~~July~~ June 7, 1972

Continuation of the descent problem for a Galois extension $F \subset E$ with Galois group π.

The naive hope is for a spectral sequence of the form

$$E_2^{pq} = H^p(\pi, K(E)_{\ell}) \Rightarrow K_{-p-q}(F).$$

The motivation: Let $X \to Y$ be a Galois covering with group π and let h^* be a generalized cohomology theory. Then the canonical map

$$P\pi \times^{\pi} X \longrightarrow Y$$

is a heq (fibre bundle with fibre $P\pi$), hence

$$h^*(Y) = h^*(P\pi \times^{\pi} X).$$

~~that~~ We can consider Y as being fibred (up to homotopy) over $B\pi$ with fibre X. Thus from skeletal decomposition of $B\pi$, we get a spectral sequence

$$E_2^{pq} = H^p(\pi, h^q(X)) \Rightarrow h^{p+q}(Y).$$

There might be convergence difficulties, but not if $B\pi$ is $\sim$ finite diml. CW complex.

However: Let us consider cases which are known. Thus take $F = \mathbb{F}_q$, $E = \bar{F}$. Then ~~the~~

$$\pi = \hat{\mathbb{Z}}$$

and the E_2-term appears:

$$
\begin{array}{lcccc}
q=0 & \mathbb{Z} & 0 & \mathbb{Q}/\mathbb{Z} & 0 \\
q=-1 & (K_1E)^\pi & 0 & 0 & 0 \\
q=-2 & 0 & 0 & 0 & \\
 & (K_3E)^\pi & 0 & 0 & \\
 & 0 & 0 & 0 &
\end{array}
$$

$$H^1(\hat{\mathbb{Z}}, \mathbb{Z}) = \mathrm{Homcont}(\hat{\mathbb{Z}}, \mathbb{Z}) = 0$$

$$H^2(\hat{\mathbb{Z}}, \mathbb{Z}) = H^1(\hat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}^{\mathrm{cont}}(\hat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}.$$

In other words the $\mathbb{Q}/\mathbb{Z}$-term destroys the effect.

Next suppose $F = \mathbb{F}_q$, $E = \mathbb{F}_{q^d}$ so that π is cyclic of order d. Then we have

$$
\begin{array}{cccccc}
\mathbb{Z} & 0 & \mathbb{Z}/d\mathbb{Z} & 0 & \mathbb{Z}/d\mathbb{Z} & 0 \quad \cdots \\
(K_1E)^\pi & \cancel{(K_1E)_\pi}\ 0 & \cancel{(K_1E)^\pi}\ 0 & \cancel{(K_1E)_\pi}\ 0 & \cdots & \\
0 & 0 & 0 & 0 & 0 & 0 \\
(K_3E)^\pi & \cancel{(K_3E)_\pi}\ 0 & \cancel{(K_3E)^\pi}\ 0 & \cancel{(K_3E)_\pi}\ 0 & \cdots & \\
0 & 0 & 0 & 0 & &
\end{array}
$$

by the periodicity of the cohomology of the cyclic group

$$H^{2i+1}(\pi, A) = \frac{\mathrm{Ker}\{N : A \to A\}}{\mathrm{Im}\{\sigma - 1\}}$$

$$H^{2i}(\pi, A) = \frac{\mathrm{Ker}\{\sigma - 1\}}{\mathrm{Im}\{N\}}$$

Now for the K_i of finite fields we know that N is surjective onto the invariants, whence

$\sigma - 1$ must map onto the kernel of N. Thus everything is OK except for the $E_2^{2i,0} = H^{2i}(\mathbb{Z}/d\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/d\mathbb{Z}$ terms.

<u>July 9, 1972.</u>

Let k be an algebraically closed field of characteristic p and $k_0 = \{x \in k \mid x^q = x\}$ the finite subfield with $q = p^d$ element. I wish to understand the non-commutative ring

$$R = k[F]$$

where F is an indeterminate $\ni$ $Fx = x^q F$ for all $x \in k$. ~~Every element is of the form~~ Elements of R are uniquely expressible as polynomials

$$a_0 + a_1 F + \dots + a_n F^n \qquad a_i \in k$$

Thus R is a graded ring without zero divisors (consider the highest degree terms).

<u>Ideal structure</u>: If L is a non-zero left ideal in R, let f be a monic polynomial of least degree contained in L. Then $L = Rf$ by division algorithm, so every left ideal is principal. Conclude

1) R left noeth (every left ideal f.g.)

2) R left regular (every monogenic R-module of form R/Rf, so either free, or of projective dim 1:

$$0 \longrightarrow R \xrightarrow{\cdot f} R \longrightarrow R/Rf \longrightarrow 0$$

as R has no zero divisors.)

Thus R being a graded left regular ring $\Rightarrow$ $K_i(k) \xrightarrow{\sim} K_i(R)$.

(Remark: The preceding holds for any ~~auto~~ endo σ of k instead of $x \mapsto x^q$ and have only used k field.
The preceding holds for right modules for an auto σ. Otherwise, it is not possible to find a monic f i.e. need

$$a_n F^n = F^n \cdot a_n^{-\sigma^2}$$

to get a monic F.)

Suppose I is a 2-sided ideal. Then if f is a monic poly of minimal degree in I, we have

$$I = Rf = fR$$

Let

$$f = F^n + a_{n-1}F^{n-1} + \cdots + a_0$$

Then

$$x^{q^n} f x^{-1} = F^n + x^{q^n - q^{n-1}} a_{n-1} F^{n-1} + \cdots + x^{q^n - 1} a_0 \ , x \in k$$

so by uniqueness of f, can conclude $a_i = 0$. (can find $x_i \ni x_i^{q^i} \neq x_i$). Therefore the only 2-sided ideals are:

$$R \cdot F^n \qquad n \geq 0$$

Module structure: Let M be a finitely generated R-module and choose a presentation for M

$$R^p \longrightarrow R^q \longrightarrow M \longrightarrow 0$$

$$(r_i) \longmapsto \left(\sum_i r_i a_{ij}\right)$$

with q minimal. Assuming M is not free, so that $a_{ij} \neq 0$, we can choose the presentation such that $a_{11} \neq 0$ and such that the degree of a_{11} is minimal. Can suppose a_{11} is monic. Then necessarily by division algorithm

$$\begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & & \\ \vdots & & \end{pmatrix}$$

we must have $a_{n1} \in Ra_{11}$, $a_{1n} \in a_{11}R$, so performing the obvious row + column operations, we can replace the matrix by

$$\begin{pmatrix} a_{11} & 0 & 0 & \cdots \\ 0 & & & \\ \vdots & & & \\ 0 & & & \end{pmatrix}$$

whence $M = R/Ra_{11} \oplus M'$. Conclude

1. Every f.g. M sum of monogenic modules.
2. Every torsion-free f.g. M is free.

Torsion f.g. R-mods (means $\forall m$, annihilator non-zero) are the same as R-modules which are f.d. over k.

This is because for f monic of degree n, R/Rf is free of rank n with basis $1, \ldots, F^{n-1}$.

<u>Structure of torsion modules.</u> A f.g. torsion R-module is simply a k-vector space V of finite dimension endowed with an operator $F: V \to V$ satisfying

$$F(xv) = x^{\delta} F v \qquad x \in k, v \in V.$$

~~The~~ We have a decreasing sequence of R-submodules (recall RF is a 2-sided ideal)

$$V \supset FV \supset F^2 V \supset \cdots\cdots\cdots$$

hence by Fitting's lemma, there is a unique splitting

$$V = V' \oplus V''$$

such that F is nilpotent (resp. bijective) on V' (resp. V'').

<u>Basic lemma</u>: If V is a f.d. v.s. $/k$ with an F which is an auto., then

$$k \otimes_{k^0} V^0 \xrightarrow{\sim} V$$

where $V^0 = \{v \mid Fv = v\}$.

Proof. Injectivity: Let e_i, $i \in I$ be a basis for V^0 and let

$$\sum x_i e_i = 0$$

be a primordial relation (set of $i \ni x_i \neq 0$ is minimal + one $x_i = 1$). Comparing this relation with its translate under F, one sees $x_i^q = x_i$, contradicting independence of the e_i.

Surjectivity: First we show $V \neq 0 \Rightarrow V^0 \neq 0$.
Can suppose V simple R-module, hence $V \simeq R/Rf$, where $f = F^n + \cdots$ is a monic polynomial of degree n say.
Claim $n=1$; will show $f = g(F-\lambda)$ for a suitable λ.
Have identity

$$F^m = \left(F^{m-1} + \lambda^{q^{m-1}} F^{m-2} + \cdots + \lambda^{q^{m-1}+\cdots+q^1}\right)(F-\lambda) + \lambda^{q^{m-1}+\cdots+q^1+q^0}$$

Hence if

$$f = \sum_{m=0}^{n} a_m F^m$$

Then

$$f = g(F-\lambda) + \left\{ \begin{array}{l} \lambda^{q^{n-1}+\cdots+1} \\ a_{m-1}\lambda^{q^{m-2}+\cdots+1} \\ \\ + a_0 \end{array} \right\}$$

Better we have that the remainder is

$$r(\lambda) = \lambda^{\frac{q^m-1}{q-1}} + a_{n-1}\lambda^{\frac{q^{n-1}-1}{q-1}} + \cdots + a_0$$

$\boxed{\text{Observe } r(\lambda) = r'(\lambda)\lambda + a_0 \text{ thus since } a_0 \neq 0,\ r(\lambda) \text{ has simple roots}}$

and since k is algebraically closed, ~~xxxx~~ there exists a root of this polynomial.

Thus must have $f = F-\lambda$, so V is 1-dimensional and for some $v \neq 0$, $Fv = \lambda v$. Now changing v to xv and arranging x so that $F(xv) = \lambda(xv)$, i.e. $x^{q-1} = \lambda$, we see $V^0 \neq 0$.

Suppose then that $W = k \otimes_{k^0} V^0 < V$. As $(V/W)^0 \neq 0$ we have a $v \in V$, $v \notin W$, such that $Fv - v = w = \sum_i y_i e_i$ where e_i is a basis of $W^0 = V^0$. To find $x_i \in k$ ∋

$$F\left(v - \sum x_i e_i\right) \stackrel{?}{=} v - \sum x_i e_i$$

i.e.

$$\underset{\textstyle\sum y_i e_i}{\underset{\|}{Fv - v}} \stackrel{?}{=} \sum (x_i^q - x_i) e_i$$

Can be done since $x_i^q - x_i = y_i$ has roots. Done with basic lemmas.

<u>Remark</u>: Above holds for k separably closed, probably for any strictly local ring in char. p. (Yes, see Oct. 18, 1971 report attached below).

<u>Cor.</u> Category of f.g. (resp. arbitrary) torsion $R = k[F]$-modules on which F acts invertibly is equivalent to the category of f.g. (resp. arb.) k^0-modules.

<u>Cor.</u> Any torsion R-module V on which F acts invertibly is an <u>injective</u> R-module

Proof. Have to show

$$\mathrm{Hom}_R(R, V) \twoheadrightarrow \mathrm{Hom}_R(L, V)$$

for any left ideal L in R. Can suppose V f.g. and $L \neq 0$, whence $L = Rf$. Can suppose $V = k$ with $Fx = x^q$. Then have

$\operatorname{Ker}\varphi = Rgf$

$$
\begin{tikzcd}
0 \arrow[r] & Rf \arrow[r] \arrow[d] \arrow[ddr, "\varphi"] & R \arrow[r] \arrow[d] & R/Rf \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Rf/gf \arrow[r] \arrow[dr, "s"'] & R/gf \arrow[r] & R/Rf \arrow[r] & 0 \\
 & & k & &
\end{tikzcd}
$$

But the bottom sequence splits by the equivalence with k_0-modules, so φ extends

July 14, 1972 — Homotopy of cats, again

Let C be a small category such that

(*) The only endos. and isos. in C are the identity maps. (Equivalently ~~...~~ for any diagram

$$X \xrightarrow{f} Y \xrightarrow{g} X$$

in C we have $Y = X$ and $f = g = id_X$.)

Let us consider a simplex in $\mathrm{Nerv}(C)$

$$X_0 \longrightarrow \cdots\cdots \longrightarrow X_p$$

For this to be non-degenerate means none of the arrows are the identity. If two of the vertices ~~...~~ coincide, say $X_j = X_k$ ~~...~~ with $j<k$, then (*) can't hold. In effect, if $k = j+1$ then the arrow $X_j \to X_k$ would be an endo., hence the identity; and if $k \geq j+2$ we have maps

$$X_j \longrightarrow X_{j+1} \longrightarrow X_k$$

so $X_j \to X_{j+1}$ would be the identity. Thus

(*) $\Longrightarrow$ all vertices of a non-degenerate simplex are distinct.

The converse is also true since

$X \xrightarrow{f} X$ would be non-degenerate if $f \neq id_X$

$X \xrightarrow{f} Y \xrightarrow{g} X$ ———————— if $f \neq id_X \neq g$.

Conclude: Suppose C satisfies $(*)$. Then.

The full sub-category $(\Delta/\mathrm{Nerv}\,C)^{nd}$ of $\Delta/\mathrm{Nerv}\text{-}C$ consisting of non-degenerate simplices is an ordered set, and it is fibred over Δ^+ (= subcat. of injective maps in Δ).

Observe that the last vertex map

$$(\Delta/\mathrm{Nerv}\,C)^{nd} \longrightarrow C$$

is pre-cofibred, and the fibre has an initial element.

Deligne's construction: Given C satisfying $(*)$ Deligne considers finite subcategories $\mathcal{F}$ of C having final objects. These form an ordered set I under inclusion and there is a functor

$$I \longrightarrow C$$

sending $\mathcal{F}$ to its final object. The functor is pre-cofibred, the fibres being ordered sets with initial element. Note that non-degenerate simplices are special cases of such functors $\mathcal{F}$, i.e.

$$(\Delta/\mathrm{Nerv}\text{-}C)^{nd} \subset I$$

Advantage of Deligne's construction: The ~~category~~ ordered set I is directed when C is filtering.

~~(*) means any face of a non degenerate simplex is non degenerate~~

The way to replace a category $\mathcal{C}'$ by a $\mathcal{C}$ satisfying $(*)$ is to let $\mathcal{C}'$ be the subcategory of $\mathcal{C} \times \mathbb{N}$ with same objects where

$$\operatorname{Hom}_{\mathcal{C}'}\big((X',m'),(X,m)\big) = \begin{cases} \phi & m' > m \\ \phi & m'=m \quad X' \neq X \\ \{id_X\} & m'=m \quad X'=X \\ \operatorname{Hom}(X',X) & m' < m \end{cases}$$

Then $\begin{array}{c}\mathcal{C}' \longrightarrow \mathcal{C} \\ (X,m)\end{array}$ is pre-cofibred: given

$$X \xrightarrow{f} Y$$

then $f_*(X,m) = \begin{cases} (X,m) & \text{if } f = id_X \\ (Y, m+1) & \text{if } f \neq id_X \end{cases}$

The fibre over X is $\mathbb{N}$ which has an initial object. Thus $\underline{\mathcal{C}^{\#} \to \mathcal{C}'}$ is a heq.

Now let $\mathcal{C}$ be an arbitrary small category and let I be the set of diagrams in $\mathcal{C} \times \mathbb{N}$ of the form

$$(X_0, n_0) \longrightarrow (X_1, n_1) \longrightarrow \cdots \longrightarrow (X_p, n_p)$$

with $n_0 < n_1 < \cdots < n_p$. Then I is an ordered set and we have a functor

$$I \longrightarrow \mathcal{C}$$

given by the last vertex. The functor is pre-cofibred and fibres have initial elements.

Given $\mathcal{C}$ let $Sd(\mathcal{C})$ be the cofibred category over $\mathcal{C}^\circ \times \mathcal{C}$ defined by the functor $(X,Y) \longmapsto Hom(X,Y)$. The objects are arrows $u: X \to Y$ and a map $(u: X \to Y) \longrightarrow (u': X' \to Y')$ is a diagram

$$\begin{tikzcd} X \arrow[r, "u"] & Y \arrow[d] \\ X' \arrow[u] \arrow[r, "u'"'] & Y' \end{tikzcd}$$

Then $Sd(\mathcal{C})$ is cofibred over $\mathcal{C}$ (and over $\mathcal{C}^\circ$) with fibres having initial objects.

Suppose $\mathcal{E} \to \mathcal{C}$ is cofibred and the fibres have initial objects: $\phi_X \in \mathcal{E}_X$. Then we define

$$(u: X \to Y) \longmapsto u_* \phi_X$$

$$\begin{tikzcd} Sd(\mathcal{C}) \arrow[rr] \arrow[dr] & & \mathcal{E} \arrow[dl] \\ & \mathcal{C} & \end{tikzcd}$$

This is a functor:

$$\begin{tikzcd} \phi_{X'} \arrow[r] & w_*\phi_{X'} \arrow[r] & (wu)_*\phi_{X'} \arrow[r] & u'_*\phi_{X'} \\ & \phi_X \arrow[u] \arrow[r] & u_*\phi_X \arrow[u] \arrow[r] & (vu)_*\phi_X \arrow[u] \end{tikzcd}$$

$$X' \xrightarrow{w} X \xrightarrow{u} Y \xrightarrow{v} Y'$$

It is a cocartesian functor (the arrow $u \to u'$ is cocartesian when $w : X' \xrightarrow{\sim} X$).

Reason for the notation $Sd(\mathcal{C})$. I conjecture $\mathcal{C} \longmapsto Sd(\mathcal{C})$ analogous to barycentric subdivision of a simplicial complex. Hopefully it will be more suited to categories.

If $\mathcal{C}$ is an ordered set, then $Sd(\mathcal{C})$ is the ordered set of layers (X, Y), $X \leq Y$ in $\mathcal{C}$, where

$$(X', Y') \leq (X, Y)$$

means

$$X \leq X' \leq Y' \leq Y.$$

Examples:

$\mathcal{C}$: $\quad 0 \leq 1$

$Sd(\mathcal{C})$: $\quad (0,0) \leq (0,1) \geq (1,1)$

$Sd^2(\mathcal{C})$

$\mathcal{C}: \quad 0 \leq 1 \leq 2$

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}$$

$Sd\,\mathcal{C}:$

$$\begin{tikzcd} & & 11 \arrow[dl] \arrow[dr] \arrow[dd] & & \\ & 01 \arrow[dr] & & 12 \arrow[dl] & \\ 00 \arrow[ur] \arrow[rr] & & 02 & & 22 \arrow[ll] \arrow[ul] \end{tikzcd}$$

<u>Conjecture:</u> $(\mathcal{C}, \mathcal{C}') \mapsto \varinjlim_n \pi_0 \underline{\mathrm{Hom}}(Sd^n \mathcal{C}, \mathcal{C}')$

carries heq's into isomorphisms. (probably need $\mathcal{C}$ finite).

<u>Question</u>: Does $\mathcal{C} \longmapsto Sd\,\mathcal{C}$ have a right adjoint Ex?

If so then

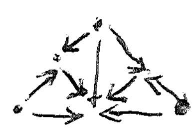
$$\begin{aligned} Ob\{Ex(\mathcal{C}')\} &= \mathrm{Hom}_{Cat}(e, Ex(\mathcal{C}')) \\ &= \mathrm{Hom}_{Cat}(Sd(e), \mathcal{C}') = \mathrm{Hom}_{Cat}(e, \mathcal{C}') \\ &= Ob\{\mathcal{C}'\} \end{aligned}$$

and

$Ar\{Ex(\mathcal{C}')\} = \mathrm{Hom}((0 \leq 1), Ex(\mathcal{C}'))$

$= \mathrm{Hom}($  $, \mathcal{C}')$

and

$Ar^{(2)}\{Ex\,\mathcal{C}'\} = \mathrm{Hom}($  $, \mathcal{C}')$

Answer. <u>NO</u>

Let $f : X \longrightarrow Y$ be a map of spaces (CW cxs say). Suppose that for every finite complex K we have that the induced map of fundamental groupoids

$$\underline{\underline{\pi}}\, \mathrm{Hom}(K,X) \longrightarrow \underline{\underline{\pi}}\, \mathrm{Hom}(K,Y)$$

is an equivalence of categories. Then f is a heq.

In effect, taking $K = pt$ we see the fundamental groupoids of X and Y are equivalent. By Whitehead we want to show that $\pi_k(X,x) \xrightarrow{\sim} \pi_k(Y, fx)$ for all k and x. But if K has a basepoint $\{*\}$, then

$$\mathrm{Hom}(K,Y) \longrightarrow \mathrm{Hom}(*,Y) = Y$$

is a fibration with fibre $\mathrm{Hom}((K,*),(Y,y))$ over $y \in Y$. Thus we have a fibration of groupoids

$$\underline{\underline{\pi}}\, \mathrm{Hom}(K,Y) \longrightarrow \underline{\underline{\pi}}\, Y$$

whose fibre over y is a groupoid with components $\pi_0 \mathrm{Hom}((K,*),(Y,y)) = [(K,*),(Y,y)]$. Thus the hypothesis implies

$$[(K,*),(X,x)] \xrightarrow{\sim} [(K,*),(Y,fx)],$$

so done.

July 15, 1971.

Remarks on Dold's paper, Partitions of unity in the theory of fibrations. Annals 1963.

Let X be a space and A, V two subspaces. Call V a halo nbd. of A if $\exists$ continuous function $\tau : X \longrightarrow [0,1]$ such that

$$\tau(A) = 1 \qquad \tau(X-V) = 0$$

Observe that if X is normal, then by Urysohn's lemma every neighborhood of a closed set is a halo neighborhood, and conversely.

Call a sheaf of sets F over X ***soft*** if for any $A \subset X$ we have surjectivity

$$F(X) \twoheadrightarrow \varinjlim_{U \supset A} F(U)$$

where U runs over the halo neighborhoods of A. It is enough to consider only A which are closed since halo nbds. of A and $\bar{A}$ are the same thing.

Observe that this agrees with the Godement definition when X is paracompact. Indeed the inductive limit above is $F(A)$ for any closed set A, (Cor. 1 p. 151), hence the condition becomes $F(X) \twoheadrightarrow F(A)$ for all closed A.

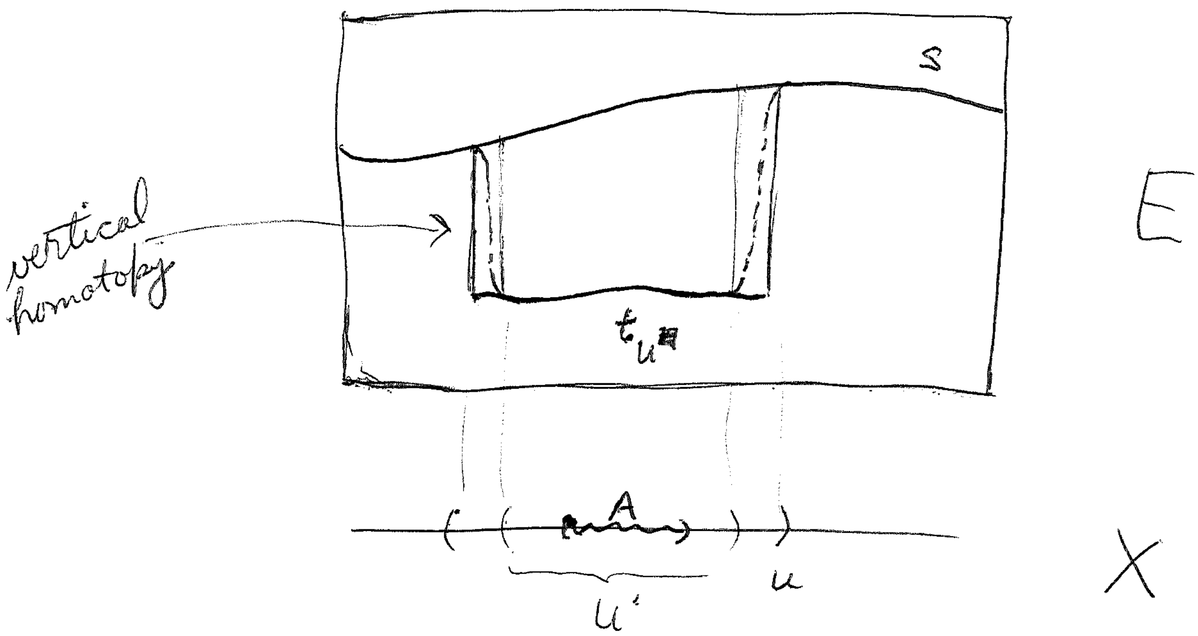
Dold's principal technical result is the following, which generalizes Godement's 3.4.1 (p. 151).

<u>Theorem:</u> Let $\{U_i\}$ be a numerable covering of X [redacted] (numerable means $\exists$ refinement of form $\rho_\lambda^{-1}(0,1]$ where ρ_λ is a locally-finite partition of 1 [redacted]), and assume that $F|U_i$ is soft for each i. Then F is soft.

<u>Examples:</u> Let $E \xrightarrow{f} X$ be a space over X and F the sheaf of its sections. Call E soft over X if F is soft.

<u>Claim:</u> If f is a fibre-homotopy-equivalence (over X) (i.e. $\exists\ s: X \longrightarrow E \ni fs = id_E$ and $sf \underset{X}{\simeq} id_E$) then E is soft over X.

<u>Proof:</u> Given a halo nbd. U of $A \subset X$ and a section t of f over U we must show that ~~[illegible]~~ t restricted to some smaller nbd. extends to all of X. Picture



The dotted arrow gives the desired extension over U'.

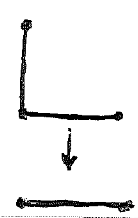
<u>Converse</u>: Assume $E \xrightarrow{f} X$ universally soft (remains soft after any base change; e.g. example above). Then f is a fibre homotopy equivalence over X.

<u>Proof</u>: First of all, soft $\Rightarrow$ $\mathcal{F}(X) \neq \phi$ since $\mathcal{F}(\phi) = pt$ (observe $\phi \subset \phi$ is a halo nbd.), hence f has a section s. Now want to construct a dotted arrow in

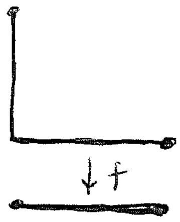
$$\begin{tikzcd} E\times \dot{I} \arrow[r, "sf + id"] \arrow[d, hook] & E \arrow[d, "f"] \\ E\times I \arrow[r, "fpr_1"] & X \end{tikzcd}.$$

Since f is universally soft, $(fpr_1)^*(E)$ is soft over $E\times I$, hence all we need ~~do~~ is extend the section over $E\times \dot{I}$ to a halo nbd. But this is clearly possible using constant homotopies near 0 and 1.

Note: We do not require that the vertical homotopy $sf \sim_X E$ preserve the section s. Thus $s(X)$ is not necessarily a strong deformation retract of E over X. In good cases one might to be able to arrange this by extending the map $E\times \dot{I} \cup X\times I \longrightarrow E$? Here is a soft map which is not a fibration:



Weak covering homotopy property: We say that $f: E \to X$ has the WCHP if given $\alpha: K \to E$ and a homotopy $K \times I \to X$ starting from $f\alpha$, there is a lifting $K \times I \to E$ whose initial position is vertically homotopic to α. Example:



Lemma: Let $f: E' \to E$ be a map of spaces over B such that there exists $g: E \to E'$ with $gf \sim_B \mathrm{id}_{E'}$. If $E \to B$ has the WCHP, then so does $E' \to B$.

Proof: Given

$$\begin{tikzcd} K \times 0 \arrow[r, "\alpha"] \arrow[d, "i"] & E' \arrow[r, "f"] \arrow[d, "p'"] & E \arrow[dl, "p"] \\ K \times I \arrow[r, "\beta"] & B & \end{tikzcd}$$

$\exists$ $H: K \times I \to E$ covering $\beta \ni Hi \sim_B f\alpha \Longrightarrow$
$gH: K \times I \to E'$ covers β and

$$gHi \sim_B gf\alpha \sim_B \alpha \qquad \text{q.e.d.}$$

Proposition: Let $f: E' \to E$ be a map of spaces over B such that both $p': E' \to B$ and $p: E \to B$ have the WCHP. If f is a homotopy equivalence, then it is a fiber homotopy equivalence.

[redacted] Note the maps with the WCHP are stable under compositon and base change. The point is that $E \to B$ has WCHP iff [redacted] the dotted arrow exists in

$$\begin{tikzcd} K\times 0 \arrow[r] \arrow[d] & X \arrow[d] \\ K\times I \arrow[r, "\beta"] \arrow[ur, dashed] & Y \end{tikzcd}$$

provided β is a constant homotopy in some interval $K\times[0,\varepsilon]$.

~~Next suppose E', E [redacted] are spaces over B with the WCHP, and let $f: E' \to E$ be a map~~

$$\cancel{\begin{tikzcd} E'\times_E (E/B)^I \arrow[r, "pr_2"] \arrow[d, "(id\times d_1)"] & (E/B)^I \arrow[d, "(d_0,d_1)"] \\ E'\times_B E \arrow[r, "f\times id"] \arrow[d, "pr_2"] & E\times_B E \\ E & \end{tikzcd}}$$

~~Since E'/B has the WCHP the [redacted] lower pr_2 does.~~

Unpleasant feature: Suppose $E \to B$ has the WCHP but not the CHP, i.e. $E^I \to E\times_B B^I$ doesn't have a section. Then as $E^I \to E\times_B B^I$ is a heq (both spaces have E as strong defm. retracts), it cannot [redacted] have the WCHP, or otherwise by [redacted] the preceding proposition it would have a section.

The preceding proposition is proved by a covering homotopy type argument which might run as follows provided we knew that $(E/B)^I \longrightarrow E \times_B E$ has the WCHP when $E \to B$ does (this is alright when B paracompact and locally contractible.) Under this condition we may factor f

$$E' \xrightarrow{i} E' \times_E (E/B)^I \xrightarrow{g} E$$

in the customary way, and the second map g has the WCHP by

$$\begin{tikzcd}
E' \times_E (E/B)^I \arrow[r, "pr_2"] \arrow[d, "id \times d_1"] & (E/B)^I \arrow[d, "(d_0, d_1) \text{ has WCHP}"] \\
E' \times_B E \arrow[r, "f \times id"] \arrow[d, "pr_2 \text{ has WCHP as } E' \to B \text{ does.}"] & E \times_B E \\
E &
\end{tikzcd}$$

Thus if f is a homotopy equivalence, the map g will be a homotopy equivalence with the WCHP. Such a map has a section:

$$\begin{tikzcd}
B \arrow[r, "s"] \arrow[d, "i_0"] & E \arrow[d, "p", "\text{WCHP}"'] \\
B \times I \arrow[r, "h"] \arrow[ur, dotted] & B
\end{tikzcd}$$

where $h : ps \sim id_B$ is any homotopy constant near 0. Since g has a section, it follows that $\exists$ $f' : E \to E'$

such that $f'f \sim_B id_{E'}$. Applying the same reasoning to f' we find $f'': E' \to E$ such that

$$f''f' \sim_B id_E$$

Then one has that $f'' \sim_B f$, so f is a fibre-homotopy equivalence.

<u>Remark</u>: If X is a space of the homotopy type of a CW complex, then its sheaf-theoretic and singular cohomology coincide. In effect both are homotopy invariants, hence reduces to case of a CW complex, where equality follows from fact that CW complexes are paracompact and locally-contractible.

Dold shows that over parac. loc. contractible spaces that a WCHP space same as a space locally fibre homotopy equivalent with a product space.

July 17, 1972

Let I be an ordered set. Its realization

$$BI = |\mathrm{Nerv}(I)|$$

is the ordered simplicial complex whose simplices are chains $X_0 < \cdots < X_p$ in I. (ordered s.cx. = s.cx. + ordering on vertices $\ni$ each simplex is lin. ordered).

$Sd(I)$ is the ordered set of layers of I. I want to interpret $BSd(I)$ as a subdivision of BI.

Example 1. $I = \{0 \leq 1\}$. Then

$$Sd\, I = \quad 00 \leq 01 \leq 11$$

so geometrically we have

BI: $\quad 0 \longrightarrow 1$

$BSd(I)$: $\quad 00 \longrightarrow 01 \longleftarrow 11$

Example 2. Suppose $\mathcal{C}'$ is a full subcat. of $\mathcal{C}$. Then $Sd\, \mathcal{C}'$ is the full subcat of $Sd\, \mathcal{C}$ consisting of arrows $u : X \to Y$ such that $X, Y \in \mathcal{C}'$. Thus if I' is a subset of I endowed with the induced ordering, (terminology: subordered set), then $Sd\, I'$ is a subordered set of $Sd\, I$.

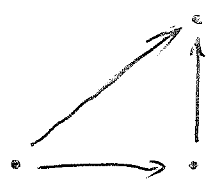
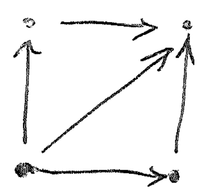
BI' is the subcomplex of BI consisting of the simplices $X_0 < \cdots < X_p$ with all X_i in I'. $B\,Sd\,I'$ is a subcomplex of $BSd\,I$.

Example 3. $\quad Sd(\mathcal{C} \times \mathcal{C}') \xrightarrow{\sim} Sd(\mathcal{C}) \times Sd(\mathcal{C}')$ In

effect
$$\mathrm{Ob}\,\{\mathrm{Sd}\,\mathcal{C}\} = \mathrm{Ar}\,\mathcal{C} = \mathrm{Hom}((0\le 1), \mathcal{C})$$
$$\mathrm{Ar}\,\{\mathrm{Sd}\,\mathcal{C}\} = \mathrm{Ar}_3\,\mathcal{C} = \mathrm{Hom}((0\le 1\le 2\le 3), \mathcal{C})$$

and these functors commute with products, in fact with arbitrary inverse limits, so we have
$$\mathrm{Sd}\,(\varprojlim \mathcal{C}_i) = \varprojlim \mathrm{Sd}\,\mathcal{C}_i .$$

Example 4. $I = (0 \le 1 \le 2)$. This can be embedded as a sub-(ordered set) of $(0\le 1)\times(0\le 1) = \bar{I}$:

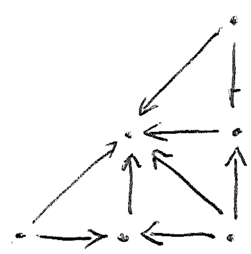
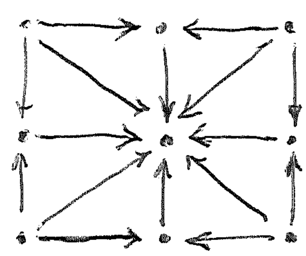
 $\subset$ 

$$\| \qquad\qquad \|$$

$$BI \quad \subset \quad B\bar{I}$$

so
$$B\,\mathrm{Sd}\,I \quad \subset \quad B\,\mathrm{Sd}\,\bar{I} = \left(B\,\mathrm{Sd}\,(0\le 1)\right)^2$$

$$\| \qquad\qquad \|$$

 $\subset$ 

Example 5. $I = [n] = (0 \le 1 \le \cdots \le n)$ which we will embed in $[1]^n$ as the sequence
$$(0,\dots,0) \le (1,0,\dots,0) \le (1,1,0,0,\dots) \le \cdots\cdots \le (1,1,\dots,1).$$

Then $BSdI$ is a subcomplex of $(BSd[1])^n$.

<u>Example 6</u>: I ordered set, we have $BSd(I)$ is a simplicial complex whose vertices are layers $X \leq Y$ in I. Define a map

$$h_t : BSd(I) \longrightarrow BI$$

$$h_t(X \leq Y) = tX + (1-t)Y \qquad 0 \leq t \leq 1$$

To show this map is well-defined we need only show that the image of the set of vertices of a simplex lie in a simplex. But a simplex in $Sd(I)$ is of the form:

$$X_p \leq \cdots\cdots (X_1 \leq (X_0 \leq Y_0) \leq Y_1) \cdots \leq Y_p$$

and

$$tX_0 + (1-t)Y_0$$

$$\cdots\cdots$$

$$\cdots\cdots$$

$$tX_p + (1-t)Y_p$$

all lie in the simplex $(X_p \leq \cdots \leq Y_p)$.

The preceding examples seem to establish the

Assertion: For any ordered set I, we have a map

$$h: BSd(I) \times [0,1] \longrightarrow BI$$
$$(X \leq Y), t \longmapsto tX + (1-t)Y$$

such that i) for $0<t<1$, h_t is a homeomorphism
ii) $h_1: BSd(I) \longrightarrow BI$ is the map induced by the target functor $Sd(I) \longrightarrow I$.
iii) h_0 is the map ~~$BSd(I)$~~ induced by source: $Sd(I) \longrightarrow I^o$ followed by the homeom $BI^o = BI$.

If I is finite, the subdivisions

$$\cdots \longrightarrow BSd^n(I) \xrightarrow{h_t} \cdots \longrightarrow BSd(I) \xrightarrow{h_t} BI$$

become arbitrarily fine, for any $0<t<1$.

Now I want to apply the simplicial approx. thm. Suppose I, J are two ordered sets, with I finite, and let

$$f: BI \longrightarrow BJ$$

be a continuous map of the associated ~~geometric~~ polyhedra. BJ has a canonical open covering - open stars of vertices $j \in J$. ~~The ... BJ.~~ For n suff. large, the composed map

$$f: BSd^n(I) \longrightarrow BJ$$

has the property that ~~every simplex~~ the open star of every vertex is contained in the inverse image of an open star of ~~a~~ BJ. Then we get a simplicial map

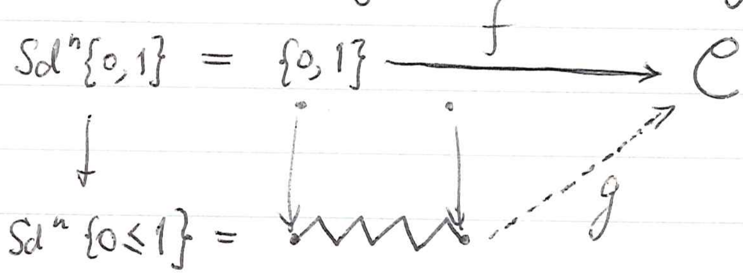
$$N\ Sd^n(I) \longrightarrow N J$$

<u>July 18, 1972</u>

The relation between what you are trying to do for categories and Kan's Ex^∞ theory:

Suppose $\mathcal{C}$ is a contractible category. Then I can solve the extension problem for the map
$$\{0,1\} \subset \{0 \leq 1\}$$

provided I subdivide enough. Precisely, suppose I have given f



Then g exists for n sufficiently large.

<u>Generalization:</u> Suppose I have maps
$$Sd^n\{0\leq 1\} \longrightarrow \mathcal{C}$$
$$Sd^n\{1\leq 2\} \longrightarrow \mathcal{C}$$
$$Sd^n\{0\leq 2\} \longrightarrow \mathcal{C}$$

which are compatible. Then can I enlarge n so as there exists an extension
$$Sd^n\{0\leq 1\leq 2\} \longrightarrow \mathcal{C}.$$

Question: Let K be a finite simplicial complex, ~~....~~ let L be a subcomplex, and let $\mathcal{C}$ be a contractible category. Given a functor $\mathrm{Cat}(L) \xrightarrow{f} \mathcal{C}$, does there exist a subdivision K' of K rel L so that f extends:

$$\begin{tikzcd} \mathrm{Cat}(L) \arrow[r, "f"] \arrow[d, phantom, "\cap"] & \mathcal{C} \\ \mathrm{Cat}(K') \arrow[ur] & \end{tikzcd} \quad ?$$

Better question: Given an ordered finite simplicial cx. K and a subcomplex L, we then have a inverse system of maps

$$\begin{array}{c} Sd^m L \\ \cap \\ Sd^m K \end{array}$$

and we can ask if given $Sd^m L \longrightarrow \mathcal{C}$, does there exists $n > m$ and an extension

$$\begin{tikzcd} Sd^n L \arrow[r] \arrow[d, phantom, "\cap"] & Sd^m L \arrow[r] & \mathcal{C} \\ Sd^n K \arrow[urr, dashed] & & \end{tikzcd} \quad ?$$

Assume the answer to the preceding is Yes. Define a functor on ordered simplicial complexes by

$$F(K) = \varinjlim_m \mathrm{Hom}(Sd^m(K), \mathcal{C})$$

Then we are asking that $F(K) \twoheadrightarrow F(L)$ if $L \subset K$. In particular if we take

$$K = \Delta(n)$$
$$L = \dot{\Delta}(n)$$

then we see that the simplicial set

$$n \longmapsto F(\Delta(n))$$

is a contractible Kan complex. Now

$$\mathrm{Hom}\left(Sd^m(\Delta(n)), \mathcal{C}\right)$$

should roughly be the same as

$$\mathrm{Hom}\left(\Delta(n), Ex^m(\mathrm{Nerv}\,\mathcal{C})\right).$$

This suggests that I am roughly aiming for a version of Ex^∞ using the elementary subdivision rather than barycentric subdivision.

~~[illegible] category~~

<u>Conjecture:</u> Let $\mathcal{C}$ be a contractible category. Then the simplicial set

$$n \longmapsto \varinjlim_m \mathrm{Hom}\left(Sd^m([n]), \mathcal{C}\right) = X(\mathcal{C})$$

is a contractible Kan complex.

Observe that if we used barycentric subdivision, then this limit would be ~~[illegible]~~ $Ex^\infty(\text{Nerv}\,\mathcal{C})$, so the conjecture would be clear.

Variations on the preceding conjecture:

1. $\mathcal{C} \xrightarrow{f} \mathcal{C}'$ ~~[illegible]~~ cofibred with contractible fibres. Then

$$X(\mathcal{C}) \longrightarrow X(\mathcal{C}')$$

is a Kan fibration with contractible fibres.

2. $\mathcal{C} \xrightarrow{f} \mathcal{C}'$ cofibred such that all cobase change functors are heq's. Then

$$X(\mathcal{C}) \longrightarrow X(\mathcal{C}')$$

is a Kan fibration (with fibre $X(\mathcal{C}_Y)$ over Y for all $Y \in \text{Ob}(\mathcal{C}')$; Observe vertices of $X(\mathcal{C})$ same as objects of $\mathcal{C}$).

July 19, 1972. On $K_*\mathbb{Z}$.

List all things that can be proved about $K_*(\mathbb{Z})$ using results about $K_*(\mathbb{F}_q)$ and the J-homom.

Claim:

$$J\{\pi_{4s-1}SO\} \hookrightarrow K_{4s-1}\mathbb{Z}$$

cyclic of order denom $(B_s/4s)$.

Proof. Diagram

$$\begin{tikzcd}
B\Sigma^+ \arrow[d] & & F \arrow[d] \\
BGL(\mathbb{Z})^+ \arrow[rru, dashed] \arrow[rr] & & BO = BGL(\mathbb{R}) \arrow[d, "(ch_{4i})_{i\geq 1}"] \\
 & & \prod_{i\geq 1} K(\mathbb{Q}, 4i)
\end{tikzcd}$$

F is the fibre of $(ch_{4i})_{i\geq1}$. Since Chern classes of a ~~flat~~ representation of a discrete group ~~G~~ are torsion ~~zero~~, the dotted arrow exists. Thus we get a diagram

$$\begin{tikzcd}
\pi^s_{4s-1} = \pi_{4s-1}B\Sigma^+ \arrow[r, "1"] \arrow[d] & \pi_{4s-1}F = \mathbb{Q}/a_s\mathbb{Z} \\
K_{4s-1}\mathbb{Z} \arrow[ru] &
\end{tikzcd}$$

$$a_s = \begin{cases} 1 & s \text{ even} \\ 2 & s \text{ odd} \end{cases}$$

I will show below that the arrow ① is $\pm$ Adams e-invariant. It is known that

$$J\{\pi_{4s-1}SO\} \xrightarrow[\sim]{e} e(\pi^s_{4s-1}) \subset \mathbb{Q}/a_s\mathbb{Z}$$

so the claim will follow.

Recall the definition of the e-invariant: Given an element of π^s_{4s-1}, represent it by a map

$$f: S^{8k+4s-1} \longrightarrow S^{8k};$$

and let X be its mapping cone. Then

$$0 \leftarrow \widetilde{KO}(S^{8k}) \leftarrow \widetilde{KO}(X) \leftarrow \widetilde{KO}(S^{8k+4s}) \leftarrow 0$$

so if we choose $x \in \widetilde{KO}(X)$ mapping onto the distinguished generator of $\widetilde{KO}(S^{8k})$, the top component of the character of x

$$ch_{8k+4s}(x) \in H^{8k+4s}(X,\mathbb{Q}) \cong H^{8k+4s}(S^{8k+4s},\mathbb{Q}) = \mathbb{Q}$$

is a rational number, determined up to $ch_{8k+4s}(\widetilde{KO}(S^{8k+4s})) = a_s\mathbb{Z}$. (This is the e-invariant) (Observe if s even, then real e-invariant = complex e-invariant, since $\widetilde{KO}(S^{8k}) \simeq \widetilde{K}(S^{8k})$)

Recast the preceding: Let $BO\langle 8k\rangle \longrightarrow BO$ induced isos. on π_j, $j \geq 8k$, with $BO\langle 8k\rangle$, $(8k-1)$-connected, and define F_{8k} by a fibration:

$$F_{8k} \longrightarrow BO\langle 8k\rangle \xrightarrow{(ch_{4i})_{i>2k}} \prod_{i>2k} K(\mathbb{Q},4i).$$

Then we have

$$\begin{tikzcd}
S^{8k+4s-1} \arrow[r, "f"] & S^{8k} \arrow[r] \arrow[dr, "\text{gen.}"'] & X \arrow[r] \arrow[d, dotted, "x"] & S^{8k+4s} \arrow[d, dotted, "ch_{8k+4s}(x)"] \\
 & & BO\langle 8k\rangle \arrow[r] & \prod\limits_{i>2k} K(\mathbb{Q},4i)
\end{tikzcd}$$

from which we see that

$$e(f) = \text{Toda bracket of}$$

$$S^{8k+4s-1} \xrightarrow{f} S^{8k} \xrightarrow{\text{gen}} BO\langle 8k\rangle \longrightarrow \prod_{i>2k} K(\mathbb{Q},4i)$$

and hence if we choose maps on the other side

$$\begin{tikzcd}
S^{8k+4s-1} \arrow[r, "f"] \arrow[d, dotted] & S^{8k} \arrow[d, dotted, "\gamma_{8k}"'] \arrow[dr, "\text{gen}"] & & \\
\Omega\prod\limits_{i>2k} K(\mathbb{Q},4i) \arrow[r] & F_{8k} \arrow[r] & BO\langle 8k\rangle \arrow[r] & \prod\limits_{i>2k} K(\mathbb{Q},4i)
\end{tikzcd}$$

we know the dotted arrow at the left $= -e(f)$ mod indeterminacy. So we find the following alternate description of the e-invariant.

$$e(f) = -f^*(\gamma_{8k}) \in \pi_{8k+4s-1}(F_{8k}) \xleftarrow{\simeq} \mathbb{Q}/a_s\mathbb{Z}$$

where $\gamma_{8k} \in \pi_{8k}(F_{8k})$ is the unique element mapped to the generator of $\pi_{8k}(BO\langle 8k\rangle)$.

Now take $8k$-fold loop spaces

$$
\begin{array}{ccccc}
\Omega^{8k}S^{8k} & & & & \\
\downarrow & & & & \\
\Omega^{8k}F_{8k} & \longrightarrow & \Omega^{8k}BO\langle 2k\rangle & \longrightarrow & \prod_{i\geq 1} K(\mathbb{Q},4i) \\
\wr\| & & \wr\| & & \| \\
\mathbb{Z}\times F & \longrightarrow & \mathbb{Z}\times BO & \longrightarrow & \prod_{i\geq 1} K(\mathbb{Q},4i)
\end{array}
$$

periodicity $\Rightarrow$

Thus it follows that the various γ_{8k} induce the map

$$
\begin{tikzcd}
\varinjlim \Omega^{8k}S^{8k} \arrow[r, "\gamma"] \arrow[dr] & \mathbb{Z}\times F \arrow[d] \\
& \mathbb{Z}\times BO
\end{tikzcd}
$$

which covers

and so it is now easy to see that the map on homotopy induced by γ in degree $4s-1$ is simply the e-invariant.

How about dimensions $8k$, $8k+1$. According to Adams the picture is that π_n^s contains direct summands

$\mathbb{Z}/2$	in	π^s_{8k}
$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	in	π^s_{8k+1}
$\mathbb{Z}/2$	in	π^s_{8k+2}

~~$\mathbb{Z}/2 \simeq J\{\pi_{8k} SO\} \simeq \pi^s_{8k}$~~

~~$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \simeq J\{\pi_{8k} SO\} \oplus \langle \eta_{8k+1} \rangle \simeq \pi_{8k+1}$~~

$$J\{\pi_{8k} SO\} \simeq \mathbb{Z}/2 \quad \text{in} \quad \pi^s_{8k}$$

$$J\{\pi_{8k+1} SO\} \oplus \langle \eta_{8k+1} \rangle \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/2 \quad \text{in} \quad \pi^s_{8k+1}$$

$$\langle \eta_{8k+2} \rangle \simeq \mathbb{Z}/2 \quad \text{in} \quad \pi^s_{8k+2}.$$

and that moreover η_{8k+i} maps non-trivially to the generator of $\pi_{8k+i} BO$, $i = 1, 2$. Thus we have direct summands

$$\mathbb{Z}/2 \quad \text{in} \quad K_{8k+1}\mathbb{Z}$$

$$\mathbb{Z}/2 \quad \text{in} \quad K_{8k+2}\mathbb{Z}.$$

(It should be true that the image J goes to zero in $\pi_* BO$, because of the fibration

$$\begin{tikzcd}
SO \arrow[r] \arrow[d, "J"'] & \mathrm{Im}\, J \arrow[r] \arrow[d] & BO \arrow[r, "\Psi^3 - 1"] \arrow[d] & BSO \arrow[d] \\
SG \arrow[r, equal] & SG \arrow[r] & * \arrow[r] & BSG
\end{tikzcd}$$

This is in fact a proof provided

$$\begin{tikzcd}
SO \arrow[rr] \arrow[dr, "J"'] & & \mathrm{Im}\, J \\
& SG & (QS^0)_0 \arrow[l, "\sim"'] \arrow[u]
\end{tikzcd}$$

commuted.)

Now bring in finite fields. The diagram

$$\begin{tikzcd} B\Sigma_n \arrow[r] \arrow[drr] & BGL_n\mathbb{Z} \arrow[r] & BGL_n(\mathbb{Z}/p) \arrow[d, "\text{Brauer}"] \\ & \cancel{BGL_n} & BU \end{tikzcd}$$

doesn't commute, however it does if we restrict to a ~~finite~~ skeleton of $B\Sigma_n$ and localize with respect to p. ~~thus we get a diagram of homotopy groups~~

$$\begin{tikzcd} & B\Sigma^{+} \arrow[dl] \arrow[dr, bend left] & & \\ BGL(\mathbb{Z}/p)^{+} \simeq F\Psi^{p} \arrow[rr] \arrow[d] & & BU \arrow[r, "\Psi^p - 1"] \arrow[d] & BU \arrow[d, "\left(\frac{1}{p^i-1} ch_{2i}\right)"] \\ F \arrow[rr] & & BU \arrow[r, "ch"] & \prod_{i\geq 1} K(\mathbb{Q},2i) \end{tikzcd}$$

so we get a homom.

$$\begin{tikzcd} \pi_{2i-1}(B\Sigma^{+}) \arrow[r] \arrow[ddr, bend right, "\text{complex } e\text{-invariant}"'] & \pi_{2i-1}(F\Psi^{p}) = K_{2i-1}(\mathbb{Z}/p) \arrow[d, equal] \\ & \mathbb{Z}\frac{1}{p^i-1}/\mathbb{Z} \arrow[d, phantom, "\cap"] \\ & \mathbb{Q}/\mathbb{Z} \end{tikzcd}$$

Some number theory

$$m(2s) = \mathrm{denom}\left(B_s/4s\right) = \prod_{\ell \text{ prime}} \ell^{m_\ell(2s)}$$

where for ℓ odd we have

$$m_\ell(t) = \nu_\ell(p^t - 1) \qquad \text{if } p \text{ gen. } \mathbb{Z}_\ell^*$$

$$= \begin{cases} 0 & \text{if } (\ell-1) \nmid t \\ \nu_\ell(t) + 1 & \text{if } (\ell-1) \mid t \end{cases}$$

and for $\ell = 2$

$$m_2(t) = \nu_2(3^t - 1)$$

$$= \begin{cases} 1 & t \text{ odd} \\ \nu_2(t) + 2 & t \text{ even} \end{cases}$$

Note that the topologists $B_s = B_{2s}$ in Borevich-Shaf.

Examples

$s = 1$, $m(2) = 2^3 \cdot 3 = 24$, $\frac{B_1}{4} = \frac{1}{4 \cdot 6}$

$s = 2$, $m(4) = 2^4 \cdot 3 \cdot 5$ $\frac{B_2}{8} = -\frac{1}{8 \cdot 30} = -\frac{1}{16 \cdot 3 \cdot 5}$

$s = 3$, $m(6) = 2^3 \cdot 3^2 \cdot 7$ $\frac{B_3}{12} = \frac{1}{12 \cdot 42} = \frac{1}{8 \cdot 9 \cdot 7}$

$s = 4$, $m(8) = 2^5 \cdot 3 \cdot 5$ $\frac{B_4}{16} = -\frac{1}{16 \cdot 30} = -\frac{1}{32 \cdot 3 \cdot 5}$

$s = 5$, $m(10) = 2^3 \cdot 3 \cdot 11$ $\frac{B_5}{20} = \frac{1}{20} \cdot \frac{5}{66} = \frac{1}{8 \cdot 3 \cdot 11}$

July 21, 1972

Observation which perhaps is important.

Let $f : C \longrightarrow C'$ be cofibred, and suppose that each fibre C_Y, $Y \in C'$, is connected. Given objects

$$X' \qquad X \qquad \text{in } C$$

and an arrow

$$fX' \xrightarrow{u} fX \qquad \text{in } C'$$

we would like to lift u to a map from X' to X.

Because f is cofibred, we can lift u to a cocartesian arrow

$$X' \longrightarrow u_* X',$$

[redacted] which is such that

$$\mathrm{Hom}_{/u}(X', Z) = \mathrm{Hom}_{C_{fX}}(u_* X', Z)$$

for all $Z \in C_{fX}$. Thus u lifts to a map $X' \longrightarrow X$ iff there is an arrow $u_* X' \longrightarrow X$ in C_{fX}.

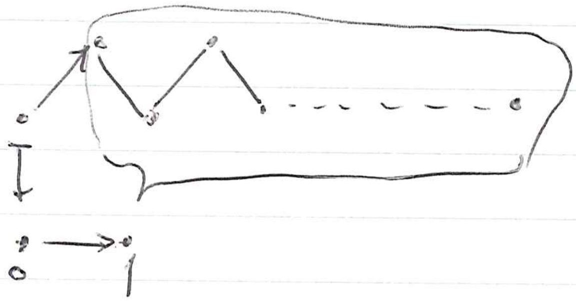
But we are given only that C_{fX} is connected, so all we have is a chain of arrows

$$u_* X' \nearrow \nwarrow \nearrow\nwarrow \cdots\cdots \nearrow \nwarrow X$$

in C_{fX}. But recall that [redacted] $Sd^m[1]$

is the category

with 2^m arrows, and that the functor $Sd^m[1] \longrightarrow [1]$ sends all the objects except 0 to 1.



Therefore for m suff. large we can find a commutative diagram

$$\begin{tikzcd}
Sd^m\{0,1\} \arrow[r, equal] \arrow[d, hook] & \{0,1\} \arrow[r, "(x',x)"] \arrow[d, hook] & \mathcal{C} \arrow[d, "f"] \\
Sd^m[1] \arrow[r] \arrow[urr, dashed] & {[1]} \arrow[r, "(a)"] & \mathcal{C}'
\end{tikzcd}$$

(It may be useful later to note that the dotted arrow may be chosen so that the first arrow goes into a cocart. arrow relative to f. The point is that $Sd^m[1] \longrightarrow [1]$ is cofibred. In effect $Sd\,\mathcal{C} \longrightarrow \mathcal{C}$ is cofibred. Thus the dotted arrow is a cocartesian functor.)

July 23, 1972 On $\underleftarrow{\text{holim}}$

Homotopy type of categories.

Up to now I have been trying to understand the homotopy groups of a small category $\mathcal{C}$ in the following way. Given a finite complex X, I want to find the set $[X, B\mathcal{C}]$. To do this I tried to construct a category $T(X,\mathcal{C})$ such that

$$\pi_0 T(X,\mathcal{C}) = [X, B\mathcal{C}]$$

Here are some potential candidates for T:

1) X compact space, then

$$T(X,\mathcal{C}) = \underline{\mathrm{Tors}}(X,\mathcal{C}) \simeq \underline{\mathrm{Homtop}}(\mathrm{Top}\,X, \mathcal{C}^{\vee})$$

2) X polyhedron

$$T(X,\mathcal{C}) = \varinjlim_{K} \underline{\mathrm{Hom}}(\mathrm{Cat}\,K, \mathcal{C})$$

where K runs over all the admissible traingulations of X.

3) X small category

$$T(X,\mathcal{C}) = \varinjlim_{m} \underline{\mathrm{Hom}}(\mathrm{Sd}^m X, \mathcal{C})$$

What is going on here?

Here is an interpretation: For 3) we have

$$T(Y, T(X, \mathcal{C})) = T(Y \times X, \mathcal{C})$$

so that $$[Y, BT(X,\mathcal{C})] = [Y \times X, B\mathcal{C}] = [Y, B\mathcal{C}^X].$$

In other words, the category $T(X, \mathcal{C})$ is playing the role of the function space

$$\begin{aligned} B\mathcal{C}^X &= \underline{\mathrm{Hom}}(X, B\mathcal{C}). \\ &= \underline{\Gamma}(X \times B\mathcal{C}/X) \end{aligned}$$

Recall Grothendieck's $\Pi_{X/S} Z$ formalism. Suppose $f: X \longrightarrow S$ is a map and Z is over X. Then Grothendieck denotes by $\Pi_{X/S} Z$ what I would write $f_* Z$. It has the property

$$\mathrm{Hom}_{/S}(T, f_* Z) = \mathrm{Hom}_{/X}(\underset{\substack{\| \\ X \times_S T}}{f^* T}, Z)$$

For example if C is over S, then

$$\begin{aligned} \mathrm{Hom}_{/S}(T, f_* f^* C) &= \mathrm{Hom}_{/X}(X \times_S T, X \times_S C) \\ &= \mathrm{Hom}_{/S}(X \times_S T, C) \end{aligned}$$

$$= \mathrm{Hom}_{/S}\left(T, \underline{\mathrm{Hom}}_S(X, C)\right)$$

so

$$\boxed{f_* f^* C = \underline{\mathrm{Hom}}_S(X, C)}$$

<u>The picture</u>: suppose we take seriously the philosophy that homotopy theory is to be constructed out of small categories. Over any X we consider the 2-category of cofibred categories over X with cocartesian functors for morphisms.

$$\underline{\mathrm{Hom}}_X(Y, Z) = \underline{\mathrm{Homcocart}}_{\mathrm{Cofcat}/X}(Y, Z)$$

Then given $f: X \longrightarrow S$ we have

$$f^*: \mathrm{Cofcat}/S \longrightarrow \mathrm{Cofcat}/X$$

and perhaps an f_* functor which when $S = e$ reduces to

$$\underline{\Gamma}(Z/X) = \underline{\mathrm{Homcocart}}_{/X}(X, Z).$$

In effect one knows that a cocart. functor

$$\begin{tikzcd} X \times T \arrow[rr, "\varphi"] \arrow[dr] & & Z \arrow[dl] \\ & X & \end{tikzcd}$$

is the same as a functor

$$T \longrightarrow \underline{\Gamma}(Z/X).$$

Now what you need to do is form the homotopy category of Cofcat/X by inverting the fibre-homotopy-equivalences. Then you wish to construct the derived functor

$$Rf_* : \mathrm{Ho}(\mathrm{Cofcat}/X) \longrightarrow \mathrm{Ho}(\mathrm{Cofcat}/S)$$

for a map $f: X \longrightarrow S$. In particular, if one takes $f: X \longrightarrow pt$, and C over pt, then

$$Rf_* f^* C = T(X, C).$$

Now perhaps you might want to use a specific construction for $Rf_* = \underrightarrow{\mathrm{holim}}$ such as

$$Rf_*(Z) = \varinjlim_m \underline{\mathrm{Hom}}^{\mathrm{cocart}}_{/X}(Sd^m X, Z).$$

July 26, 1972. To understand $\underleftarrow{\text{holim}}$.

Let $\mathcal{F} \longrightarrow S$ be a fibred category. I want to understand $\underset{S}{\underleftarrow{\text{holim}}}\, \mathcal{F}$.

Example: Suppose $\mathcal{F}$ is the fibred category in groupoids defined by a complex of abelian group functors of length 2

$$K^\cdot : \quad K^0 \xrightarrow{d} K^1 \longrightarrow 0 \cdots$$

Compute $\underset{S}{\underleftarrow{\lim}}\, \mathcal{F}$, i.e. the category of cartesian section of $\mathcal{F}/S$. Now since the fibres are groupoids, every arrow is cartesian. Thus we want sections of $\mathcal{F}/S$. Such a thing consists of

$$\text{Ob } S \ni y \longmapsto s(y) \in \text{Ob}(\mathcal{F}_y) = K^1(y)$$

$$\text{Ar } S \ni (u : y' \to y) \longmapsto t(u) \in \underset{\mathcal{F}_{y'}}{\text{Hom}}(s(y'), u^*s(y)) \subset K^0(y')$$

$$dt(u) = s(y') - u^*s(y)$$

such that for $y \xleftarrow{v} y' \xleftarrow{u} y''$ we have

$$t(vu) = t(v) + v^*t(u).$$

Thus a section of $\mathcal{F}/S$ is a 1-cocycle in the complex

$$\begin{tikzcd}
C^0(S, K^0_\bullet) \arrow[r, "\delta"] \arrow[d, "d"] & C^1(S, K^0_\bullet)^t \arrow[r] \arrow[d, "d"] & C^2(S, K^0_\bullet) \\
C^0(S, K^1) \arrow[r, "\delta"] \arrow[d] & C^1(S, K^1) \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots &
\end{tikzcd}$$

where $$C^q(S, F) = \prod_{y_0 \leftarrow \cdots \leftarrow y_q} F(y_q) \qquad \text{as usual.}$$

Thus it is clear that $\varprojlim_S \mathcal{F}$ is the category belonging to the complex

$$(C^\cdot(S, K^\cdot))^0 \longrightarrow Z^1 C^\cdot(S, K^\cdot) \longrightarrow \cdots\cdots$$

What should the homotopy ~~inverse~~ limit be? According to Kan, one wants an object $s(y)$ over y, $\forall y \in \mathrm{Ob}\, C$, and for every arrow $u : y' \to y$ a path from $s(y')$ to $u^* s(y)$, etc. Since the fibres of $\mathcal{F}$ are groupoids, it follows that every path must be an isomorphism. Thus in general we have

$$\boxed{\begin{array}{c} \varprojlim_S \mathcal{F} \xrightarrow{\sim} \underset{\overleftarrow{S}}{\mathrm{holim}}\, \mathcal{F} \\ \text{when } \mathcal{F} \longrightarrow S \text{ is fibred in groupoids} \end{array}}$$

July 26, 1972

Dear Jack,

As I wrote you earlier, the assertion in your note that I can prove the injectivity of the map

$$J(\pi_i O) \subset \pi_i^S \longrightarrow K_i\mathbb{Z}$$

is inaccurate with respect to the 2-torsion. Unfortunately, the corrections I sent are also incorrect. Since Kervaire has requested some details, I am sending the following account of what I know about the above map, in order to clear the confusion.

1. First consider the dimensions $i = 8k,\ 8k+1$, where $J(\pi_i O) = \mathbb{Z}/2$. I do not know whether this group injects into $K_*\mathbb{Z}$, and suspect that it does not, except of course when $k = 0$.

However, Adams has produced elements of order 2, $\eta_j \in \pi_j^S$, $j = 8k+1$, $8k+2$, closely related to the image of J in the preceding dimensions, which do map non-trivially into $K_*\mathbb{Z}$. To see this, consider the square

(1)
$$\begin{tikzcd} B\Sigma_\infty^+ \arrow[r] \arrow[d] & BO \arrow[d, "\simeq"] \\ BGL(\mathbb{Z})^+ \arrow[r] & BGL(\mathbb{R}) \end{tikzcd}$$

induced by the various group inclusions. Passing to homotopy groups, we obtain homomorphisms $\pi_j^S \cong \pi_j B\Sigma_\infty^+ \longrightarrow K_j\mathbb{Z} \longrightarrow \pi_j BO$ whose composition is the degree map for KO-theory. Since Adams has shown that the degree map carries η_j to the generator of $\pi_j BO = \mathbb{Z}/2$, the image of η_j in $K_j\mathbb{Z}$ is non-trivial. In fact, we have

$$K_j\mathbb{Z} = \mathbb{Z}/2 \oplus ?, \qquad j = 8k+1,\ 8k+2.$$

I should mention that this observation appears already in one of Gersten's papers.

2. Next consider the dimension $i = 4s-1$, where $J(\pi_i O)$ is cyclic of

order $\operatorname{denom}(B_s/4s)$. I shall prove the injectivity:

$$J(\pi_{4s-1}0) \hookrightarrow K_{4s-1}\mathbb{Z}$$

by showing that the Adams e-invariant on π^s_{4s-1}, which detects $J(\pi_{4s-1}0)$, comes from an invariant defined on $K_{4s-1}\mathbb{Z}$.

Following Sullivan, consider the fibration

$$F \longrightarrow BO \xrightarrow{(ch_{4i})} \prod_{i \geq 1} K(\mathbb{Q}, 4i)$$

where $K(\mathbb{Q},j)$ is an Eilenberg-Maclane space and ch_j represents the j-th component of the Chern character. Since $B\Sigma^+_\infty$ has trivial rational cohomology, the degree map $B\Sigma^+_\infty \to BO$ lifts by obstruction theory, uniquely up to homotopy, to a map

(2) $\quad B\Sigma^+_\infty \longrightarrow F$

which induces a homomorphism

$$\pi^s_{4s-1} \longrightarrow \pi_{4s-1}F \simeq \mathbb{Q}/a_s\mathbb{Z}$$

where a_s is 1 or 2 depending on whether s is even or odd.

I claim this homomorphism is the negative of the Adams e-invariant. Assuming this for the moment, consider the diagram

$$\begin{tikzcd} B\Sigma^+_\infty \arrow[r] & BGL(\mathbb{Z})^+ \arrow[d, "w"] \arrow[dl, dashed] & \\ F \arrow[r] & BO \arrow[r, "ch"] & \prod K(\mathbb{Q}, 4i) \end{tikzcd}$$

with the map w obtained from (1). Since the Chern classes of representations of discrete groups are torsion classes, the map (ch)w is null-homotopic, and the dotted arrow exists. The induced map from $B\Sigma^+_\infty$ to F must be (2). Thus we obtain a commutative diagram

$$\begin{tikzcd} \pi^s_{4s-1} \arrow[r] \arrow[d, "-e"'] & K_{4s-1}\mathbb{Z} \arrow[dl, dashed] \\ \mathbb{Q}/a_s\mathbb{Z} & \end{tikzcd}$$

as desired.

3. To prove the claim about the e-invariant, consider the map

$$BO(8k) \longrightarrow \prod_{i \geq 1} K(\mathbb{Q}\ ,\ 8k+4i)$$

with components ch_{8k+4i}, where $BO(8k)$ is the $(8k-1)$-connected covering of BO. Denote this map briefly by $c : BO(8k) \longrightarrow E(8k)$ and let $F(8k)$ be its fibre. Let $b : S^{8k} \longrightarrow BO(8k)$ represent the generator of $\pi_{8k}BO(8k) = \pi_{8k}BO$ provided by Bott periodicity.

Now suppose given a map $f : S^{8k+4s-1} \longrightarrow S^{8k}$ representing an element $\bar{f}$ of π^s_{4s-1}. We compute the Toda bracket $\{c,b,f\}$ by forming the diagram

$$\begin{tikzcd}
S^{8k+4s-1} \arrow[r, "f"] \arrow[d, "u"] & S^{8k} \arrow[r] \arrow[d, "v"] \arrow[dr, "b"] & \text{Cone } f \arrow[r] \arrow[d, "x"] & S^{8k+4s} \arrow[d, "y"] \\
\Omega E(8k) \arrow[r] & F(8k) \arrow[r] & BO(8k) \arrow[r, "c"] & E(8k)
\end{tikzcd}$$

in which the arrows x,y and v,u can be filled in as bf and cb are null-homotopic. By definition, the Toda bracket is the element represented by y in

$$\pi_{8k+4s}E(8k) \,/\, c_*\pi_{8k+4s}BO(8k) + f^*\pi_{8k}\Omega E(8k) \quad = \mathbb{Q}/\,a_s\mathbb{Z}\ .$$

Now Adams defines the e-invariant of $\bar{f}$ by choosing an element z of $\widetilde{KO}(\text{Cone } f)$ restricting to the generator of $\widetilde{KO}(S^{8k})$, and forming

$$ch_{8k+4s}(z) \in H^{8k+4s}(\text{Cone } f,\mathbb{Q}) \simeq H^{8k+4s}(S^{8k+4s},\mathbb{Q}) \simeq \mathbb{Q}\ .$$

The image of this rational number in $\mathbb{Q}/a_s\mathbb{Z}$ is then $e(\bar{f})$. Clearly z and $ch_{8k+4s}(z)$ may be identified with the maps x and y in the diagram, hence we have the formula

$$e(\bar{f}) \;=\; \{c,b,f\}\,.$$

On the other hand, from the theory of Toda brackets one knows that the map u in the diagram represents the negative of $\{c,b,f\}$. Thus we have the formula

$$(3) \qquad e(\bar{f}) \;=\; -\,f^*(v_k) \in \pi_{8k+4s-1}F(8k) \;=\; \mathbb{Q}/a_s\mathbb{Z}$$

where $v_k = v$ is the unique element of $\pi_{8k}F(8k)$ mapping to the generator of $\pi_{8k}BO(8k)$. Now by periodicity we have $\Omega^{8k}F(8k) \simeq \mathbb{Z}\times F$. The maps v_k fit together to induce a map

$$\bar{v} : \lim_k \Omega_o^{8k}S^{8k} \longrightarrow F$$

which covers the degree map into BO. Thus $\bar{v}$ is the map (2). The formula (3) shows that its effect on homotopy groups is the negative of the e-invariant, which proves the claim.

4. Additional information on the image of $J(\pi_{4s-1}O)$ in $K_{4s-1}\mathbb{Z}$ can be obtained from the computation of the K-groups of finite fields as follows. Let p be a prime number and $\mathbb{F}_p$ the field with p elements, and consider the obvious homomorphisms

$$\pi^s_{4s-1} \longrightarrow K_{4s-1}\mathbb{Z} \longrightarrow K_{4s-1}\mathbb{F}_p \ .$$

I will show below that this composition is essentially the part of the complex e-invariant which is prime to p. More precisely, there is a commutative diagram

$$(4) \qquad \begin{array}{ccc} \pi^s_{4s-1} & \longrightarrow & K_{4s-1}\mathbb{F}_p \simeq \mathbb{Z}/(p^{2s}-1)\mathbb{Z} \\ {\scriptstyle -e}\downarrow & & \downarrow{\scriptstyle \theta} \\ \mathbb{Q}/a_s\mathbb{Z} & \longrightarrow & \mathbb{Q}/\mathbb{Z}[p^{-1}] \end{array}$$

where θ is injective with image the unique subgroup of order $p^{2s}-1$. Here $\mathbb{Z}[p^{-1}]$ denotes the ring of rational numbers with powers of p in the denominator.

Assuming this, let ℓ be an odd prime, and choose p to be a topological generator of the group $\mathbb{Z}_\ell^\cdot$ of ℓ-adic units. According to Adams, the e-invariant is injective on $J(\pi_{4s-1}O)$, and the ℓ-primary component $J(\pi_{4s-1}O)_{(\ell)}$ is cyclic of order ℓ^n, $n = v_\ell(p^{2s}-1)$, v_ℓ = ℓ-adic valuation. We have therefore an isomorphism

$$J(\pi_{4s-1}O)_{(\ell)} \xrightarrow{\simeq} (K_{4s-1}\mathbb{F}_p)_{(\ell)}.$$

It follows that the odd part of $J(\pi_{4s-1}0)$ is isomorphic to a direct summand of $K_{4s-1}\mathbb{Z}$.

Suppose now that $\ell = 2$ and take $p = 3$. Using Adams work, both the source and target of the map

$$J(\pi_{4s-1}0)_{(2)} \longrightarrow (K_{4s-1}\mathbb{F}_3)_{(2)}$$

are cyclic of order 2^n, $n = v_2(3^{2s}-1)$; and the map is essentially multiplication by a_s. It follows that for s even, when $a_s = 1$, $J(\pi_{4s-1}0)_{(2)}$ is isomorphic to a direct summand of $K_{4s-1}\mathbb{Z}$.

Finally, observe that the diagram (4) shows the unique element of order 2 of $J(\pi_{4s-1}0)$, when s is odd, goes to zero in $K_{4s-1}\mathbb{F}_p$ for all p.

Summarizing:

Proposition: *The homomorphism* $\pi^s_{4s-1} \longrightarrow K_{4s-1}\mathbb{Z}$ *induces an injection of* $J(\pi_{4s-1}0)$ *into* $K_{4s-1}\mathbb{Z}$. *For even* s, *the image of* $J(\pi_{4s-1}0)$ *is a direct summand. For odd* s, *the odd-torsion part of the image is a direct summand. For odd* s, *the unique element of order* 2 *of the image is in the kernel of the homomorphism* $K_{4s-1}\mathbb{Z} \longrightarrow K_{4s-1}\mathbb{F}_p$ *for all primes* p.

I do not know whether or not the image of $J(\pi_{4s-1}0)_{(2)}$ is a direct summand of $K_{4s-1}\mathbb{Z}$ when s is odd. The first case is $s=1$, where

$$\mathbb{Z}/24 = J(\pi_3 0) = \pi^s_3 \hookrightarrow K_3\mathbb{Z} = H_3(St(\mathbb{Z}),\mathbb{Z}) .$$

Here $K_3\mathbb{F}_3 = \mathbb{Z}/8$ and the map $J(\pi_3 0) \longrightarrow K_3\mathbb{F}_3$ has a kernel of order 6.

5. It remains to construct the diagram (4). Consider the diagram

$$
\begin{tikzcd}
F \arrow[r] \arrow[d] & BO \arrow[r, "ch"] \arrow[d] & \prod_{i \geq 1} K(\mathbb{Q}, 4i) \arrow[d] \\
F' \arrow[r] & BU[p^{-1}] \arrow[r, "ch"] & \prod_{j \geq 1} K(\mathbb{Q}, 2j) \\
F\Psi^p \arrow[r, dash] \arrow[u] & BU \arrow[r, "\Psi^p - 1"] \arrow[u] & BU \arrow[u, "((p^j-1)^{-1} ch_{2j})"']
\end{tikzcd}
$$

where F' and $F\Psi^p$ are defined so that the rows are fibrations. Here $BU[p^{-1}]$ is the localization of BU which represents the functor $K(?) \otimes \mathbb{Z}[p^{-1}]$. Examining the homotopy sequences of these fibrations, we obtain isomorphisms

$$
(5) \qquad
\begin{tikzcd}
\pi_{4s-1} F \arrow[r, phantom, "\simeq"] \arrow[d] & \mathbb{Q}/a_s\mathbb{Z} \arrow[d] \\
\pi_{4s-1} F' \arrow[r, phantom, "\simeq"] & \mathbb{Q}/\mathbb{Z}[p^{-1}] \\
\pi_{4s-1} F\Psi^p \arrow[r, phantom, "\simeq"] \arrow[u] & (p^{2s}-1)^{-1}\mathbb{Z} / \mathbb{Z} \arrow[u, phantom, "\cup"]
\end{tikzcd}
$$

where the maps at the right are the obvious ones.

From the computation of the K-groups of a finite field, there is a homotopy equivalence

$$BGL(\mathbb{F}_p)^+ \simeq F\Psi^p$$

induced by lifting representations of finite groups over $\mathbb{F}_p$ to virtual complex representations by means of the Brauer theory. I claim that the diagram

$$
\begin{tikzcd}
B\Sigma_\infty^+ \arrow[r] \arrow[d] & BGL(\mathbb{F}_p)^+ \arrow[r, phantom, "\simeq"] & F\Psi^p \arrow[d] \\
BO \arrow[rr] & & BU[p^{-1}]
\end{tikzcd}
$$

is commutative. The upper right path is obtained by lifting the obvious representation of Σ_n on $\mathbb{F}_p^n$ to a virtual complex representation, while the lower right path comes from the obvious action of Σ_n on $\mathbb{C}^n$.

These two virtual representations are not the same in general. However, it is known that their characters agree on elements of Σ_n of order prime to p, because both the representations $\mathbb{F}_p^{\,n}$ and $\mathbb{C}^n$ come from the integral representation $\mathbb{Z}^n$. Thus the two virtual representations agree on the Sylow ℓ-subgroups Σ_n^{ℓ} for all primes $\ell \neq p$. By a standard transfer argument, one has

$$[B\Sigma_n, BU[p^{-1}]] \hookrightarrow \prod_{\ell \neq p} [B\Sigma_n^{\ell}, BU[p^{-1}]].$$

Consequently, the above diagram commutes as claimed.

Since $B\Sigma_\infty^+$ has trivial rational cohomology, it follows by obstruction theory that the diagram

$$\begin{tikzcd} B\Sigma_\infty^+ \arrow[r] \arrow[d] & BGL(\mathbb{F}_p)^+ \simeq F\Psi^p \arrow[d] \\ F \arrow[r] & F' \end{tikzcd}$$

is commutative, where the vertical arrow at the left is the one inducing minus the e-invariant. The desired commutative diagram (4) now results by taking homotopy groups, and using the isomorphisms (5).

This concludes the account of the map $J(\pi_* O) \rightarrow K_*\mathbb{Z}$. To the best of my knowledge, nothing more is known about $K_i\mathbb{Z}$ for $i>2$ beyond what this and Borel's theorem provide.

Best wishes,

Dan Quillen

July 31, 1972. On stability

Let k be a field and consider $\mathcal{M} = \mathrm{Mod}_f(k)$. Let C_n be the full subcategory of $Q(\mathcal{M})$ consisting of M of dimension $\leq n$.

Let $f: C_{n-1} \hookrightarrow C_n$ be the inclusion. Then f/V is equivalent to the ordered set of layers in V of dimension $\leq n-1$. This clearly has a final object ~~thus~~ if $\dim V < n$, so suppose $\dim(V) = n$.

Let $\bar{X}(V)$ be the simplicial complex whose simplices are chains $0 \subset W_0 < \cdots < W_p \subset V$ such that W_p/W_0 is of dim $< n$, i.e. either $W_0 > 0$ or $W_p < W$. Then $\bar{X}(V)$ is clearly the suspension of the building $X(V)$. Thus since we know

$$X(V) \text{ is } (n-3)\text{-connected} \qquad (\text{begins dim } n-2)$$

$$\Rightarrow \bar{X}(V) \text{ is } (n-2)\text{-connected.} \qquad (\text{begin dim } n-1)$$

~~But if I_V is the ordered set equivalent to f/V, we have a homotopy we know that then I_V is the ordered set of 1-simplices in $\bar{X}(V)$, and we have a homotopy equ~~

Let I_V be the ordered set above which is equivalent to f/V, i.e. the ordered set of 1-simplices in the ordered simplicial complex $\bar{X}(V)$. Then

$$\mathrm{Cat}\left[\bar{X}(V)\right] \longrightarrow I_V$$

$$(W_0 < \cdots < W_p) \longmapsto (W_0, W_p)$$

is a homotopy equivalence.

(cofibred: $(W_0 < \cdots < W_p)$ + $(W_0, W_p) \leq (W', W'') \mapsto (W' < W_0 < \cdots < W_p < W'')$ which is clearly the smallest simplex with ends W', W'' + which contains $W_0 < \cdots < W_p$. fibre ~~is contractible~~ has initial object.)

(<u>General lemma</u>: Let X be a simplicial complex, ~~with~~ a (partial) ordering on the vertices such that each simplex is linearly ordered, and such that any chain is a simplex provided its bottom and top form a 1-simplex. Then (i) 1-simplices in X form an ordered set I_X (ii)

$$\mathrm{Cat}(X) \longrightarrow I_X$$
$$(x_0 < \cdots < x_p) \longmapsto (x_0, x_p)$$

is a homotopy equivalence (iii) the nerve of I_X is a subdivision of X.)

Thus we can conclude that f/V is $(n-1)$-conn. for each V in $\mathcal{C}_n$. And further that the h-fibre of

$$\mathcal{C}_{n-1} \longrightarrow \mathcal{C}_n$$

is $(n-2)$-connected. Thus the h-fibre of

$$\mathcal{C}_n \hookrightarrow Q(\mathcal{M})$$

has homotopy groups begining in dimension n. (e.g. $n=0$, the fibre is $\Omega Q(\mathcal{M})$ which begins in dim 0)

Suppose now that A is a Dedekind domain with fraction field K. Let $\mathcal{M} = \mathcal{P}_A$, and define again the filtration

$$\cdots \subset \mathcal{C}_{n-1} \subset \mathcal{C}_n \subset \cdots\cdots \subset Q(\mathcal{M})$$

by: $\mathcal{C}_n$ consists of M of rank $\leq n$. Again if $f: \mathcal{C}_{n-1} \to \mathcal{C}_n$ is the inclusion, then f/M is the ordered set of admissible layers in M of rank $< n$. But, there is a 1-1 correspondence between subbundles of M and subspaces of $M \otimes K$:

$$N \subset M \ni M/N \text{ is in } \mathcal{P}_A \iff N = M \cap (N \otimes K) \subset M \otimes K$$

Therefore the fibres f/M are all $(n-2)$-connected.

~~The next thing is the try to show that the homotopy groups of C_n are~~

Suppose A is the ring of integers in a number field K, whence it is known that the groups $GL_n A$ have fin. gen. homology in each degree. I want to try now to prove that C_n has finitely generated homology in each degree. Then by the above stability considerations, we have that $Q(\mathcal{M})$ has f.g. homology, and so, as it is an H-space, its homotopy groups are finitely generated.

Another way of thinking about C_n. Consider the fibred category over Δ whose fibre over $[p]$ is the groupoid of p-filtered objects

$$0 \subset M_1 \subset \cdots\cdots \subset M_p$$

~~of~~ P_A such that $\operatorname{rank}(M_p) \leq n$. Call this cat. $\mathcal{F}_n$.
Then we have a functor

$$f: \mathcal{F}_n \longrightarrow C_n$$

and f/M is the fibred cat. $/\Delta$ consisting of

$$0 \subset M_1 \subset \cdots \subset M_p \quad + \quad M_p < M$$

i.e. it is the simplicial set of

$$M_0 \subset M_1 \subset \cdots \subset M_p \subset M$$

$\operatorname{rank}(M_p/M_0) \leq n$
no condition if $\operatorname{rank}(M) \leq n$

which is contractible.

Thus we can use $\mathcal{F}_n$ to calculate the homology of C_n. We get usual spec. sequence

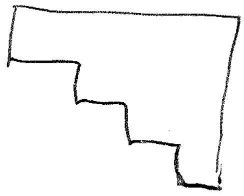
$$E^2_{pq} = \check{H}_p(\nu \longmapsto H_q((\mathcal{F}_n)_\nu, \Lambda)) \Longrightarrow H_{p+q}(C_n, \Lambda)$$

But observe: $(\mathcal{F}_n)_\nu$ is the groupoid of ν-filtered vector bundles

(*) $$0 \subset M_1 \subset \cdots\cdots \subset M_\nu$$

with rank $M_\nu \leq n$. Thus the non-degenerate part occurs with $\nu \leq n$, and $E^2_{pq} = 0$, $p > n$. But also, the isom. classes of sequences (*) is finite (finiteness of class number), and the group of autos of (*) has f.g. homology. Thus E^2_{pq} is fin. gen, and we conclude $H_q(C_n, \Lambda)$ is f.g.

(Checkable case: A a P.I.D. Then every projective is free, and a filtered object is determined up to isomorphism by the ranks of M_i/M_{i-1}, $\forall i$. The group of autos. is then an arithmetic group:



)

<u>Conclusion</u>: K number field, S finite set of places including the arch. ones, A = ring of S-integers. Then $K_i A$ is finitely generated for each $i \geq 0$.