October 6, 1972: Products in exact sequence K-theory.

$\mathcal{M}$ = an additive cat with exact sequences. Then I recall that $Q(\mathcal{M})$ is homotopy equivalent to the fibred cat. over Δ, $[n] \mapsto \mathcal{M}_n$, $\mathcal{M}_n$ = groupoid of n-filtered objects of $\mathcal{M}$. It would be better to write $\mathcal{M}_n^{is}$ the "is" standing for the sub-groupoid of the additive category $\mathcal{M}_n$ of n-filtered objects. Now $\mathcal{M}_n$ is also an additive category with a notion of exact sequences, whence we have defined $(\mathcal{M}_n)_q$ = additive category of q-filtered objects in $\mathcal{M}_n$

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products:

$n \longmapsto \mathcal{M}_n^{is}$ fibred over Δ.

---

relation between filtered and bifiltered obj.

idea $\mathcal{M}_{p,*}$ $Q(\mathcal{M}_p)$

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back to cobordism theory. universal functors.

Suppose I can make a theory operate over a base k. Then

Example: $\mathcal{C}$ monoid category

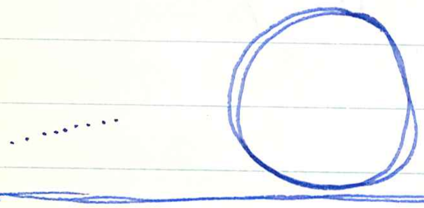
$B\mathcal{C}$ s. category.

why is $\mathcal{C} = \Omega B\mathcal{C}$ if $\mathcal{C}$ connected or better invertible.

Segal model for BM — real. of nerve of $M \simeq N$.

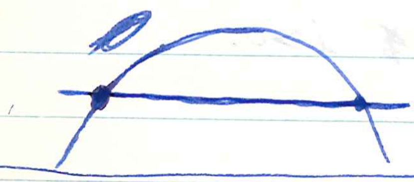
points: $\sum_{i\geq 0} t_i = 1$ & m_{ij} all $i \leq j$ & $t_i t_j > 0$.

topology: vector field on the circle.



fibre $\longrightarrow F\Gamma_1 \longrightarrow K(\mathbb{R},3)$.

contractible?



gh moves a small amount.

An observation: M = monoid of maps $S^1 \to S^1$ which are submersions

$M_1 = \mathrm{Diff}(S^1)$

Then can form

$$\begin{tikzcd}
PM_1 \times^{M_1} S^1 \arrow[r] & PM \times^{M} S^1 \arrow[r, "(?)"] & F\Gamma_1 \\
BM_1 \times S^1 \arrow[u, "s"] \arrow[r, "\mathrm{id} \times (z \mapsto z^p)"] & BM_1 \times S^1 \arrow[u] &
\end{tikzcd}$$

$$BG \times S^1 \xrightarrow{\mathrm{id} \times (z \mapsto z^p)} BG \times S^1$$

The relation: $BG \times S^1$

~~[scribbled]~~ Is W a F-subspace of V and is E/W a lattice in $E\otimes F/W$?

$\mathcal{O}$ discrete valuation ring.

$$\bigcap_n \pi^n E = W$$

Then
$$\bigcup_{n\geq 0} \pi^{-n} E = E \otimes_{\mathcal{O}} F$$
is a subspace. But what about the intersection

Is W divisible. $\pi W \supset W$.

But $x \in \bigcap_n \pi^n E \Rightarrow$ ~~[scribbled]~~ $\pi^{-n} x \in E$ all n

$\Longrightarrow \pi^{-n}\pi^{-1}x \in E$ all $n \Rightarrow \pi^{-1}$

Thus W is a subspace

now suppose
$$0 = \bigcap \pi^n E$$
$$V = \bigcup \pi^{-n} E$$

Is E a lattice? So let L be a lattice

to show $\pi^{-n} L \supset E$ for some n.

$$L \supset \pi^n E$$

[illegible]

BGL$(A)^+$ interpretation has following virtues

(i) relation with cohomology of GL(A):

~~$H_i(BGL(A)^+, A) = H_i(GL(A), A)$ any $K_1(A) = GL(A)/E(A)$-module~~

shows thing coincides with Bass + Milnor

$$K_3 A = H_3(St(A), \mathbb{Z}).$$

$$K_n A \otimes \mathbb{Q} \xrightarrow{\sim} \mathrm{Prim}\, H_n(GL(A), \mathbb{Q})$$

(ii) Computability: ~~leads to following~~

(ii) relation the theory of permutative categories and generalized cohomology theories.

Define: Permutative category P

Segal and Anderson construct a spectrum

$$B(P) = \{B_n P\}_n \qquad \Omega B_{n+1} P = B_n P$$

Group-completion thm. of Barratt-Priddy:

$$K_0 A \times BGL(A)^+ \cong B_0(P_A) \qquad P_A = \text{proj f.g. } A\text{-modules + isos.}$$

such that the filtrations

$$\begin{tikzcd}
 & M_0 & & M_p & \\
P_{0q} \arrow[r, phantom, "\subset"] & P_{1q} \arrow[u] \arrow[r, phantom, "\subset \cdots \subset"] & & P_{pq} \arrow[u] \arrow[r] & P_\varepsilon \arrow[ddd, dash, dashed] \\
 & & & P_{p,q-1} \arrow[u, phantom, "\cup"] & \\
 & & & \vdots \arrow[u, phantom, "\cup"] & \\
 & & & P_{p0} \arrow[u, phantom, "\cup"] &
\end{tikzcd}$$

are transverse. (Recall diagram

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & A/A\cap B \arrow[r] \arrow[u] & X/B \arrow[r] \arrow[u] & X/A+B \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & A \arrow[r, phantom, "\subset"] \arrow[u] & X \arrow[r] \arrow[u] & X/A \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & A\cap B \arrow[r] \arrow[u] & B \arrow[r] \arrow[u, phantom, "\cup"] & B/A\cap B \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

$\exists$ equivalence between

$$A+B=X$$

$$X \xrightarrow{\sim} X/A \times X/B$$

$$B \twoheadrightarrow X/A$$

$$A \twoheadrightarrow X/B.$$

in which case we say that A, B ~~[illegible]~~ are transverse in X.) The other requirement is that the quotients for the horizontal filtration are in $\mathcal{M}(r-1)$ and for the vertical filtration are in $\mathcal{M}(r)$.

The problem is now to show that ~~...~~ ~~...~~ the functor
$$\mathcal{E} \longrightarrow Q(\mathcal{M})^2$$
is homotopy equivalent to
$$\Delta : Q(\mathcal{M}) \longrightarrow Q(\mathcal{M})^2 .$$

Claim that
$$\mathcal{E} \longrightarrow Q(\mathcal{M})^2 \xrightarrow{pr_1} Q(\mathcal{M})$$
$$(E \to M \times N) \longmapsto M$$
is a heq. In effect, it is fibred, so we only have to show the fibre is contractible. Denote the fibre by $\mathcal{E}(M,*)$. We have a functor
$$\mathcal{E}(M,*) \longrightarrow \mathcal{J}_M$$
$$(E \twoheadrightarrow M \times N) \longmapsto (E \twoheadrightarrow M)$$
where $\mathcal{J}_M$ is the category of admissible surj. $E \twoheadrightarrow M$ ~~and~~ with morphisms: ~~...~~
$$\begin{tikzcd} E' \arrow[r, hook] \arrow[dr] & E \arrow[d] \\ & M \end{tikzcd}$$
$\mathcal{J}_M$ is contractible by the cone construction:

$$\begin{tikzcd} E \arrow[r, hook] \arrow[dr] & E\oplus E' \arrow[d, two heads] & E' \arrow[l, hook'] \arrow[dl] \\ & M & \end{tikzcd}$$

~~so~~ it suffices to show the functor is cofibred with contractible fibres. ~~[illegible]~~ ~~[illegible]~~ The fibre over $(E \twoheadrightarrow M)$ is the ordered set of quotients N of E which are transversal to M, and this has a least element 0. As for cofibredness, observe that for any map $(E' \to M\times N')$ to $(E \to M\times N)$ can be uniquely factored ~~=~~

$$\begin{tikzcd} E' \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & E \arrow[d] \\ M\times N_1 \arrow[r, hook] \arrow[d] & M\times N \\ M\times N' & \end{tikzcd}$$

$$\begin{tikzcd} E' \arrow[r, hook] \arrow[dd] & E \arrow[dd] \arrow[dr] & \\ & & M\times N \arrow[dl] \\ M\times N' \arrow[r, hook] & M\times(N' \amalg^{E'} E) & \end{tikzcd}$$

~~[illegible]~~ and so it's relatively clear.

Similarly

$$\mathcal{E} \longrightarrow Q(m)^2 \xrightarrow{pr_2} Q(m)$$

is a heq. Thus what I am trying to do is to show that the resulting self-heq of $Q(m)$ is the identity.

Application to the localization problem. Suppose A, K, as usual (d.v.r.) and denote by $\mathcal{T}$ = Tors f.g. A-mods
$\mathcal{P}$ f.g. projective A-modules, $\mathcal{A}$ all fin. gen. A-modules
$\mathcal{A}/\mathcal{T}$ = finit. gen. K-modules.

Now I want to consider the groupoid $\mathcal{G}(Q,M)$ consisting of surjections $P \twoheadrightarrow Q\times M$ where $M\in\mathcal{T}$ and $P, Q \in \mathcal{P}$ and their isos. over $Q\times M$.
Observe that

$$\begin{tikzcd} P' \arrow[r, phantom, "\subset"] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & P \arrow[d] \\ Q_1\times M_1 \arrow[r, phantom, "\subset"] \arrow[d] & Q\times M \\ Q'\times M' & \end{tikzcd}$$

$$P/P' \simeq Q/Q_1 \times M/M_1$$
$$\Longrightarrow P' \in \mathcal{G}(Q',M')$$

so we can form a fibred category $\mathcal{G}$ over $Q(\mathcal{P})\times Q(\mathcal{T})$. Fix $M\in\mathcal{T}$. Then as before $\mathcal{G}_M$ will be cofibred with contractible fibres over the cat of $P\twoheadrightarrow M$ with maps

$$\begin{tikzcd} P \arrow[r, hook] \arrow[dr] & P' \arrow[d] \\ & M \end{tikzcd} \qquad P'/P \in \mathcal{P}$$

which is contractible as before. ~~If $Q\in\mathcal{P}$, then $\mathcal{G}_P$ will be cofibred with contractible fibres~~ Careful: The fibre over $P\to M$ is the set of quotients Q of P which are in $\mathcal{P}$ and transversal to M; again 0 is least.

$$\begin{tikzcd} P' \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & P \arrow[d] \\ Q_1\times M \arrow[r, hook] \arrow[d] & Q\times M \arrow[d] \\ Q'\times M \arrow[r, hook] & Q/Q_0\times M \end{tikzcd} \qquad \begin{tikzcd} P' \arrow[r, hook] \arrow[d] & P \arrow[d] \\ Q' \arrow[r, hook] & Q/Q_0 \end{tikzcd}$$

OKAY.

Fix $Q \in \mathcal{P}$. And map $\mathcal{G}_Q$ to

$$\begin{tikzcd} P \arrow[r, hook] \arrow[dr, two heads] & P' \arrow[d, two heads] \\ & Q \end{tikzcd} \qquad P'/P \in \mathcal{T}$$

Fibre over $P \twoheadrightarrow Q$ is clearly the set of $\mathcal{T}$-quotients M of P transversal to Q. Cofibred:

$$\begin{tikzcd} P' \arrow[r, hook] \arrow[d] & P \arrow[d] \\ Q \times M_1 \arrow[r, hook] \arrow[d] & Q \times M \arrow[d] \\ Q \times M' \arrow[r, hook] & Q \times M/M_0 \end{tikzcd} \qquad\qquad \begin{tikzcd} P' \arrow[r, hook] \arrow[d] & P \arrow[d, two heads] \\ M' \arrow[r, hook] & \bar{M}. \end{tikzcd}$$

Thus it appears that

$$\begin{aligned} \mathcal{G} &\longrightarrow Q(\mathcal{T}) \qquad \text{is a heq} \\ (P \twoheadrightarrow Q \times M) &\longmapsto M \end{aligned}$$

and

$$\mathcal{G} \longrightarrow Q(\mathcal{P})$$

is fibred, the fibre over Q being the groupoid of exact sequences over A

$$0 \longrightarrow V \longrightarrow E \longrightarrow Q \longrightarrow 0$$

where V is a K-vector space.

So now assume we know that

$$* \qquad \begin{tikzcd} \mathcal{G} \arrow[r, "\text{cart}"] \arrow[d] & (\text{fibred cat of extensions}) \arrow[d] \\ Q(\mathcal{P}) \arrow[r] & Q(\mathcal{A}/\mathcal{T}) \end{tikzcd}$$

is homotopy-cartesian and one sees that we have another proof of the localization theorem.

Except that we do not know that

$$\begin{tikzcd} \mathcal{G} \arrow[r, "heq"] \arrow[d] & Q(\mathcal{T}) \arrow[d, phantom, "\cap"] \\ Q(\mathcal{P}) \arrow[r] & Q(\mathcal{A}) \end{tikzcd}$$

is commutative, so we cannot as yet identify the ~~map~~ heq. of $Q(\mathcal{T})$ with the homotopy fibre of $Q(\mathcal{P}) \to Q(\mathcal{A}/\mathcal{G})$ with the transfer. By naturality, it would be enough to solve our initial problem

$$\begin{tikzcd} \mathcal{E}(\mathcal{A}) \arrow[r] \arrow[d] & Q(a) \arrow[d, equal] \\ Q(a) \arrow[r, equal] & Q(a) \end{tikzcd}$$

Idea: Let me reserve $\mathcal{E}$ for the ~~category of short exact sequences in $\mathcal{M}$~~ fibred cat over $Q(\mathcal{M})$ with fibre

$$\mathcal{E}(M) = \text{groupoid of } \quad 0 \to P \to E \to M \to 0.$$

and $\mathcal{G}$ for the fibred cat ~~of~~ $Q(\mathcal{M})^2$ with fibre

$$\mathcal{G}(M, N) = \mathcal{E}(M \times N)$$

so that we have a ~~....~~ cartesian square:

$$(**) \qquad \begin{tikzcd} \mathcal{G} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ Q(\mathcal{M})^2 \arrow[r, "+"] & Q(\mathcal{M}). \end{tikzcd}$$

Assuming (**) is homotopy-cartesian we see therefore that we have an H-space situation

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & EM \arrow[d] \\ M\times M \arrow[r, "+"] & M \end{tikzcd}$$

so $Z \sim M$ but embedded via $x \longmapsto (x,-x)$.

Conclude: The problem on page 1 is OKAY provided we modify it so as to say that $\mathcal{G} \to Q(M)^2$ is the difference map.

It can be proved simply by observing that the composites

$$\begin{tikzcd} \mathcal{G} \arrow[r, "f"] & Q(M)^2 \arrow[r, shift left=2, "pr_1"] \arrow[r, "+" description] \arrow[r, shift right=2, "pr_2"'] & Q(M) \end{tikzcd}$$

are

$$\begin{matrix} pr_1 f \simeq \\ pr_2 f \simeq \end{matrix} \quad \text{and} \quad +f = 0 \;.$$

Dec 1, 1972

first proof of ~~[illegible]~~ resolution problem

~~March 28, 1972~~ to go with Dec. 1, 1972

~~To fix the ideas suppose~~ Let $\mathcal{M}$ be the category ~~of~~ whose objects are projective f.t. modules over a ~~[illegible]~~ given ring R and whose morphisms are isomorphisms. For each $p \geq 0$, let $S_p\mathcal{M}$ be the category whose objects are ~~[illegible]~~ R-modules endowed with a filtration of length p:

$$0 \subset M_1 \subset \cdots \subset M_p$$

such that M_j/M_i is ~~[illegible]~~ in $\mathcal{M}$ for all $0 \leq i \leq j \leq p$. Morphisms in $S_p\mathcal{M}$ are isomorphisms.

Then $[p] \longmapsto S_p\mathcal{M}$ is a pseudo-functor from Δ° to Cat, so we can form a cofibred category $S\mathcal{M}$ over Δ° whose fibre over $[p]$ is $S_p\mathcal{M}$.

Problems:

1. Find a good description of ~~[illegible]~~ the loop space of $S\mathcal{M}$.

2. Suppose R is regular and noetherian. Let $\mathcal{M}$ be the category of f.t. R-modules and their isomorphisms, let $\mathcal{M}(n)$ denote those of projective dimension $\leq n$. Then we have inclusion functors

$$\mathcal{M}(0) \longrightarrow \mathcal{M}(1) \longrightarrow \mathcal{M}(2) \longrightarrow \cdots\cdots \longrightarrow \mathcal{M}$$

Show these induces homotopy equivalences

$$S\mathcal{M}(0) \longrightarrow S\mathcal{M}(1) \longrightarrow \cdots\cdots \longrightarrow S\mathcal{M}.$$

Idea for problem 2. Fix r and denote objects of $\mathcal{M}(r)$ by M, M', etc. and objects of $\mathcal{M}(r-1)$ by P, P' etc. Given M_0, P_0 I consider the category $\mathcal{C}(M_0, P_0)$ consisting of surjective maps

$$P \twoheadrightarrow M_0 \times P_0$$

and isomorphisms over $M_0 \times P_0$. I note that given

$$M_1 \hookrightarrow M_0 \qquad \text{with cokernel in } \mathcal{M}(r)$$

we have a functor

$$\mathcal{C}(M_0, P_0) \longrightarrow \mathcal{C}(M_1, P_0)$$

$$\begin{array}{ccc} P & & M_1 \times_{M_0} P \\ \downarrow & \longmapsto & \downarrow \\ M_0 \times P_0 & & M_1 \times P_0 \end{array}$$

This is well-defined because $P' = M_1 \times_{M_0} P$ fits into

$$\begin{tikzcd} 0 \arrow[r] & P' \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "\wr"] & 0 \\ 0 \arrow[r] & M_1 \arrow[r] & M_0 \arrow[r] & M_0/M_1 \arrow[r] & 0 \end{tikzcd}$$

hence $P \in \mathcal{M}(r-1)$, $C \in \mathcal{M}(r) \implies P' \in \mathcal{M}(r-1)$.
Moreover, given

$$M_0 \twoheadrightarrow M_1 \qquad \left(\begin{array}{c}\text{kernel necessarily} \\ \text{in } \mathcal{M}(r)\end{array}\right)$$

we have

$$\mathcal{C}(M_0, P_0) \longrightarrow \mathcal{C}(M_1, P_0)$$

$$\begin{array}{c} P \\ \downarrow \\ M_0 \times P_0 \end{array} \longmapsto \begin{array}{c} P \\ \downarrow \\ M_0 \times P_0 \\ \downarrow \\ M_1 \times P_0 \end{array}$$

Similarly, given

$$P_1 \hookrightarrow P_0 \qquad \text{with} \qquad P_0/P_1 \ \text{ in } \ \mathcal{M}(r-1)$$

$$\mathcal{C}(M_0, P_0) \longrightarrow \mathcal{C}(M_0, P_1)$$

$$\begin{array}{c} P \\ \downarrow \\ M_0 \times P_0 \end{array} \longmapsto \begin{array}{c} P \times_{P_0} P_1 \\ \downarrow \\ M_0 \times P_1 \end{array}$$

and we have

$$\begin{tikzcd} 0 \arrow[r] & P \times_{P_0} P_1 \arrow[r] \arrow[d] & P \arrow[r] & C \arrow[r] \arrow[d, phantom, "\wr"] & 0 \\ 0 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & P_0/P_1 \arrow[r] & 0 \end{tikzcd}$$

so it is well-defined. Given

$$P_0 \twoheadrightarrow P_1$$

have functor

$$\mathcal{C}(M_0, P_0) \longrightarrow \mathcal{C}(M_0, P_1)$$

in which P doesn't change.

Now I construct a cofibred category over $\Delta^\circ \times \Delta^\circ$ whose fibre over $[p],[q]$ is what might be denoted

$$\mathcal{S}_p \mathcal{M}(r) \times_{\mathcal{M}(r)} \mathcal{C} \times_{\mathcal{M}(r-1)} \mathcal{S}_q \mathcal{M}(r-1)$$

and it consists of diagrams

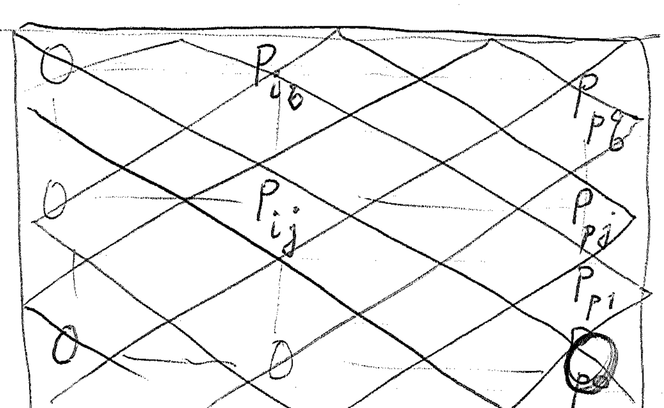
$$\begin{tikzcd}[column sep=small] & & & & P \arrow[dl] \arrow[dr] & & & & \\ 0 \subset M_1 \subset M_2 \subset \cdots & \subset & & M_p & & P_q \supset \cdots & \supset & P_1 \supset 0 \end{tikzcd}$$

up to isomorphism. Perhaps the best way to describe such a thing is as a bifiltered object

$$P_{ij} = M_i \times_{M_p} P \times_{P_q} P_j$$

$$= (M_i \times P_j) \times_{(M_p \times P_q)} P$$

$$0 \leq i \leq p \,, \quad 0 \leq j \leq q$$



December 17, 1972 Exact categories:

When I do K-theory, I work with a full subcat^m $\mathcal{M}$ of an abelian cat $\mathcal{A}$ which is closed under extensions and contains 0. But the cat $Q(\mathcal{M})$ depends only upon $\mathcal{M}$ and the exact sequences in $\mathcal{M}$, the abelian cat. $\mathcal{A}$ really serving only to define the notion of exactness. So what I want to develop is an intrinsic notion of additive category with exact sequences, (exact category for short).

The idea will be to start with $\mathcal{M}$ (which I will assume to be small), put $\mathcal{A} = \mathrm{Homadd}(\mathcal{M}^\circ, \mathcal{Ab})$, let $\mathcal{L} \subset \mathcal{A}$ be the full subcat of left exact functors. I want to find conditions on the exact sequences in $\mathcal{M}$ which will imply $\mathcal{L}$ is abelian and that $h : \mathcal{M} \longrightarrow \mathcal{L}$ embeds $\mathcal{M}$ as a full subcategory closed under extensions.

First we must know that $h(\mathcal{M}) \subset \mathcal{L}$ i.e. that

(1) $$0 \to M' \to M \to M'' \to 0 \quad \text{exact in } \mathcal{M}$$

$$\Longrightarrow \quad 0 \to \mathrm{Hom}(M'', N) \longrightarrow \mathrm{Hom}(M, N) \longrightarrow \mathrm{Hom}(M', N)$$

is exact in $\mathcal{Ab}$. Thus we must know that M'' is the cokernel of $M' \longrightarrow M$ for any exact sequence

The next point will be to show that the inclusion $\mathcal{L} \subset \mathcal{A}$ has a left adjoint R°. I am following the model for a small abelian category, where I know (Gabriel) what's happening.

Introduce $\mathcal{B} \subset \mathcal{A}$ the subcategory of effaceable functors, i.e. those F such that $\forall\, \xi \in F(M)$ $\exists$ an ~~[illegible]~~ admissible epim. $M' \xrightarrow{u} M$ (i.e. one occurring in an exact sequence) such that $F(u)(\xi) = 0$. I want $\mathcal{B}$ to be a Serre subcategory of $\mathcal{A}$ which requires the following. Given

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$$

with F', F'' effaceable, and $\xi \in F(M)$ I can kill ~~[illegible]~~ the image of ξ in $F''(M)$ by an adm. epi $M' \to M$; and then I can kill the result. elt of $F'(M')$ by an adm. epi $M'' \to M'$. So I need

(2) Composition of admissible epis. is an admissible ~~[illegible]~~ epim.

Now I will eventually want to identify $\mathcal{L}$ with the quotient category $\mathcal{A}/\mathcal{B}$. Given F in $\mathcal{A}$ let

$$F_0(M) = \{ \xi \in F(M) \mid u^*(\xi) = 0 \text{ for some } u : M' \twoheadrightarrow M \}$$

I want F_0 to be a functor of M. Thus given a map $N \to M$ and an adm. epi $M' \twoheadrightarrow M$, I will want the pull-back

$$\begin{tikzcd} N \times_M M' \arrow[r] \arrow[d] & M' \arrow[d] \\ N \arrow[r] & M \end{tikzcd}$$

to exist and be an admissible epim.

(3) Admissible epis are stable under base-change.

It follows from (2) & (3) that they are closed under fibre products, so that $F_0(M)$ is a subgroup of $F(M)$.

Clearly F_0 is effaceable, and the largest effaceable subfunctor of F. I will call F separated if $F_0 = 0$, or equivalently if $M' \twoheadrightarrow M \Rightarrow F(M) \hookrightarrow F(M')$. Clearly F/F_0 is separated.

Now let F be separated, choose an injective functor I and an injection $F \hookrightarrow I$, and let F_1 be the maximal subfunctor of $I \ni F \subset F_1$ and $F_1/F \in \mathcal{B}$, i.e. $F_1/F \cong (I/F)_0$. Let $\bar{F} = F_1/(F_1)_0$; since F separated, $F \hookrightarrow \bar{F}$. I claim $\bar{F}$ is left exact: suppose given an exact sequence in $\mathcal{M}$.

$$0 \to M' \to M \to M'' \to 0$$

Because $\bar{F}$ is separated, $\bar{F}(M'') \hookrightarrow \bar{F}(M)$. What we must show is that given u

$$\begin{tikzcd} h_{M'} \arrow[r] \arrow[dr, "0"'] & h_M \arrow[d, "u"] \\ & \bar{F} \end{tikzcd}$$

Suppose F separated and let I be an injective hull of F in $\mathcal{a}$. Then $I_0 \cap F = F_0 = 0$ so $I_0 = 0$, and I is separated. I want to show I is an exact functor. Suppose

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is exact in $\mathcal{M}$. Then need

(1') $\quad 0 \to M' \to M \to M'' \to 0$ exact in $\mathcal{M}$

$$\Rightarrow \quad 0 \to \mathrm{Hom}(N, M') \to \mathrm{Hom}(N, M) \to \mathrm{Hom}(N, M'') \text{ exact.}$$

Granted this we have an exact sequence in $\mathcal{a}$

$$0 \longrightarrow h_{M'} \longrightarrow h_M \longrightarrow h_{M''} \longrightarrow h_{M''}/\mathrm{Im}\, h_M \longrightarrow 0$$

where $h_{M''}/\mathrm{Im}\, h_M$ is effaceable by (3). Thus we get an exact sequence

$$0 \longleftarrow I(M') \longleftarrow I(M) \longleftarrow I(M'') \longleftarrow \underset{\substack{\shortparallel \\ 0}}{I(h_M/\mathrm{Im}\, h_M)} \longleftarrow 0$$

because I is separated.

So now define $F' \subset I$ by $F'/F = (I/F)_0$, so that I/F' is separated, and so I/F' can be embedded in an injective exact functor I'. Thus we have an exact sequence

$$0 \longrightarrow F' \longrightarrow I \longrightarrow I'$$

which yields that F' is left exact.

Thus I have shown that for any functor F $\exists$ map $F \longrightarrow F'$ where F' is left exact which is a B-isomorphism. Now suppose L is left exact and ~~we~~ embed L in a separated injective J. Then from

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
0 & \to & L(M'') & \to & J(M') & \to & J/L(M'') & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & L(M) & \to & J(M) & \to & J/L(M) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & L(M') & \to & J(M'') & \to & J/L(M'') & \to & 0 \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

one sees that J/L is separated, whence it can be embedded in a separated injective J'. ~~Suppose given $u: F \longrightarrow L$~~

$$\begin{array}{ccccccccc}
F & \twoheadrightarrow & F/F_0 & \hookrightarrow & I & \twoheadrightarrow & I/F_1 & \hookrightarrow & I' \\
 & {\scriptstyle u}\searrow & \downarrow & & \downarrow{\scriptstyle \exists} & & \downarrow & \swarrow{\scriptstyle \exists} & \\
 & 0 \to & L & \longrightarrow & J & \longrightarrow & J' & &
\end{array}$$

~~whence we get a map $F' \longrightarrow L$~~

$$\begin{array}{ccc}
F & \longrightarrow & F' \\
 & {\scriptstyle u}\searrow & \swarrow \\
 & L &
\end{array}$$

~~commutes. Better we have~~

~~[illegible]~~

We have

$$\begin{array}{ccccccc}
0 \to \mathrm{Hom}(F,L) & \longrightarrow & \mathrm{Hom}(F,I) & \longrightarrow & \mathrm{Hom}(F,I') \\
\uparrow & & \uparrow \wr & & \uparrow \wr \\
0 \to \mathrm{Hom}(F',L) & \longrightarrow & \mathrm{Hom}(F',I) & \longrightarrow & \mathrm{Hom}(F',I')
\end{array}$$

which shows that the inclusion of $\mathcal{L}$ in $\mathcal{A}$ has a left adjoint. We denote this $F \mapsto R^0F$.

Now the facts that $\mathcal{L}$ is abelian and R^0 is exact should be formal. The point is the explicit description of maps in $\mathcal{A}/\mathcal{B}$. Thus I have seen that given F the category of objects $F \to G$ under F which are $\mathcal{B}$-isom. to F has a final object $F \to R^0F$. So an $\mathcal{A}/\mathcal{B}$-map

$$\begin{array}{ccc}
F_1 & & F_2 \\
& \searrow & \downarrow \scriptstyle{\mathcal{B}\text{-iso}} \\
& & G
\end{array}$$

will be just a map $F_1 \longrightarrow R^0F_2$. Thus we can identify

$$\mathcal{L} \underset{\sim}{\xleftarrow{R^0}} \mathcal{A}/\mathcal{B}$$

and a sequence of left exact functors is exact in $\mathcal{L}$ iff its homology is in $\mathcal{B}$.

Finally I want to see what I need to conclude that h embeds $\mathcal{M}$ in $\mathcal{L}$, etc. (1) guarantees $h(\mathcal{M}) \subset \mathcal{L}$, (1') that h is left exact, and (3) that h is exact.

Suppose $0 \to M' \to M \to M'' \to 0$ is a sequence in $\mathcal{M}$ such that $0 \to h_{M'} \to h_M \to h_{M''} \to 0$ is exact in $\mathcal{L}$. Then M' is the kernel of $M \to M''$. Since $h_M \to h_{M''}$ is onto in $\mathcal{L}$, it follows that $\exists$ $N \twoheadrightarrow M''$ admiss. epi such that the induced sequence

$$0 \longrightarrow M' \longrightarrow M \times_{M''} N \longrightarrow N \longrightarrow 0$$

splits. Thus we want

(4) If $0 \to M' \to M \to M'' \to 0$ is a sequence which is left exact and such that $\exists$ adm. epi $N \to M''$ and

$$\begin{tikzcd} & M \arrow[d] \\ N \arrow[ur, dashed] \arrow[r, two heads] & M'' \end{tikzcd}$$

then the sequence is exact.

With this axiom I know now that a sequence is exact in $\mathcal{M}$ iff it is exact in $\mathcal{L}$.

Next I want $\mathcal{M}$ to be closed under extensions in $\mathcal{L}$. So suppose I have an exact sequence in $\mathcal{L}$

$$0 \longrightarrow h_{M'} \longrightarrow F \longrightarrow h_{M''} \longrightarrow 0$$

Then this is left exact in $\mathcal{A}$ and the cokernel of $F \to h_{M''}$ is effaceable, so we can find $N \twoheadrightarrow M''$ admiss epi $\ni$

$$\begin{tikzcd} & F \arrow[d] \\ h_N \arrow[ur] \arrow[r] & h_{M''} \end{tikzcd}$$

commutes.

~~Thus $F \times_{h_{M''}}$~~

$$\begin{tikzcd}
0 \arrow[r] & h_{M'} \arrow[r] & F \arrow[r] & h_{M''} \arrow[r] & 0 \\
0 \arrow[r] & h_{M'} \arrow[r] \arrow[u, equal] & F \times_{h_{M''}} h_N \arrow[r] \arrow[u] \arrow[d, phantom, "\cong" rotate=90] & h_N \arrow[r] \arrow[u] & 0 \\
 & & h_{M'} \oplus h_N = h_{M' \oplus N} & &
\end{tikzcd}$$

Thus

$$F \times_{h_{M''}} h_N \simeq h_{M'} \times h_N = h_{M' \oplus N}$$

and we have the exact diag. in $\mathcal{L}$

$$\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & h_{M'} \arrow[r] & F \arrow[r] \arrow[u] & h_{M''} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & h_{M'} \arrow[r] \arrow[u, equal] & h_{M' \oplus N} \arrow[r] \arrow[u] & h_N \arrow[r] \arrow[u] & 0 \\
 & & h_{N'} \arrow[r, equal] \arrow[u] & h_{N'} \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Thus we need the following condition

(5) Call a map an admissible mono. if it is the kernel of an admissible epim. Then admissible monos. are closed under composition.

Thus we have exact sequences

$$0 \longrightarrow M' \oplus N' \longrightarrow M' \oplus N \longrightarrow M'' \longrightarrow 0$$

$$0 \longrightarrow N' \longrightarrow M' \oplus N' \longrightarrow M' \longrightarrow 0$$

so that (5) implies we have an exact sequence

$$0 \longrightarrow N' \longrightarrow M \oplus N' \longrightarrow M \longrightarrow 0$$

for some M. It follows then that $F = h_M$ by diagram chasing in $\mathcal{L}$.

Thus we have proved.

<u>Theorem</u>: Let $\mathcal{M}$ be an additive category endowed with a class of short exact sequences:

(*) $$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

satisfying the following conditions:

a) any sequence isom. to an exact sequence is exact

b) for any exact sequence (*) M' is the kernel of $M \longrightarrow M''$ and M'' is the cokernel of $M' \longrightarrow M$.

(Thus the class of exact sequences is determined by the class of arrows which occur as the first (resp. second) arrow in a s.e.s. Call these admissible monos + epis. resp.)

c) Any $0 \to M' \xrightarrow{in_1} M' \oplus M'' \xrightarrow{pr_2} M'' \to 0$ is exact.

d) Admissibles epis are stable under composition. They are also quarrable and stable under base changes.

e) If $M' \xrightarrow{u} M$ is such that $\exists$ an admissible epim. $N \twoheadrightarrow M$ with $N \times_M M' \to N$ an admissible epim, then $M' \to M$ is an admissible epim.

f) Admissible monos are stable under composition

Then the full subcat $\mathcal{L}$ of $\mathcal{A} = \mathrm{Homad}(\mathcal{M}^\circ, \mathcal{Ab})$ consisting of the left exact functors is abelian, the Yoneda functor

$$h: \mathcal{M} \longrightarrow \mathcal{L}$$

is a full embedding such that a sequence E is exact iff $h(E)$ is, and further $\mathcal{M}$ is closed under extensions in $\mathcal{L}$.

December 21, 1972.

Consequences of the homotopy ~~[illegible]~~ theorem

The homotopy theorem says that for a ~~[illegible]~~ left noetherian ring A one has

$$K_i'(A[T]) = K_i'(A)$$

the isom. being induced by ~~[illegible]~~ the maps $A \longrightarrow A[T]$ ~~[illegible]~~ which is flat and hence induces a map on K'.

Suppose that $A = \bigoplus_{n \geq 0} A_n$ is a graded ring with is left noeth, ~~[illegible]~~ and suppose that A is of finite Tor dim as a right A_0-module, so that we have a map

$$K_i'(A_0) \longrightarrow K_i'(A) \qquad M \longmapsto A \otimes_{A_0} M$$

and that A_0 is of finite Tor dim as a right A-mod, whence we have a map

$$K_i'(A) \longrightarrow K_i'(A_0) \qquad N \longmapsto A_0 \otimes_A N$$

Then we know that the composition

$$K_i'(A_0) \longrightarrow K_i'(A) \longrightarrow K_i'(A_0)$$

is the identity, by functorality considerations. For the other composition, observe we have a "homotopy

$$A \xrightarrow{\ h\ } A[T]$$
$$\sum a_n \longmapsto \sum a_n T^n$$

and commutative diagrams

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d] & A[T] \arrow[d, "T\mapsto 0"] \\ A_0 \arrow[r] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "h"] \arrow[dr, equal] & A[T] \arrow[d, "T\mapsto 1"] \\ & A \end{tikzcd}$$

so that provided h induces maps on K' we get

$$\begin{tikzcd} K'_i(A) \arrow[r, "h_*"] \arrow[d] & K'_i(A[T]) \arrow[d, "(T\mapsto 0)_*"] \\ K'_i(A_0) \arrow[r] & K'_i(A) \end{tikzcd} \qquad \begin{tikzcd} K'_i(A) \arrow[r, "h_*"] \arrow[dr, equal] & K'_i(A[T]) \arrow[d, "(T\mapsto 1)_*"] \\ & K'_i(A) \end{tikzcd}$$

But $(T\mapsto 0)_*$ and $(T\mapsto 1)_*$ are inverses for the isom $K'_i(A) \longrightarrow K'_i(A[T])$, so $(T\mapsto 0)_* = (T\mapsto 1)_* = (h_*)^{-1}$ so we would be done. It remains to check that h makes $A[T]$ into a right A-module of fin. Tor dim.

~~This doesn't seem to work.~~ hA is the subring of $A[T]$ consisting of $\sum a_n T^n$ with $a_n \in A_n$. Any $f \in A[T]$ can be decomposed into homogeneous polys

$$\sum_{n+k\geq 0} b_n T^{n+k} \qquad b_n \in A_n$$

for different k. Call this sub. $h(A)$-module P_k. For $k \geq 0$, P_k is free over hA with basis T^k, but when $k<0$, P_k is the ideal

$$P_k \cong \bigoplus_{n\geq -k} A_n$$

in A. So I have to know this ideal is of finite

Tor dim as a right A-module. But this is OKAY by induction:

Call $J_m = \bigoplus_{n \geq m} A_n$. Then we have

$$0 \longrightarrow J_{m+1} \longrightarrow J_m \longrightarrow A_m \longrightarrow 0$$

where A_m is regarded as an A-module via the map $A \to A_0$ and the obvious A_0-module structure of A_m. By hyp. A_m is finite Tor dim as a right A_0-module, and A_0 is of finite Tor dim. as a right A-module, ~~so~~ A_m is of finite Tor dim as a right A-module:

$$E^2_{pq} = \mathrm{Tor}_p^{A_0}(A_m, \mathrm{Tor}_q^{A}(A_0, X)) \Longrightarrow \mathrm{Tor}_{p+q}^{A}(A_m, X)$$

$$A_m \otimes_A X = A_m \otimes_{A_0} (A_0 \otimes_A X)$$

Thus by induction on m, we see J_m has finite Tor dim as a right A-module.

Thus we have proved

<u>Prop.</u> $A = \bigoplus_{n \geq 0} A_n$ graded noeth ring. A (resp. A_0) of finite Tor dim over A_0 (resp. A) as right modules. Then

$$(A_0 \to A)^* : K_i'(A_0) \longrightarrow K_i'(A)$$

$$(A \to A_0)^* : K_i'(A) \longrightarrow K_i'(A_0)$$

are isomorphisms inverse to each other.

<u>Filtered rings.</u> Now suppose A is a ring with an increasing filtration:

$$A = \bigcup_{n \geq 0} F_n A, \qquad (F_i A)(F_j A) \subset F_{i+j} A, \quad 1 \in F_0 A$$

I want to assume A is of f. Tordim as a right $F_0 A$-module, so that I have a homomorphism

$$K_i'(F_0 A) \longrightarrow K_i'(A).$$

Form the graded ring $A' = \coprod_{n \geq 0} (F_n A) T^n \subset A[T]$.

There doesn't seem to be anyway of getting hold of $K_i'(A)$ using the homotopy axiom. My original idea was to relate the K-groups of A' and of A, but I don't seem to be able to produce a map of A to a free A'-algebra. I can map A to $A'/(T-1) \subset A'_T = A[T, T^{-1}]$, but without applying some version of the localization thm. to A'_T I can't get anywhere.

December 23, 1972 Gersten's theorem & coherence

$A = A_0 \oplus A_1 \oplus \ldots\ldots$ a graded ring. Let M be a graded A-module ~~bounded above~~ such that $A_0 \otimes_A M$ is projective over A_0, and $\mathrm{Tor}_1^A(A_0, M) = 0$. Choose a section for the map

$$M \longrightarrow A_0 \otimes_A M = T_0 M$$

as graded A_0-modules, whence we get a map

$$A \otimes_{A_0} T_0 M \longrightarrow M$$

which on applying T_0 gives an isomorphism. Thus if M is bdd above the cokernel must be zero, and then as $\mathrm{Tor}_1^A(A_0, M) = 0$, the kernel is zero.

$A = k\langle X_1, \ldots, X_n \rangle = T(V)$ V vector space over k. Then I have the resolution

$$0 \to V \otimes T(V) \longrightarrow T(V) \longrightarrow k \longrightarrow 0$$

of A_0 as a right A-module. Let $J \subset T(V)$ be a homogeneous ideal (left), whence using the above resolution we get

$$0 \to \mathrm{Tor}_1^A(A_0, J) \longrightarrow V \otimes J \longrightarrow J \longrightarrow A_0 \otimes_A J \longrightarrow 0$$

Thus $\mathrm{Tor}_1^A(A_0, J) = 0$ and so J is free.

Let J be any ideal in $T(V) = A$. Filter A by $F_n A = T^0 + \cdots + T^n$, and put $F_n J = F_n A \cap J$. Then $\mathrm{gr}\, J$ is a homogeneous ideal which is free, hence J is free as an A-module. (Check this: Let $\mathrm{gr}\, J$ have generators $x_{ni} \in gr_n J$, $i \in I_n$, and lift x_{ni} to $y_{ni} \in F_n J$. Then we get a homom.

$$\bigoplus_n A(-n)[I_n] \longrightarrow J$$

of filtered A-modules whose gr is an isom, hence it is an isom.

The point: Let M be an A-module endowed with a filtration $0 \subset F_0 M \subset \cdots$, $\bigcup F_n M = M$, $F_i A \cdot F_j M \subset F_{i+j} M$ such that $gr\, M$ is projective over $gr\, A$, i.e.

$$\coprod_{m \geq 0} gr(A)(-m) \otimes_{A_0} E_m \simeq \coprod_{m \geq 0} F_m M / F_{m-1} M$$

where E_m are proj. A_0-modules. Then we can lift $E_n \to F_n M$ and define a map of filtered mods

(*) $$\coprod_{m \geq 0} A(-m) \otimes_{A_0} E_m \longrightarrow M$$

where $A(-m) = A$ with shifted filtration: $F_n(A(-m)) = F_{n-m} A$. The map (*) is an isomorphism because the associated graded map is.)

Sorites on finitely presented modules and coherent rings:

Def: A module M is _finitely presented_ if $\exists$ a presentation

$$A^q \longrightarrow A^p \longrightarrow M \longrightarrow 0$$

Prop: Given $0 \to M' \to M \to M'' \to 0$
a) M'' f.p. + M f.t. $\Longrightarrow$ M' f.t.
b) M f.p. + M' f.t. $\Longrightarrow$ M'' f.p.
c) M' + M'' f.p. $\Longrightarrow$ M f.p.

Proof: Form

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K \arrow[r, phantom, "="] \arrow[d] & K \arrow[d] & \\
0 \arrow[r] & M' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & A^p \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

a) If M'' f.p., then can choose $A^p \to M''$ so that K is f.t, whence if M f.t., then X is f.t. But M' is a direct summand of X, so M' is f.t. whence a).

b) M f.p. $\Rightarrow$ M'' f.t, so $\exists$ $A^p \to M''$. Then M', A^p f.t. $\Rightarrow$ X f.t., whence by a) applied to $0 \to K \to X \to M \to 0$, K is f.t. whence M'' f.p, whence b).

c) If M', M'' f.p., then $X = M' \oplus A^p$ is f.p., so by b) applied to $0 \to K \to X \to M \to 0$ get M f.p.

Prop. TFAE for a ring A
i) Every f.g. left ideal $I \subset A$ is f.p.
ii) The kernel, image, + cokernel of a map of f.p. modules is f.p.
iii) Any f.g submodule M' of a f.p. module M is f.p.

iii) $\Rightarrow$ i) trivial

Proof. ii) $\Rightarrow$ iii) $\quad 0 \to \underset{f.t.}{M'} \to \underset{f.p.}{M} \to M'' \to 0 \quad \Rightarrow M''$ f.p. b)

so M' is the kernel of a map of f.p. modules, so is f.p.

iii) $\Rightarrow$ ii)

$$0 \to K \to M \to I \to 0 \qquad M \text{ f.p.} \Rightarrow I \text{ f.t.}$$

$$0 \to I \to M_0 \to C \to 0 \qquad I \text{ f.t.} + M_0 \text{ f.p.} \Rightarrow C \text{ f.p.}$$

but I f.t. + iii) $\Rightarrow$ I f.p. and then a) $\Rightarrow$ K f.t.
whence by iii) K is f.p.

i) $\Rightarrow$ iii). suppose have M' f.t. $\subset M$ f.p.

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] & N \arrow[r] \arrow[d, phantom, "\cap"] \arrow[dr, phantom, "\text{cart}"] & M' \arrow[r] \arrow[d, phantom, "\cap"] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[u, phantom, "\parallel"] & A^p \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

K, M' f.t. $\Rightarrow$ N f.t. If can prove N f.p. then b) $\Rightarrow$ M' f.p. So reduce to N f.t. $\subset A^p$, Then

$$\begin{tikzcd}
0 \arrow[r] & N' \arrow[r] \arrow[d, phantom, "\cap"] & N \arrow[r] \arrow[d, phantom, "\cap"] & I \arrow[r] \arrow[d, phantom, "\cap"] & 0 \\
0 \arrow[r] & A^{p-1} \arrow[r] & A^p \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

N f.t. $\Rightarrow$ I f.t. $\Rightarrow$ by i) I f.p.
$\Rightarrow$ by a) N' f.t. Induction $\Rightarrow$ N' is of f.p. $\overset{c)}{\Rightarrow}$ N f.p. done.

Def: such a ring A is called coherent. The f.p. modules form a fully-abelian subcategory of all A-mod.

<u>Example.</u> $T(V)$ is coherent because we've seen that every ideal is free.

I want to show that the filtered ring theorem applies with noetherian replaced by coherent. So let $A = \bigcup F_n A$ be a filtered ring, and form the graded ring

$$A' = \coprod_{n \geq 0} (F_n A) t^n$$

so that we have

$$A'/A't = gr(A)$$

$$A'/A'(t-1) = A.$$

Now I want to describe the K-theory of A in terms of the K-theory of graded modules over A'.

Assume $\bar{A} = gr\, A$ is graded-coherent, i.e. the finitely presented graded $\bar{A}$-modules form a fully-abelian subcategory of all graded $\bar{A}$-modules. I want to show then that A is coherent. So let $J \subset A$ be a finitely generated left ideal, and consider $gr\, J \subset \bar{A}$. To show $gr\, J$ is fin. gen. ?

However, suppose $A = T(V) = A_0 \oplus A_1 \oplus \cdots\cdots$ so we know A is coherent, as well as $A[t]$ presumably. Then

$$A' = \coprod_{j \leq n} A_j t^n \cong A[t]$$

is coherent so everything should work.