September 1, 1972.

I have discovered that the homology
$$H_*(\mathrm{Aut}\, M,\ St(M))$$
is extremely important. **Notation**: M f.g. projective module over a Dedekind domain A and $St(M)$ is the Steinberg module belonging to the vector space $K \otimes_A M$, K = fraction field of A.

Examples:

1) arithmetic case: Let X be the symmetric space and $\bar{X}$ is cornered completion. Then
$$H_i(\bar{X}, \partial\bar{X}; \mathbb{Z}) = \begin{cases} 0 & i \neq n-1 \\ St(M) & i = n-1 \end{cases} \qquad n = \mathrm{rank}\, M$$
$(\partial\bar{X} \sim \vee S^{n-2})$. Thus if $\Gamma = \mathrm{Aut}(M)$, then
$$H_i^\Gamma(\bar{X}, \partial\bar{X}; \mathbb{Z}) = H_{i-n+1}(\Gamma, St(M))$$
and these are f.g. abelian groups (passing to a net subgroup Γ' one has $(\bar{X}/\Gamma', \partial\bar{X}/\Gamma')$ is a compact manifold + bdry.)

2) The function field case should be similar.

3) This homology arises when one filters $Q(\mathcal{P}_A)$:
$$H_i(F_n Q(\mathcal{P}_A), F_{n-1} Q(\mathcal{P}_A); \mathbb{Z}) = \bigoplus_\alpha H_{i-n}(\mathrm{Aut}(M_\alpha), St(M_\alpha))$$

Sept. 7, 1973 2
Seattle Conf.

Acyclic maps

X acyclic space : $H_i X = \begin{cases} \mathbb{Z} & i=0 \\ 0 & i>0 \end{cases}$

$(X \sim pt \iff \pi_1 X = 0.$
$H_1 X = \pi_1 X / (\pi_1 X, \pi_1 X) = 0$, $\pi_1 X$ is perfect.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & X \times_Y Y^I \arrow[d, "p"] \\ & Y \end{tikzcd}$$

i heq
p fibration

The homotopy-fibre of f over $y \overset{\text{defn}}{=}$ fibre of p over y
$\sim$ fibre of f over y if f is a fibration (quasi-fibration).

Def + prop: $X \xrightarrow{f} Y$ acyclic if equiv:
(i) The homotopy-fibres of f are acyclic
(ii) $H_i(X, f^*L) \xrightarrow{\sim} H_i(Y, L)$ all L on Y.

Example: homology spheres: If $f: X \longrightarrow S^n$ induces isom $H_*(X) \xrightarrow{\sim} H_*(S^n)$, then f acyclic for $n \geq 2$.

Remarks 1) Whitehead thm.
Remark 2) From now on work with conn. CW complexes with basepoint. Suppose $f: X \to Y$ acyclic, F h-fibre over basepoint of y.

$$\pi_1 F \longrightarrow \pi_1 X \longrightarrow \pi_1 Y \longrightarrow 0$$

so $\pi_1 f$ is surjective and $\mathrm{Ker}(\pi_1 f) = \mathrm{Im}(\pi_1 F)$ is perfect.

Universal property: Given $\ni$ $\mathrm{Ker}\, \pi_1 f \subset \mathrm{Ker}\, \pi_1 u$

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Z \\ Y \arrow[ur, dotted, "v"'] \end{tikzcd}$$

$\exists$ dotted arrow $v \ni vf \sim u$.
and v is unique up to $\sim$

Proof:

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Z \arrow[d, "f'"] \\ Y \arrow[r, "u'"] & Y \amalg^X Z \end{tikzcd}$$

$$\underset{\overset{\shortparallel}{0}}{H_i(Y, X; u^*L)} \simeq H_i(Y \amalg^X Z, Z; L)$$

f acyclic $\Longrightarrow$ f' acyclic

$$\pi_1(Y \amalg^X Z) = \pi_1(Y) *^{\pi_1 X} \pi_1 Z = \pi_1 Z$$

~~old problem was why~~ Whitehead thm. $\Rightarrow$ f' h.eq.

Cor: (!ness of an acyclic map $X \to Y$ with $\operatorname{Ker} \pi_1(f)$ fixed).

Prop (Classification of acyclic maps with given ~~target~~ source)
For each perfect normal subgroup N of $\pi_1 X$, $\exists$ an acyclic map $X \xrightarrow{f} Y$, ~~with~~ unique up to $\sim$, $\ni$

$$\operatorname{Ker} \pi_1(f) = N.$$

Example: A ring 1

$GL(A)$

$E(A) = (GL(A), GL(A)) = (E(A), E(A))$

$\exists$ acyclic map, unique up to homotopy

$$BGL(A) \xrightarrow{f} BGL(A)^+$$

with $\operatorname{Ker} \pi_1(f) = E(A)$.

<u>Defn:</u> $K_i^Q A = \pi_i BGL(A)^+ \qquad i \geq 1$.

$$K_1^Q A = GL(A)/E(A) = K_1^B A$$

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & BE(A) \arrow[r] \arrow[d] & \widetilde{BGL(A)^+} \arrow[d, "\text{universal cover}"] \\
F \arrow[r] & BGL(A) \arrow[r] & BGL(A)^+
\end{tikzcd}$$

acyclic

$$K_2^Q A = \pi_2 BGL(A)^+ = \pi_2 \widetilde{BGL(A)^+}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad \|$$

$$K_2^M A = H_2 BE(A) = H_2 \widetilde{BGL(A)^+}$$

<u>Prop:</u> $BGL(A)^+$ is a homotopy (comm. + ass.) H-space

<u>Prop.</u>

$$\begin{tikzcd}
BGL(A) \arrow[r] \arrow[d] & Z \\
BGL(A)^+ \arrow[ur, dashed] &
\end{tikzcd}$$

$\pi_1 Z$ has no perfect subgps

<u>Definition:</u> $K_i A = \pi_i(BGL(A)^+)$ $\quad i \geq 1$

<u>Relation with the homology of $GL(A)$:</u>

$K_1 A = \pi_1(BGL(A)) = GL(A)/E(A) = \text{Bass } K_1 A$

$K_2 A = H_2(E(A)) = \text{Milnor } K_2 A$

$K_3 A = H_3(St(A))$

Because $BGL(A)^+$ is an H-space with
$H_*(BGL(A)^+) = H_*(GL(A))$.

$$K_n A \otimes \mathbb{Q} \cong \text{Primitive subspace of } H_n(GL(A), \mathbb{Q}).$$
$$= \{x \mid \Delta x = x \otimes 1 + 1 \otimes x \quad \Delta : GL(A) \to GL(A)^2\}.$$

<u>Computations:</u>

<u>Borel-Garland theorem:</u> A ring of integers in a number field F with r_1 real and r_2 complex places. Then

$\dim K_n A \otimes \mathbb{Q} =$	0	$n \equiv 0$	
	$r_1 + r_2$	$n \equiv 1$	(4)
	0	$n \equiv 2$	
	r_2	$n \equiv 3$	

vanishing ~~death~~ for n even due to Garland

except for $n = 1$, where $\dim K_1 A \otimes \mathbb{Q} = r_1 + r_2 - 1$

Finite fields: $\mathbb{F}_q \subset F$ alg. closure

Choose $F^\circ \hookrightarrow \mathbb{C}^\circ$

then for any finite group it induces a map of Groth rings.

$$R_{\mathbb{F}_q}(G) \longrightarrow R_{\mathbb{C}}(G)^{\Psi^q}$$

$$\begin{tikzcd}
 & F\Psi^q \arrow[d] \\
BGL(\mathbb{F}_q) \arrow[ur, dashed] \arrow[r] & BU \arrow[d, "\Psi^q - 1"] \\
 & BU
\end{tikzcd}$$

Th. $BGL(\mathbb{F}_q) \longrightarrow F\Psi^q$ induces isom

Theorem:

i) $$K_{2i}(\mathbb{F}_q) = 0 \qquad i \geq 1$$
$$K_{2i-1}(\mathbb{F}_q) \cong \mathbb{Z}/(q^i-1)\mathbb{Z}$$

ii) If $F \longrightarrow F'$ is a map of finite fields then $K_{2i-1}F \hookrightarrow K_{2i-1}F'$

iii) If $fr : F \longrightarrow F$ is the Frobenius $fr(x) = x^p$ then $fr_* : K_{2i-1}F \longrightarrow K_{2i-1}F$ is multiplication by p^i.

Cor: $\bar{F}$ algebraic closure of $\mathbb{F}_p$, then

$$K_{2i}\bar{F} = 0$$

$$K_{2i-1}\bar{F} \cong \bigoplus_{\substack{\ell \neq p \\ \ell \text{ prime}}} \mathbb{Q}_\ell/\mathbb{Z}_\ell$$

and fr acts on $K_{2i-1}\bar{F}$ as mult. by p^i.

Let $\mathbb{F}_q \subset \bar{F}$ alg. closure & choose $\bar{F}^\times \hookrightarrow \mathbb{C}^\times$. Then Brauer theory of modular characters provides a lifting ρ:

$$\begin{tikzcd}
R_{\bar{F}}(G) \arrow[r, "\rho"] & R_{\mathbb{C}}(G) \\
R_{\mathbb{F}_q}(G) \arrow[u] \arrow[r] & R_{\mathbb{C}}(G)^{\Psi^q} \arrow[u, phantom, "\cup"] = \{\alpha \mid \Psi^q\alpha = \alpha\}
\end{tikzcd}$$

$$\begin{tikzcd}
& F\Psi^q \arrow[d] \\
BGL(\mathbb{F}_q) \arrow[ur, dashed] \arrow[r, "``\rho"''] & BU \arrow[d, "\Psi^q - 1"] \\
& BU
\end{tikzcd}$$

Thm: $BGL(\mathbb{F}_q) \longrightarrow F\Psi^q$ Homology isom.

Let $\bar{F}$ be an alg. closure of $\mathbb{F}_q$ & choose $\bar{F}^{\cdot} \overset{\rho}{\hookrightarrow} \mathbb{C}^{\cdot}$.
Then the Brauer theory gives

$$\begin{tikzcd}
 & F\Psi^q \arrow[d] \\
BGL(\mathbb{F}_q) \arrow[ur, dashed] \arrow[r, "``\rho"", Rightarrow] & BU \arrow[d, "\Psi^q - 1"] \\
 & BU
\end{tikzcd}$$

<u>Theorem</u>: $BGL(\mathbb{F}_q) \longrightarrow F\Psi^q$ is a homology isom.

$$\begin{tikzcd}
BGL(\mathbb{F}_q) \arrow[r] \arrow[d] & F\Psi^q \\
BGL(\mathbb{F}_q)^+ \arrow[ur, dashed] &
\end{tikzcd}$$

<u>Cor</u>: $BGL(\mathbb{F}_q)^+ \longrightarrow F\Psi^q$ heq

so $K_i(\mathbb{F}_q) = \pi_i F\Psi^q$

[Relation with cohomology theories derived from
[permutative (or Γ-) categories (Anderson + Segal).

A ring, $\mathcal{P}_A^\circ$ = projective f.g. A-modules + their isos. Then their is a Ω-spectrum

$$B_0(\mathcal{P}_A^\circ),\ B_1(\mathcal{P}_A^\circ),\ \ldots\ldots,\ B_n(\mathcal{P}_A^\circ),\ \ldots$$

Thm: $$K_0A \times BGL(A)^+ \cong B_0(\mathcal{P}_A^\circ)$$

Cor: $BGL(A)^+$ is an infinite loop space.

Analogue: Take instead of $\mathcal{P}_A^\circ$ the category $\mathcal{F}$ of finite sets and their autos. Then

Thm. (Barratt-Priddy-Segal)

$$\mathbb{Z} \times B\Sigma_\infty^+ \simeq \varinjlim_n \Omega^n S^n$$

"K-theory of symmetric groups = stable homotopy theory"

$$e_i : K_{2i-1}\mathbb{C} \longrightarrow \mathbb{C}^*$$

$$e_1 : \mathbb{C}^* \xrightarrow{id} \mathbb{C}^*$$

$$e_i(\bar{E}) = (-1)^{i+1} e_i(E)$$

$$\begin{tikzcd}
K_{2i-1}\mathbb{R} \arrow[r] \arrow[drr, dashed] & K_{2i-1}\mathbb{C} \arrow[r, "e_i"] & \mathbb{C}^* \\
 & & \{\pm 1\} \arrow[u, phantom, "\subset" rotate=90]
\end{tikzcd}$$

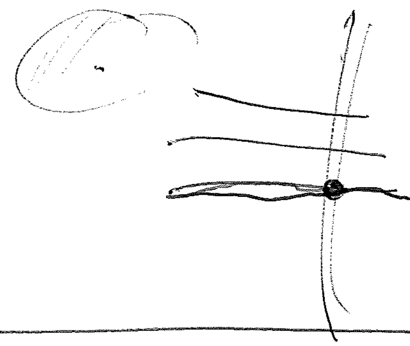
if i is odd.

$G \subset GL(n, \mathbb{R})$

$$\begin{array}{c} E \\ \downarrow G \\ X \end{array} \quad \xi \text{ assoc vect bundle}$$

q: $H^q(X; \underline{\xi})$ sheaf of ge. of cont cross sect. of ξ

$H_c^*(G)$



X quasi-foliation

Ω_X

E

$S \subset T_X \otimes \mathbb{C}$

i) $[S, S] \subset S$

$\mathbb{P}E$

ii) $[S+\bar{S}, S+\bar{S}] \subset [S+\bar{S}]$

~~Frobenius~~ Nirenberg's thm:

$\Longrightarrow$ locally

$$X \xrightarrow{\text{submersion}} Y \xrightarrow[\text{family}]{\text{holom}} Z$$

$$X \longrightarrow X/(S \cap \bar{S}) \longrightarrow X/S+\bar{S}$$

Acyclic maps and algebraic K-theory.

1. Acyclic maps + definition of $K_i A$

2. Relation with homology of $GL(A)$.

3. Computations

   $K_i \mathbb{F}_q$, Borel's theorem

4. Relation with permutative categories

   Barratt–Priddy–Segal, etc.

$$\mathbb{Z} \times B\Sigma_\infty^+ \xrightarrow{\sim} \lim_{n\to\infty} \Omega^n S^n$$

K-theory of symmetric groups $=$ stable homotopy theory.

---

permutative category $\longmapsto$ connected generalized coh. theory

5. Results on $K_*(\mathbb{Z})$.

$$\begin{array}{ccc} \pi_i^s & \longrightarrow & K_i\mathbb{Z} \\ \cup & & \\ J\{\pi_i O\} & & \end{array}$$

$J\{\pi_{4s-1} O\} \subset \pi^s_{4s-1}$ is cyclic of order denom$(B_s/4s)$

September 7, 1972 (groggy since Sept. 3)

Review of ideas during the Seattle K-theory conference.

1. Homology with coefficients in the Steinberg module.

Recall that if A is a Dedekind domain with quotient field K, we have

$$H_i\left(F_n Q(P_A), F_{n-1} Q(P_A); \mathbb{Z}\right) \cong \bigoplus_{\alpha \in \operatorname{Pic} A} H_{i-n}\left(\operatorname{Aut}(P_\alpha), St(K \otimes_A P_\alpha)\right) \qquad n \geq 1$$

where P_α is "the" proj. A-module of rank n with $c_1(P_\alpha) = cl(\Lambda^n P_\alpha) = \alpha$.

In the case where A = ring of integers in a no. field K, then one knows that $\operatorname{Pic} A$ is finite and that $\forall \alpha$, $H_i(\operatorname{Aut} P_\alpha, St(K \otimes P_\alpha))$ is f.g. $\forall i$. The proof of the last statement proceeds most honestly by using the fact that if $\Gamma \subset GL(V)$ is arithmetic, V of dim n over K then

$$H_i(\Gamma, St(V)) = H^{\Gamma}_{i+n-1}(\overline{X}(V), \partial \overline{X}(V); \mathbb{Z})$$

where $\overline{X}(V)$ is the Borel-Serre ~~[illegible]~~ completion of the symmetric space of $GL(V \otimes_{\mathbb{Z}} \mathbb{R})$. One knows, replacing Γ by a torsion-free subgp. of finite index that $(\overline{X}/\Gamma, \partial\overline{X}/\Gamma)$ is a compact manifold with boundary

Function field case: Let A = coordinate ring of a complete non-singular curve over a finite field minus one point. One conjectures that $H_i(\operatorname{Aut}(P_\alpha), St(K \otimes P_\alpha))$ is also finitely generated. I demonstrated (Aug 31) the following:

<u>Prop:</u> We have the implications

i) for all f.g. proj A-modules M

$$H_i(\mathrm{Aut}(M), \mathrm{St}(K \otimes M)) \quad \text{f.g. } \mathbb{Z}\text{-mod} \quad \forall i$$

$\Downarrow$

ii) for all M as above

$$H_i(\mathrm{Aut}(M), \mathbb{Z}[\tfrac{1}{p}]) \quad \text{f.g. over } \mathbb{Z}[\tfrac{1}{p}] \quad \forall i$$

$\downarrow$

iii) for all M as above

$$H_i(\mathrm{Aut}(M), \mathrm{St}(K \otimes M) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[\tfrac{1}{p}]) \quad \text{f.g. over } \mathbb{Z}[\tfrac{1}{p}]$$

Intuitively (i) is reasonable because the homology is "the part of the compactification modulo boundary."

(giozzy)

September 8, 1972: sfields

Let k be a finite field and k' the quadratic extension of k'; denote by $z \mapsto \bar{z}$ the generator of $\mathrm{Gal}(k'/k)$. Let

$$A = k' \otimes_k k[X]$$

with twisted multiplication given by

$$Xz = \bar{z}X \qquad \forall z \in k'.$$

From my theorem on the K of filtered rings one have

$$K_i(A) \xrightarrow{\sim} K_i(k).$$

The isomorphism is given by base change with respect to $k \subset A$ or the augmentation $A \to k$ with kernel AX.

I will assume that A has a ring of quotients D which is a sfield, and that $\mathrm{Modf}(D)$ is the quotient of $\mathrm{Modf}(A)$ by the Serre subcategory $\mathcal{B}$ of A-modules which are f.g. over k.

Granted this, every object of $\mathcal{B}$ is of finite length, whence

$$K_i \mathcal{B} = \bigoplus_\alpha K_i(D_\alpha)$$

where α runs over the rep. for the iso classes of simple objects of $\mathcal{B}$ and D_α = skew field of endos of α. In this situation D_α is a finite extension of k, so we should be able to compute $K_* D$ using the localization long exact sequence.

Suppose char $k \neq 2$, whence $k' = k[Y]$, $Y^2 = \lambda$, $\lambda \in k^{\cdot} - k^{\cdot 2}$. Thus A is a quaternion algebra:

$$A = k + kX + kY + kXY$$

$$X^2 = T \in A$$

$$Y^2 = \lambda$$

$$XY = -YX$$

i.e. the Clifford algebra of the quadratic form

$$\begin{pmatrix} T & 0 \\ 0 & \lambda \end{pmatrix}$$

over $k[T]$.

Thus if we have a homo. $k[T] \longrightarrow \Omega$ such that T becomes invertible, A_Ω is an Azumaya algebra over Ω. Now consider a simple A-module E which is f.d. over k. Then we have

$$\begin{tikzcd} k[T] \arrow[r] \arrow[dr, two heads] & \mathrm{End}_A(E) \\ & k[T]/\mathfrak{m} \arrow[u] \end{tikzcd}$$

where $\mathfrak{m}$ is a maximal ideal of $k[T]$.

<u>Case 1:</u> $T \notin \mathfrak{m}$. Then E is an $A/\mathfrak{m}A$-module, where $A/\mathfrak{m}A$ is an Azumaya alg over the finite field $k[T]/\mathfrak{m}$, so conclude that $A/\mathfrak{m}A$ is a matrix ring and

$$\mathrm{Modf}(k[T]/\mathfrak{m}) \simeq \mathrm{Modf}(A/\mathfrak{m}A)$$

$$A/\mathfrak{m}A = \mathrm{End}_A(E) \quad .$$

<u>Case 2:</u> $T \in \mathfrak{m}$, whence $\mathfrak{m} = (T)$

$$A/\mathfrak{m}A \;=\; k'\bar{X} \qquad \bar{X}^2 = 0$$
$$y\bar{X} = -\bar{X}y.$$

In this case we have by dévissage

$$K_*(k') \xrightarrow{\sim} K_*(A/TA).$$

Now we have to compute the map

$$K_*(A/\mathfrak{m}A) \longrightarrow K_*(A)$$

In the first case we have

$$\begin{tikzcd}
K_*(k[T]/\mathfrak{m}) \arrow[r, "0"] \arrow[d, "\wr"] & K_*(k[T]) \arrow[d] \\
K_*(A/\mathfrak{m}A) \arrow[r] & K_*(A)
\end{tikzcd}$$

where the zero comes from the fact that since $T \notin \mathfrak{m}$ we have transversality

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & \mathrm{Sp}(k[T]/\mathfrak{m}) \arrow[d, "i"] \\
\mathrm{Sp}(k) \arrow[r, "j"] & \mathrm{Sp}(k[T])
\end{tikzcd}$$

whence $j^* i_* = 0$, so $i_* = 0$, as j^* is an isom.

In the case $\mathfrak{m} = (T)$ we want the composite

$$\begin{tikzcd}
K_*(k') \arrow[r] & K_*(A/TA) \arrow[r] & K_*(A) \arrow[d] \\
 & & K_*(k')
\end{tikzcd} \qquad \text{(transfers)}$$

base change with $A \to A/TA$

But if $V \in \mathrm{Modf}(k')$, then can resolve it over A as follows:

$$0 \longrightarrow \underset{\substack{\| \\ \bigoplus_{i\geq 0} X^i(XV)}}{A\otimes_{k'}\bar{V}} \longrightarrow \underset{\substack{\| \\ \bigoplus_{i\geq 0} X^{i}V}}{A\otimes_{k'}V} \longrightarrow V \longrightarrow 0$$

Thus the composite $K_*(k') \longrightarrow K_*(k')$ is

$$V \longmapsto V - \bar{V}$$

i.e. $\quad id - ($[redacted] conjugation$)$.

Now from the long exact sequence, we get

$$\longrightarrow \bigoplus_{m} K_i^{coh}(A/mA) \longrightarrow K_iA \longrightarrow K_i D \xrightarrow{\partial} \cdots\cdots.$$

$$K_{2i}D \cong K_{2i-1}k \oplus \bigoplus_{\substack{m\subset k[T] \\ \text{[redacted]} \\ T\notin m}} K_{2i-1}(k[T]/m).$$

$$0 \longleftarrow K_{2i-1}D \longleftarrow K_{2i-1}k' \xleftarrow{1-\sigma} K_{2i-1}k' \qquad i \geq 2$$

$$K_1k' \xrightarrow{1-\sigma} K_1k' \longrightarrow K_1D \longrightarrow \bigoplus_{\text{all } m} \mathbb{Z} \longrightarrow 0$$

$$\begin{tikzcd}
\arrow[r] & K_i(k[T,T^{-1}]) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "(*)"] & K_i\, k(T) \arrow[r] \arrow[d] & \bigoplus_{\mathfrak{m} \neq T} K_{i-1}(k[T]/\mathfrak{m}) \arrow[d, "\cong"'] \\
\arrow[r] & K_i(A[T^{-1}]) \arrow[r] & K_i\, D \arrow[r] & \bigoplus_{\mathfrak{m} \neq T} K_{i-1}(A\,[\text{redacted}]/\mathfrak{m}A)
\end{tikzcd}$$

This gives rise to a bicartesian square $*$. On the other hand one has from

$$K_*(A/T) \longrightarrow K_* A \longrightarrow K_*(A[T^{-1}]) \xrightarrow{\partial} \cdots$$

and the preceding computations, exact sequences

$$0 \longrightarrow K_{2i}(A[T^{-1}]) \longrightarrow K_{2i-1} k' \xrightarrow{1-\sigma} K_{2i-1} k' \longrightarrow K_{2i-1} A[T^{-1}] \twoheadrightarrow \begin{cases} 0 & i \geq 2 \\ \mathbb{Z} & i = 1 \end{cases}$$

Generalize: Suppose k' is the cyclic extension of k of degree n, and $A = k'[X]$ with

$$Xz = z^\sigma X \qquad z \in k'$$

σ the distinguished [redacted] generator of $Gal(k'/k)$, e.g. $z^\sigma = z^q$, $q = card(k)$. At least if k contains μ_n and n is prime to q, then A is the "cyclic" algebra:

$$X^n = T$$
$$Y^n = \lambda \qquad \lambda \in k^\bullet - (k^\bullet)^n$$
$$XY = \zeta YX$$

which [redacted] is an Azumaya algebra over $k[T,T^{-1}]$.

Again we have that for $\mathfrak{m} \neq (T)$, $A/\mathfrak{m}A$ is a

$n \times n$ matrix ring over $k[T]/\mathfrak{m}$, so again we will have the same formulas as on the top of page 5.

Special case of K_2:

$$0 \longrightarrow K_2(A[T^{-1}]) \longrightarrow K_2 D \longrightarrow \bigoplus_{\substack{\mathfrak{m} \\ T \notin \mathfrak{m}}} K_1(k[T]/\mathfrak{m}) \longrightarrow 0$$

$$0 \longrightarrow K_2(A[T^{-1}]) \longrightarrow (k')^{\cdot} \xrightarrow{1-\sigma} (k')^{\cdot} \longrightarrow K_1(A[T^{-1}]) \longrightarrow \mathbb{Z} \longrightarrow 0$$

$$\therefore \quad K_2(A[T^{-1}]) = k^{\cdot}$$

$$K_1(A[T^{-1}]) = \mathbb{Z} \oplus k'^{\cdot}/(k'^{\cdot})^{\sigma-1}$$

Ex. 1: $F^2 \to 0$. Then $\mathfrak{m} = (F)$ and $E = k$.

$$k^\sigma[F^2] \longrightarrow k[F]$$

generated by elements x, F such that

$$F^2 = F^2$$
$$\bar{x} F = F x$$

So I have this algebra and I want to reduce it modulo a max. ideal $\mathfrak{m}$ of $k^\sigma[F^2]$ such a maximal ideal is a map

$$\begin{array}{ccc} k^\sigma[F^2] & \longrightarrow & \Delta \\ \downarrow & & \downarrow \text{rank } 4 \\ k[F] & \longrightarrow & - \end{array}$$

a finite extension

---

want the simple modules for $k[F]$.

Suppose E is a simple module over $k[F]$ then there is a unique maximal ideal $\mathfrak{m} \subset k^\sigma[F^2]$ such that $\mathfrak{m}E = 0$.

so $\Delta = k^\sigma[F^2]$ is a finite field contained in the field of endos of E.

$$D = \mathrm{End}_{k[F]}(E)$$

$$\begin{array}{l} k^\sigma \subset \Delta = k^\sigma(\lambda) \subset D \\ \cap \\ k \end{array}$$

λ = image of F^2 in D.

k finite field, σ an automorphism
$k_0 = k^\sigma$ fixed field. Suppose $[k : k^\sigma] = d$
and $\sigma^d = 1$. Then form

$$k^{\sigma}[F] \qquad \text{with} \qquad Fx = x^\sigma F$$

and I consider its quotient field D. Then

$$K_i(\text{tors Mod}_f\, k[F]) \longrightarrow K_i(k[F]) \longrightarrow K_i(D) \xrightarrow{\partial}$$

now what are the simple $k[F]$-modules

a) $F = 0$

~~[illegible]~~

example: Suppose $k^\sigma \to k$ is a quadratic extension. Then I want to classify simple $k[F]$-modules. where $Fx = \bar{x}F$. ~~[illegible]~~ So consider $k[F^2]$. The first point is to realize it must be isotropic ~~[illegible]~~ over the center $k^\sigma[F^2]$ whence we have a homomorphism. Let E be a simple $k[F]$-module, ~~[illegible]~~ and let $D = \mathrm{End}_{k[F]}(E)$. Then D is ~~[illegible]~~ a finite field.

Now suppose we consider

$$k^\sigma[F^2] \longrightarrow D$$

$$E = k[F]/L \qquad L = \text{maximal left ideal}$$

$k[F]$ is of rank 4 over $k^\sigma[F^2]$.

$\therefore$ E is of rank ≤ 4 over $k^\sigma[F^2]/$

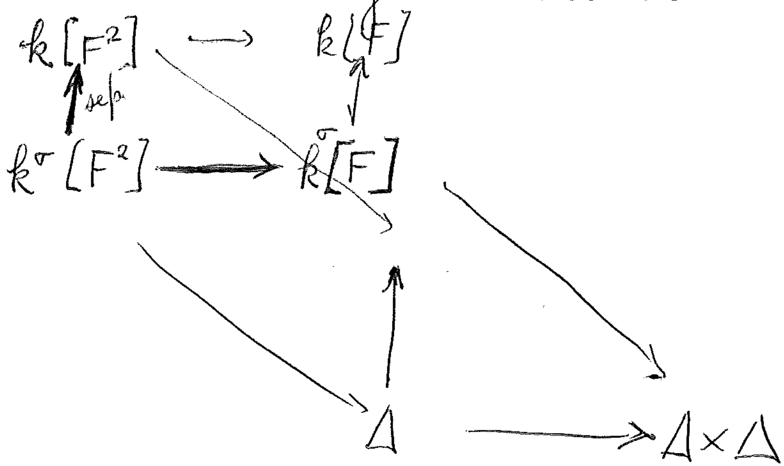
$A = k[F] \otimes_{k^\sigma[F^2]} \Delta$ algebra of rank 4 over Δ

E is a simple A-modules.

possibilities for A

A 2×2 ~~matrix~~ algebra over Δ

Suppose Δ is the algebraic closure



Since k is separable over k^σ
we have that $k[F^2]$ is etale over $k^\sigma[F^2]$
hence

$$\begin{tikzcd} k[F^2] \arrow[r] & \bar{k}\times\bar{k} \\ k^\sigma[F^2] \arrow[u] \arrow[r] & \bar{k} \arrow[u] \end{tikzcd}$$

$Fx = \bar{x}F$

Now one also $F^2 \longrightarrow \lambda \in \Delta$

so we ought to be able to see what happens

September 10, 1972

Basic problems:

Moore's theorem generalized: A rings of integer in a number field F, show

$$K_{2i}F \xrightarrow{\partial} \bigoplus K_{2i-1}(A/\mathfrak{p})$$

the sum being taken over all primes $\mathfrak{p}$ of A. From the exact sequence

$$0 \to K_{2i}A \to K_{2i}F \xrightarrow{\partial} \bigoplus K_{2i-1}(A/\mathfrak{p}) \to K_{2i-1}A \to K_{2i-1}F \to 0$$

and the finite generation of K_*A, one knows that the cokernel of ∂ is finite.

Ideas that don't work:

1. $$K_1F \otimes K_{2i-1}F \longrightarrow K_{2i}F$$ is not ~~onto~~:

If it were we would be able to generate $K_{2i}F$ by products $\alpha \cdot \beta$ with $\alpha \in K_1F$, $\beta \in K_{2i-1}A$. So

$$\partial(\alpha \cdot \beta) = \partial\alpha \cdot \beta$$

would generate the image of ∂. But $(\partial\alpha)_\mathfrak{p} \in K_0(A/\mathfrak{p}) = \mathbb{Z}$, so $(\partial\alpha \cdot \beta)_\mathfrak{p}$ is a multiple of the image of β under the reduction map

$$K_{2i-1}A \longrightarrow K_{2i-1}(A/\mathfrak{p}).$$

But by Galois considerations (see below), most of these maps are not onto, so we get a contradiction, since $\operatorname{Im}\partial$ is of finite index).

Galois considerations: Fix ℓ and consider

$$\begin{tikzcd} \cancel{K_{2i-1}(A)} \arrow[r] \arrow[d] & \cancel{K_{2i-1}(A/\wp)_{(\ell)}} \\ \cancel{K_{2i-1}(A[\tfrac{1}{\ell}, \mu_{\ell^\infty}])} \arrow[r] & \cancel{K_{2i-1}(} \end{tikzcd}$$

the image H of $\mathrm{Gal}(F)$ in $\mathrm{Aut}(\mu_{\ell^\infty}) = \mathbb{Z}_\ell^*$.
For each prime $\wp \nmid \ell$ one also has a subgroup $H_\wp \subset H$, and a distinguished generator (Frobenius) of $H_\wp$. By Chebotarev density, one can get any open subgroup infinitely often. Done, because the image of $K_{2i-1}(A) \longrightarrow K_{2i-1}(A/\wp)_{(\ell)} = (T_\ell)^{H_\wp}$ is fixed by H.

2. $K_2F \otimes K_{2i-2}F \longrightarrow K_{2i}F$ not onto.
(follows from preceding as $F^\bullet \otimes F^\bullet \twoheadrightarrow K_2F$.)

3. According to Gersten the transfer map

$$R_{A/\wp}(G) \longrightarrow R_A(G)$$

is not zero for G finite. e.g. $A = \mathbb{Z}$, $G = \mathbb{Z}/p\mathbb{Z}$, $\wp = p\mathbb{Z}$.

September 17, 1972 (mind almost clear) (clear on 19th)

Let X be a space. I want to describe $B(\coprod_k (X^k)_{\Sigma_k})$ as a category.

When $X = pt$ we take $\mathcal{Q} = Q(\text{finite sets})$. $\mathcal{Q}$ has finite sets for objects, and a map $S' \to S$ is an isom. of S' with a layer of S:

$$S' \cong S_1 - S_0 \quad , \quad S_0 \subset S_1 \subset S .$$

In general $\mathcal{Q}(X) = Q(\text{finite sets over } X)$ should be the topological category fibred over $\mathcal{Q} = \mathcal{Q}(pt)$ defined by the functor $S \longmapsto X^S$.

Given $X \to Y$, one has $\mathcal{Q}(X) \to \mathcal{Q}(Y)$ necessarily fibred (at least on the "discrete" level). The fibre over $S \to Y$ consists of all liftings to X, so the fibres have different homotopy types.

Mystery: Anderson proves quite simply that

$$\boxed{\cancel{\mathcal{Q}(A) \to \mathcal{Q}(X) \to \mathcal{Q}}}$$

for a pair (X, A) of simplicial sets

$$\begin{tikzcd}
B\left(\coprod_k (A^k)_{\Sigma_k}\right) \arrow[r] \arrow[d] & B\left(\coprod_k (X^k)_{\Sigma_k}\right) \arrow[d] \\
B\left(\coprod B\Sigma_k\right) \arrow[r] & B\left(\coprod_k (X/A)^k_{\Sigma_k}\right)
\end{tikzcd}$$

is h-cartesian.

Segal's machinery for loop spaces:

Segal calls a s. space $n \mapsto X(n)$ special if $X(0) = pt$ and if

$$X(n) \longrightarrow X(1)^n$$

is a heq for all n. There are adjoint functors

$$\begin{pmatrix}\text{reduced} \\ \text{s. spaces}\end{pmatrix} \underset{\Omega_*}{\overset{|\ |}{\rightleftarrows}} \begin{pmatrix}\text{ptd.} \\ \text{spaces}\end{pmatrix}$$

where $(\Omega_* X)(n)$ = maps $\Delta(n) \to X$ ~~carrying~~ vertices to basepoint.

He shows

i) $|\Omega_* Y| \longrightarrow Y$ heq $\Longleftrightarrow$ Y conn.

ii) $X \longrightarrow \Omega_* |X|$ heq $\Longleftrightarrow$ X special.

<u>Consequence</u>: If A is any space, then $A(*) : n \mapsto A^n$ is special. Up to homotopy, it is the ~~[illegible]~~ special s. space generated by A, ~~so~~ so

$$\boxed{\cancel{[A(*), X] \simeq [A(*), \Omega_* |X|]}}$$

$$[A(*), X] = [A, X(1)].$$

$$\therefore [\Sigma A, Y] = [A, \Omega Y] = [A(*), \Omega_* Y] = [|A(*)|, Y]$$

$\therefore$ James: $\quad \Sigma A = B\left(\coprod_n A^n\right).$

<u>Criticism</u>: It is not clear that the machine is justified. In fact one has to interpolate a step justifying $A(*) =$ free special simplicial space.
It seems one should have a model in which $A(*) =$ free gadget gen. by A. Therefore maybe one should work with monoids.

Goals of the theory of n-fold loop spaces:

Computation of $H_*(\Omega^n S^n X)$ in terms of $H_*(X)$.

Description of $\Omega^n S^n X$ in small terms – the free invertible gadget generated by X.

Recognizing when $Y = \Omega^n X$ for some X.

Symmetries of $\Omega^n X$, operation on $H_*(\Omega^n X)$.
Descent for the functor $X \longmapsto \Omega^n X$.

Basically one theorem: James type theorems

$$\Omega B\{\amalg X^k\} = \Omega \Sigma X$$

$$\Omega B\{\amalg (X^k)_{\Sigma_k}\} = \Omega^\infty S^\infty X$$

for X pointed and unpointed.

<u>September 19, 1972:</u>

A [redacted] commutative ring, to understand $K_*(A[\varepsilon])$ where $\varepsilon^2=0$, commutes with A.

$$GL_n A[\varepsilon] = GL_n A \tilde{\times} \mathrm{End}(A^n)$$

$$0 \longrightarrow A^n\varepsilon \longrightarrow A[\varepsilon]^n \longrightarrow A^n \longrightarrow 0$$

Segal pointed out yesterday that

$$GL_n A[\varepsilon] \subset (GL_n A \times GL_n A)\tilde{\times} \mathrm{End}\, A^n.$$

is the centralizer of the matrix $\begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}$ so perhaps your results on block groups can be related to this situation.

<u>Example</u>: Suppose we are interested in rational homology and that A is an algebra over $\mathbb{Z}[\ell^{-1}]$ for some prime ℓ. Then consider the spectral sequence of the extension

$$1 \longrightarrow \mathrm{End}\, A^n \longrightarrow GL_n A[\varepsilon] \longrightarrow GL_n A \longrightarrow 1$$

$$E^2_{pq} = H_p(GL_n A, \Lambda_q(\mathrm{End}\, A^n)\underset{\mathbb{Z}}{\otimes}\mathbb{Q}) \Longrightarrow H_{p+q}(GL_n A[\varepsilon], \mathbb{Q}).$$

Now let $\mathbb{Z}[\ell^{-1}]^*$ act as autos. of the ring $A[\varepsilon]$ by $\theta_\lambda(a+b\varepsilon) = a + b\lambda\varepsilon$.

Then $\mathbb{Z}[\ell^{-1}]^*$ acts on the spectral sequences.

Let $\quad X+\varepsilon Y \in GL_n A[\varepsilon], \quad X \in GL_n A, \; Y \in \operatorname{End} A^n.$

$$\theta_\lambda(X+\varepsilon Y) = X + \lambda\varepsilon Y.$$

Thus on the normal subgroup $\operatorname{End} A^n$, θ_λ acts as multiplication by λ. Precisely, $\operatorname{End} A^n$ as an abelian group is a $\mathbb{Z}[\ell^{-1}]$-module and θ_λ acts as multiplication by λ. Thus on $\Lambda_q[(\operatorname{End} A^n)\otimes_{\mathbb{Z}} \mathbb{Q}]$ we have that θ_λ multiplies by λ^q. So

$$\theta_\lambda = \lambda^q \quad \text{on} \quad E_2^{pq}$$

and the spectral sequence therefore must degenerate.

Recall that there are canonical invariant forms

$$e_q : \Lambda^{2q-1} \operatorname{End}(A^n) \longrightarrow A$$

e.g. $\quad e_1 = \operatorname{tr} : \operatorname{End}(A^n) \longrightarrow A$

"Basic generators of the Lie algebra cohomology of $\mathfrak{gl}_n$".

Now over $\mathbb{Q}$ there is a standard decomposition of

$$\Lambda^{\bullet} \mathrm{End}(A^n) \otimes \mathbb{Q}$$

into irreducible representations of $GL_n A$. This is OKAY at least if A is a field extension of $\mathbb{Q}$; we will assume this from now on to simplify. Now one knows

$$H_+(\mathfrak{gl}_n A, M) = 0$$

for any simple non-trivial f.d. $\mathfrak{gl}_n A$ module, so this motivates:

<u>Conjecture:</u> $$E^2_{\bullet} = H_{\bullet *}(GL_n A, \Lambda^{*} \mathrm{End}\, A^n \otimes \mathbb{Q}) \xleftarrow{\sim} H_*(GL_n A) \otimes \Lambda[\check{e}_1, \dots, \check{e}_n]$$

where ~~the generators~~

$$\Lambda[\check{e}_1, \dots, \check{e}_n] = \Lambda^{*}(\mathrm{End}\, A^n \otimes \mathbb{Q}) / GL_n A .$$

<u>Consequence:</u> $\exists$ basic non-trivial maps

$$K_{2i-1}(A[\varepsilon]) \longrightarrow A \otimes_{\mathbb{Z}} \mathbb{Q}$$

This should be verifiable directly, since we have direct summands by the above.

<u>September 21, 1972</u>

Given a space X, why is $B(\coprod_{k\geq 0} X^k) = S(X_+)$?

<u>Example</u>: Suppose X is a set. Then $S(X_+)$ is the realization of the category having objects $X \cup \{0\}$ with

$$\mathrm{Hom}(0, x) = \{\alpha_x, \beta_x\}$$
$$\mathrm{Hom}(0,0) = \{id_0\}$$
$$\mathrm{Hom}(x, x') = \begin{cases} \phi & x \neq x' \\ id_x & x = x' \end{cases}$$

Also $\coprod_{k\geq 0} X^k$ is the free monoid generated by X, and $B(\coprod X^k)$ is the realization of the category defined by the monoid. We have the [redacted] map

(*) $$S(X_+) \longrightarrow B(\coprod_k X^k)$$

induced by the functor

(**) $$\begin{aligned} 0 &\longmapsto 0 \\ x &\longmapsto 0 \\ (\alpha_x : 0 \to x) &\mapsto (id : 0 \to 0) \\ (\beta_x : 0 \to x) &\mapsto (x : 0 \to 0). \end{aligned}$$

<u>Assertion</u>: (*) is a heq.

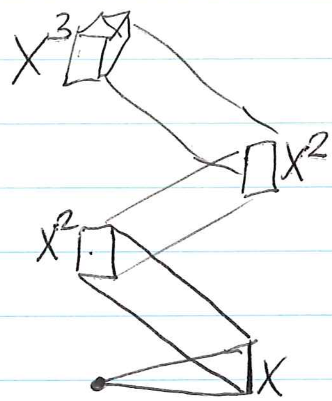
Generalize: Suppose we have a functor $f: \mathcal{C}' \longrightarrow$ [redacted] $\mathcal{C}$, and we form the factorization

$$\mathcal{C}' \xrightarrow{i} \underset{\{(X, Y, fX \to Y)\}}{\overset{\mathcal{E}_f}{\shortparallel}} \xrightarrow{p} \mathcal{C}$$

where $\mathcal{E}_f$ is a (left) $\mathcal{C}$-torsor over $\mathcal{C}'$. (The fibres of $\mathcal{E}_f \to \mathcal{C}'$

are "representable functors" on $\mathcal{C}$.) Then we can show f is a heq by proving that the fibres of p are contractible. In the particular case where $\mathcal{C}$ has one object O, this amounts to proving that f/O is contractible.

In the case of the functor (**), we want to prove contractibility of the category ~~[illegible]~~ fibred over $\mathcal{C}'$ (represents $S(X_+)$) whose objects are pairs (O, m) or (x, m), $m \in \coprod X^k$, $O, x \in \mathcal{C}'$. Picture:

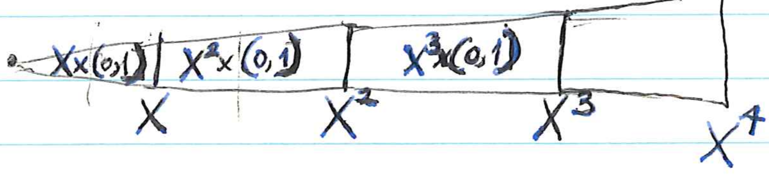


$$\begin{tikzcd} (O,x) \arrow[r] \arrow[dr] & (x,1) \\ (O,1) \arrow[r] & (x,1) \end{tikzcd}$$

$$\mathcal{C}' : \quad O \overset{\beta_x}{\underset{\alpha_x}{\rightrightarrows}} x$$

The contractibility is evident.

In general the "torsor" over $S(X_+)$ appears to be a telescope:



So possibly the general method is this. Let $T(X)$ be the above telescope with the obvious action of $M(X) = \coprod_{k \geq 0} X^k$ on it. Then we can form the simplicial space over $S(X^+)$

$$\begin{tikzcd}
\cdots & T \times M^2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & T \times M \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & T \arrow[r] \arrow[d] & S(X_+) \\
\cdots & M^2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & M \arrow[r, shift left] \arrow[r, shift right] & pt &
\end{tikzcd}$$

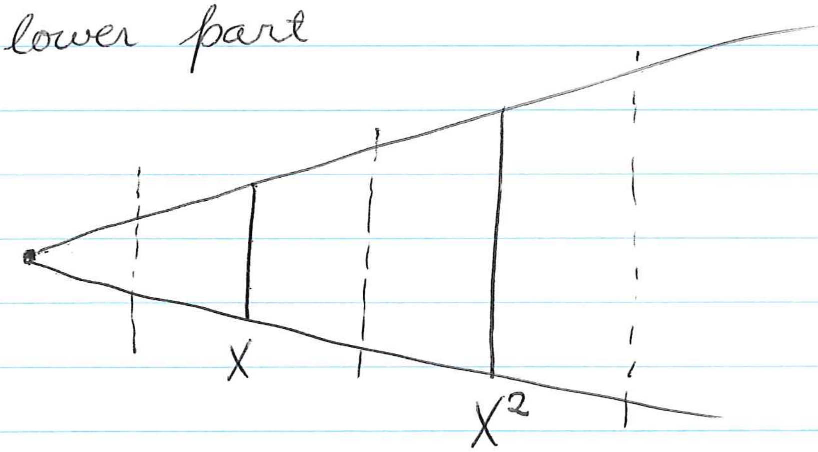
together with a map to Nerv(M). Claim this sets up a heqs

$$|\mathrm{Nerv}(M)| \longleftarrow |\mathrm{Nerv}(T, M)| \longrightarrow S(X_+)$$

The former comes from the fact that T is contractible; for the latter we break up $S(X_+)$.

~~Over the first part T is homotopic to M [illegible] X×M. So it's more~~

Over the lower part



T is equivariantly homotopic to M. Over the upper part it is equivariantly homotopic to $X \times M$. Thus

$$|\mathrm{Nerv}(T, M)| = |\mathrm{Nerv}(M, M)| \cup |\mathrm{Nerv}(X \times M, M)|$$

should be heq to $S(X_+)$.

September 22, 1970

Let F be a functor from (comm.) rings to [redacted] Ab. Can we define an A-module structure on

$$\mathrm{Ker}\,\{F(A[\varepsilon]) \longrightarrow F(A)\}\ ?$$

Example: Assume $F(A) = \mathrm{Hom}_{\text{rings}}(R, A)$. [redacted]

[redacted] We know a map $f : R \longrightarrow A[\varepsilon]$ is a pair consisting of a map $\bar{f} : R \longrightarrow A$ and a derivation $D : R \longrightarrow A$ of R with values in A considered as an R-module via $\bar{f}$. Precisely

$$f(r) = \bar{f}(r) + D(r)\varepsilon$$

Thus if I fix g [redacted] $\in F(A)$, [redacted] we have

$$\{ f \in F(A[\varepsilon]) \mid \bar{f} = g \} = \mathrm{Der}(R, A_g)$$

which is an A-module.

Example: Suppose we fix A (and $\alpha \in F(A)$) and consider all infinitesimal extensions $A \twoheadleftarrow B$ of A and the functor

$$F_\alpha(B) = \{ \beta \in F(B) \mid \beta \mapsto \alpha \}.$$

Suppose F_α is pro-[redacted] representable, i.e. $\exists\ (B_i)$ pro-object $\ni$

$$F_\alpha(B) = \varinjlim\ [redacted]\ \mathrm{Hom}(B_i, B)$$

Then

$$F_\alpha(A[\varepsilon]) = \varinjlim \operatorname{Hom}(B_i, A[\varepsilon])$$

$$= \varinjlim \operatorname{Der}(B_i, A)$$

is an A-module.

The addition on $F_\alpha(A[\varepsilon])$

$$(*) \qquad \begin{array}{ccc} F_\alpha(A[\varepsilon]) \times F_\alpha(A[\varepsilon]) & \xleftarrow{\sim} & F_\alpha(A[\varepsilon] \times_A A[\varepsilon]) \\ & & \downarrow \\ & & F_\alpha(A[\varepsilon]) \end{array}$$

where the vertical arrow is induced by the homomorphism

$$\begin{array}{ccc} A[\varepsilon] \times_A A[\varepsilon] & \longrightarrow & A[\varepsilon] \\ (\varepsilon, 0) & \longmapsto & \varepsilon \\ (0, \varepsilon) & & \end{array}$$

Now for the K-functor one cannot expect the isomorphism $(*)$ to be true by Swan.

<u>Problem:</u> For each element $a \in A$ we can define an endomorphism θ_a of the abelian group

$$\operatorname{Ker}\{K_i A[\varepsilon] \longrightarrow K_i A\}$$

namely by means of the ring homo: $A[\varepsilon] \to A[\varepsilon]$ sending $\varepsilon \mapsto a\varepsilon$. Is

$$\theta_{a+b} = \theta_a \oplus \theta_b \quad ?$$

September 23, 1972.

Let C be a complete n.s. curve over $k = \overline{\mathbb{F}}_p$. Then we have the long exact sequence

$$0 \to K_{2i}C \to K_{2i}F \xrightarrow{\partial} \bigoplus_{x \in C} K_{2i-1}k \to K_{2i-1}C \to \underbrace{K_{2i-1}F \to 0}_{i \geq 2.}$$

Now using cup products (still to be worked out)

$$\begin{tikzcd}
0 \arrow[r] & K_{2i}C \arrow[r] & K_{2i}F \arrow[r] & K_{2i-1}k \otimes D \arrow[r] & K_{2i-1}C & \\
0 \arrow[r] & \mathrm{Tor}(K_{2i-1}k, P) \arrow[r] \arrow[u] & K_{2i-1}k \otimes F^*/k^* \arrow[r] \arrow[u] & K_{2i-1}k \otimes D \arrow[r] \arrow[u, equal] & K_{2i-1}k \otimes P \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Now $\quad K_{2i-1}k \otimes P = K_{2i-1}k \quad$ as $\quad P = \mathbb{Z} \oplus P^0$ torsion

On the other hand we have Gysin map $K_{2i-1}C \to K_{2i-1}k$, which is onto as there are rational points.

Therefore we conclude there are exact sequences

$$0 \to K_{2i}C \to K_{2i}F \xrightarrow{\partial} K_{2i-1}k \otimes D \xrightarrow{\Sigma} K_{2i-1}k \to 0$$

$$K_{2i-1}C \cong K_{2i-1}k \oplus K_{2i-1}F \qquad i > 1$$

$$K_1C \cong k^* \oplus k^*$$

Conjectures: $\quad \mathrm{Tor}_1(K_{2i-1}k, P) \xrightarrow{\sim} K_{2i}C \qquad i \geq 1$

$$K_{2i-1}k \oplus K_{2i-1}k \xrightarrow{\sim} K_{2i-1}C \qquad i \geq 1$$

$$K_{2i-1}k\otimes F^{\cdot} \xrightarrow{\sim} K_{2i-1}k\otimes F^{\cdot}/k^{\cdot} \xrightarrow{\sim} K_{2i}F \qquad i\geq 1$$

$$K_{2i-1}k \xrightarrow{\sim} K_{2i-1}F \qquad i>1$$

Here is another way of seeing that for $i_x: \mathrm{Sp}(k)\to C$ we have

$$(i_x)_* = (i_y)_*$$

for any two points $x, y\in C$. Namely [redacted] if $f: C\longrightarrow \mathrm{Sp}(k)$ is the canon. map, then

$$(i_x)_*\alpha = (i_x)_*(i_x)^* f^*\alpha$$

$$= (i_x)_* 1\cdot f^*\alpha$$

where $(i_x)_* 1 \in \tilde{K}_0 C = \mathrm{Pic}(C)$. Thus

$$(i_x)_*\alpha - (i_y)_*\alpha = ((i_x)_* 1 - (i_y)_* 1)\cdot f^*\alpha$$

is in the image of a map

$$\mathrm{Pic}^0(C)\otimes K_{2i-1}k \longrightarrow K_{2i-1}C$$

and the former is zero because $\mathrm{Pic}^0(C)$ is [redacted] torsion and $K_{2i-1}k$ is divisible.

Observe in any case that we have established a diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & K_{2i}C \arrow[r] & K_{2i}F \arrow[r] & K_{2i-1}k \otimes D^{\circ} \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Tor}(K_{2i-1}k, \mathrm{Pic}^{\circ}) \arrow[r] \arrow[u] & K_{2i-1}k \otimes F^{\circ} \arrow[r] \arrow[u] & K_{2i-1}k \otimes D^{\circ} \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

for all $i \geq 1$.

Suppose now k is a finite field and A is the coordinate ring of an affine curve over k and that i: $\mathrm{Spec}\, k \to \mathrm{Spec}\, A$ is a rational point. I would like to show that the trace

$$i_* : K_{2i-1}k \longrightarrow K_{2i-1}A$$

is zero.

Since we have maps

$$k \xrightarrow{f} A \xrightarrow{i} k$$

with composite the identity $\alpha = i^* f^* \alpha$, so

$$i_* \alpha = i_* i^* f^* \alpha = i_* 1 \cdot f^* \alpha$$

where $i_* 1 \in \tilde{K}_0 A = \mathrm{Pic}\, A$. Here $\mathrm{Pic}\, A$ is not divisible, but we can hope to make $i_* 1$ arbitrarily divisible in some extension.

So let k' be a finite extension of k. ~~We are assuming k is integrally closed in A (this is what we mean by a curve over k)~~ It is known, I believe, that $k' \otimes_k A$ is the coordinate ring of an affine curve over k'. ($k'\otimes_k A$ is a Dedekind domain = the integral closure of A in $k'\otimes_k F$ which is a field.) In any case, we have a commutative diagram

$$\begin{array}{ccc} K_{2i-1}k' & \xrightarrow{i'_*} & K_{2i-1}A' \\ \downarrow{\scriptstyle tr} & & \downarrow{\scriptstyle tr} \\ K_{2i-1}k & \xrightarrow[i_*]{} & K_{2i-1}A \end{array} \qquad A' = k'\otimes_k A$$

Where the tr at the left is surjective by my computation. So what we have to do is to show i'_* is zero.

What we know is that

$$K_{2i-1}\bar{k} \xrightarrow{\bar{i}_*} K_{2i-1}\bar{A}$$

is zero because $\operatorname{Pic}\bar{A}$ is divisible and $K_{2i-1}\bar{k}$ is torsion. To simplify, suppose A obtained by removing one rational point from a complete curve C over k, so that

$$\operatorname{Pic}(A) \simeq \operatorname{Pic}^\circ(C)$$

$$\operatorname{Pic}(A') \simeq \operatorname{Pic}^\circ(C') \qquad C' = k'\times_k C$$

$$\operatorname{Pic}(\bar{A}) \simeq \operatorname{Pic}^\circ(\bar{C})$$

Now one knows the l-primary component of $\operatorname{Pic}^\circ(\bar{C})$ is $(\mathbb{Q}_l/\mathbb{Z}_l)^{2g}$.

The hope is that $\iota'_* 1$, which is the image of $\iota_* 1$ under the map $\operatorname{Pic} A \longrightarrow \operatorname{Pic} A'$, becomes divisible at a faster rate than the order of the element $\beta \in K_{2i-1} k'$ needed such that $tr(\beta) = \alpha$ = a given element of $K_{2i-1} k$. <u>Doesn't work.</u>

Observe that if the conjectures <u>on</u> page **1** are true, then for F = alg. closure $\overline{k(T)}$ we have

$$W^{(i)} = K_{2i-1} k \xrightarrow{\sim} K_{2i-1} \overline{F} \qquad i > 1$$

$$K_{2i} \overline{F} = 0 \qquad i \geq 1.$$

Set $k_1 = \overline{F}$ and let C be a curve over k_1. Then $\operatorname{Pic}^\circ C$ divisible and $K_{2i-1} k_1$ torsion for $i \geq 2$ makes

$$\begin{tikzcd}
0 \arrow[r] & K_{2i} C \arrow[r] & K_{2i} F_C \arrow[r] & K_{2i-1} k \otimes D_C^{\deg 0} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Tor} \arrow[r] \arrow[u] & K_{2i-1} k \otimes F_C^\circ \arrow[r] \arrow[u] & {} \arrow[r] & 0
\end{tikzcd}$$

Therefore if $k_2 = \overline{k(T_1, T_2)}$, it is reasonable to conjecture that

$$K_{2i}(k_2) = 0 \qquad i \geq 2$$

$$W^{(i)} = K_{2i-1}(k_2) \qquad i > 2$$

September 26, 1972

A = Dedekind domain, $F_r Q$ = full subcat of $Q = Q(\mathcal{P}_A)$ consisting of modules of rank $\leq r$.

Suppose A a field to simplify, let [redacted] $G_r = GL_r A$ and X_r = building of A^r = simplicial complex whose simplices are the chains of proper subspaces of A^r. [redacted] Let J_r = the ordered set of layers $0 < W \leq W' < A^r$; [redacted] then G_r acts on J_r, so we obtain a cofibred category (J_r, G_r) over G_r with fibre J_r.

Now there is an evident functor

$$(J_r, G_r) \longrightarrow F_{r-1} Q$$

$$(W_0 \leq W_1) \longmapsto W_1/W_0$$

$$\begin{array}{l} (W_0', W_1') \xrightarrow{g} (W_0, W_1) \\ (\text{i.e. } W_0 \leq gW_0' \leq gW_1' \leq W_1) \end{array} \longmapsto \left(W_1'/W_0' \longrightarrow W_1/W_0 \right)$$

and there are two ways of contracting this functor to a point.

$$(W_0 \leq W_1) \longrightarrow [redacted] \qquad \begin{array}{c} W_1/W_0 \\ \Downarrow \\ W_1/0 \\ \Uparrow \\ 0/0 \end{array} \qquad \begin{array}{c} W_1/W_0 \\ \Downarrow \\ V/W_0 \\ \Uparrow \\ V/V . \end{array}$$

Thus we get a map

$$S(J_r, G_r) \longrightarrow F_{r-1} Q .$$

Claim

$$\begin{tikzcd} S(J_r, G_r) \arrow[r] \arrow[d] & S(G_r) \arrow[d] \\ F_{r-1}Q \arrow[r] & F_rQ \end{tikzcd}$$

commutes. (The second vertical arrow comes from the functor $G_r \to F_r$ given by G_r acting on $A^r \cong V$, plus the two contractions

$$0 \xrightarrow{\text{inj}} V \xleftarrow{\text{surj}} 0.)$$

So first have

$$\begin{tikzcd} (J_r, G_r) \arrow[r, "{(W_0,W_1)\ \mapsto\ (0,V)}"] \arrow[d] & G_r \arrow[d] \\ F_{r-1}Q \arrow[r, "(W_1/W_0)"] & F_rQ^{\,V} \end{tikzcd}$$

and this commutes up to the natural transf.

$$\begin{tikzcd} (W_0, W_1) \arrow[r, mapsto] \arrow[d, mapsto] & V \\ W_1/W_0 \arrow[ur, Rightarrow, "\text{evident map}"'] & \end{tikzcd}$$

I have to check this is compatible with the two contractions

$$\begin{tikzcd} W_1/W_0 \arrow[r] \arrow[d, leftrightarrow] & V \\ W_1/0 \arrow[ur, hook] & 0 \arrow[u, hook] \\ 0/0 \arrow[u, hook] \arrow[r, equal] & 0 \end{tikzcd}$$

OKAY and ditto for other one.

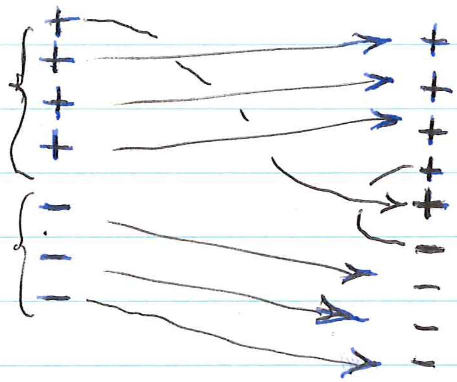
So here is what I have found. There is a cocartesian square (up to homotopy)

$$\begin{array}{ccc} S(J_r, G_r)_+ & \longrightarrow & S(G_r)_+ \\ \downarrow & & \downarrow \\ F_{r-1}Q & \longrightarrow & F_r Q . \end{array}$$

in which the cofibration is the suspension of the cofibre of $(J_r, G_r) \longrightarrow G_r$, which is analogous to a Thom space.

September 28, 1972

Recall the model for $\Omega^\infty S^\infty$: the category whose objects are pairs of finite sets and whose morphisms $(S^+,S^-) \longrightarrow (T^+,T^-)$ [redacted] consist of a pair $S^+ \hookrightarrow T^+$, $S^- \hookrightarrow T^-$ of injections together with an isomorphism of the complements.



This category divides up into components card $S^+ -$ card $S^- = k$. In addition, there is an obvious filtration on each component. Take the [redacted] degree 0 component, where card $S^+ =$ card S^-, and consider the [redacted] filtration by cardinality. Then it will be desirable to understand the category $f/(T^+,T^-)$. This may be identified with the ordered set consisting of

$$S^+ \subset T^+,\ S^- \subset T^- \quad \text{and } \theta: T^+ - S^+ \xrightarrow{\sim} T^- - S^-$$

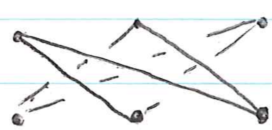
with [redacted] $(S'_+, S'_-, \theta') \leq (S_+, S_-, \theta)$ if $S'_+ \subset S_+$ and θ' restricts to θ. It is simply to describe the opposite ordered set which consists of a subset C of T^+ and an injection $\theta : C \hookrightarrow T^-$ where $(C', \theta') \leq (C, \theta)$ iff $C' \subset C$ and θ restricts to θ'.

<u>Problem:</u> If C is a finite set of card $\geq n$, describe the homotopy type of the ordered set of pairs (σ, θ) where σ is a non-empty ordered subset of $\{1, \ldots, n\}$ and where $\theta : \sigma \hookrightarrow C$.

This is a simplicial complex consisting of those non-empty subsets of $\{1, \ldots, n\} \times C$ which map injectively under both projections. It is a s. cx. of dim $n-1$.

Ex. <u>$n=2$</u>

card $C = 2$  not conn.

card $C > 2$  connected, bouquet of S^1.

circle with three pairs of points collapsed, so this is of the homotopy type of $\bigvee^4 S^1$

It is possible that the $n-n$ configuration, i.e. simplices in $\{1, \ldots, n\}^2$ projecting non-degenerately for each projection, is $\sim \frac{n}{2}$-connected.

September 28, 1972:

Consider the unitary groups U_n and form [redacted] Segal's simplicial space corresponding to Q:

$$\coprod_{m,n} BU_{mn} \;\overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}}\; \coprod BU_n \rightrightarrows pt$$

We can filter this in the same way with F_r:

$$\coprod_{m+n\le r} BU_{mn} \;\overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}}\; \coprod_{n\le r} BU_n \rightrightarrows pt$$

The cofibre F_r/F_{r-1} is

$$\begin{matrix} \coprod_{m+n\le r} BU_{mn} & \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} & [redacted]\; BU_r & \rightrightarrows pt \\ \cup & & \cup & \\ pt & & pt & \end{matrix}$$

which has non-degenerate stuff

$$\coprod_{\substack{i+j+k=r\\ i,j,k>0}} BU_{ijk} \rightrightarrows \coprod_{\substack{i+j=r\\ i,j>0}} BU_{ij} \longrightarrow BU_r \qquad pt$$

with the rest of the faces $=$ "0". This suggests that

$$F_r/F_{r-1} = S\left(\mathrm{Cone}\left((X_r)_{U_r} \longrightarrow BU_r\right)\right)$$

where X_r is the simplicial space consisting of flags in $\mathbb{C}^r$:

$$(X_r)_k = \coprod_{\substack{[redacted]\\ a_0+\cdots+a_{k+1}=r\\ a_0,\,a_{k+1}>0}} U_r/U_{a_0\cdots a_{k+1}}$$

September 30, 1972

spherical fibrations

Let $S^n = \mathbb{R}^n \cup \{\infty\}$ be the n-sphere with basepoint, ~~and~~ so that $\Omega^n S^n$ is the space of maps $S^n \to S^n$ preserving the basepoint. Then

$$G_n = (\Omega^n S^n)_{\pm 1}$$

is the monoid of self-heq's of S^n. BG_n classifies fibrations $Y \to X$ with section such that the fibre has the homotopy type of S^n. Since G_n is ~~[illegible]~~ a top. monoid such that $\pi_0 G_n$ is a group, we have

$$G_n = \Omega B G_n .$$

Fix a prime p and consider the monoid

$$\coprod_{k \geq 0} (\Omega^n S^n)_{p^k}$$

of maps $S^n \to S^n$ of degree p^k. Question: Is

$$B \coprod_{k \geq 0} (\Omega^n S^n)_{p^k} = B(SG_n)\left[\tfrac{1}{p}\right]$$

and does it classify fibrations with fibre $S^n[\frac{1}{p}]$?

This is already wrong because of the fundamental group. Modifications:

$$B\left\{ \coprod_{k \geq 0} (\Omega^n S^n)_{p^k} \amalg (\Omega^n S^n)_{-p^k} \right\}$$ classifies fibrations with fibre $S^n[\frac{1}{p}]$ (+section). ?

$$\Omega B\left\{\coprod_k (\Omega^n S^n)_{\pm p^k}\right\} = \{\pm p^k\} \times SG_n\left[\tfrac{1}{p}\right] \quad ?$$