August 1, 1972

Let A be a Dedekind domain, $\mathcal{M} = \mathcal{P}_A$, $\mathcal{C}' \subset \mathcal{C} \subset Q(\mathcal{M})$ full subcategories consisting of projective modules of ranks $< r$ and $\leq r$ resp.,

$$j : \mathcal{C}' \longrightarrow \mathcal{C}$$

the inclusion functor. Observe that $\mathcal{C}'$ is a crible in $\mathcal{C}$, so in some sense, $\mathcal{C}'$ is open in $\mathcal{C}$.

Let $M \in \mathcal{C}$. Then

$$(R^q j_* \Lambda)(M) = H^q(j/M, \Lambda)$$

if we use contravariant functors. If $M \in \mathcal{C}'$, the cat. j/M has a final object, and the cohomology is trivial, so suppose $\operatorname{rank}(M) = r$.

j/M is equivalent to the ordered set of admissible layers in M of rank $< r$, which is the same for the K-module $M \otimes K$. We know that j/M has the homotopy type of the suspension of the building of $M \otimes K$. Thus

$$j/M \sim \text{bouquet of } S^{r-1}$$

and precisely

$$\tilde{H}^q(j/M, \Lambda) = \begin{cases} 0 & q \neq r-1 \\ \operatorname{Hom}(\tilde{H}_{r-1}(j/M), \Lambda) & q = r-1. \end{cases}$$

~~The homology group~~ $\cancel{\tilde{H}_{r-1}(j/M) = \tilde{H}_{r-2}(X(M \otimes K))}$

More accurately: Let $\bar{X}(M \otimes K)$ be the s. complex of chains $0 \subset V_0 < \cdots < V_p \subset M \otimes K \ni \dim(V_p/V_0) < r$. Then

$$r = 1, \qquad \bar{X}(M \otimes K) = \text{two points}$$

$$r = 2, \qquad \bar{X}(M \otimes K) = \sum \mathbb{P}_1 K.$$

The point is that

$$\tilde{H}_{r-1}(\bar{X}(V), \mathbb{Z})$$

is a free $\mathbb{Z}$-module on which $\mathrm{Aut}(V)$ acts. It is called the Steinberg representation, and will denote it by $St(V)$. It will perhaps be important to recall that if we fix a flag $0 < V_1 < \cdots < V_{r-1} < V$ in V and let B be the Borel subgroup associated to this flag, then as a B-module

$$St(V) \cong \mathbb{Z}[B/T].$$

Observe that $M \mapsto \Lambda^r M \in \mathrm{Pic}(A)$ determines a module of rank r up to isomorphism. Let M_α, $\alpha \in \mathrm{Pic}(A)$ be representatives. Then we have an open-closed situation.

$$\mathcal{C}_{r-1} \xrightarrow{\ j\ } \mathcal{C}_r \xleftarrow{\ i = \coprod i_\alpha\ } \coprod_\alpha \mathrm{Aut}(M_\alpha)$$

and we have seen that

$$R^{r-1} j_*(\Lambda) = i_*\left(\alpha \mapsto \mathrm{Hom}(St(M_\alpha \otimes K), \Lambda)\right)$$

Thus we have a triangle

$$\Lambda \longrightarrow \mathbb{R}j_*(\Lambda) \longrightarrow \prod_\alpha (i_\alpha)_*\left(\mathrm{Hom}(St(M_\alpha \otimes K), \Lambda)\right)[-r+1]$$

and a long exact sequence in cohomology

$$\longrightarrow \prod_{\alpha} H^{q-r}(\mathrm{Aut}\, M_\alpha, \mathrm{Hom}(St(M_\alpha \otimes K), N)) \longrightarrow H^q(\mathcal{C}_r, \Lambda) \longrightarrow H^q(\mathcal{C}_{r-1}, \Lambda) \longrightarrow$$

which homologically should amount to a long exact sequence

$$\xrightarrow{\partial} H_q(\mathcal{C}_{r-1}, \mathbb{Z}) \longrightarrow H_q(\mathcal{C}_r, \mathbb{Z}) \longrightarrow \sum_{\alpha \in \mathrm{Pic}(A)} H_{q-r}(\mathrm{Aut}\, M_\alpha, St(M_\alpha \otimes K))$$

I can give a better derivation by using homology as follows.

$$\mathcal{C}_{r-1} \xrightarrow{j} \mathcal{C}_r \xleftarrow{i} \coprod_{\alpha} \mathrm{Aut}(M_\alpha)$$

If we use covariant functors

$$L_* j_!(\mathbb{Z})(N) = H_*(j/N, \mathbb{Z}) = \mathbb{Z}[0]$$

if $N \in \mathcal{C}_{r-1}$ and

$$L_* j_!(\mathbb{Z})(M_\alpha) = H_*(j/M_\alpha, \mathbb{Z})$$

$$= \begin{cases} \mathbb{Z} & * = 0 \\ 0 & * \neq 0, r-1 \\ St(M_\alpha \otimes K) & * = r-1 \end{cases}$$

hence we have a triangle

$$\coprod_{\alpha} (i_\alpha)_! \, St(M_\alpha \otimes K)^{[n-1]} \longrightarrow \mathbb{L} j_!(\mathbb{Z}) \longrightarrow \mathbb{Z}$$

which leads to the above exact sequence.

<u>August 2, 1972</u>

Review Borel-Serre, at least in the case of SL_n over $\mathbb{Q}$. They complete

$$X = SL_n\mathbb{R}/SO_n$$

to a cornered manifold (non-compact) $\bar{X}$ by adjoining

$$\partial\bar{X} \sim \text{building belonging to the parabolics in } SL_n$$

Thus
$$\partial\bar{X} \sim \vee S^{n-2}$$
and
$$\tilde{H}_{n-2}(\partial\bar{X}) = St(\mathbb{Q}^n)$$

Thus
$$(\bar{X},\partial\bar{X}) \sim \vee S^{n-1}$$
$$H_{n-1}(\bar{X},\partial\bar{X}) = St(\mathbb{Q}^n).$$

Now if $\Gamma \subset SL_n\mathbb{Z}$ is torsion-free of finite index, Borel-Serre prove Γ acts freely on $\bar{X}$ and $\bar{X}/\Gamma$ is compact (topologically it is a manifold with ∂; from C^∞-point of view it is a cornered manifold). One has spec. sequence for any Γ-module M

$$E^2_{pq} = H_p(\Gamma, H_q(\bar{X},\partial\bar{X};\tilde{M})) \Longrightarrow H_{p+q}(\bar{X}/\Gamma, \partial\bar{X}/\Gamma;\tilde{M})$$

which degenerates. By Poincare duality + compactness of $\bar{X}_\Gamma$ have

$$H_i(\bar{X}/\Gamma,\partial\bar{X}/\Gamma;\tilde{M}) \xleftarrow[\cap\mu\in H_d(\bar{X}_\Gamma,\partial\bar{X}_\Gamma;\mathbb{Z})]{\sim} H^{d-i}(\bar{X}/\Gamma;\tilde{M}) = H^{d-i}(\Gamma,M)$$

and thus we get the Borel-Serre duality formula

$$H_p(\Gamma, I \otimes M) = H^{(d-\ell)-p}(\Gamma, M)$$

where

$d = \dim X$

$\ell = \operatorname{rank}_{\mathbb{Q}} G = n-1$ for SL_n

$I =$ Steinberg repn.

But what is fascinating about the above is that we have a geometric ~~[illegible]~~ interpretation of $H_*(\Gamma; St(\mathbb{Q}^n))$, namely

$$H_{i}(\Gamma, St(\mathbb{Q}^n)) = H_{n-1+i}(\bar{X}_\Gamma, \partial \bar{X}_\Gamma; \mathbb{Z})$$

This should be true for $\Gamma = GL_n\mathbb{Z}$. But we also saw that

$$H_{\underset{n-1+i}{\overset{\shortparallel}{j}}}(\mathcal{C}_n, \mathcal{C}_{n-1}) = H_{\underset{i}{\overset{\shortparallel}{j-n+1}}}(GL_n\mathbb{Z}, St(\mathbb{Q}^n)).$$

This suggests that there should be a close relation between ~~[illegible]~~ the pairs

$$(\mathcal{C}_n, \mathcal{C}_{n-1}) \text{ and } (\bar{X}_\Gamma, \partial\bar{X}_\Gamma)$$

August 2, 1972

I want to achieve an understanding of homotopy inverse limits and especially how they relate to the descent problem in algebraic K-theory.

1. The Zariski descent problem in algebraic K-theory: Let X be a noetherian scheme. For each open set U of X we consider the abelian category

$$\mathcal{M}(U) = \mathrm{Modf}(\mathcal{O}_U).$$

These categories are the fibres of a fibred category $\mathcal{M}$ over the cat. Open (X). $\mathcal{M}$ is a stack over the Zariski site of X.

Let

$$K_i(U) = K_i(\mathcal{M}(U)) = \pi_{i+1} Q(\mathcal{M}(U)).$$

These are presheaves on X; let $\mathcal{K}_i$ be the associated sheaves. Then we should have

$$(\mathcal{K}_i)_x = K_i(\mathrm{Modf}(\mathcal{O}_x))$$

One formulation of the Zariski descent problem is to construct a spectral sequence of the form

$$(1) \qquad E_2^{pq} = H^p(X, \mathcal{K}_{-q}) \Longrightarrow K_{-p-q}(X).$$

Slightly more general formulation: If $\mathcal{U} = \{U_n\}$ is a Zariski hypercovering of X, then there is a spectral sequence

$$(2) \qquad E_2^{pq} = \check{H}^p(\nu \mapsto K_{-q}(U_\nu)) \Longrightarrow K_{-p-q}(X).$$

(2) $\Longrightarrow$ (1) by Verdier's theorem.

~~We can try to generalize (2) as follows. Suppose given a category I and a torsor P under I over X. Thus for each object $i \in I$ we have a sheaf P_i on X and for each map $i' \to i$ a map $P_i \to$~~

We can try to generalize (2) as follows. Suppose I is a category and we have a contravariant functor $i \mapsto U_i$ from I to sheaves on X. Then we get a functor

$$\begin{array}{ccc} \mathrm{Top}(X) & \xrightarrow{f_*} & I^\vee \quad (\text{cov. functors}) \\ F & & (i \mapsto F(U_i)) \end{array}$$

I want to assume for each x, that the category of couples (i, η), $i \in I$, $\eta \in (U_i)_x$ is contractible, whence it should be the case that there is a spectral sequence

$$E_2^{pq} = H^p(I, i \mapsto H^q(U_i, F)) \Longrightarrow H^{p+q}(X, F).$$

for any abelian sheaf F. Now (2) is a special case of a spectral sequence of the form

(3) $$E_2^{pq} = H^p(I, i \mapsto K_{-q}(U_i)) \Longrightarrow K_{-p-q}(X).$$

<u>Examples:</u> of such functors $I^\circ \to Top(X)$.

a) cribles. Take I° to be a covering crible R with the evident functor $R \to Open(X)$. Given x, the category of couples (U, η) is the directed set of open sets in R containing x (they exist since R is covering). Thus the cat of couples (i, η) fibred over I is filtering, hence contractible. In this case the spectral sequence in question is the Leray spectral sequence for the canonical morphism of topoi

$$Top(X) \longrightarrow R^\wedge .$$

(I recall for any site, $\exists$ can. morph. $\tilde{R} \longrightarrow R^\wedge$ whose inverse image is "associated sheaf".)

b) Generalize a) by taking R to be a presheaf whose associated sheaf is e, the final object of $Top(X)$. Again we have that the spectral sequence is the Leray spectral sequence for the morphism of topoi:

$$Top(X) \longrightarrow R^\wedge .$$

<u>Question:</u> Can cohomology be computed using such presheaves R? Observe that the category of such presheaves (i.e. $\ni$ $R \to e$ is bicovering) is cofiltering. Thus we can take the limit over R in the spectral sequence

$$E_2^{pq} = H^p(R, \mathcal{H}^q(F)) \Longrightarrow H^{p+q}(X, F)$$

and hopefully E_2^{p+} will be zero in the limit, yielding an isomorphism

$$\varinjlim_{R} H^p(R, F) = H^p(X, F).$$

c) Suppose X covered by two open sets U, V and consider the functor

$$\begin{tikzcd} & \cdot \arrow[dl] \\ \cdot & \\ & \cdot \arrow[ul] \end{tikzcd} \longmapsto \begin{tikzcd} & U \\ U \cap V \arrow[ur, hook] \arrow[dr, hook] & \\ & V \end{tikzcd}.$$

Here I = category of simplices in 1-simplex. More generally we can consider n open sets.

descent problem

August 4, 1972

To understand flask sheaves.

X top. space, F sheaf (abelian) on X. One says F is <u>flask</u> if whenever we have open sets $U \subset V$, then

$$F(V) \longrightarrow F(U)$$

is surjective. Let's make a list of possibilities:

1) ~~[illegible]~~ $U \subset V \Longrightarrow F(V) \twoheadrightarrow F(U)$

1)' $\exists$ crible R covering X such that

$$U \subset V \subset R \Longrightarrow F(V) \twoheadrightarrow F(U).$$

~~2) For any covering $\mathcal{U}$ we shall have $H^1(\mathcal{U}, F) = 0$~~

3) $H^1_Z(X, F) = 0$ for all locally closed Z in X.

3) For any pointed sheaf of sets S, i.e. endowed with a section, we have

$$\mathrm{Ext}^+(\mathbb{Z}S, F) = 0$$

3)' For any abelian sheaf G whose stalks are free over $\mathbb{Z}$ we have

$$\mathrm{Ext}^+(G, F) = 0$$

4) For any sheaf of sets S, we have

$$\mathrm{Ext}^+(\mathbb{Z}S, F) = 0.$$

4') For any open set U, $H^+(U, F) = 0$.

Now we have the following relations:

1) $\Longleftrightarrow$ 1') See Godement for local character of flaskness.

2) If $Z = U - V$, then

$$F(U) \to F(V) \to H^1_Z(X, F) \to H^1(U, F)$$

so 1) $\Longleftrightarrow$ 2) (using fact that flask $\Longrightarrow H^1(U, F) = 0$ all U).

Clearly 3) $\Longrightarrow$ 2) since given $V \subset U$ ~~we choose~~ can take S to be ~~two~~ a ~~copies~~ copy of U glued to X along V, whence $\bar{\mathbb{Z}}S = \mathbb{Z}U + {}^{\mathbb{Z}V}\!\!\!\!\!\!\!\!\!\!\!\! \qquad = \mathbb{Z}_Z$.

Conversely, given S and F flask

$$\mathbb{Z}X \longrightarrow \mathbb{Z}S \longrightarrow \bar{\mathbb{Z}}S$$

we have to prove $H^+(S, F) = 0$ and that $F(S) \twoheadrightarrow F(X)$. The latter is clear as we have a map $S \to X$, so we have to prove $H^+(S, F) = 0$. But we can replace X by S, so this results from the local character of flaskness.

I know that 4') $\not\Rightarrow$ 1). (Soft sheaves such as C^∞ fns on a C^∞ manifold).

<u>Proposition:</u> An abelian sheaf F on a top. space X is flask $\Longleftrightarrow$ for all [redacted] etale spaces $f: S \to X$ we have $H^+(S, f^*F) = 0$.

Proof. ($\Rightarrow$) [redacted] Let R be the crible over S such that an open $U \subset S$ is in $R \Longleftrightarrow U \xrightarrow{f_u} X$ is an open immersion. Then for $V \subset U$ in R we have

$$\Gamma(U, f^*F) = \Gamma(U, f_u^*(F)) = \Gamma(f_u(U), F)$$

so by flaskness of F, $\Gamma(U, f^*F) \twoheadrightarrow \Gamma(V, f^*F)$. ~~So~~ by the local character of flaskness, f^*F is flask on S, whence

$$H^+(S, f^*F) = 0$$

($\Leftarrow$). Given $V \subset U \subset X$ and let S be the sheaf of sets

$$S = U \amalg^V X$$

Then we have a split exact sequence

$$0 \longrightarrow \mathbb{Z}_X \rightleftarrows \mathbb{Z}_S \longrightarrow \mathbb{Z}_U / \mathbb{Z}_V \longrightarrow 0$$

so $0 = H^1(S, F) = \mathrm{Ext}^1(\mathbb{Z}_S, F) = \mathrm{Ext}^1(\mathbb{Z}_U/\mathbb{Z}_V, F)$.

Since we have the long exact sequence

$$H^0(U, F) \longrightarrow H^0(V, F) \xrightarrow{\delta} \mathrm{Ext}^1(\mathbb{Z}_U/\mathbb{Z}_V, F)$$

it follows that F is flask.

The preceding proposition works for a topos.
Thus in SGA4, we have acyclic $\Leftrightarrow$ flask.

<u>Def</u>: A sheaf F (abelian) in a topos is <u>flask</u> if

$$H^{+}(S, F) = 0$$

for all S in F.

<u>Examples</u>: 1) G-sets. Then a G-module is flask $\Leftrightarrow$

$$H^{+}(H, M) = 0$$

for all $H \subset G$. Same as cohomological triviality.

2) I ordered set, $E =$ topos $I^{\wedge}$. $F : I^{\circ} \to ab$ is flask provided for every $x \in I$ and crible $U \subset I/x$ we have

$$F(x) \twoheadrightarrow \varprojlim_{U} F$$

(Here I is a topological space in which the open sets are cribles in I, so what we are giving is the local criterion for flaskness for all opens contained in an open of the form I/x). Small example:

$$\begin{tikzcd} \alpha \arrow[dr] & \\ & \gamma \\ \beta \arrow[ur] & \end{tikzcd}$$

Then the condition amounts to $F(\gamma) \twoheadrightarrow F(\alpha) \times F(\beta)$.
Another:

$$\begin{tikzcd} & U \\ U \cap V \arrow[ur] \arrow[dr] & \\ & V \end{tikzcd}$$

then the condition is simply surjectivity: $F(U) \twoheadrightarrow F(U \cap V)$, $F(V) \twoheadrightarrow F(U \cap V)$.

3. Something similar for a category without loops?

Conjecture: Let X be a top space and let $\mathcal{A}$ be a stack over X. Assume that $\forall\, V \subset U$ the functor

$$\mathcal{A}(U) \longrightarrow \mathcal{A}(V)$$

is h-flat, meaning that 2-base change = h-base change, i.e. any 2-cartesian ~~sq~~ square

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{A}(U) \arrow[d] \\ \mathcal{C}' \arrow[r] & \mathcal{A}(V) \end{tikzcd}$$

($\mathcal{C} \to \mathcal{C}' \overset{2}{\underset{\mathcal{A}(V)}{\times}} \mathcal{A}(U)$ equivalence) is h-cartesian. Then the conjecture asserts that the canonical functor

$$\Gamma(X, \mathcal{A}) \longrightarrow h\Gamma(X, \mathcal{A})$$

is a heq.

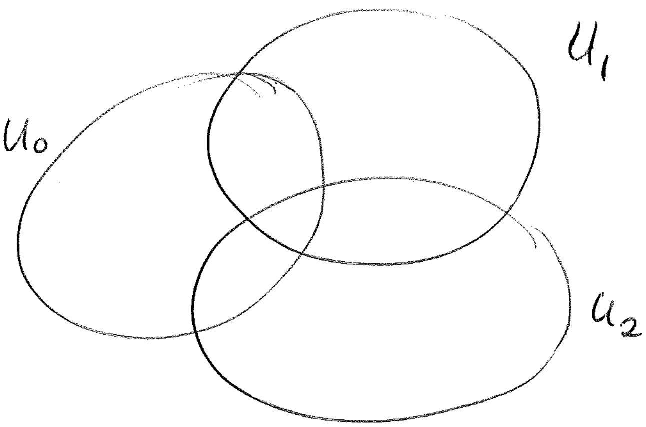
Question: Given a top space X and a functor from pairs (U, V), $V \subset U$, of open sets to long exact sequences

$$F^q(U,V) \to F^q(U) \longrightarrow F^q(V) \xrightarrow{\delta} F^{q+1}(U,V)$$

such that excision holds: $F^*(U,V) \xrightarrow{\sim} F^*(U',V')$ if $U'-V' = U-V$. Does there then exist a spectral sequence

$$E_2^{pq} = H^p(X, \widetilde{F^q}) \Longrightarrow F^{p+q}(X)?$$

It does not seem possible to produce ~~[illegible]~~ the spectral sequence from such limited data ~~[illegible]~~ ~~[illegible]~~ even in the special case of a covering by ~~[illegible]~~ 3 open sets



It seems one needs the **skeleta** of Segal's space

$$\bigcup U_\sigma \times \Delta(\dim \sigma)$$

which are not of the homotopy type of any open set.

Typical problem: Given $\mathcal{A}$ fibred over a group G, when is $\varprojlim \mathcal{A} = \operatorname{holim} \mathcal{A}$? Want a kind of freeness condition which would hold in the Galois situation.

August 7, 1972

flaskness for simplicial sets:

Let Y be a simplicial set, $F : (\Delta/Y)^\circ \to \mathcal{Ab}$ an abelian sheaf over Y, i.e. of the induced topos $\Delta^\wedge/Y$. I claim that

$$\boxed{F \text{ flask} \iff \forall y \in \Delta/Y, \quad F(y) \twoheadrightarrow F(\mathring{y}).}$$

Proof. Let Z be a simplicial set over Y; I will show $H^1(Z, F) = 0$. Let P be a torsor for F over Z, and let U be a simplicial subset over which $\exists$ section s of P, and such that (U, s) is maximal. If $U < Z$, let y be a simplex of Z not in U of least dimension k. Then we have

$$\begin{tikzcd}
\Delta(k)^\circ \arrow[r, hook] \arrow[d] & \Delta(k) \arrow[d] \arrow[dr, "y"] & \\
U \arrow[r, hook] \arrow[rr, bend right] & U \amalg_{\Delta(k)^\circ} \Delta(k) \arrow[r, "\alpha"] & Z
\end{tikzcd}$$

I claim that the arrow α is injective; this is a consequence of the Eilenberg-Zilber lemma which implies

$$\Delta(k)_m - \Delta(k)^\circ_m \hookrightarrow Z_m \qquad \forall m$$

as y is a non-degenerate simplex of Z.
Now $\exists$ section s' of P_y; comparing restrictions to $\Delta(k)^\circ$ of s and s', we get an element $\gamma \in F(\mathring{y})$ which extends to $\gamma' \in F(y)$ by hyp.

Thus modifying s' via γ', we get a section over the larger $U \coprod^{\Delta(k)^\circ} \Delta(k) = U \cup \{y\} \subset Z$.

Thus $U = Z$ by maximality, proving $H^1(Z, F) = 0$ for all Z.

Now given

$$0 \to F' \to F \to F'' \to 0$$

~~with~~ with F' satisfying the condition we have

$$\begin{tikzcd} 0 \arrow[r] & F'(y) \arrow[r] \arrow[d] & F(y) \arrow[r] \arrow[d] & F''(y) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & F'(\dot{y}) \arrow[r] & F(\dot{y}) \arrow[r] & F''(\dot{y}) \arrow[r] & H^1(\dot{y}, F') \arrow[d, phantom, "\shortparallel"] \\ & & & & 0 \end{tikzcd}$$

so F satisfies the condition $\iff$ F'' does. From this it follows that for all Z, $H^+(Z, F) = 0$ and so F is flask.

August 10, 1972

Subdivision of a groupoid. If $\mathcal{G}$ is a groupoid so is $Sd\,\mathcal{G}$

$$\begin{tikzcd}
\cdot \arrow[r, "u"] & \cdot \arrow[d, "\beta"] \\
\cdot \arrow[u, "\alpha"] \arrow[r, "v"] & \cdot \arrow[d, "\beta^{-1}"] \\
\cdot \arrow[u, "\alpha^{-1}"] \arrow[r, "u"] & \cdot
\end{tikzcd}
\qquad
\begin{aligned}
&v = \beta u \alpha \\
\Rightarrow\ &\beta^{-1} v \alpha^{-1} = u
\end{aligned}$$

Thus the canonical functor

$$Sd\,\mathcal{G} \longrightarrow \mathcal{G}$$

must be an equivalence of groupoids, the inverse functor being

$$\begin{array}{ccc}
X & \longmapsto & \begin{tikzcd} X \arrow[r, "id"] & X \arrow[d, "f"] \\ Y \arrow[u, "f^{-1}"] \arrow[r] & Y \end{tikzcd} \\
\downarrow f & \longmapsto & \\
Y & &
\end{array}$$

.

Consequence: I conjectured previously that for $\mathcal{C}$ finite

$$[\mathcal{C}, \mathcal{C}'] = \varinjlim_m \pi_0\, \underline{\mathrm{Hom}}(Sd^m \mathcal{C}, \mathcal{C}')$$

The heuristic proof went as follows:

$$\begin{aligned}
N\,\underline{\mathrm{Hom}}(Sd^m\mathcal{C}, \mathcal{C}') &= \underline{\mathrm{Hom}}(N(Sd^m\mathcal{C}), N\mathcal{C}') \\
&\overset{?}{=} \underline{\mathrm{Hom}}(Sd^m(N\mathcal{C}), N\mathcal{C}') \\
&= \underline{\mathrm{Hom}}(N\mathcal{C}, Ex^m(N\mathcal{C}'))
\end{aligned}$$

Thus $\varinjlim_m \pi_0 \underline{\mathrm{Hom}}(Sd^m C, C') \overset{NC \text{ finite}}{=} \pi_0 \underline{\mathrm{Hom}}(NC, Ex^\infty(NC'))$
$$= [C, C'].$$

(Recall $N(Sd\, C) = Sd(NC)$ never established). The last step seems to require NC to be finite, which signifies that $\mathrm{Ar}\, C$ is finite and C has no loops.

The above shows that the conjecture is false unless NC is finite. In effect if G is a finite group then we have seen that $Sd^m G$ and G are equivalent categories. But there are non-trivial $x \in [G, K(\pi, n)]$, not coming from a functor, e.g. $K(\pi,n)$ might be a simplicial complex.

August 8, 1972.

Consider the category of simplicial G-sets, where G is a group. We try to make it into a model category by taking

$$\begin{aligned} \text{cofibrations} &= \text{injective maps} \\ \text{heq's} &= \text{heq's of underlying s. sets} \\ & \quad \text{(but not equivariant homotopies)} \end{aligned}$$

The corresponding fibrations will be called flask maps to avoid confusion. Thus a map of simp. G-sets $X \to Y$ is <u>flask</u> provided it has the RLP

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

with respect to all injective $A \hookrightarrow B$ which are heq's.

<u>Example</u>: Let $Q \to Q'$ be a map of s. G-sets whose underlying map of s.sets is a Kan fibration, and let P be a free s. G-set. Claim

$$\underline{\mathrm{Hom}}(P,Q) \longrightarrow \underline{\mathrm{Hom}}(P,Q')$$

is a flask map. In effect we have to check

$$\begin{tikzcd} A \times P \arrow[r] \arrow[d, hook] & Q \arrow[d] \\ B \times P \arrow[r] \arrow[ur, dashed] & Q' \end{tikzcd}$$

but $A \times P \hookrightarrow B \times P$ will be G-cofibration in the strict sense, which is also a heq (obtained by successive $G \times \Delta(k)^\circ \subset G \times \Delta(k)$ attaching), so the lifting has been proved ~~by~~ in your earlier work. (Actually if $P' \subset P$ is an injection of free s. G-sets we have ~~[illegible]~~ more generally that

$$\underline{\mathrm{Hom}}(P,Q) \longrightarrow \underline{\mathrm{Hom}}(P,Q') \times_{\underline{\mathrm{Hom}}(P',Q')} \underline{\mathrm{Hom}}(P',Q)$$

is flask.)

Consequences: Suppose $X \to Y$ flask.

1.) for any $H \subset G$, $X^H \to Y^H$ is a fibration. This is the RLP wrt

$$G/H \times V(n,k) \subset G/H \times \Delta(n).$$

~~2.) When $Y = pt$. Then for any G-set S the map~~

$$\cancel{PG \times S \longrightarrow S}$$

~~is a heq. [illegible] If $S' \to S$ is a heq [illegible] of G sets, we can factor it~~

$$\cancel{S' \subset S' \times \Delta(1) \sqcup_{S' \times \{1\}} S \longrightarrow S}$$

~~where the second map is a strict G-equivariant homotopy equivalence. Thus since the first~~

<u>Definition</u>: Two maps of G-sets $X \rightrightarrows Y$ are strictly homotopic if they are in the same component of $\underline{\mathrm{Hom}}_G(X,Y)$. (analogous to equivariant homotopy of Gspaces.)

2.) Suppose $A \hookrightarrow B$ is an injective heq. Then so is

$$(A \times \Delta(n)) \cup (B \times \Delta(n)^{\circ}) \hookrightarrow B \times \Delta(n)$$

whence we can conclude for $X \to Y$ flask that

$$\underline{\mathrm{Hom}}_G(B,X) \longrightarrow \underline{\mathrm{Hom}}_G(A,X) \times_{\underline{\mathrm{Hom}}_G(A,Y)} \underline{\mathrm{Hom}}_G(B,Y)$$

is a heq + fibration. (also when $A \to B$ is just injective it is a fibration.)

3.) Suppose $S' \xrightarrow{f} S$ is a heq of G-sets + factor it

$$S' \subset \overset{M_f}{\overset{\parallel}{S' \times \Delta(1) \underset{S' \times 1}{\cup} S}} \longrightarrow S.$$

First map is an injective heq, second map a <u>strict</u> heq. Thus if $X \to pt$ is flask

$$\underline{\mathrm{Hom}}_G(S,X) \xrightarrow[\text{strict heq}]{} \underline{\mathrm{Hom}}_G(M_f,X) \xrightarrow[\text{fibn heq}]{} \underline{\mathrm{Hom}}_G(S',X)$$

In particular, applying this to the map $PG \times S \to S$, we have a heq

$$\underline{\mathrm{Hom}}_G(S,X) \longrightarrow \underline{\mathrm{Hom}}_G(PG \times S, X)$$

for flask X. Special case $S = pt$

$$\begin{array}{ccc} X^G & \xrightarrow[\text{heq}]{} & \underline{\mathrm{Hom}}_G(PG, X) \\ \text{fixpts} & & \text{homotopy fixpoints} \end{array}$$

<u>Example</u>: Suppose we work with topological G-spaces. Then the condition that for all G-spaces S

$$\pi_0 \underline{\mathrm{Hom}}_G(S, X) \simeq \pi_0 \underline{\mathrm{Hom}}_G(PG \times S, X)$$
$$\qquad\qquad\qquad\qquad \| $$
$$\qquad\qquad\qquad\qquad \pi_0 \underline{\mathrm{Hom}}_G(S, \underline{\mathrm{Hom}}(PG, X))$$

is equivalent to

$$X \longrightarrow \underline{\mathrm{Hom}}(PG, X)$$

being a G-homotopy equivalence. If I remember correctly, Segal showed that a map of (reasonable) G-spaces $X \to Y$ is a G-homotopy equivalence iff for all $H \subset G$, $X^H \to Y^H$ is a homotopy equivalence. Maybe he assumed G compact.

<u>Conclude</u>: If we work with reasonable G-spaces then flaskness for a G-space signifies that

$$X \longrightarrow \underline{\mathrm{Ham}}(PG, X)$$

is a G-homotopy equivalence, and this under suitable conditions (e.g. G finite) means that for each $H \subset G$, $X^H =$ homotopy H-fixpts of X.

More carefully: Suppose $X \longrightarrow \underline{\mathrm{Hom}}(PG, X)$ is a G-heq. and let $A \subset B$ be a injective heq of G-spaces. Then given

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[r] & \underline{\mathrm{Hom}}(PG, X) \arrow[l, bend right] \\ B \arrow[urr, "\exists"'] & & \end{tikzcd}$$

and using the homotopy inverse, we know a G-homotopic map to $A \to X$ extends to B. Thus we have to establish the HEP

$$\begin{tikzcd} A \times I \cup B \times 0 \arrow[r] \arrow[d] & X \\ B \times I \arrow[ur, dashed] & \end{tikzcd}$$

but this is clear by standard argument, i.e. $A \times I \cup B \times 0$ has a nbd. which equivariantly deforms down, etc.

Characterization of flask heq's:

Prop. TFAE for a map of s. G-sets: $X \to Y$

(i) RLP wrt all injective $A \hookrightarrow B$.

(ii) RLP wrt maps

$$G/H \times \Delta(n)^{\circ} \subset G/H \times \Delta(n)$$

for all $n \geq 0$ and $H \subset G$

(ii)' $\forall\, H \subset G$, $X^H \to Y^H$ is a fibration + heq.

Proof: $\mathrm{Hom}_G(G/H \times Z, X) = \mathrm{Hom}(Z, X^H)$ for all s. sets Z, hence (ii) and (ii)' are equivalent. Clearly (i) $\Rightarrow$ (ii) and to prove converse we use skeletal decomposition of $A \hookrightarrow B$. The point is that if $b \in B_k$ is a simplex of minimal dim. in B but not in A, and H is the stabilizer of b, then we have

$$\begin{tikzcd}
G/H \times \Delta(k)^{\circ} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\text{cocart}"] & G/H \times \Delta(k) \arrow[d] \\
A \arrow[r, hook] & A \cup b \subset B
\end{tikzcd}$$

In effect, we must show

$$G/H \times \Delta(k)_m - G/H \times \Delta(k)^{\circ}_m \hookrightarrow B_m .$$

i.e. that the elements $g_i s_\alpha b$, $G = \coprod g_i H$ are distinct. The point is that $s_\alpha b$ clearly has stabilizer H, and by E-Z lemma for $G\backslash B$ we know the orbits $s_\alpha G b$ are distinct. So clear.

~~Proposition: Suppose X is a simplicial G-set such that for all H ⊂ G~~

Proposition: X is a flask simplicial G-set iff (i) $\forall H \subset G$, X^H is a Kan complex

(ii) $X \longrightarrow \underline{\mathrm{Hom}}(PG, X)$ is G-heq.

<u>Proof:</u> ~~Necessity~~ done already. Conversely, given an injective heq $A \subset B$, we get from (ii):

$$\begin{tikzcd} A \arrow[r] \arrow[d, phantom, "\cap" rotate] & X \arrow[r, shift left] & \underline{\mathrm{Hom}}(P_G, X) \arrow[l, shift left] \\ B \arrow[urr, "\exists"'] & & \end{tikzcd}$$

and we have to establish an HEP:

$$\begin{tikzcd} A \arrow[r] \arrow[d, phantom, "\cap" rotate] & X^I \arrow[d] \\ B \arrow[r] \arrow[ur, dotted] & X \end{tikzcd}$$

But (i) $\Rightarrow (X^I)^H \longrightarrow X^H$ is a fibration $\forall H$, so done by first proposition.

Special case: $X = K(M, q)$ where M is a G-module, ~~th~~ More generally suppose X is a simplicial G-module. Then in testing

$$\begin{tikzcd} A \arrow[r] \arrow[d, phantom, "\cap" rotate] & X \\ B \arrow[ur, dotted] & \end{tikzcd}$$

we can replace A by $\mathbb{Z}A$, and replace everything by chain complexes of G modules

$$\begin{tikzcd} C_\bullet^N(A) \arrow[r] \arrow[d] & \mathrm{Norm}(X) \\ C_\bullet^N(B) \arrow[ur, dotted] & \end{tikzcd}$$

By similar arguments to the above, flaskness should amount to a G-heq

$$N \longrightarrow \underline{Hom}(P, N) \quad \text{truncated}$$

where $P \to \mathbb{Z}$ is a free G-resolution, here N is a chain complex of G-modules, and the $\underline{Hom}$ has to be truncated in degree 0:

$$\cdots \longrightarrow \underline{Hom}^{-1}(P,N) \longrightarrow Z^0\,\underline{Hom}(P,N) \longrightarrow 0 \longrightarrow 0 \cdots$$

so if $X = K(M,q)$, $N = Norm(X) = M[q]$, and we are considering the complex of cochains

$$\cancel{C^0(G,M) \to C^1(G,M) \to \cdots \to Z^q(G,M) \to 0}$$

~~Conclude: If M is a G-module, $K(G,q)$ is flask iff the complex of~~

$$\cancel{0 \to M \to C^0(G,M) \to \cdots \to Z^q(G,M) \to 0 \cdots}$$

~~has a contracting homotopy~~

$$\begin{array}{ccccccc}
Map(G,M) & \longrightarrow & Map(G^2,M) & \longrightarrow & \cdots \longrightarrow & Z^q Map(WG, M) & \longrightarrow 0 \cdots \\
\| & & \| & & & & \\
I^0(G,M) & \longrightarrow & I^1(G,M) & \longrightarrow & \cdots \longrightarrow & Z^q(G,M) & \longrightarrow 0 \\
q & & & & & 0 &
\end{array}$$

<u>Conclude</u>: $K(M,q)$ is a flask s. G-set iff the complex

$$0 \to M \to I^0(G,M) \to I^1(G,M) \to \cdots \to Z^q(G,M) \to 0$$

has an equivariant contracting homotopy. Observe that for $q \geq 1$ this implies M is a direct summand of $I^0(G,M) = Map(G,M)$. I claim this means that the complex

$$0 \to M \to I^0(G,M) \to I^1(G,M) \to \cdots\cdots$$

is contractible. Indeed to ~~test~~ contractibility of a complex all one has to show is that it remains exact for the functors $Hom(J, ?)$ for all J. This will be so for $I^0(G,M)$, so if M is a direct summand of $I^0(G,M)$, it's clear.

<u>Check</u>: Take PG = Nerv of category (G,G) with objects $g \in G$ and a unique morphism $g \to g'$. Then

$$\underline{Hom}(PG, K(M,1)) = \underline{Hom}(\text{Nerv}(G,G), \text{Nerv } M)$$

$$= \text{Nerv}(\underline{Hom}((G,G), M))$$

Now any object of $\underline{Hom}((G,G),M)$ consists of a function $f: G^2 \to M$ such that

$$f(g,g) = 0 \quad \forall g$$

$$f(g_1,g_2) + f(g_2,g_3) = f(g_1,g_3)$$

i.e. $f \in Z^1(G,M)$. And a natural transf. from f to f' is a function $h: G \to M \ni$

$$f(g,g') + h(g') = f'(g,g') + h(g)$$

$$(f-f')(g,g') = h(g)-h(g').$$

Thus $\underline{\mathrm{Hom}}((G,G),M)$ is the category associated to the complex

$$I^0(G,M) \longrightarrow Z^1(G,M).$$

We have an evident map of the complex

$$M \longrightarrow 0$$

into this representing the functor

$$M \overset{\leftarrow\cdots}{\longrightarrow} \underline{\mathrm{Hom}}((G,G),M).$$

To say this admits a G-invariant retraction $\leftarrow\cdots$ would signify that M is a G-module direct summand of $I^0(G,M)$.

<u>Conclude</u>: TFAE for a G-module:

(i) $K(M,1)$ is a <u>flask</u> simplicial G-set

(ii) $K(M,q)$ ———————————— for all $q \geq 0$

(iii) M is a direct summand of the (co)induced module $\mathrm{Map}(G,M)$.

This is a very strong condition on M, and probably not the same as cohomological triviality, except if G is finite.

Generalize the preceding to small categories.

Let I be a small category; call objects of $\underline{\mathrm{Hom}}(I, \mathrm{Sets})$ simply I-sets. We then can consider simplicial I-sets which are the same thing as functors $i \mapsto X_i$ from I to simplicial sets. In my HA notes I made simplicial I-sets into a model category by calling the fibrations maps $X \to Y$ such that $X_i \to Y_i$ is a fibration for each i. Here I want to do something different.

Analogue of PG. PI is the simplicial I-set with

$$(PI)_n = Ar_{n+1} I = \{\underbrace{\leftarrow \leftarrow \cdots \leftarrow}_{n+1 \text{ arrows}}\}$$

and evident left I action. As a functor $I \to \Delta^\wedge$ it sends

$$i \longmapsto \mathrm{Nerv}(I/i).$$

Thus we see that

$$(*) \qquad \underline{\mathrm{Hom}}_I (PI, X)$$

is what Kan-Bousfield denote

$$\underset{\underset{I}{\longleftarrow}}{\mathrm{holim}}(X).$$

Meaning of $(*)$.

$$\underline{\mathrm{Hom}}_I (PI, X) = pr_{2*} \, \underset{(I^\circ \times \Delta)^\wedge}{\mathrm{Hom}} (PI, X)$$

Conjecture: A simplicial I-set is flask provided the canonical map

$$X \longrightarrow \mathrm{Hom}(PI, X)$$

is a I-heq, and also $\mathrm{Hom}_I(S, X)$ is a Kan complex for all I-sets S.

(unknown quantity: whether an injection $A \subset B$ of simplicial I-sets can be broken down as in the prop on p. 5.)

August 9, 1972

Recall Postnikov systems: Suppose $X \to Y$ is a Kan fibration and define a tower

$$X \twoheadrightarrow \cdots \twoheadrightarrow F^m X \twoheadrightarrow F^{m-1}X \twoheadrightarrow \cdots \twoheadrightarrow F^{-1}X \subset Y$$

as follows. $F^m X$ is quotient of X by the equivalence relation: $x, x' \in X_k$ are equivalent $\iff$ they have same image in Y and the same m-skeleton. It is known that the above is a tower of fibrations. Moreover the map $F^m X \to F^{m-1}X$ maybe factored canonically

$$F^m X \xrightarrow[\text{fibn.}]{\text{asph.}} \bar{F}^m X \xrightarrow[\text{fib.}]{\text{min}} F^{m-1}X$$

by identifying simplices which are homotopic relative to Y and their $(m-1)$-skeletons. The minimal fibration has fibres $K(\pi_m, m)$, where $\pi_m = \pi_m(\text{fibre of } X \to Y)$.

The preceding Postnikov factorizations are canonical and so can be applied as follows. Let X be a simplicial G-set which is a Kan complex. We have already seen that given a fibration of pointed simplicial G-sets

$$F \longrightarrow E \longrightarrow B$$

that we get a fibration of simplicial sets

$$\underline{\mathrm{Hom}}_G(PG, F) \longrightarrow \underline{\mathrm{Hom}}_G(PG, E) \longrightarrow \underline{\mathrm{Hom}}_G(PG, B)$$

More generally let I be a small category and $X \to Y$ a map of simplicial I-sets, such that $X_i \to Y_i$ is a fibration for all i. Then I want to show the induced map

$$(*) \qquad \underline{\mathrm{Hom}}_I(PI, X) \longrightarrow \underline{\mathrm{Hom}}_I(PI, Y).$$

is a fibration.

Recall PI is the simplicial I-set, $n \mapsto Ar_{n+1} I$; alternatively the functor $i \mapsto \mathrm{Nerv}(I/i)$. If we denote

$$\underline{\mathrm{Hom}}(PI, X)$$

the internal hom in $(I^{\circ} \times \Delta)^{\wedge}$, then

$$\underline{\mathrm{Hom}}_I(PI, X) = pr_{2*}\, \underline{\mathrm{Hom}}(PI, X).$$

From this formula it is clear that if Y is a pointed s. I-set, whence a map $pt \to Y$, (pt = final ob. of $(I^{\circ} \times \Delta)^{\wedge}$), then

$$\begin{tikzcd}
 & \underline{\mathrm{Hom}}_I(PI, pt \times_Y X) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_I(PI, X) \arrow[d] \\
\Delta(0) = & \underline{\mathrm{Hom}}_I(PI, pt) \arrow[r] & \underline{\mathrm{Hom}}_I(PI, Y)
\end{tikzcd}$$

is cartesian. (Better: $X \mapsto \underline{\mathrm{Hom}}_I(PI, X)$ commutes with $\varprojlim$'s).

Let $A \subset B$ be an injective heq in $\Delta^{\wedge}$. Then we have

$$\mathrm{Hom}_{\Delta}(B, \underline{\mathrm{Hom}}_I(PI, X)) = \mathrm{Hom}_{(I^{\circ} \times \Delta)^{\wedge}}(pr_2^* B, \underline{\mathrm{Hom}}(PI, X))$$

$$= \mathrm{Hom}_{(I^{\circ} \times \Delta)^{\wedge}}(PI, \underline{\mathrm{Hom}}(pr_2^* B, X))$$

$$\underline{Hom}(pr_2^* B, X) = X^B$$

But since $X_* \to Y_*$ is a fibration (object-wise)

$$X^B \longrightarrow X^A \times_{Y^A} Y^B$$

is a fibration + heq (object-wise). Thus (*) will be a fibration with fibre $\underline{Hom}_I(PI, pt \times_Y X)$ provided we show:

$$\begin{tikzcd} & X \arrow[d, "f"] \\ PI \arrow[ur, dashed] \arrow[r] & Y \end{tikzcd}$$

whenever f is a **fib + heq** (object-wise). The proof of this seems to require a skeletal induction.

<u>Example</u>: $I = G$. Then $PG = \text{Nerve}(G, G)$. In order to construct the lifting we proceed by induction constructing the lifting over $PG^{(k)} =$ the inverse image of the k-skeleton of $BG = \text{Nerve } G$.

$$\begin{tikzcd} \coprod G \times \Delta(k)^\circ \arrow[r] \arrow[d] & \coprod G \times \Delta(k) \arrow[d] \\ PG^{(k-1)} \arrow[r] & PG^{(k)} \end{tikzcd}$$

Would work for any free simplicial G-set.

Concept of a free simplicial I-set P. For each n, P_n can be interpreted as an I-set, i.e. a covariant functor $i \mapsto P_n(i)$ from I to sets. We can speak of the degenerate I-subset $P_n^{deg} \subset P_n$

and to say P is free means that

$$P_n = P_n^{deg} \amalg \text{representable functors.}$$

So it's all pretty clear. (*)

<u>Conclusion</u>: Suppose X is a pointed simplicial I-set ~~which~~ such that each X_i is a Kan complex. Then the Postnikov tower

$$X \cdots \longrightarrow F^m X \longrightarrow F^{m-1} X \longrightarrow \cdots \longrightarrow pt$$

of X will give rise to a tower of fibrations

$$\longrightarrow \underline{\mathrm{Hom}}_I(PI, F^m X) \longrightarrow \underline{\mathrm{Hom}}_I(PI, F^{m-1} X) \longrightarrow \cdots$$

and hence to a spectral sequence (Kan-Bousfield style).

(*) We can probably define a free simplicial I-set to be the ~~total~~ s.set of an I-torsor

$$\begin{tikzcd} & P \arrow[dl] \arrow[d, "f"] \\ \mathrm{Ob}\, I & B \end{tikzcd} \qquad \mathrm{Ar}\, I \times_{\mathrm{Ob}\, I} P \longrightarrow P$$

such that each stalk P_x, $x \in B$ is representable.

<u>Situation</u>: The problem (where I is a group G): Given a category $\mathcal{A}$ fibred over $G_\wedge$ with a cartesian section, to find agreeable conditions which imply that there is a spectral sequence

$$E_2^{pq} = H^p(G, \pi_{-q}\mathcal{A}) \Longrightarrow \pi_{-p-q}\left(\varprojlim_G \mathcal{A}\right)$$

(the spectral sequence lives in the range $0 \leq p \leq -q$.)

Present program:

1. Replace $\mathcal{A}$ by an equivalent pointed G-category, then by a pointed simplicial G-set, then by a pointed simplicial G-set X satisfying the extension condition (forgetting the G-action).

2. Then we have the s. set

$$\operatorname{holim}_{\overleftarrow{G}} X = \underline{\mathrm{Hom}}_G(PG, X)$$

and the spectral sequence of Bousfield-Kan

$$H^p(G, \underset{\underset{\pi_{-q}\mathcal{A}}{\parallel}}{\pi_{-q}X}) \Longrightarrow \pi_{-p-q}\left(\operatorname{holim}_{\overleftarrow{G}} X\right).$$

Thus what we need now is something which will allow us to identify

$$\varprojlim_G \mathcal{A} \quad \text{and} \quad \operatorname{holim}_{\overleftarrow{G}} X$$

up to homotopy. Now this seems hard, and a better idea perhaps is to understand the spectral sequence and its construction with the hope that it could be done more directly working with $\mathcal{A}$.

3. First approach was to form the Postnikov tower of X (defined because X is a Kan complex)

$$X \quad \cdots \longrightarrow F^m X \longrightarrow F^{m-1} X \longrightarrow \cdots\cdots$$

and then use the associated tower of filtrations

$$\longrightarrow \underline{\mathrm{Hom}}_G(PG, F^m X) \longrightarrow \underline{\mathrm{Hom}}_G(PG, F^{m-1}X) \longrightarrow \cdots$$

to construct the spectral sequence.

This approach has the defect of requiring one to work with X. It might be better to understand:

4. Bousfield–Kan approach using a filtration of PG.

Review B-K theory:

Examples: 1. P, X two simplicial sets, then

$$p,q \longmapsto \mathrm{Hom}(P_p, X_q) = Y_{pq}$$

is a cosimplicial s. set. The total s. set is

$$\mathrm{Tot}(Y) = \underline{\mathrm{Hom}}_\Delta(\Delta, Y) = pr_{2*}\, \underline{\mathrm{Hom}}(\Delta, Y)$$

where $\Delta : p \longrightarrow \Delta(p)$. (Here Δ is an efficient version of $P\Delta$.)

~~Thus~~

$$\cancel{\operatorname{Hom}_{\Delta^\wedge}(\Delta(n), \operatorname{Tot}(Y)) = \operatorname{Hom}_{(\Delta^\circ \times \Delta)^\wedge}\left(pr_2^* \Delta(n), \underline{\operatorname{Hom}}(\Delta, Y)\right)}$$

$$\cancel{= \operatorname{Hom}\left(\Delta \times pr_2^* \Delta(n), Y\right)}$$

An n-simplex of $\operatorname{Tot}(p \mapsto \underline{\operatorname{Hom}}(P_p, X))$ is a compatible family of maps

$$\Delta(p) \times \Delta(n) \longrightarrow \underline{\operatorname{Hom}}(P_p, X) \qquad \forall p$$

i.e. a map

$$P \times \Delta(n) \longrightarrow X.$$

Thus we have

$$\operatorname{Tot}(p \mapsto \underline{\operatorname{Hom}}(P_p, X)) = \underline{\operatorname{Hom}}(P, X).$$

2. Let P, X be simplicial G-sets and calculate

$$\operatorname{Tot}(p \mapsto \underline{\operatorname{Hom}}_G(P_p, X)).$$

An n-simplex is a compatible family of G-maps

$$P_p \times \Delta(p) \times \Delta(n) \longrightarrow X$$

so we get

$$\boxed{\operatorname{Tot}(p \mapsto \underline{\operatorname{Hom}}_G(P_p, X)) = \underline{\operatorname{Hom}}_G(P, X).}$$

Skeleta: Denote by $P^{[s]}$ the s-skeleton of the simplicial set P. B-K define

$$\begin{aligned}
\mathrm{Tot}_s(Y) &= \underline{\mathrm{Hom}}_{\hat{\Delta}}(\Delta^{[s]}, Y) \\
&= \mathrm{Ker}\Big\{ \prod_p \underline{\mathrm{Hom}}(\Delta(p)^{[s]}, Y_p) \rightrightarrows \prod_{p\to q} \mathrm{Hom}(\Delta(p)^{[s]}, Y_q) \Big\}
\end{aligned}$$

Thus an n-simplex of $\mathrm{Tot}_s(p \mapsto \underline{\mathrm{Hom}}(P_p, X))$ is a compatible family

$$\Delta(p)^{[s]} \times \Delta(n) \longrightarrow \underline{\mathrm{Hom}}(P_p, X)$$

i.e. $\quad P_p \times \Delta(p)^{[s]} \times \Delta(n) \longrightarrow X$

i.e. a map

$$P^{[s]} \times \Delta(n) \longrightarrow X.$$

Thus

$$\boxed{\mathrm{Tot}_s(p \mapsto \underline{\mathrm{Hom}}(P_p, X)) = \underline{\mathrm{Hom}}(P^{[s]}, X).}$$

Similarly we have for simplicial G-sets

$$\mathrm{Tot}_s(p \mapsto \underline{\mathrm{Hom}}_G(P_p, X)) = \underline{\mathrm{Hom}}_G(P^{[s]}, X)$$

where here $P^{[s]}$ is the inverse image of the s-skeleton of the orbits P/G.

<u>Conclusion</u>: Since problems with extension condition can *perhaps* be circumvented by suitable subdivision, it [redacted] appears that the essential problem of whether

$$\varprojlim_{G} a \qquad \underset{G}{\text{holim}}\, {}^{\prime}a^{\prime}$$

coincide can not be treated by [redacted] Bousfield-Kan methods.

It seems that the correct yoga is this: On the category of G-spaces (polyhedra?) we have defined an [redacted] analogue of KR-theory. We must then prove that

$$KR(X) \xrightarrow{\sim} KR(PG \times X).$$

It seems that this requires something like periodicity.

You should determine why this works in the Zariski case.

August 12, 1972

The barycentric subdivision of a simplicial set.

Start with

$Sd\,\Delta(n)$ = the nerve of the category of simplices of standard n-simplex
= the s. set belonging to the barycentric subdivision of $\Delta(n)$.

Thus a simplex of $Sd\,\Delta(n)$ is a chain

$$\sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_q$$

of non-empty subsets of $\{0, \dots, n\}$.

Note that any map $\Delta(m) \longrightarrow \Delta(n)$ induces a map $\{0, \dots, m\} \longrightarrow \{0, \dots, n\}$ and hence a map

$$Sd\,\Delta(m) \longrightarrow Sd\,\Delta(n),$$

whence we have a functor

$$\begin{tikzcd} \Delta \arrow[r, "Sd"] \arrow[d] & \Delta^{\wedge} . \\ \Delta^{\wedge} \arrow[ur, dashed] & \end{tikzcd}$$

Claim Sd extends ~~uniquely~~ to $Sd: \Delta^{\wedge} \longrightarrow \Delta^{\wedge}$ in a unique way compatible with arbitrary $\varinjlim$'s.

Proof: Define $Ex: \Delta^{\wedge} \longrightarrow \Delta^{\wedge}$ by the formula

(*) $$\mathrm{Hom}(\Delta(n), \mathrm{Ex}(X)) = \mathrm{Hom}(\mathrm{Sd}\,\Delta(n), X).$$

Then define Sd as we must by the formula

(**) $$\mathrm{Sd}\, Y = \varinjlim_{\Delta(n)\to Y} \mathrm{Sd}\,\Delta(n) \qquad \text{(Kan extension)}$$

the ind. limit being taken over the cat. Δ/Y. Passing to the limit in (*) we get

$$\mathrm{Hom}(Y, \mathrm{Ex}\, X) = \mathrm{Hom}(\mathrm{Sd}\, Y, X)$$

which proves (**) defines a functor Sd with a right adjoint, hence compatible with arbitrary $\varinjlim$'s.

Now let us use the skeletal decomposition of X

$$\begin{tikzcd} \coprod_{X_k^{nd}} \Delta(k)^\circ \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\text{cocart}"] & \coprod_{X_k^{nd}} \Delta(k) \arrow[d] \\ \mathrm{sk}_{k-1} X \arrow[r, hook] & \mathrm{sk}_k X \end{tikzcd}$$

$$X = \bigcup \mathrm{sk}_k X.$$

Then we get cocartesian squares

(+) $$\begin{tikzcd} \coprod \mathrm{Sd}\,\Delta(k)^\circ \arrow[r, hook] \arrow[d] & \coprod \mathrm{Sd}\,\Delta(k) \arrow[d] \\ \mathrm{Sd}(\mathrm{sk}_{k-1} X) \arrow[r, hook] & \mathrm{Sd}(\mathrm{sk}_k X) \end{tikzcd}$$

Let z be a non-degenerate simplex of $Sd(X)$. Then there is a least k such that $z \in Sd(sk_k X)$. There is then a unique non-deg. k-simplex $x : \Delta(k) \to X$ such that z is in the image of $Sd\,\Delta(k) \longrightarrow Sd\,X$, and not in the image of the composite map

$$Sd\,\Delta(k)^{\circ} \subset Sd\,\Delta(k) \xrightarrow{Sd(x)} X$$

Better, there is a unique non-deg simplex $x : \Delta(k) \to X$ such that z is in the image of the map

$$Sd\,\Delta(k) - Sk\,\Delta(k)^{\circ} \xhookrightarrow{Sd(x)} X$$

the injectivity + uniquenes of x being evident from (+). Thus we see that z may be identified with a chain of faces

$$x_0 \xrightarrow{<} x_1 \xrightarrow{<} \cdots \xrightarrow{<} x_q = x$$

($<$ means proper face).

<u>Conclude</u>: If X has the property that any face of a non-degenerate simplex is non-degenerate, then $Sd\,X$ is the nerve of the category of non-degenerate simplices of X (full subcat of Δ/X consisting of non-deg. simplices).

In general, there is a map

$$\text{Nerve}(\text{non-deg. s. of } X) \hookrightarrow Sd\,X.$$

which is injective, but not necessarily onto, e.g.

if $X = \Delta(2)/\Delta(2)^{\circ}$, then there are only two non-degenerate simplices ~~a~~ forming a category:

$$\begin{tikzcd} o \arrow[r, bend left=50, "d_0"] \arrow[r, "d_1"] \arrow[r, bend right=50, "d_2"'] & \sigma \end{tikzcd}$$

with different homotopy type.

<u>Idea</u>: To modify X in a fashion analogous to replacing a category by a category without loops.

Given a simplicial set X, let $\tilde{X}$ be the simplicial subset of $X \times \Delta(\infty)$ consisting of pairs (x,y) of the same dimension such that: If $y = \eta^* y'$ with η a surjective map, then $x = \eta^* x'$ for some x'. In other words the non-degenerate simplices of $\tilde{X}$ are ~~the~~ pairs (x,y) with y non-degenerate.

I want to prove that $\tilde{X} \longrightarrow X$ is a heq. It will suffice to show for each non-degenerate simplex $z : \Delta(n) \longrightarrow X$, that the fibre $\tilde{X}_z$ is contractible. $\tilde{X}_z \subset \Delta(n) \times \Delta(\infty)$ consists of pairs (φ, y) such that $(\varphi^* z, y) \in \tilde{X}$. Suppose $y = \eta^* y'$ with η surj. and y' non-deg. Then we have $\varphi^* z = \eta^* x'$ for some x'. Write $\varphi = \varepsilon' \eta'$, whence

$$\eta'^* \varepsilon'^* z = \eta^* x'$$

$$\eta'^* \eta_1^* w \qquad ?$$

can't conclude $\varphi = \tau \eta$ because the faces of z might be degenerate.

August 13, 1972

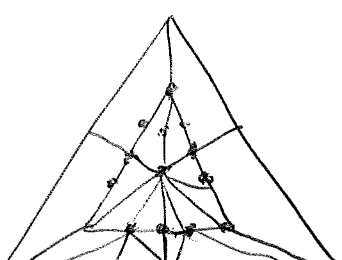
Why any s. set is replaceable by a simplicial complex.

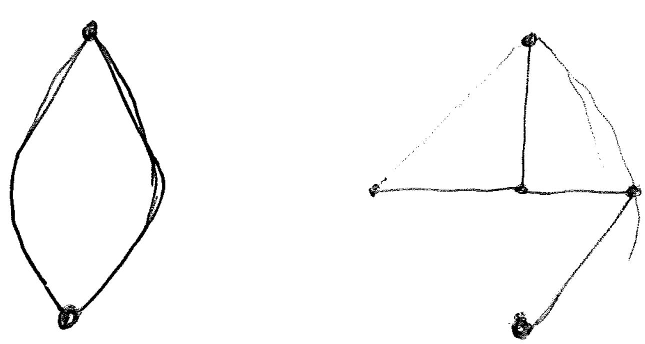
Conjecture (perhaps proved by Whitehead): The geometric realization of a s. set is triangulable in the following way: ~~Given $\Delta(n) \xrightarrow{x} X$ non-deg., the image of $|\Delta(n)| \to |X|$ is homeomorphic nicely with the realization of the second barycentric subdivision modulo the~~

Suppose given a non-degenerate simplex $x : \Delta(n) \to X$, and let $R = \Delta(n) \times_X \Delta(n)$. [redacted] Then the claim is that if we identify vertices of the 2nd baryc. subdivision of $\Delta(n)$ according to R, we have a simplicial complex.

Ex. 1.

Ex. 2.





?

The point of the ~~[illegible]~~ construction: Ordered simplicial complexes (compatible linear orderings on ~~each~~ simplices) form a full subcategory of $\Delta^\wedge$ closed under products, subobjects, and containing $\Delta(n)$. What one should show is that if R is an equivalence relation on K, then in the category the quotient of the 2nd barycentric subdivision of K by R exists.

/ Remains to understand the reduced subdivision functor Sdr on simplicial sets, the one related to subdivision of a category. Any non-deg. simplex of $\mathrm{Sdr}(X)$ would consist of a non-deg. simplex $x \in X_k$ + a simplex in $\mathrm{Sdr}(\Delta(k)) - \mathrm{Sdr}(\Delta(k)^{\cdot})$ which would be a chain of intervals

$$\sigma_0 < \sigma_1 < \cdots < \sigma_q \subset [k]$$

with $\sigma_q = \{0, k\}$. Thus any non. deg. simplex of $\mathrm{Sdr}(X)$ is ~~[illegible]~~ uniquely representable as a chain

$$x_0 \longrightarrow x_1 \longrightarrow \cdots\cdots \longrightarrow x_q$$

where x_q is non-deg. in X, and where each arrow is an interval face which is proper.

Observe it will not generally be the case that the face of a non-degenerate simplex is non-degenerate. However if $X = \mathrm{Nerv}(\mathcal{C})$, then any interval face of a non-degenerate simplex is non-degenerate

August 13, 1972: Proof that $\boxed{\mathrm{Sd}_r(\mathrm{Nerv}\,\mathcal{C}) = \mathrm{Nerv}(\mathrm{Sd}_r\,\mathcal{C})}$

Reduced subdivision:

$$\mathrm{Sd}_r\,\Delta(n) = \mathrm{Sd}_r\,\mathrm{Nerv}([n]) = \mathrm{Nerv}(\mathrm{Sd}_r\,[n])$$

Thus a non-deg. simplex in $\mathrm{Sd}_r\,\Delta(n)$ is a chain of intervals

$$\sigma_0 < \cdots < \sigma_q$$

of the ordered set $[n]$. Now the observation is that

$$\Delta(n) \longmapsto \mathrm{Sd}_r\,\Delta(n)$$

is a functor $\Delta \longrightarrow \Delta^\wedge$

hence extends to a pair of adjoint functors

$$\Delta^\wedge \underset{\mathrm{Ex}_r}{\overset{\mathrm{Sd}_r}{\rightleftarrows}} \Delta^\wedge$$

given by formulas:

$$\mathrm{Hom}(\Delta(n), \mathrm{Ex}_r X) = \mathrm{Hom}(\mathrm{Sd}_r\,\Delta(n), X)$$

$$\mathrm{Sd}_r(X) = \varinjlim_{\Delta(n) \to X} \mathrm{Sd}_r\,\Delta(n).$$

Since Sd_r commutes with $\varinjlim$'s it will be compatible with skeletal decomposition, so we have a cocartesian square

$$\begin{tikzcd} \coprod \mathrm{Sd}_2\, \Delta(n)^\circ \arrow[r] \arrow[d] & \coprod \mathrm{Sd}_2\, \Delta(n) \arrow[d] \\ \mathrm{Sd}_2\,(sk_{n-1} X) \arrow[r] & \mathrm{Sd}_2\,(Sk_n X) \end{tikzcd}$$.

$\coprod$ taken over X_n^{nd}.

First have to check $\mathrm{Sd}_2\, \Delta(n)^\circ \hookrightarrow \mathrm{Sd}_2\, \Delta(n)$.
Start with cartesian square

$$\begin{tikzcd} \coprod_{i,j} \partial_i \Delta(n-1) \times_{\Delta(n)} \partial_j \Delta(n-1) \arrow[r] \arrow[d] & \coprod_j \partial_j \Delta(n-1) \arrow[d] \\ \coprod_i \partial_i \Delta(n-1) \arrow[r] & \Delta(n) \end{tikzcd}$$

Suppose can prove

(?) $\qquad \mathrm{Sd}_2\ \partial_i \Delta(n-1) \times_{\Delta(n)} \partial_j \Delta(n-1) = \mathrm{Sd}_2(\partial_i \Delta(n-1)) \times_{\mathrm{Sd}_2 \Delta(n)} \mathrm{Sd}_2(\partial_j \Delta(n-1))$

Abbreviate the square to

$$\begin{tikzcd} X \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$.

so that (?) implies

$$\begin{tikzcd} \mathrm{Sd}_2(X \times_Y X) \arrow[r] \arrow[d] & \mathrm{Sd}_2(X) \arrow[d] \\ \mathrm{Sd}_2(X) \arrow[r] & \mathrm{Sd}_2(Y) \end{tikzcd}$$

is cartesian. ~~It then follows that if~~

If $Z = \mathrm{Im}(X \to Y)$, then we have cokernel situation

$$X \times_Y X \rightrightarrows X \longrightarrow Z$$

and so since Sdr is right exact

$\Rightarrow$ exact:
$$\begin{array}{ccccc} \mathrm{Sdr}(X \times_Y X) & \rightrightarrows & \mathrm{Sdr}(X) & \longrightarrow & \mathrm{Sdr}(Z) \\ \| & & \wr\| & & \\ \mathrm{Sdr}\,X \times_{\mathrm{Sdr}\,Y} \mathrm{Sdr}\,X & \rightrightarrows & \mathrm{Sdr}\,X & \longrightarrow & \mathrm{Im}(\mathrm{Sdr}\,X \to \mathrm{Sdr}\,Y). \end{array}$$

Thus we conclude that

$$\mathrm{Sdr}\,Z = \mathrm{Im}(\mathrm{Sdr}\,X \to \mathrm{Sdr}\,Y)$$

i.e. $\qquad \mathrm{Sdr}\,\Delta(n)^{\circ} \hookrightarrow \mathrm{Sd}\,\Delta(n)$ as desired.

So consider (?). OKAY for $i = j$ because $\mathrm{Sdr}\,\Delta(n-1) \hookrightarrow \mathrm{Sd}\,\Delta(n)$. The left side is $\mathrm{Sdr}\,\Delta(n-2)$.

~~Suppose we have a chain of intervals~~

~~contained in the subset~~

A simplex on the right may be identified with a chain of 1-simplices (possibly degenerate)

$$\sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_q$$

in $\Delta(n)$, i.e. arrows in $[n]$, such that each σ_i is also a 1-simplex in $\partial_i \Delta(n-1)$ and $\partial_j \Delta(n-1)$, i.e. neither source or target of σ_i equals i or j. Thus σ_i is an arrow in the full subcategory

$\{0, \ldots \hat{i}, \ldots \hat{j}, \ldots n\} \subset [n]$, so its all clear.

Conclude that Sdr preserves injections.

Returning to skeletal decomposition:

$$\begin{tikzcd} \coprod \mathrm{Sdr}\,\Delta(n)^\circ \arrow[r, hook] \arrow[d] & \coprod \mathrm{Sdr}\,\Delta(n) \arrow[d] \\ \mathrm{Sdr}(sk_{n-1}X) \arrow[r, hook] & \mathrm{Sdr}(sk_n X) \end{tikzcd}$$

let z be a simplex of $\mathrm{Sdr}(X)$. Let n be least such that $z \in \mathrm{Sdr}(sk_n X)$. Since the above diagram shows

$$\mathrm{Sdr}(sk_n X) - \mathrm{Sdr}(sk_{n-1}X) \cong \coprod \left(\mathrm{Sdr}\,\Delta(n) - \mathrm{Sdr}\,\Delta(n)^\circ\right)$$

we see there is a unique non-deg. n-simplex x and a unique simplex γ of $\mathrm{Sdr}\,\Delta(n) - \mathrm{Sdr}\,\Delta(n)^\circ$ such that

$$\begin{array}{ccc} \mathrm{Sdr}\,\Delta(n) - \mathrm{Sdr}\,\Delta(n)^\circ & \xhookrightarrow{x} & \mathrm{Sdr}\,X \\ & & \psi \\ \gamma & \longmapsto & z \end{array}$$

~~[illegible]~~ If γ is a chain

$$\sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_q$$

of ~~[illegible]~~ 1-simplices in $\Delta(n)$, the fact that it is not in $\mathrm{Sdr}\,\Delta(n)^\circ$ means that every i, $0 \le i \le n$ is either source or target of some σ_i. In particular we must have $\sigma_q = (0 \le n)$.

Thus we arrive at a canonical form for any simplex of $Sdr(X)$: z may be identified with a pair (x, γ); x non-degenerate $\Delta(n) \to X$, γ in $Sdr\Delta(n) - Sdr\Delta(n)^{\circ}$.

~~Suppose X is the s.set associated to an ordered simplicial set K. If z is a non-degenerate simplex of $Sdr(X)$ represented by x, γ, then we have~~

$$\Delta(n) \hookrightarrow X$$

$$\Delta(q) \hookrightarrow Sdr\,\Delta(n) \hookrightarrow Sdr\,X$$

So now let C denote a small category. For each $[n] \to C$ of $\Delta/\text{Nerv}\,C$ we have a map

$$Sdr\,\Delta(n) = \text{Nerv}(Sdr[n]) \longrightarrow \text{Nerv}(Sdr\,C)$$

hence a map

$$(*) \qquad Sdr(\text{Nerv}\,C) = \varinjlim_{\Delta/\text{Nerv}\,C} Sdr\,\Delta(n) \longrightarrow \text{Nerv}(Sdr\,C).$$

Observe first that the map is surjective. In effect a ~~non degenerate~~ q-simplex in $\text{Nerv}(Sdr\,C)$ is a diagram of the form

$$\begin{array}{ccccccc} X_0 & \longrightarrow & X_1 & \longrightarrow & & \longrightarrow & X_q \\ \downarrow & & \downarrow & & & & \downarrow \\ Y_0 & \longleftarrow & Y_1 & \longleftarrow & \cdots & \longleftarrow & Y_q \end{array}$$

The simplex is non-degenerate if and only if for each i not both $X_{i-1} \to X_i$ and $Y_{i-1} \leftarrow Y_i$

are the identity maps. Now this comes from a functor $[2q+1] \longrightarrow \mathcal{C}$.

We know prove injectivity. Given a simplex (x,γ) in $\mathrm{Sdr}(\mathrm{New}\,\mathcal{C})$, we shall identify its image in $\mathrm{New}(\mathrm{Sdr}\,\mathcal{C})$ and show the image determines x and γ. So let x be the diagram

$$(1) \qquad X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_n$$

in $\mathcal{C}$ where no map is an identity. Suppose

$$\gamma : \quad \sigma_0 \subset \cdots\cdots \subset \sigma_q .$$

$$\sigma_i = (\lambda_i \leq \mu_i)$$

$$\lambda_0 \leq \cdots \leq \lambda_q \leq \mu_q \leq \cdots \leq \mu_0$$

By the condition that $\gamma \notin \mathrm{Sdr}\,\Delta(n)^{\circ}$ we know that the sequence $\lambda_0, \ldots\ldots, \mu_0$ exhausts $0, 1, \ldots, n$. The image of (x,γ) is then the diagram

$$\begin{array}{ccccc} X_{\lambda_0} & \longrightarrow & X_{\lambda_1} & \longrightarrow \cdots\cdots \longrightarrow & X_{\lambda_q} \\ \downarrow & & \downarrow & & \downarrow \\ X_{\mu_0} & \longleftarrow & X_{\mu_1} & \longleftarrow \cdots\cdots \longleftarrow & X_{\mu_q} \end{array} .$$

But this diagram determines x; it is the unique non-degenerate simplex associated to

$$X_{\lambda_0} \longrightarrow \cdots\cdots \longrightarrow X_{\mu_0} .$$

And it determines the sequence $\lambda_0, \ldots\ldots, \mu_0$, hence it determines γ.

Conclude:

$$\boxed{\mathrm{Sdr}(\mathrm{Nerv}\,\mathcal{C}) \xrightarrow{\sim} \mathrm{Nerv}(\mathrm{Sdr}\,\mathcal{C})}$$

Application:

$$\mathrm{Sdr}(\Delta(p)\times\Delta(q)) = \mathrm{Sdr}\ \mathrm{Nerv}([p]\times[q])$$
$$= \mathrm{Nerv}(\mathrm{Sdr}[p]\times\mathrm{Sdr}[q])$$
$$= \mathrm{Sdr}\,\Delta(p)\times\mathrm{Sdr}\,\Delta(q)$$

Conclude, taking limit over $\Delta(p)\in\Delta/X$, $\Delta(q)\in\Delta/Y$,

$$\boxed{\mathrm{Sdr}(X\times Y) = \mathrm{Sdr}(X)\times\mathrm{Sdr}(Y)}$$

More generally

$$\mathrm{Sdr}(X\times_{\mathrm{Nerv}\,\mathcal{C}} Y) = \mathrm{Sdr}(X)\times_{\mathrm{Nerv}(\mathrm{Sdr}\,\mathcal{C})}\mathrm{Sdr}(Y)$$

Observe however that Sdr cannot commute with fibred products, since then the adjoint functors

$$\Delta^\wedge \underset{\mathrm{Ex}}{\overset{\mathrm{Sd}}{\rightleftarrows}} \Delta^\wedge$$

would ~~constitute~~ a morphism of topos which (would) means that

$$\Delta(m) \longmapsto (\mathrm{Sd}^{\Delta(m)})_n$$

would be pro-representable.

Summary. We originally wanted the formula

$$\mathrm{Sdr}(\mathrm{Nerv}\,\mathcal{C}) = \mathrm{Nerv}(\mathrm{Sdr}\,\mathcal{C})$$

~~[illegible]~~ or better a ~~[illegible]~~ good theory of Sdr, in order to do the Ex^∞ theory nicely on the category level. In particular

$$[\mathcal{C},\mathcal{C}'] \overset{(?)}{=} \pi_0 \underline{\mathrm{Hom}}(N\mathcal{C}, \mathrm{Exr}^\infty N\mathcal{C}')$$

$$= \varinjlim_{m} \pi_0 \underline{\mathrm{Hom}}_{\Delta^\wedge}(\mathrm{Sdr}^m N\mathcal{C}, N\mathcal{C}')$$

$$= \varinjlim_{m} \pi_0 \underline{\mathrm{Hom}}_{Cat}(\mathrm{Sdr}^m \mathcal{C}, \mathcal{C}')$$

provided $N\mathcal{C}$ is finite. The point which might be useful later is that certain ~~[illegible]~~ constructions turn out nicely. Example: I conjecture that when $\mathcal{C}' \longrightarrow \mathcal{C}$ is (cofibred) ~~[illegible]~~ with all base changes heq's, then

$$\mathrm{Exr}^\infty(N\mathcal{C}') \longrightarrow \mathrm{Exr}^\infty(N\mathcal{C})$$

should be a Kan fibration, not just a q-fibn.

It remains to establish that $\mathrm{Exr}^\infty(N\mathcal{C})$ is a Kan complex, among other things. This requires explicitly retracting $\Delta(n)$ to $V(n,k)$ after subdividing.

Idea: Use instead

$$\begin{array}{l} \Delta(n)\times 0 \cup \dot\Delta(n)\times\Delta(1) \subset \Delta(n+1) \\ \Delta(n)\times 1 \cup \dot\Delta(n)\times\Delta(1) \end{array}$$

This suffices (see Gabriel-Zisman).

August 23, 1971. Mumford conjecture

Let $k = \overline{\mathbb{F}}_p$ and let V be a representation of $B =$ Borel subgroup of GL_n over k. I want to compute

$$H^*(B(k), V) = \varinjlim_{k_d \subset k} H^*(B(k_d), V)$$

where k_d denotes the subfield of k with p^d elements. First of all

$$H^*(B(k_d), V) \xrightarrow{\sim} H^*(U(k_d), V)^{T(k_d)}$$

as $T(k_d)$ is of order prime to p. Secondly, by Borel's fixpoint theorem $\exists$ a flag in V stable under B

$$V = V_0 \supset V_1 \supset \cdots\cdots \supset V_N \supset 0$$

hence a spectral sequence

$$E_1^{pq} = H^{p+q}(U(k_d), V_p/V_{p+1}) \Longrightarrow H^{p+q}(U(k_d), V).$$

$$\parallel$$

$$H^{p+q}(U(k_d)) \otimes V_p/V_{p+1}$$

This gives an estimate

$$P.S.\{H^*(U(k_d), V)\} \ll P.S.\{H^*(U(k_d))\} \cdot P.S.\{V\}$$

where the Poincare series is defined to be

$$\sum t^n \, [H^n(U(k_d), V)] \in R(T(k_d))[[t]]$$

$R(T(k_d)) =$ the character ring of $T(k_d)$.

To simplify suppose $p = 2$ and $n = 2$. Then

$$H^*(U(k_d)) = S\left[\bigoplus_{a=0}^{d-1} \check{L}^{p^a}\right]$$

where L is the obvious ~~one~~-diml. repn. of k_d^* on k. So

$$P.S.\{H^*(U(k_d))\} = \prod_{0 \le a < d} \frac{1}{(1 - t\,\check{L}^{p^a})}$$

~~Now~~ $\cancel{P.S.\{V\} = \sum_{-N \le i \le +N} m_i L^i}$

~~where the~~ $\cancel{m_i}$ ~~are integers~~ $\cancel{\ge 0}$

Now $P.S.\{V\}$ is a fixed sum of characters L_α of $T(k_d)$. If H^1 contains an invariant, then we have that

$$L_\alpha = L^{p^a}$$

for some α and a.

August 26, 1972

$\mathcal{C}$ cat, $\mathrm{Sub}\,\mathcal{C}$ = cat with objects $(X \to Y)$ in $\mathcal{C}$ and maps $(X' \to Y') \to (X \to Y)$ being diagrams "

$$\begin{tikzcd} X' & X \arrow[l] \arrow[d] \\ Y' \arrow[u, leftarrow] \arrow[r] & Y \end{tikzcd}$$

What is $\mathrm{Sub}(\mathrm{Sub}\,\mathcal{C})$? Objects are squares.

maps

$$\begin{tikzcd}
 & U \arrow[d] & & X \arrow[ll] \arrow[d] \arrow[dll] \\
U' \arrow[ur] \arrow[d] & X' \arrow[l] \arrow[d] & V \arrow[r] \arrow[dll] & Y \\
V' \arrow[r] & Y' \arrow[urr] & &
\end{tikzcd}$$

Thus objects are

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & U \arrow[r] & V \arrow[r] \arrow[d] & Y \\
X' \arrow[r] & U' \arrow[r] \arrow[u] & V' \arrow[r] & Y' \arrow[u]
\end{tikzcd}$$

and maps are

$\Rightarrow$ we get a chain of length 8.

August 27, 1972

C proj n.s. curve over $k = \Gamma(C, \mathcal{O}_C)$ finite, F = fn. field
∞ point of C, $\mathcal{O} = \mathcal{O}_\infty$
$A = \Gamma(C - \infty, \mathcal{O}_C)$
M f.g. proj. A-module, $\Gamma = \mathrm{Aut}(M)$

$X = X(M \otimes_A F)$ building consisting of $\mathcal{O}$-lattices in the F-vector space $M \otimes_A F$. A vertex L of X represents an extension of M to a vector bundle over C.

The problem is to compactify X/Γ. What Borel-Serre do in the arithmetic case is to define

$$\overline{X}(V) = \coprod_{0 \subset W \subset W' \subset V} X(W'/W)$$

with a suitable topology. Perhaps I can do the same thing here.

There are perhaps problems associated with GL_2. For example, Nagao's theorem

$$GL_2(k[X]) = GL_2(k) *_{B_2(k)} B_2(k[X])$$

shows that $H_1(GL_2\, k[X]) = k^* \oplus k[X]$ which is not finitely generated.

What I want to do is understand whether these compactifications might be relevant to K-theory.

<u>Review</u>: Recall that we have this filtration

$$\cdots\cdots \subset F_{r-1}Q(\mathcal{P}_A) \subset F_r Q(\mathcal{P}_A) \subset \cdots\cdots$$

and that it leads to a long exact sequence

$$\xrightarrow{\partial} H_q F_{r-1} \longrightarrow H_q F_r \longrightarrow \bigoplus_\alpha H_{q-r}\big(\mathrm{Aut}(M_\alpha),\ \mathrm{St}(M_\alpha \otimes F)\big) \xrightarrow{\partial}$$

where M_α runs over reps for the iso classes of pr.f.g. A-modules. (Thus $\alpha \in \mathrm{Pic}\,A$).

I refresh my memory: One has

$$F_{r-1} \xrightarrow{j} F_r \xleftarrow{i_M} \mathrm{Aut}(M)$$

and for any M of rank r we know that j/M is equivalent to the suspension of the building $X(M \otimes F)$.

GL_n ~~[illegible]~~

$GL_n \supset GL_{n-1}$

GL_n

$H_*(\Gamma, St(V))$

equivariant cohomology with compact supports.

$GL_n \longleftarrow GL_{n-1} \longleftarrow GL_{n-2} \longleftarrow \cdots$

I am interested in $H_*(\Gamma, St(V))$. Homology vanishes

~~[illegible]~~ $H_*(\Gamma, St(V))$

G group

consider the category of G-sets

and the crible formed of those ~~[illegible]~~ whose stabilizers are the unipotent groups. Note a subobject of the final object. Can we make sense of relative coh.

---

cat of G/u, u unipotent

cat. of G/e. Result is

~~examples~~

~~G P nilpotent~~

~~then get H^*~~

$$H_*(G) \longrightarrow H_*(\{G/u\})$$

$P \lhd G$

$H^*(BG, BP)$

$$\boxed{Hom_G(G/P, G/P)} = \{gP \mid Pg P = gP\} = Norm(P)/P.$$

$g^{-1}Pg \subset P$

$Hom_G(G/P, G/P)$

$\boxed{\text{idea}}$ Take cochains in $C^{\cdot}(G,A)$ which vanish on all p-subgroups. Thus one wants the relative cohomology

$$\bigcup_{\substack{P\subset G\\ P\ p\text{-grp}}} BP \subset BG$$

But if Sylow p-subgroup P is normal in G get $H^*(BG, BP)$

this is the reduced cohomology of G with $\mathbb{Z}'$ coeffs.
But for $\mathbb{Z}/p\mathbb{Z}$ it is wrong, since we should expect 0

$$H^*(BG, \mathbb{F}_p) = H^*(BP, \mathbb{F}_p)^{G/P}$$

In the case of $GL_2(\mathbb{F}_p)$ one has a P for each line so that $\bigcup BP = \bigvee_{\mathbb{P}_1} BP$

and the homology is gigantic

$H_*(\Gamma, St)$

1 part of homology Γ which is primitive

for ex. if we restrict the Steinberg to a parabolic P, then St is free over the unipotent part so it collapses the homology:

$$H^*(P, St(G)) = H^*(P/U, St(P/U)) = \text{tensor product}$$

$H^*(GL_2, St)$.

$$\begin{tikzcd} {} \arrow[r] & H_*(B_2) \arrow[r] \arrow[d] & H_*(G_2) \arrow[r] \arrow[d] & H_*(G_2, St(k^2)) \\ & H_*(G_1 \times G_1) \arrow[r] & \bigcirc \arrow[r] & '' \end{tikzcd}$$

Goal was fairly simple. To modify $GL_2 A$ so that its homology

---

$GL_2(K)$ $\qquad$ K local field, valn. ring A

act on parahoric building of K^2.

simple orbits

The category of p-subgroups of G

Consider the cat of irred. G-sets whose stabilyzers are p-groups. One object for each conjugacy class of p-groups, and funny maps.

G/H

parahoric building $X(V)$
has three kinds of simplices

0	L		$GL_2 A$
1	$L<L'$	$\dim L'/L = 1$	P
"	$L<L'$	——— $=2$	GL_2
2	$L_0<L_1<L_2$		P

$$\begin{tikzcd} & P \arrow[r, shift left, "in"] \arrow[r, shift right, "\cancel{\text{twisted}}"'] & G\cancel{A} \\ P \arrow[ur, shift left] \arrow[ur, shift right] \arrow[r] & G \arrow[ur, shift left] \arrow[ur, shift right, "{''}"'] & \end{tikzcd}$$

what about

$0 \qquad H_*(P) \longrightarrow H_*(G) \longrightarrow ?$

mod p cohomology

$$H^*(G) \hookrightarrow H^*(P)$$

$$\cancel{H^*(GL_2, H^*(\mathbb{C})) \hookrightarrow H^*(P, H^*(\mathbb{C}))}$$

$$P \cap x P x^{-1} = T \tilde{\times} \mathbb{C}$$

$$\boxed{P \cap x P x^{-1} \underset{x^{-1} \cdot x}{\overset{in}{\rightrightarrows}} P}\ .$$

$x \in G$. But we can modify x ~~[illegible]~~ by an element of P but ~~[illegible]~~ one can only change x by an element of ~~[illegible]~~ $P \cap \operatorname{cent}(P \cap x P x^{-1})$.

GL_3

<u>orbits</u>:

0	G_3	L
1	$G_3,\ G_{12},\ G_{21}$	$L_0 \quad L_1$
2	$G_{12},\ G_{21},\ G_{111}$	$L_0 \cdot L_1 \cdot L_2$
3	$G_{111}.$	$L_0 < L_1 < L_2 < L_3$

So the idea, it would appear, is to systematically introduce the "Steinberg **homology**" into the calculations.

To prove Moore's theorem

$$\bigoplus_m K_{2i-1}(\mathcal{O}/m) \longrightarrow K_{2i-1}\mathcal{O} \longrightarrow K_{2i-1}F \longrightarrow 0$$

want this to be zero

Thus A Dedekind, we want $K_{2i-1}(A/m) \to K_{2i-1}A$ to be zero. Representation

$$\boxed{\bigoplus_{m} R_{A/m}(G) \longrightarrow R_A(G) \longrightarrow R_F(G) \longrightarrow 0}$$

$$R_{A/m}(G) \longrightarrow R_A(G) \xrightarrow{\sim} R_{A[S^{-1}]}(G) \longrightarrow 0$$

for G prime to p.

$$\begin{array}{c} E \\ \downarrow \\ A \end{array}$$

Steinberg homology for GL_2. $\quad H_*(GL_2, St(k^n))$

Now we let $GL_n K$ acts on $X(K^n)$

an orbit will $L_0 < \cdots < L_q \qquad q < n$

and there are two kinds: $\dim(L_q/L_0) < n$
$\dim(L_q/L_0) = n$.

~~And so if~~ we have integers

$$\pi L_q \subset L_0 < L_1 \cdots < L_q$$
$$n_0, \quad n_1 \quad n_q \qquad \text{where} \quad n_i \geq 0 \quad i > 0$$

and stabilizer is essentially

$$Gl_{n_0} \times G_{n_1} \times \cdots \times G_{n_q}$$

so our E_1-term is as follows: Take

$$R \otimes \underbrace{\tilde{R} \otimes \cdots \otimes \tilde{R}}_{q}$$

But the boundary is more complicated: In addition to deleting the L_i ~~we must produce~~

d_0 deletes L_0, so its adds $n_0 + n_1$

but d_q deletes L_q, so it adds n_q to n_0

i.e. this means we have

$$(R \otimes \tilde{R} \otimes \cdots \otimes \tilde{R} \otimes R) \overset{R \otimes R}{\otimes} R$$

and so we get $\mathrm{Tor}^{R \otimes R}(R, R) = R \otimes \mathrm{Tor}^R(\Lambda, \Lambda)$

for the algebra. So in degree n we should have

$$\oplus \; H_*(G_i) \otimes \mathrm{Tor}_*^R(\Lambda, \Lambda)_{n-i}$$

which is most messy.

Review the building

$GL(V)$ acts on $X(V)$, $[V:K]=n$, K local field with valuation ring A, residue field k.

Want then to understand the orbit spectral sequence for the mod l cohomology, $l \neq \operatorname{char} k$.

$$\bigoplus_{n\geq 0} H_*(GL_n(k)) = R$$

It will be very important to understand the Steinberg homology $H_*(GL_n k, St(k^n))$

$$\begin{array}{ccccccc} n & & & 2 & & 1 & & 0 \\ BG_{\underbrace{1,\dots,1}_{n}} & \quad \coprod \quad & & \coprod_{\substack{i+j=n \\ i,j>0}} BG_{ij} & \overrightarrow{\underrightarrow{\rightarrow}} & BG_n & \rightrightarrows & pt \end{array}$$

$$\tilde{R} = \bigoplus_{n>0} H_* G_n \qquad\qquad (\tilde{R}\otimes\tilde{R})_n \qquad \tilde{R}_n$$

The point is that

$$H_i(F_n, F_{n-1}) = \operatorname{Tor}_i^{\tilde{R}}(\Lambda,\Lambda)_n = H_{i-n}(G_n, St(k^n))$$

$n=2$. $\qquad n = n_0 + n_1 + \cdots + n_q \qquad q \le 2$

$q=0 \qquad G_2$ $\qquad$ (n_0)

$q=1 \qquad G_{11}$ ($n_0 n_1$), $\; G_2$ (n_1)

$q=2 \qquad G_{111}$

$n=3$

$q=0 \qquad G_3$

$q=1 \qquad G_{21} \qquad G_{12} \qquad G_3$

$q=2 \qquad G_{111}$ (n_0) $\qquad G_{12}$ ($n_1 n_2$) $\qquad G_{21}$ ($n_1 n_2$)

$q=3 \qquad G_{111}$ ($n_0 n_1 n_2$)

$n=4$

$\boxed{n_0 > 0}$

$q=0 \qquad G_4$

$q=1 \qquad G_{31} \qquad G_{22} \qquad G_{13} \qquad G_4$

$q=2 \qquad G_{211} \qquad G_{121}$ (n_0) $\qquad G_{112} \qquad G_{31} \qquad G_{22} \qquad G_{13}$

$q=3 \qquad G_{1111} \qquad G_{211} \qquad G_{121} \qquad G_{112}$

$q=4 \qquad G_{1111}$

$n_0 = 0$

rank 2 bundles
compute $H_*(\mathrm{Aut}\, E, St(E \otimes K))$

E indecomposable — no good
E decomposable · should be possible

assume $E = L \oplus L'$ with $\deg L \gg \deg L'$

then $\mathrm{Aut}(E) = \begin{pmatrix} k^* & \mathrm{Hom}(L', L) \\ & k^* \end{pmatrix}$

subgroup of the ~~unipotent~~ Borel group $B = \begin{pmatrix} K^* & K \\ & K^* \end{pmatrix}$

recall that Steinberg restricted to the Borel subgroup is $\mathbb{Z}[B/T]$

If $k = \mathbb{F}_2$ for example, $k^* = 1$, so $\mathbb{Z}[B/T] = \mathbb{Z}[U]$ is ~~an~~ ~~induced representation~~ direct sum of infinitely many copies of the regular repn of $\mathrm{Aut}(E)$. $\therefore$ $H_*(\mathrm{Aut}\, E, St(E \otimes K))$ is infinite

$\boxed{\text{The Vilodin game}}$ ~~suppose that Aut E works~~

~~Aut E~~ $\bigcup B(pTp^{-1}) \longrightarrow BG$

Volodin
fibre $\longrightarrow BU \vee BU' \longrightarrow BGL_2$

Serre's idea
$B(\text{Borel}) \longrightarrow BGL_2 \longrightarrow$ cofibre

my idea:

$$\begin{array}{ccc} B(\text{Bor.}) & \longrightarrow & BGL_2 \\ \downarrow & & \downarrow \\ B(GL_1 \times GL_1) & \longrightarrow & \text{pushout} \end{array}$$

$$0 \longrightarrow L \longrightarrow E \longrightarrow L' \longrightarrow 0$$

$$h^0(E) - h^1(E) = \deg E + 2(1-g)$$

$$h^0(E(n)) \geq \deg E + 2n + 2(1-g) > 0$$

$$\Rightarrow \deg L(n) \geq 0 \qquad \text{as } L \text{ is maximal.}$$

$aF = Fa$

$$\deg L + n \geq 0$$

$$2n > 2(g-1) - \deg E$$

$$n \geq g - \tfrac{1}{2} \deg E$$

$$\deg L + g - \tfrac{1}{2} \deg E \geq 0$$

$$\deg L + g - \tfrac{1}{2}(\deg L + \deg L') \geq 0$$

$$\tfrac{1}{2}(\deg L - \deg L') + g \geq 0$$

i.e. $\boxed{\deg L' - \deg L \leq 2g}$

If indecomposable, then

$$H^1(\mathrm{Hom}(L', L)) \geq 0$$

$$\shortparallel$$

$$H^0(\omega \otimes L^* \otimes L')$$

$$2g - 2 - \deg L + \deg L' \geq 0$$

$$\boxed{\deg L' - \deg L \geq -(2g-2)}$$

<u>The problem:</u> To find for each n a map

$$BGL_n \longrightarrow X_n$$

such that

i) exact sequences add

$$\begin{tikzcd}
BGL_{p,q} \arrow[r] \arrow[d] & BGL_{p+q} \arrow[dr] & \\
BGL_p \times BGL_q \arrow[dr] & & X_{p+q} \\
& X_p \times X_q \arrow[ur, "\exists"] &
\end{tikzcd}$$

ii) For ℓ prime to the characteristic, the map

$$BGL_n \longrightarrow X_n$$

should induce isos. on mod ℓ homology

iii) X_n should be nice with respect to stability.

iv) X_n should have no mod p cohomology for a finite field of characteristic p.

~~at~~ stable splitting theorem.
has to be understood at same time as stability theorem.

operations on extensions

$$0 \to V_0 \to V \to 1 \to 0$$

sum, product, symm $S^n V$.

point perhaps is that G is a perfect group and BG^+ has no p-torsion. The problem is that G acts on V_0 so the cohomology is non-trivial, but

---

$$V_0 \to V \to 1$$

~~$SL_2(k)$~~

$$\boxed{0 \to V_0 \to V \to 1 \to 0.}$$

$$0 \to \frac{V_0 \cdot S^{n-1} V}{S^2 V_0 \cdot S^{n-1} V} \to S^n V \to 1 \to 0$$

$$0 \to I \to SV \to k[T] \to 0$$

$$I/I^2 \simeq V_0$$

$$I/I^2 \to SV/I^2 \to k \qquad 0$$

Conjecture

OKAY for $p=2$

$$0 \to SV_0 \to S^p V \to 1 \to 0$$

$$S^2 V_0 \subset \underline{V_0 \cdot V} \hookrightarrow \underline{S^2 V} \to 1$$

Conjecture is clear

$$\begin{tikzcd} & V_0 S^{p-1}V \arrow[r] & S^pV \arrow[r] & S^p1 \arrow[r] & 0 \\ 0 \arrow[r] & V_0^{(p)} \arrow[r] \arrow[u] & V^{(p)} \arrow[r] \arrow[u] & 1^{(p)} \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

this shows that the operation comes from a map

$$V_0^{(p)} \longrightarrow V_0 S^{p-1} V.$$

<u>which dies in</u>

$p=2$

$$\begin{tikzcd} H^1(S_2V_0) \arrow[d] & H^1(V_0^{(2)}) \arrow[l] \\ H^1(V_0V) \arrow[d] & \\ H^1(V_0) & H^1(V_0) \arrow[l, "0"'] \arrow[uu, "\cong"'] \end{tikzcd}$$

No the thing to prove is that the class if it dies in $H^1(V_0V)$ then it dies in $H^1(S_2V_0)$.

(OKAY if V_0 has no invariants)

$$\begin{tikzcd} S_2V_0 \arrow[r] & \boxed{V_0V} \arrow[r] & V_0 \\ & S_2V \arrow[u, phantom, "\cap"] \arrow[d] & \\ & S_2(1) & \end{tikzcd}$$

$V_0 \subset V \to 1$

$x_0 \cdot z$

wrong because

$\mathbb{Z}_2$ $\tau x = x+y$ $\tau y = y$

canonical reason?

$S^2V_0 = V_0^{(2)}$ (doesn't split here.)

$$\begin{array}{ccc} & F & \\ & \downarrow & \\ Ar_2 C & Ar C & \coprod_{y \leftarrow x} F_y \end{array}$$

$$\left(F' \times_{Ob C} Ar C\right) \times_{Ar C} \left(Ar C \times_{Ob C} Ar C\right)$$

$$Ar C \rightrightarrows Ob C$$

$$V_0^{\pm}, V_1, \dots, V_n \qquad \left(V_0^{\pm} + \Delta V_1 + \dots + \Delta V_n\right) + \Delta V_{n+1}$$

$$S^2 V^{\pm}$$

$$S^n(V_0 \oplus 1) = S^n V_0 + S^{n-1} V_0 + \dots + \boxed{S^1 V_0} + S^0 V_0$$

$$\cancel{\;\;} \quad \mathrm{Filt}^p(S^n V) = \mathrm{Im} \quad \cancel{\;\;}$$

$$S^p V_0 \otimes S^{n-p} V \to S^n V$$

$$\underbrace{S^n V_0 + S^{n-1} V_0 \cdot V}_{} \qquad \underbrace{V_0 S^{n-1} V}_{}$$

$$S^2 V_0 S^{n-2} V \hookrightarrow V_0 S^{n-1} V$$

August 31, 1972

Suppose C is a proj. n.s. curve over k finite, $k = H^0(C, \mathcal{O}_C)$. Let ∞ be a point of C and

$$\Lambda = H^0(C - \infty, \mathcal{O}_C)$$

the coordinate ring of the affine curve $C - \infty$. Suppose k of char. p.

Let M be a proj. Λ-module of rank r. I wish to prove that

$$H_i(\operatorname{Aut} M, \mathbb{Z}[\tfrac{1}{p}])$$

is a finitely generated $\mathbb{Z}[\frac{1}{p}]$-module ~~abelian group~~ for each i.

Suppose I know this is true for all ~~[illegible]~~ proj Λ-modules of rank $< r$. Let us be given a simplex

$$\sigma: \quad 0 < (M_0 < \cdots < M_q) < M$$

of the building of M (i.e. of $F \otimes_\Lambda M$, $F =$ fn. field). Let Γ_σ be the stabilizer of σ. Then if we choose complements M_j in M_{j+1}, we have an exact sequence

$$1 \longrightarrow \Gamma_\sigma' \longrightarrow \Gamma_\sigma \longrightarrow \prod_{j=0}^{q+1} \operatorname{Aut}(M_j/M_{j-1}) \longrightarrow 1.$$

It is clear that $H_*(\Gamma_\sigma', \mathbb{Z}[\frac{1}{p}]) = \mathbb{Z}[\frac{1}{p}]$. By induction this reduces to the fact that the additive group $\operatorname{Hom}(M,N)$ will have trivial homology because it is a k-module.

Now by our induction hypothesis, the big product on the right has fin. type homology over $\mathbb{Z}[\frac{1}{p}]$, hence so does Γ_σ for any simplex σ of $X(M)$.

Now there only finitely many orbits of $\mathrm{Aut}(M)$ on $X(M)$. In effect once we give [redacted] [redacted] two filtrations

$$0 < M_1 < \cdots < M_\nu = M$$
$$0 < M_1' < \cdots < M_\nu' = M$$

with [redacted] $M_i/M_{i-1} \simeq M_i'/M_{i-1}'$ for all i, then these simplices are conjugate under $\Gamma = \mathrm{Aut}(M)$. Once the jump ranks are fixed, the iso. classes are in 1-1 corresp. with elts. of $\mathrm{Pic}\,\Lambda$ which is finite.

We conclude therefores by the induction hypothesis that

$$H_i^{[redacted]}(X(M)_\Gamma, \mathbb{Z}[\tfrac{1}{p}]) \qquad \Gamma = \mathrm{Aut}(M)$$

is finitely generated for each i. So now the problem is to show that the relative group

$$H_i([redacted]\, pt_\Gamma, X(M)_\Gamma ; \mathbb{Z}[\tfrac{1}{p}]) \qquad \forall i$$

is finitely generated. Then we have that $H_i(\Gamma, \mathbb{Z}[\frac{1}{p}])$ is f.g. and we can continues the induction.

Notice[redacted]: [redacted]

$$0 \longrightarrow \bar{C}_\cdot(X(M)) \longrightarrow C_\cdot(X(M)) \longrightarrow \mathbb{Z} \longrightarrow 0$$

and $$H_i(\bar{C}_\cdot(X(M), \mathbb{Z}) = \begin{cases} 0 & i \neq r-2 \\ St(F \otimes M) & i = r-2 \end{cases}$$

Therefore the relative group is:

$$H_i(\mathrm{St}_\Gamma, X(M)_\Gamma ; \mathbb{Z}[\tfrac{1}{p}]) = H_{i-1}(\Gamma, \bar{C}(X(M))\otimes\mathbb{Z}[\tfrac{1}{p}])$$

$$= H_{i-h+1}(\Gamma, St(M\otimes F)\otimes \mathbb{Z}[\tfrac{1}{p}])$$

<u>Conjecture</u> (possibly proved by Serre)

$$H_i(\mathrm{Aut}(M), St(M\otimes F)) \quad \text{is f.g.} \quad \forall i$$