^\nu \longrightarrow I \backslash G^\nu$$

$$(S_i, g_i)_{i=1,\dots,\nu} \longmapsto \Big(S_1 + \dots + S_\nu \,;\, \underset{\substack{\| \\ in_1(g_1)}}{g_1 + \varepsilon + \cdots}, \; \underset{in_2(g_2)}{\varepsilon + g_2 + \varepsilon \cdots}, \; \cdots\cdots \Big)$$

The problem is to compare the simplicial cats

$$(I \backslash G)^2 \;\substack{\rightarrow\\\rightarrow\\\rightarrow}\; (I \backslash G) \Longrightarrow pt$$

$$I \backslash G^2 \;\substack{\rightarrow\\\rightarrow\\\rightarrow}\; I \backslash G \Longrightarrow pt$$

where in the former ~~the~~ one uses that $\oplus$ and in the latter the product in G. Now I claim that there is a map from the former to the latter. ~~Indeed~~ To see this consider the case $\nu = 2$. ~~Then~~ Let J be the cat whose objects are inj: $(S_1 + S_2 \hookrightarrow T)$

and whose arrows are commutative squares

$$\begin{array}{ccc} S_1'+S_2' & \hookrightarrow & T' \\ \updownarrow & & \downarrow \\ S_1+S_2 & \hookrightarrow & T . \end{array}$$

Then J maps both to I and to $I\times I$. ~~The~~ Observe that ~~[illegible]~~ the fibre of J over (S_1, S_2) has an initial object, and that J is both fibred and cofibred over S_1, S_2 so $J \to I\times I$ is a heq. But further the cofibred category over J defined by the functor $(S_1+S_2 \hookrightarrow T) \mapsto G_{S_1}\times G_{S_2}$ will be cofibred over $(I\backslash G)^2$ with contractible fibres; Call this category $J_2 G$ so that we have the heq

$$J_2 G \longrightarrow (I\backslash G)^2$$
$$\begin{pmatrix} g_1 & g_2 \\ S_1+S_2 \hookrightarrow T \end{pmatrix} \longmapsto \begin{pmatrix} g_1 & g_2 \\ S_1 , & S_2 \end{pmatrix}$$

Now observe this generalizes to a map of simp cats:

$$J_\nu G \longrightarrow (I\backslash G)^\nu$$

which is a heq for each ν. On the other hand we have a functor

$$J_2 G \longrightarrow I\backslash G^2$$
$$\begin{pmatrix} g_1 & g_2 \\ S_1+S_2 \hookrightarrow T \end{pmatrix} \longmapsto (T\, ;\, in_1(g_1), in_2(g_2))$$

which is compatible with face operations, so we have map of simplicial categories

(*) $\qquad J_\nu G \longrightarrow I\backslash G^\nu$.

Thus we get our map

$$B(I/G) \xleftarrow{\text{heq}} J_\bullet G \longrightarrow I\backslash BG$$

as claimed.

The problem now is to show that the functor (*) above is a heq. This might not be true; thus for $\nu = 2$, why should it be the case that $I\backslash G^2$ is heq to the product of $I\backslash G$ with itself? For example take an object (T, g_1, g_2) where $g_1, g_2 \in G_T$ do not have disjoint support. Clearly by means of injective maps in T one can't separate them. Thus I see no reason now why $I\backslash G^\nu$ should be heq to $(I\backslash G)^\nu$. Nevertheless it might still be so that $I\backslash BG$ heq. $B(I\backslash G)$.

(Observe that this is OKAY if G is topological such as the unitary group. In effect, by arguments of long ago it should be possible to replace I by the category of infinite countable sets and injections. Under these conditions $S \mapsto G_S^\nu$ is "locally constant" with respect to homotopy type. Thus we have a map of fibrations

$$\begin{tikzcd}
G^\nu \arrow[r] \arrow[d] & G^\nu \arrow[d] \\
(G\backslash I)^\nu \arrow[r] \arrow[d] & G^\nu\backslash I \arrow[d] \\
I^\nu \arrow[r, "+"] & I
\end{tikzcd}$$

with contractible bases and a heq on the fibres, so done.)

In preceding, I should write $G\backslash I$, $G^2\backslash I$, etc.

(October 25). Consider $B(\coprod U_n)$ = the simp cat $\nu \mapsto (\coprod U_n)^\nu$. Make $\coprod_n U_n$ act on $\mathbb{N}$ by $m \cdot U_n = m+n$. Then we have a functor

$$(\mathbb{N}, \coprod_n U_n) \longrightarrow U$$

sending the objects $m \mapsto$ unique obj and the arrow $m \xrightarrow{g} m+n$, $g \in U_n$ into $\varepsilon^m \oplus g$ in U.

$$\begin{tikzcd} m \arrow[r, "g"] \arrow[rr, bend right, "g\oplus g'"'] & m+n \arrow[r, "g'"] & m+n+n' \end{tikzcd} \qquad \begin{array}{c} (\varepsilon^m \oplus g)\cdot(\varepsilon^{m+n} \oplus g') \\ \parallel \\ \varepsilon^m \oplus g \oplus g' \end{array}$$

where ε^m denotes the identity in U_m.

(October 26) Let U be the group of doubly-infinite unitary matrices almost equal to the identity. Thus $U = \varinjlim U_{2n}$ where $U_{2n} \longrightarrow U_{2n+2}$ sends g to $\varepsilon \oplus g \oplus \varepsilon$. We can also think of this as those unitary matrices $(a_{ij})_{i,j\in\mathbb{Z}}$ almost equal to the identity. Now let $\mathbb{Z}$ act on U by translation

$$n*(a_{ij}) = a_{i-n,\, j-n}$$

and form the semi-direct product $\mathbb{Z} \tilde{\times} U$ with product
$$(g, n)(g', n') = (g\,(n*g'),\, n+n').$$

Then we have a map

$$\coprod_n U_n \longrightarrow \mathbb{Z} \tilde{\times} U$$
$$g \in U_n \longmapsto (g, n)$$

which is a monoid homomorphism since if $g \in U_n$, $g' \in U_{n'}$ then $g \oplus g' = g(n * g') \in U_{n+n'}$.

<u>Prop.</u> $\coprod_n U_n \longrightarrow \mathbb{Z} \tilde{\times} U$ is a group-completion.

One applies the group-completion theorem to the monoid $\coprod_n U_n$.

<u>Cor.</u> $B(\coprod U_n) = B(\mathbb{Z} \tilde{\times} U) \sim S^1 \times BU$

Comments (Oct 29). I decided that it is unreasonable to expect $B(G\backslash I)$ to be $BG\backslash I$. The problem is that no way exists to relate a product $g_1 g_2 \in G_S$ with $g_1 \oplus g_2$ in G_{S+S}. One can never get the supports of g_1, g_2 disjoint by maps in S.

Then I ~~got~~ got interested in trying to define the boundary map

$$BGL(k) \longrightarrow B(\coprod_n BGL_n(\mathbb{F}_q)).$$

point: G acts ~~on~~ on G/H

$$\left(G^2 \times G/H \substack{\rightarrow\\\rightarrow\\\rightarrow} G \times G/H \rightrightarrows G/H\right) \longrightarrow BH$$

a heq. Thus we have more available than just a map $G_S \longrightarrow BG_S^\sigma$

compatible with $\oplus$.

$x\, g_1 g_2\, x^{-\sigma}$

$$g_1 g_2 \sim x g_1 g_2 x^{-\sigma} = x g_1 x^{-1} \cdot \underbrace{x g_2 x^{-\sigma}}_{\text{can be made } = 1}$$

$$g_1 g_2 g_3 = (g_1 \oplus g_2 \oplus g_3)\gamma \qquad \gamma \text{ commutator}$$

$$x\gamma x^{-\sigma} = 1$$

$$x g_1 g_2 g_3 x^{-\sigma} = \cancel{x g_1 x^{-1} \oplus x g_2 x^{-1} \oplus}$$

$$= x g_1 x^{-1} \cdot x \bar{g}_2 x^{-1} \cdot x \bar{g}_3 x^{-1} \; \underset{1}{\underset{\shortparallel}{x\gamma x^{-\sigma}}}$$

HOPE.

October 24, 1972 Group-completion theorem

M topological monoid (in compactly gen. spaces)
P an M-space over X.

Suppose that the augmentation $\mathrm{Nerv}(P,M) \to X$ induces a homotopy equivalence $|\mathrm{Nerv}(P,M)| \to X$ and that this remains true after base change by any map $Y \to X$.

Example: $X = BM = |\mathrm{Nerv}(M)|$, $P = PM = |\mathrm{Nerv}(M,M)|$ where M acts to the left on itself. Then $|\mathrm{Nerv}(PM,M)|$ is the realization of the bisimplicial space $(p,q) \mapsto M^p \times M \times M^q$. For p fixed it is contractible fibre-wise over M^p. Thus by the lemma of May-Tornehave:

Lemma: $U_\cdot \to V_\cdot$ a map of simplicial spaces such that $U_p \to V_p$ is a heq for all $p \Rightarrow |U| \to |V|$ is a heq.

we see at least that $|\mathrm{Nerv}(PM,M)| \to BM$ is a heq. To conclude the same is true for arbitrary base change we need

Conjectural lemma: Assume in addition that $U_p \to V_p$ is a universal heq for all $p \Rightarrow |U| \to |V|$ is also a universal heq.

(Actually this perhaps is a consequence of Segal's argument proving his lemma.) see p. 3

Let $\mathcal{C}$ be the full subcat. of (right) M-spaces over P of the form $\Delta(n)\times M \longrightarrow P$, $n\geq 0$. Then we can consider the bisimplicial space with two augmentations

$$\begin{tikzcd}
\coprod_{\mathrm{Ar}\,\mathcal{C}} \Delta(n)\times M\times M \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \coprod_{\mathrm{Ob}\,\mathcal{C}} \Delta(n)\times M\times M \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & P\times M \arrow[d, shift left] \arrow[d, shift right] \\
\coprod_{\mathrm{Ar}\,\mathcal{C}} \Delta(n)\times M \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \coprod_{\mathrm{Ob}\,\mathcal{C}} \Delta(n)\times M \arrow[r] \arrow[d] & P \arrow[d] \\
\coprod_{\mathrm{Ar}\,\mathcal{C}} \Delta(n) \arrow[r, shift left] \arrow[r, shift right] & \coprod_{\mathrm{Ob}\,\mathcal{C}} \Delta(n) \arrow[r] & X
\end{tikzcd}$$

Clearly vertically things are ~~acyclic~~ aspherical. If the horizontal row is asph. in degree 0, then things will be also horizontally acyclic, and so we will obtain a heq ~~of the category~~ $B\mathcal{C} \longrightarrow X$. On the other hand from the asph. of the horizontal row in degree 0, we get a spectral sequence

$$E^2_{pq} = H_p\left(\mathcal{C}, (\Delta(n)\times M\to P) \mapsto H_q(\Delta(n)\times M)\right) \Longrightarrow H_{p+q}(P).$$

which can be localized.

Lemma: For any M-space P

$$\coprod_{\mathrm{Ar}\,\mathcal{C}} \Delta(n) \times M \rightrightarrows \coprod_{\mathrm{Ob}\,\mathcal{C}} \Delta(n) \times M \longrightarrow P$$

is aspherical.

Proof: By cone construction

$$\coprod_{\mathrm{Ar}\,\mathcal{C}} \Delta(n) \times \mathrm{Sing}(M) \rightrightarrows \coprod_{\mathrm{Ob}\,\mathcal{C}} \Delta(n) \times \mathrm{Sing}\, M \longrightarrow \mathrm{Sing}\, P$$

is horizontally ~~[illegible]~~ aspherical. Now realize vertically + then horizontally + use that $|\mathrm{Sing}\, M| \to M$ is a heq.

~~On page 1, given $Y \to BM$, then $Y \times_{BM} |\mathrm{New}(PM, M)| = |\mathrm{New}(Y \times_{BM} PM, M)|$~~

$$\begin{tikzcd} Y \arrow[d] & Y \times_{BM} PM \arrow[l] \arrow[d] & (Y \times_{BM} PM) \times M \arrow[l, shift left] \arrow[l, shift right] \arrow[d] \\ BM & PM \arrow[l] & PM \times M \arrow[l, shift left] \arrow[l, shift right] \end{tikzcd}$$

If $Y \to BM$ is a fibn (e.g. $Y =$ path space), then Segal's fibration lemma shows that ~~$|\mathrm{New}(Y \times_{BM} PM, M)|$ fibres over $|\mathrm{New}(PM, M)|$ with same fibre~~ we have a map of fibns.

$$\begin{tikzcd} F \arrow[d] \arrow[r, equal] & F \arrow[d] \\ Y \arrow[d] & {|\mathrm{New}(PM \times_{BM} Y, M)|} \arrow[l] \arrow[d] \\ BM & {|\mathrm{New}(PM, M)|} \arrow[l] \end{tikzcd}$$

so total spaces are heq as desired.

Group-completion theorem – simplicial case

1. M simplicial monoid
   P a (right) simplicial M-set.
   $\mathcal{C}$ = full subcat of s. M-sets over P consisting of $\Delta(n)\times M \longrightarrow P$, $n \geq 0$.

Then have standard resolution

$$\cdots \coprod_{\mathrm{Ar}\,\mathcal{C}} \Delta(n)\times M \rightrightarrows \coprod_{\mathrm{Ob}\,\mathcal{C}} \Delta(n)\times M \longrightarrow P$$

this is a simp object in simp. M-sets; regarding it as a bisimp set it is horizontally aspherical. Thus the spectral sequence of this bisimplicial set takes the form

$$E^2_{pq} = H_p\left(N(\mathcal{C}), (\Delta(n)\times M\to P)\mapsto H_q(\Delta(n)\times M)\right) \Longrightarrow H_{p+q}(P)$$

2. Assuming now that $\pi_0 M$ acting on itself on the right is a filtering category, we may localize obtaining a spectral seq.

$$E^2_{pq} = H_p\left(N\mathcal{C}, (\Delta(n)\times M\to P)\mapsto H_q(M)[\pi_0 M^{-1}]\right) \Longrightarrow H_{p+q}(P)[\pi_0 M^{-1}].$$

($\exists$ obvious right action of $\pi_0 M$ on the above spectral sequence).
If moreover the left action of $\pi_0(M)$ on $H_*(M)[\pi_0 M^{-1}]$ is invertible, then the functor $(\Delta(n)\times M\to P)\mapsto H_q(M)[\pi_0 M^{-1}]$ from $\mathcal{C}$ to $\mathcal{Ab}$ is morphism-inverting.

3. Suppose now that P is a s. set over X, and that M acts fibrewise so that we have an augmented simplicial object

$$P\times M^2 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; P\times M \rightrightarrows P \longrightarrow X$$

in $\Delta^\wedge$. Now form the bisimp gadget in $\Delta^\wedge$:

Then $P \sim \Omega BM$, (ΩBM fibre of $X \to BM$, P fibres over PM which is contractible). Next note that $\forall q$, P_q is a free M_q set ~~Nerv(P, M)~~ with $P_q/M_q \xrightarrow{\sim} X_q$; hence $|\mathrm{Nerv}(P,M)| \to X$ heq because it is so horizontally. Thus X cont.

$$E^2_{pq} = H_p(X, \mathcal{L}_q) \Longrightarrow H_{p+q}(P)[\pi_0 M^{-1}]$$

$\Longrightarrow$ degenerates yielding the isom.

$$H_*(M)[\pi_0 M^{-1}] \simeq H_*(\Omega BM)[\pi_0 M^{-1}]$$

(necessary to go into the details of the action.)

$$\begin{tikzcd}
\coprod_{\mathrm{ar}\,\mathcal{C}} \Delta(n)\times M\times M \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \coprod_{\mathrm{ob}\,\mathcal{C}} \Delta(n)\times M\times M \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & P\times M \arrow[d, shift left] \arrow[d, shift right] \\
\coprod_{\mathrm{ar}\,\mathcal{C}} \Delta(n)\times M \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \coprod_{\mathrm{ob}\,\mathcal{C}} \Delta(n)\times M \arrow[r] \arrow[d, "\Delta(n)"] & P \arrow[d, dotted] \\
\coprod_{\mathrm{ar}\,\mathcal{C}} \Delta(n) \arrow[r, shift left] \arrow[r, shift right] & \coprod_{\mathrm{ob}\,\mathcal{C}} \Delta(n) \arrow[r, dotted] & X
\end{tikzcd}$$

Vertically it is the nerve of M right acting on the standard resolution. Horizontally aspherically by the above; Vertically aspherical because $\mathrm{Nerv}(M;\text{right } M)$ is contractible. Thus we get a heq of $N\mathcal{C}$ and the total ~~gadget~~ gadget belonging to $\mathrm{Nerv}(P,M)$. Thus

<u>Prop</u>: Suppose that $\Delta\mathrm{Nerv}(P,M) \longrightarrow X$ is a heq. Then there is a spectral sequence

$$E^2_{pq} = H_p(X, \mathcal{L}_q) \Longrightarrow H_{p+q}(P)[\pi_0 M^{-1}]$$

where $\mathcal{L}_q$ is a loc. coeff. system on X with stalks $\cong H_q(M)[\pi_0 M^{-1}]$. (In fact ~~[illegible]~~ $\mathcal{L}_q$ obtained by descending an explicit const. local system on P; ~~[illegible]~~ UGLY POINT.)

4. <u>Group-completion</u>: Apply preceding to

$$\begin{tikzcd}
P \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart.}"] & PM \arrow[d] \\
X \arrow[r, "\text{fibn}"] & BM
\end{tikzcd}$$

X contractible

October 26, 1972

Let k be an algebraically closed field of characteristic p. I consider polynomial maps $f: V \longrightarrow W$ between fin. dimb. vector spaces over k which are additive, i.e.

$$f(v_1+v_2) = f(v_1)+f(v_2)$$

The set of polynomial maps is

$$\mathrm{Hom}_{k\,\mathrm{algs}}\left(SW^*, S(V^*)\right) = W \otimes S(V^*)$$

and the additive ones are clearly

$$\bigoplus_{i\geq 0} W \otimes (V^*)^{(p^i)} \subset \bigoplus_{i\geq 0} W \otimes S_{p^i}(V^*).$$

Put another way, any such map f can be uniquely decomposed

$$f = \sum f_i \qquad \text{finite sum}$$

where $f_i : V \longrightarrow W$ satisfies $f_i(\lambda x) = \lambda^{p^i} f(x)$, i.e. f_i is homogeneous of degree p^i.

Suppose now that $f : V \longrightarrow W$ is bijective. Then $f^*: S(W^*) \longrightarrow S(V^*)$ is necessarily an F-isomorphism; ~~In effect f^* is clearly injective; also normalization style arguments will show that~~ this should be a consequence of ZMT. Thus for n sufficiently large

$$\begin{tikzcd} & S(V^*)^{(q)} \arrow[dl, dotted] \arrow[d, phantom, "\cap" description] \\ S(W^*) \arrow[r, hook] & S(V^*) \end{tikzcd} \qquad (q) = p^n$$

so we obtain $g: W \longrightarrow V^{(q)}$ such that

$$V \xrightarrow{f} W \xrightarrow{g} V^{(q)}$$

is the canonical map $v \longmapsto v^{(q)}$. Applying the same argument to g we have an $h: V^{(q)} \longrightarrow W^{(q')}$ such that

$$W \xrightarrow{g} V^{(q)} \xrightarrow{h} W^{(q')}$$

is the canonical map $w \longmapsto w^{(q')}$. Clearly then h must be $f^{(q)}$. Thus f is bijective, iff we have $g: W \to V^{(q)}$ such that $gf: V \to V^{(q)}$, $fg: W \to W^{(q)}$ are the canonical maps

To simplify suppose

$$V = V_1 \oplus V_2$$

$$W = W_1 \oplus W_2$$

and that $f = f_0 + f_1$ where $f_0: V_1 \xrightarrow{\sim} W_1$, $f_1: V_2^{(q)} \xrightarrow{\sim} W_2$. In addition suppose we are given an isomorphism $\Theta: V \xrightarrow{\sim} W$. If I choose an inverse $g: W \to V$ i.e. such that gf and fg are homogeneous of degree q. then $gf: V \to V$ defines an $\mathbb{F}_q$-reduction of V and $fg: W \longrightarrow W$ defines an $\mathbb{F}_q$-reduction of W. Then we have two $\mathbb{F}_q$-reductions of V

$$gf \qquad \Theta^{-1} fg \Theta$$

which we know are conjugate via an auto φ of V:

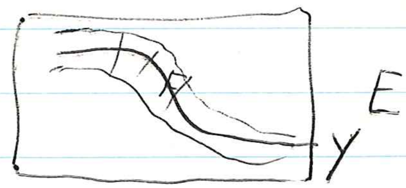
$$gf = (\Theta\varphi)^{-1} fg \Theta\varphi$$

?

October 27, 1972

Suppose E and F ~~are~~ vector bundle over a manifold X and that $f: E \to F$ is ~~a fibre hop of the sphere bundles,~~ a proper map which is fibrewise. ~~is invertible in category of these maps.~~ Define Y as the inverse image of a section s

$$\begin{array}{ccc} Y & \longrightarrow & E \\ \downarrow & & \downarrow f \\ X & \xrightarrow{s} & F \end{array}$$



of F transversal to f. Then $Y \to X$ is proper with stable normal bundle $F-E$. Conversely if we give $g: Y \to X$ proper with $\nu_g = F-E$ of dim. 0, then at least when Y can be embedded in E, we get such a map $f: E \to X$. In particular, when $E=F$ Y is a framed manifold over X

Suppose now that $E=F$ and that $f: E \to E$ is ~~an Adams~~ Frobenius style map, say for example in each fibre E_x there is a basis such that

$$f(\textstyle\sum \lambda_i e_i) = \sum \lambda_i^q f(e_i)$$

where $f(e_i)$ is also a basis. Better: Assume $f(\lambda e) = \lambda^q f(e)$ where $q \geq 2$. Then f is proper-homotopic to $f-\mathrm{id}$. Thus if $f-\mathrm{id}$ is transversal to the zero section, the framed manifold Y over X (well-defined cobordism class) will be the set of fixpoints of f.

<u>Program:</u>

Suppose X is a variety over an alg. closed field k of characteristic p, V is a vector bundle over X, and $f: V \to V$ is a polynomial map over X which is $\mathbb{F}_q$-linear (in particular additive) and radicial surjective. Then I would like if possible to associate to (V,f) a map of X_{et} to $BGL(\mathbb{F}_q)^+$. For example if $f(\lambda x) = \lambda^q f(x)$, then the fixpoints of f form a $GL_n(\mathbb{F}_q)$-bundle over X.

To begin, one ~~should~~ might try to understand the composition

$$X \longrightarrow BGL(\mathbb{F}_q)^+ \longrightarrow G$$

where the second map is induced by $GL_n \mathbb{F}_q \to \Sigma_{q^n}$, forgetting the $\mathbb{F}_q$ structure.

For example suppose we work with spaces. A map $X \to G$ may be interpreted as a framed proper map $Y \to X$, or better as a natural transf of all gen. coh. theories on spaces over X. Suppose given a gen. coh. theory h and $f: V \to V$ proper over X with V a vector bundle over X. If V is trivial we have

$$h(X) \xrightarrow{\sim} h_c(V) \xrightarrow{f^*} h_c(V) \xleftarrow{\sim} h(X)$$

$$x \longmapsto xU \longmapsto x f^*(U) \longmapsto x \frac{f^*U}{U}$$

where U is a Thom class for V. Observe this map is independent of the choice of U. In the general case one adds a bundle to V to make it trivial, or else uses $h_c(V, \alpha)$, α some kind of coefficients.

<u>Question:</u> Given V vector space over k with a $\mathbb{F}_q$ linear radicial surjective endo. $f: V \to V$, what is the degree of f?

More generally, suppose $f: V \to W$ is a proper map between vector spaces of the same dimension. Then f is finite, so $S(W^*) \hookrightarrow S(V^*)$ and $S(V^*)$ is a finitely ~~free~~ generated projective module over $S(W^*)$. (Should be known: suppose $A \to B$ is a local homo. of reg. local rings of the same dimension, which is quasi-finite, i.e. $B/m_A B$ is fin. diml. over A/m_A. Then generators for m_A form a ~~regular seq~~ system of parameters in B, hence a regular sequence, showing that $\mathrm{Tor}_1^A(A/m_A, B) = 0$. Now use ~~quasi~~ local criterion of flatness to conclude B flat over A.). Thus the degree of f will be the rank of $S(V^*)$ as an $S(W^*)$-module.

Suppose now that I have a family $f_t: V \to W$ parameterized by T: ~~then have as proper map~~

$$\begin{tikzcd} V \times T \arrow[rr, "f"] \arrow[dr] & & W \times T \arrow[dl] \\ & T & \end{tikzcd}$$

Assuming f proper one gets an induced map on coh. with supports proper over T. Say we compactify V to $\bar{V}$ and W to $\bar{W}$ so that f extends to $\bar{f}$; then we are talking about the map

$$\bar{f}^*: H^*((\bar{W}, \partial\bar{W}) \times T) \longrightarrow H^*((\bar{V}, \partial\bar{V}) \times T)$$

If T should be the affine line, we then conclude that
$$f_{t_1}^* = f_{t_2}^* \; : \; H_c^*(V) \longrightarrow H_c^*(W)$$
because $H^*((\overline{W}, \partial\overline{W}) \times T) = H^*(\overline{W}, \partial\overline{W})$.

Now suppose we take $f : V \longrightarrow V$ such that $f(\lambda x) = \lambda^q f(x)$. Then because f is homogeneous of degree $q > 1$, we know that $f_t = f + t \cdot id$ is a proper family. In fact using Lang's theorem we ~~can~~ reduce to seeing that
$$k[t, x] \longleftarrow k[t, y]$$
$$y = tx + x^q \qquad\qquad y$$
is finite, which is clear. (In general if $V \longrightarrow W$ is such that the leading terms of a basis of W^* form a regular sequence in $S(V^*)$, then the lower terms can be altered at will.)

So f will be properly homotopic to $f - id$ for which 0 is a regular value. Thus if V is now over X, and $Y \longrightarrow X$ is the covering of fix points of f, then we have a cart. square
$$\begin{array}{ccccc} Y & \xrightarrow{j} & V & & \\ g\downarrow & & \downarrow f-id & & \\ X & \xrightarrow{0} & V & \xrightarrow{\pi} & X \end{array}$$
and so
$$h(X) \xrightarrow{0_*} h_c(V) \xrightarrow{(f-id)^*} h_c(V) \xrightarrow{\pi_*} h(X)$$
$$\pi_* 0_* (f-id)^* = \pi_* j_* g^* = g_* g^* = g_* 1 \cdot (?)$$

showing the map $h(X) \longrightarrow h(X)$ is just multiplying by the covering g_*1.

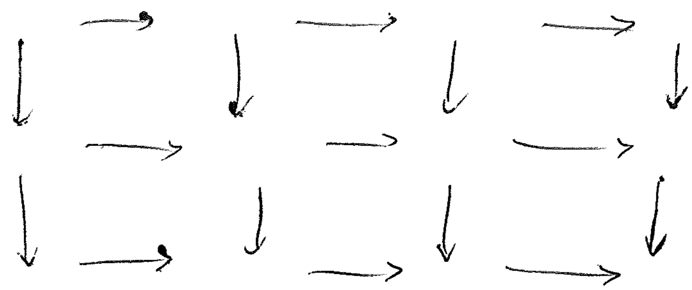
October 28, 1972. Cobordism and K-theory

Example: Let G be a finite group and consider equivariant complex cobordism for G-manifolds, possibly with supports. I want to localize it with respect to the Euler classes of the non-trivial irreducible representation and then restrict ~~it~~ to the category consisting of these representations not containing the trivial representation, and equivariant linear maps. Call the functor h. I recall that for the purposes of this theory a G-manifold is essentially the same as its fixpoint set + the normal bundle, which is a direct sum of vector bundles one for each non-trivial irreducible representation of G. So we are in effect considering the localized theory as the category of G-manifolds with fixpoint set a point.

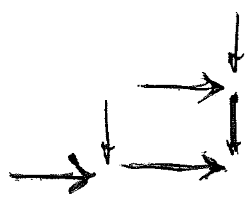
So for any map $f: V \to W$ we have $f^*: h(W) \to h(V)$, and for any proper (i.e. injective) f we have $f_*: h(V) \to h(W)$ such that the functorial and transversal conditions hold. On the other hand, because we have inverted Euler classes we can define f_* for arbitrary maps.

This example suggests that it should be possible to extend any h-fibration over $Q(\mathcal{M})$ to a larger category which would have inverse images f^* and Gysin maps f_* defined for all arrows subject only to the requirements of functorality and transversality.

Conjecture: Define a bisimplicial set with $X_{pq} = \mathrm{Funct}([p]\times[q], \mathcal{M})$ such that each 1-1 square is bicartesian:



Thus the Artin-Mazur total simplicial set has 2-simplices



which is exactly the way we want to compose things.

The conjecture is that this bisimplicial set has the homotopy type of $Q(M)$.

October 28, 1972

Basic geometric facts:

1) $X \to Y$ a map of simp. spaces $\ni$ $X_k \to Y_k$ a heq.
$\Rightarrow$ $|X| \to |Y|$ an heq.

2) (Segal) $X \to Y$ $\ni$
$$\begin{tikzcd} X_k \arrow[r] \arrow[d] & X_l \arrow[d] \\ Y_k \arrow[r] & Y_l \end{tikzcd}$$
homotopy-cartesian for all $[k] \leftarrow [l]$ $\Rightarrow$
$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & {|X|} \arrow[d] \\ Y_0 \arrow[r] & {|Y|} \end{tikzcd}$$
homotopy-cartesian.

Clearly 2) $\Rightarrow$ 1). 2) also implies your result on quasi-fibrations (~~[illegible]~~ If $X \to Y$ a q-f. then one looks at the bisimplicial

$$\begin{tikzcd} \cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \coprod_{y_0 \to y_1} X_{y_0} \arrow[d] \arrow[r, shift left] \arrow[r, shift right] & \coprod_{y} X_y \arrow[d] \\ \cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \coprod_{y_0 \to y_1} pt \arrow[r, shift left] \arrow[r, shift right] & \coprod_{y} pt \end{tikzcd}$$
$y \in \Delta / Y$

I tried without success to deduce the heq's are preserves by base change wrt q-fibns. using 1). Thus given

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d] & X \arrow[d, "\text{q-fibn.}"] \\ Y' \arrow[r, "g", "\text{heq}"'] & Y \end{tikzcd}$$

one can prove that g' is a heq if g is a strict heq, and also if g is a q-fibn with contractible fibres. Is any heq factorizable into these two things?

October 29, 1972: Lang problem:

Recall that this consists in showing that the square

$$\begin{tikzcd}
\coprod_n BGL_n(\mathbb{F}_q) \arrow[r] \arrow[d, "\phi"] & \coprod_n BGL_n(k) \arrow[d, "\Delta"] \\
\coprod_n BGL_n(k) \arrow[r, "\Gamma"'] & \coprod_n BGL_n(k)^2
\end{tikzcd}$$

remains h-cartesian after group-completion. The h-fibres of ϕ and Δ can be determined using the following principle:

Let $M' \longrightarrow M$ be a homomorphism of top. monoids, and let $B(M/M')$ be the classifying space of the top. cat obtaining by letting M' right act on M, whence the composite

$$(*) \qquad B(M/M') \longrightarrow BM' \longrightarrow BM$$

is null-homotopic. Observe M left acts on $B(M/M')$. If M acts invertibly on $B(M/M')$, then $(*)$ is a homotopy-fibration. In effect

$$BM' \sim B(B(M\backslash M)/M') = B(M \backslash B(M/M'))$$

and the latter fibres over BM with fibre $B(M/M')$.

Applying this to $M' = \coprod_n BGL_n(\mathbb{F}_q)$, $M = \coprod_n BGL_n(k)$, we have that the fibre of ϕ is the classifying space of the double category formed by letting the groupoid of $\mathbb{F}_q$ vector spaces act on the groupoid of k vector spaces. But the action maps $GL_n(k) \times GL_m(\mathbb{F}_q) \longrightarrow GL_{n+m}(k)$ are injective, so the double category is equivalent to ~~the~~ the ~~single~~ category $\mathcal{F}_\phi$ in which the objects are k-vector spaces and in which a map $V \longrightarrow V'$ is a complemented injection together with an $\mathbb{F}_q$-reduction of the complement.

Similarly the fibre of Δ is ~~equiv~~ equivalent to the category $\mathcal{F}_\Delta$ of pairs (V_1, V_2) in which a map is a complemented injection together with a reduction of the complement to the diagonal. The map from $\mathcal{F}_\phi$ to $\mathcal{F}_\Delta$ sends V to $(V, \sigma V)$.

~~Now I want to describe the categories~~

Now Segal said that $Q(B\mathbb{F}_q^\bullet)$ is the homotopy fixpoints of σ on $Q(Bk^\bullet)$; the proof could be done by cohomology means. This suggests that the square of monoids

$$\begin{tikzcd}
\coprod_n B(\Sigma_n \wr \mathbb{F}_q^\bullet) \arrow[r] \arrow[d, "\phi"] & \coprod_n B(\Sigma_n \wr k^\bullet) \arrow[d, "\Delta"] \\
\coprod_n B(\Sigma_n \wr k^\bullet) \arrow[r, "\Gamma"] & \coprod_n B(\Sigma_n \wr (k^\bullet \times k^\bullet))
\end{tikzcd}$$

NO

should remain h-cartesian after group completion.
The fibre for ϕ can be represented by the category $\mathcal{F}_\phi$ consisting of k-vector spaces V with axes in which a map is a complemented injection with $\mathbb{F}_q$

$\mathcal{F}_\phi$ equivalent to the category whose objects are $\mathbb{F}_q$ vector spaces L and in which a map from L' to L consists of an isomorphism

$$(*) \qquad L'\otimes k \oplus L''\otimes k \xrightarrow{\sim} L\otimes k$$

$\mathcal{F}_\Delta$ equivalent to the cat. with the same objects but in which a map from L' to L consists of an isom.

$$(**) \qquad L'\otimes k \oplus V'' \xrightarrow{\sim} L\otimes k$$

together with an automorphism of $L\otimes k$.

The functor from $\mathcal{F}_\phi$ to $\mathcal{F}_\Delta$ may be interpreted as assigning to $(*)$ the isom $(**)$ together with the auto of $L\otimes k$ which measures the difference between the two Frobenius maps on both sides. If we consider all maps $(*)$ which give rise to the same isomorphism $(**)$, the autos. of $L\otimes k$ we obtain are
NO → precisely those which preserve the decomposition and act as the identity on $L'\otimes k$, i.e. $\mathrm{Aut}(V'')$, whereas we would like to have $\mathrm{Aut}(L\otimes k)$.

But observe that the subcategory of $\mathcal{F}_\Delta$ we are getting has nothing to do with σ. Thus let $\mathcal{C}$ be the category whose objects are $\mathbb{F}_q$-vector spaces L and in which a map from L' to L is a complemented injection

$$L'\otimes k \oplus V'' \xrightarrow{\sim} L\otimes k$$

together with an automorphism of V''.

The above ~~isn't~~ correct unless the Frobenius on $L \otimes k$ induces that on $L' \otimes k$ via the isom. (**). ~~Thus~~ Nevertheless, it is very suggestive. Thus you should try to see ~~[illegible]~~ whether the subcategory of $\mathcal{F}_\Delta$ you have defined is homotopy equivalent to $\mathcal{F}_\Delta$.

October 28, 1972

Basic geometric facts:

1) $X \to Y$ a map of simp. spaces $\ni$ $X_k \to Y_k$ a heq.
$\Rightarrow$ $|X| \to |Y|$ an heq.

2) (Segal) $X \to Y$ $\ni$
$$\begin{tikzcd} X_k \arrow[r] \arrow[d] & X_l \arrow[d] \\ Y_k \arrow[r] & Y_l \end{tikzcd}$$
homotopy-cartesian

for all $[k] \leftarrow [l]$ $\Rightarrow$
$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & {|X|} \arrow[d] \\ Y_0 \arrow[r] & {|Y|} \end{tikzcd}$$
homotopy-cartesian.

Clearly 2) $\Rightarrow$ 1). 2) also implies your result on quasi-fibrations (~~[illegible]~~ If $X \to Y$ a q-f. then one looks at the bisimplicial •

$$\begin{array}{ccccc} \Rrightarrow & \coprod_{y_0 \to y_1} X_{y_0} & \rightrightarrows & \coprod_y X_y & \\ & \downarrow & & \downarrow & \\ \Rrightarrow & \coprod_{y_0 \to y_1} pt & \rightrightarrows & \coprod_{y_0} pt & \end{array}$$

$y \in \Delta/Y$

I tried without success to deduce the heq's are preserves by base change wrt q-fibns. using 1). Thus given

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d] & X \arrow[d, "\text{q-fibn.}"] \\ Y' \arrow[r, "g", "\text{heq}"'] & Y \end{tikzcd}$$

one can prove that g' is a heq if g is a strict heq, and also if g is a q-fibn with contractible fibres. Is any heq factorizable into these two things?