<u>May 4, 1972.</u> Conjectures on $K_*(F)$, F function field 1 var/$k = \bar{k}$.

k an alg. closed field, X curve $/k$, $K =$ function field.

$$(*) \qquad 0 \longrightarrow K^*/k^* \longrightarrow D \longrightarrow \mathrm{Pic}(X) \longrightarrow 0$$

where $D = \bigoplus_{x \in X} \mathbb{Z}$ is the divisor group. I conjecture that there are exact sequences

$$\cancel{0 \to \mathrm{Tor}_1(K_0(X), K_{i-1}(k)) \longrightarrow K_i(X) \longrightarrow K_0(X) \otimes_{\mathbb{Z}} K_i}$$

$$0 \longrightarrow K_0 X \otimes_{\mathbb{Z}} K_i k \longrightarrow K_i X \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(K_0 X, K_{i-1} k) \to 0.$$

Since $K_0 X = \mathbb{Z} \oplus \mathrm{Pic}(X)$, this amounts to exact sequences

$$0 \longrightarrow K_i k \oplus \mathrm{Pic} \otimes K_i k \longrightarrow K_i X \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(\mathrm{Pic}, K_{i-1} k) \to 0$$

Since $(*)$ is a free resolution of Pic, it gives

$$0 \to \mathrm{Tor}_1(\mathrm{Pic}, K_i k) \to K^*/k^* \otimes K_i k \longrightarrow \bigoplus_{x \in X} K_i k \to \mathrm{Pic} \otimes K_i k \to 0.$$

On the other hand, we conjecture a long exact sequence

$$K_{i+1} X \longrightarrow K_{i+1} K \longrightarrow \bigoplus_{x \in X} K_i k \longrightarrow K_i X \cdots$$

So putting things together, we conjecture that there should be an exact sequence

$$\boxed{0 \longrightarrow K_i k \longrightarrow K_i K \longrightarrow K^*/k^* \otimes K_{i-1} k \longrightarrow 0}$$

Reformulation: The diagram

$$\begin{tikzcd} K_{i-1}k \otimes k^* \arrow[r] \arrow[d] & K_i k \arrow[d] \\ K_{i-1}k \otimes K^* \arrow[r] & K_i K \end{tikzcd}$$

is cocartesian; (horizontal maps come from product, vertical from inclusion of k in K).

Injectivity of $K_i k \longrightarrow K_i K$ is easy: Let $x \in X$, let $\mathcal{O}_x$ be the local ring then have the composition

$$k \xrightarrow{i} \mathcal{O}_x \xrightarrow{u} k$$

$= id$, so $K_i k$ is a direct summand of $K_i \mathcal{O}_x$. By the exact sequence (conjectural at the moment):

$$K_i k \xrightarrow{u_*} K_i \mathcal{O}_x \longrightarrow K_i K$$

it will follow that $K_i \mathcal{O}_x \hookrightarrow K_i K$ provided u_* is zero. But $u^* : K_i \mathcal{O}_x \longrightarrow K_i k$ is surjective as $u^* i^* = (ui)^* = id$, and

$$u_*(u^* z) = u_* 1 . z = 0$$

because $u_* 1 = 0$.

Precise proof uses fact that $K_i(K) = \varinjlim_{S \subset X} K_i(X-S)$ as S runs over the finite subsets of X, and the fact that $K_i k \hookrightarrow K_i(X-S)$ as any point of $X-S$ gives a map the other way.

We have not used X complete.

Thus if X is a (not nec. complete) non-singular connected curve over k alg. closed, then we have the exact sequence (conj.)

$$\longrightarrow K_i(X) \longrightarrow K_i K \xrightarrow{\partial} \mathrm{Div}(X)\otimes K_{i-1}k \longrightarrow K_{i-1}X \longrightarrow \cdots$$

Denote by $\tilde{K}_i(X) = K_i(X)/K_i k$, etc., so that we have

$$(*) \qquad \longrightarrow \tilde{K}_i X \longrightarrow \tilde{K}_i K \xrightarrow{\partial} \mathrm{Div}(X)\otimes K_{i-1}k \longrightarrow \tilde{K}_{i-1}X \longrightarrow \cdots$$

If the formula at the top of page 2:

$$\tilde{K}_i K = K^*/k^* \otimes K_{i-1}k$$

is correct, and if ∂ in $(*)$ can be identified with

$$K^*/k^* \otimes K_{i-1}k \xrightarrow{\text{can} \otimes \text{id}} \mathrm{Div}(X)\otimes K_{i-1}k$$

then we obtain from $(*)$ short exact sequences

$$\boxed{0 \to \mathrm{Pic}\,X \otimes K_i k \longrightarrow \tilde{K}_i X \longrightarrow \mathrm{Tor}_1(\mathrm{Pic}\,X, K_{i-1}k) \longrightarrow 0}$$

for any such X.

Precisely we want $(*)$ to be enlarged to a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}_1(P, K_{i-1}k) \arrow[r] & K^*/k^* \otimes K_{i-1}k \arrow[r] \arrow[d] & \mathrm{Div}(X) \otimes K_{i-1}k \arrow[r] \arrow[d, equal] & \mathrm{Pic}(X) \otimes K_{i-1}k \arrow[r] \arrow[d] & 0 \\
{} \arrow[r] & \tilde{K}_i X \arrow[r] & \tilde{K}_i(K) \arrow[r, "\partial"] & \mathrm{Div}(X) \otimes K_{i-1}k \arrow[r] & \tilde{K}_{i-1} X \arrow[r] & {}
\end{tikzcd}$$

which should be formal hopefully. This diagram shows equivalence of the formula: $\tilde{K}_*K = K^*/k^* \otimes K_{*-1}k$ with earlier conjectures concerning K_*X.

Another consequence of

$$\tilde{K}_i(K) = (K^*/k^*) \otimes K_{i-1}k \ :$$

Let K run over all subfields of $\overline{k(T)}$ finite over $k(T)$, and take the limit. One obtains that

$$\tilde{K}_i(\overline{k(T)}) = (\overline{k(T)}^*/k^*) \otimes K_{i-1}k$$

is a $\mathbb{Q}$-vector space. Thus, we can prove by induction on transcendence degree that

$$K_i(k) \cong \underset{K_i(k_0)}{} \oplus \ (\mathbb{Q} \text{ vector space})$$

where k_0 is the algebraic closure of the prime field.

Question: Is $K_*(K) = K_*(k) \otimes_{M_*(k)} M_*(K)$ where $M_*(K)$ is the Milnor ring.

May 5, 1972. Conjectures on $K_*(\overline{\mathbb{Q}})$.

Let F be a number field of degree d, $\pi = \mathrm{Gal}(\overline{\mathbb{Q}}/F)$, and for the moment suppose F is totally imaginary. Then I believe it is known that π has cohomological dimension 2, i.e. $H^i(\pi, M) = 0$ for $i \geq 3$ and all continuous π modules M. (Is "strict" cd π $\cancel{\leq} 2$? Need to check this)

According to Borel's theorem $K_{2i}F$ is torsion and $K_{2i-1}F$ has rank $d/2 = r_2$ for $i \geq 1$, except for $K_1F = F^*$ which has infinite rank. If $\mathcal{O} \subset F$ is the ring of integers of F, we expect exact sequences

$$0 \longrightarrow K_{2i}\mathcal{O} \longrightarrow K_{2i}F \longrightarrow \bigoplus_v K_{2i-1}(\mathcal{O}/m_v) \longrightarrow 0 \qquad i \geq 1$$

$$K_{2i-1}\mathcal{O} \xrightarrow{\sim} K_{2i-1}F \qquad i \geq 2$$

It is also conjectured that $K_*\mathcal{O}$ are f.g. abelian groups &

$$(K_{2i-1}\mathcal{O})_{tors} \simeq (\mu_\infty^{\otimes i})^\pi \qquad i \geq 1$$

$$K_{2i-1}\mathcal{O}/(K_{2i-1}\mathcal{O})_{tors} \simeq \mathbb{Z}^{d/2} \qquad i \geq 2$$

$$K_{2i}\mathcal{O} \quad \text{finite.} \qquad i \geq 1.$$

Now ~~[illegible]~~ consider the limit as $F \nearrow \overline{\mathbb{Q}}$. Generalizing what is known for K_2F one conjectures that

$$K_{2i}\overline{\mathbb{Q}} = 0 \qquad i \geq 1$$

and generalizing what is known for K_1F, one conjectures that there should be an exact sequence

$$0 \longrightarrow \mu_\infty^{\otimes i} \longrightarrow K_{2i-1}\overline{\mathbb{Q}} \longrightarrow V^{(i)} \longrightarrow 0 \qquad i \geq 1$$

where $V^{(i)}$ is a $\mathbb{Q}$-vector space, ($= \overline{\mathbb{Q}}^*/\mu_\infty$ when $i=1$).

The structure of $V^{(i)}$ should be predictable from Borel's theorem: Thus

$$K_{2i-1}(F) \otimes [redacted] \mathbb{R} \xrightarrow{\sim} \prod_{\mathrm{Hom}(F,\mathbb{C})/\mathbb{Z}/2} \mathbb{R}$$

$\mathbb{Z}/2$ acts as conjugation

Assuming $F \supset \mathbb{Q}[i]$, then $\mathrm{Hom}(F,\mathbb{C})/\mathbb{Z}/2 = \mathrm{Hom}_{\mathbb{Q}[i]}(F,\mathbb{C}) \cong \mathrm{Gal}(F/\mathbb{Q}[i])$, [redacted] provided F is Galois over $\mathbb{Q}[i]$. Thus $K_{2i-1}(F) \otimes \mathbb{Q}$ is the regular representation of [redacted] $\mathrm{Gal}(F/\mathbb{Q}[i])$. So passing to the limit, it becomes "clear" that

$$V^{(i)} \simeq \mathrm{Homcont}\Big(\underbrace{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})/\mathrm{Gal}(\overline{\mathbb{Q}}/\overline{\mathbb{Q}}\cap\mathbb{R})}_{\text{infinite places}}, \mathbb{Q}\Big)$$

[redacted] In any case, $K_{2i-1}\overline{\mathbb{Q}}$ divisible $\Longrightarrow$

$$H^0(\pi, V^{(i)}) = K_{2i-1}F \otimes \mathbb{Q}$$

$$H^+(\pi, V^{(i)}) = 0$$

Now I hope eventually for Galois descent, which should yield a spectral sequence

$$E_2^{pq} = H^p(\pi, K_{-q}\bar{\mathbb{Q}}) \Longrightarrow K_{-p-q}(F)$$

in negative dimensions. Assume π has $\mathrm{scd} \leq 2$, the differentials must vanish

$$\begin{tikzcd}
H^0(\pi, K_{2i-1}\bar{\mathbb{Q}}) \arrow[drr] & & & \\
0 & 0 & 0 & \\
& & H^2(\pi, K_{2i+1}\bar{\mathbb{Q}}) & 0
\end{tikzcd}$$

and we get the following formulas:

$$0 \to H^2(\pi, K_{2i+1}\bar{\mathbb{Q}}) \longrightarrow K_{2i-1}F \longrightarrow H^0(\pi, K_{2i-1}\bar{\mathbb{Q}}) \longrightarrow 0$$

$$K_{2i}F = H^1(\pi, K_{2i+1}\bar{\mathbb{Q}}).$$

Consider the long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & H^0(\pi, \mu_\infty^{\otimes i}) \arrow[r] & H^0(\pi, K_{2i-1}\bar{\mathbb{Q}}) \arrow[r] & K_{2i-1}(F)\otimes\mathbb{Q} \arrow[dll, bend left] \\
& H^1(\pi, \mu_\infty^{\otimes i}) \arrow[r] & H^1(\pi, K_{2i-1}\bar{\mathbb{Q}}) \arrow[r] & 0
\end{tikzcd}$$

$$H^2(\pi, \mu_\infty^{\otimes i}) \xrightarrow{\sim} H^2(\pi, K_{2i-1}\bar{\mathbb{Q}})$$

It follows then that

$$K_{2i}F = H^1(\pi, \mu_\infty^{\otimes(i+1)}) \Big/ \underbrace{H^1(\pi, \mu_\infty^{\otimes(i+1)})_{div}}_{}\ \overset{(\mathbb{Q}/\mathbb{Z})^{d/2}}{\overset{\wr\|}{}}$$

which agrees with Tate's results for K_2F. But on the other hand the exactness of

$$0 \longrightarrow H^0(\pi, \mu^{\otimes i}) \longrightarrow H^0(\pi, K_{2i-1}\overline{\mathbb{Q}}) \longrightarrow \mathbb{Z}^{d/2} \longrightarrow 0$$

leaves no room in $K_{2i-1}F$ for $H^2(\pi, K_{2i-1}\overline{\mathbb{Q}})$, and thus implies

(*) $$\boxed{H^2(\pi, \mu_\infty^{\otimes i}) = 0 \qquad i \geq 2}$$

$$K_{2i-1}(F) \xrightarrow{\sim} H^0(\pi, K_{2i-1}\overline{\mathbb{Q}})$$

<u>Summary of conjectures:</u>

A. $K_{2i}\overline{\mathbb{Q}} = 0 \qquad i \geq 1$

$$0 \to \mu_\infty^{\otimes i} \longrightarrow K_{2i-1}\overline{\mathbb{Q}} \longrightarrow V^{(i)} \longrightarrow 0$$

where $V^{(i)}$ is a $\mathbb{Q}$-vector space.

B. $H^2(\pi, \mu_\infty^{\otimes i}) = 0 \qquad i \geq 2$ ← OK except for 2 torsion possibly.

$H^1(\pi, \mu_\infty^{\otimes i})_{div} \cong (\mathbb{Q}/\mathbb{Z})^{d/2} \qquad i \geq 2$

C. $K_{2i-1}F = H^0(\pi, K_{2i-1}\overline{\mathbb{Q}}) \qquad i \geq 1$

$K_{2i-2}F = H^1(\pi, K_{2i-1}\overline{\mathbb{Q}}) \qquad i \geq 2$

$$= H^1(\pi, \mu_\infty^{\otimes i}) / H^1(\pi, \mu_\infty^{\otimes i})_{div}$$

<u>May 6, 1972</u>

~~One problem with the preceding conjecture is that $V^{(i)}$ is nearly an induced Γ-module if it is divisible~~

I now want to extend yesterday's conjectures about $K_*\bar{\mathbb{Q}}$ and $K_* F$ to the ring of integers $\mathcal{O}$ in F. The main point should be ~~the point~~ that in the exact sequence

$$\longrightarrow K_j \mathcal{O} \longrightarrow K_j F \xrightarrow{\partial} \bigoplus_v K_{j-1}(\mathcal{O}/m_v) \longrightarrow \cdots$$

∂ is surjective. For $j=2$, this is Calvin Moore's result on the uniqueness of reciprocity laws. From this one derives

$$K_{2i-1}\mathcal{O} \xrightarrow{\sim} K_{2i-1} F \qquad i \geq 2$$

$$0 \longrightarrow K_{2i}\mathcal{O} \longrightarrow K_{2i} F \longrightarrow \bigoplus_v K_{2i-1}(\mathcal{O}/m_v) \longrightarrow 0 \qquad i \geq 1.$$

On the other hand, I have made conjectures concerning $K_*\mathcal{O}$ and Iwasawa theory, based on an analogue of the Atiyah-Hirzebruch spectral sequence. Suppose therefore I fix a prime number ℓ. Then

$$K_i \mathcal{O} \otimes \mathbb{Z}_\ell \xrightarrow{\sim} K_i(\mathcal{O}[\ell^{-1}]) \otimes \mathbb{Z}_\ell \qquad i \geq 1$$

and the latter should ~~[illegible]~~ be accessible from a spectral sequence

$$E_2^{pq} = H^p\left(\mathcal{O}[\ell^{-1}], \begin{cases} T_\ell^{\otimes i} & q = -2i \\ 0 & q \text{ odd} \end{cases}\right) \Longrightarrow K_*(\mathcal{O}[\ell^{-1}]) \otimes \mathbb{Z}_\ell$$

Now $\mathcal{O}[\ell^{-1}]$ has $cd = 2$, so the spectral sequence ought to degenerate yielding

$$K_{2i-1}\mathcal{O} \otimes \mathbb{Z}_\ell = H^1(\mathcal{O}[\ell^{-1}], T_\ell^{\otimes i})$$
$$K_{2i-2}\mathcal{O} \otimes \mathbb{Z}_\ell = H^2(\mathcal{O}[\ell^{-1}], T_\ell^{\otimes i})$$

for $i \geq 2$ at least. As Tate remarks, the etale cohomology should be Galois cohomology for the Galois group of $\mathcal{O}[\ell^{-1}]$; in effect $\mathcal{O}[\ell^{-1}]$ should be a $K(\pi, 1)$ because $\mathcal{O}[\ell^{-1}, \mu_{\ell^\infty}]$ has cd 1, hence is a $K(\pi,1)$ (?). In any case we can compute these groups (say ℓ odd) by the spectral of the covering $\mathcal{O}[\ell^{-1}] \longrightarrow \mathcal{O}[\ell^{-1}, \mu_\ell^\infty]$ with Galois group $\tilde{\Gamma} = \Delta \times \Gamma$

$$H^p(\tilde{\Gamma}, H^q(\mathcal{O}[\ell^{-1}, \mu_\ell^\infty], T_\ell^{\otimes i})) \Longrightarrow H^{p+q}(\mathcal{O}[\ell^{-1}], T_\ell^{\otimes i})$$

$$H^q(\mathcal{O}[\ell^{-1}, \mu_\ell^\infty], T_\ell^{\otimes i}) = \begin{cases} T_\ell^{\otimes i} & q = 0 \\ \mathrm{Hom}(X, T_\ell^{\otimes i}) & q = 1 \\ 0 & q \geq 2 \end{cases}$$

Hom = Homcont

where $X = H_1(\mathcal{O}[\ell^{-1}, \mu_\ell^\infty])$ is the Iwasawa module for $F[\mu_\ell]$. Then we get exact sequences

$$K_{2i-2}\mathcal{O} \otimes \mathbb{Z}_\ell = \mathrm{Hom}(X, T_\ell^{\otimes i})_{\tilde{\Gamma}} \qquad \text{coinvariants}$$

$$0 \longrightarrow (T_\ell^{\otimes i})_{\tilde{\Gamma}} \longrightarrow K_{2i-1}\mathcal{O} \otimes \mathbb{Z}_\ell \longrightarrow \mathrm{Hom}(X, T_\ell^{\otimes i})^{\tilde{\Gamma}} \longrightarrow 0.$$

Granted Iwasawa's conjecture (say $F \supset \mu_\ell$):

$$X = \Lambda^{d/2} \oplus X_{tors} \qquad X_{tors} \simeq \mathbb{Z}_\ell^{\lambda}, \quad \Lambda = \mathbb{Z}_\ell[\Gamma]$$

one finds

$$\boxed{\begin{gathered} K_{2i-2}\mathcal{O} \otimes \mathbb{Z}_\ell = \mathrm{Hom}_{\mathbb{Z}_\ell}(X_{tors}, T_\ell^{\otimes i})_\Gamma \\ 0 \to (T_\ell^{\otimes i})_\Gamma \longrightarrow K_{2i-1}\mathcal{O} \otimes \mathbb{Z}_\ell \longrightarrow \mathbb{Z}_\ell^{d/2} \longrightarrow 0 \end{gathered}}$$

for $i \geq 2$ at least.

Return to conjecture $H^2(\pi, \mu_\infty^{\otimes i}) = 0$ for $i \geq 2$. I think this results from Tate's letter to Iwasawa. So let's introduce Tate's notation

$$W^{(r)} = \mu_\infty^{\otimes r}$$ ← observe my notation is lousy as strictly $\mu_\infty^{\otimes r} = 0$.

So I want to show that

$$H^2(F, W^{(i)}) = 0 \qquad \text{for } i \geq 2.$$

Since

$$H^*(\pi, \varinjlim M_\alpha) = \varinjlim H^*(\pi, M_\alpha)$$

for the cohomology of profinite group, hence it suffices to consider the ℓ-primary component and show

$$H^2(F, W_\ell^{(i)}) = 0.$$

Let $K = F(W)$ and let $\tilde{\Gamma} = Gal(K/F)$. Then we have

$$E_2^{pq} = H^p(\tilde{\Gamma}, H^q(K, W_\ell^{(i)})) \Longrightarrow H^{p+q}(F, W_\ell^{(i)})$$

and so we first need to know about $H^*(K, W_\ell^{(i)})$. Since $W_\ell \subset K$, $W_\ell^{(i)}$ is a trivial $Gal(\overline{\mathbb{Q}}/K)$-module. There is an exact sequence

$$0 \longrightarrow W \longrightarrow \overline{\mathbb{Q}}^{\cdot} \longrightarrow V \longrightarrow 0$$

where V is a $\mathbb{Q}$-vector space, hence

$$0 \longrightarrow H^2(k, W) \xrightarrow{\sim} \underset{\underset{Br(k)}{\parallel}}{H^2(k, \overline{\mathbb{Q}}^{\cdot})} \longrightarrow 0$$

for any subfield $k \subset \overline{\mathbb{Q}}$. (This is very general). Thus

$$H^2(k, W_\ell) = Br(k)_\ell \qquad \ell\text{-primary component.}$$

Now for a number field k one knows that

$$0 \longrightarrow Br(k) \longrightarrow \bigoplus_v Br(k_v) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

where k_v runs over all completions of k. I am going to use this to show that

(*) $$Br(K)_\ell = 0.$$

~~(i) $Br(k_v) = \mathbb{Q}/\mathbb{Z}$ gets killed in the maximal unramified extension of k_v, hence $Br(k_v)_\ell$ gets killed in the~~

~~max. unram. l-extension. But the latter is contained in $k_v[W]$ provided $v \nmid l$. (The max. l-extension of a finite field k is contained in $k(\mu_{l^\infty})$). Thus as $k = F(\mu_{l^n}) \nearrow K$, the ...)~~

In Serre's Corps locaux one finds that if k' is a finite extension of a local field k, then

$$\begin{tikzcd} Br(k) \arrow[r, "\sim"] \arrow[d] & \mathbb{Q}/\mathbb{Z} \arrow[d, "{[k':k]}"] \\ Br(k') \arrow[r, "\sim"] & \mathbb{Q}/\mathbb{Z} \end{tikzcd}$$

commutes. It follows from this that as $k = F(\mu_{l^n}) \nearrow K$, that the residue field extensions $k(v)$ for $v \nmid l$ have degree divisible by higher + higher powers of l. For $v \mid l$, the situation is totally ramified, so again the local degrees $e_v f_v$ are increasing divisible by l. Thus in the limit all of the l-torsion must be killed, proving $(*)$.

~~Returning to the spectral sequence on page 12 and using the fact that K is of $cd \leq 2$ and ... we have~~

$$\cancel{H^2(F, W_l^{(i)}) = H^1(F,}$$

Thus

$$H^2(K, W_l^{(i)}) = 0$$

and since totally imaginary no fields have $cd \leq 2$, we have

$$H^j(K, W_l^{(i)}) = 0 \qquad j \geq 2$$

Also

$$H^1(K, W_\ell) = \varinjlim_n H^1(K, \mu_{\ell^n})$$

$$= \varinjlim_n K^\circ/(K^\circ)^n = K^\circ \otimes \mathbb{Q}/\mathbb{Z}$$

so
$$H^1(K, W_\ell^{(i)}) = K^\circ \otimes W_\ell^{(i-1)}.$$

Also
$$H^0(K, W_\ell^{(i)}) = W_\ell^{(i)}.$$

Thus the spectral sequence at the top of page 12 ~~has~~ only two non-zero rows. Assume now that $\tilde{\Gamma}$ has cd ≤ 1, i.e. that [redacted] ℓ is odd or that $\ell = 2$ and $\sqrt{-1} \in F$. Then the spectral sequence yields

$$H^2(F, W_\ell^{(i)}) = H^1(\tilde{\Gamma}, K^\circ \otimes W_\ell^{(i-1)})$$

$$H^1(F, W_\ell^{(i)}) \xrightarrow{\sim} [K^\circ \otimes W_\ell^{(i-1)}]^{\tilde{\Gamma}}$$

appears in Tate's letter p.3

Now we can use the lemma of Tate's letter to Iwasawa which implies that

$$H^1(\Gamma, W_\ell^{(i-1)} \otimes N) = 0 \qquad i-1 \neq 0$$

for any discrete Γ module N on which Γ acts continuously. Note that $|\tilde{\Gamma}/\Gamma|$ divides $\ell - 1$, so

$$H^1(\tilde{\Gamma}, K^\circ \otimes W_\ell^{(i-1)}) = 0 \qquad \text{for } i \geq 2 \qquad (i \neq 1)$$

which completes the proof that

$$H^2(F, W^{(i)}) = 0 \qquad \text{for} \qquad i \geq 2 \qquad (i \neq 1).$$

at least when $\sqrt{-1} \in F$, or modulo 2 torsion.

May 17, 1972: Compactification of $GL_n\mathbb{R}/O_n$

Let V be a (f.d.) vector space over $\mathbb{R}$. Let $X(V)$ be the space of positive definite quadratic forms on V. ~~such forms~~ ~~map~~ ~~$Q: V \to \mathbb{R}$~~ The topology on $X(V)$ is uniform convergence on compact sets. $X(V)$ is convex hence contractible (this might be of use in describing the e_i).

I want to compactify $X(V)$ in such a way that

$$\overline{X(V)} = \coprod_{0 \subset W \subset W' \subset V} X(W'/W)$$

set-theoretically (perhaps even as a stratified set). The intuition is as follows. Let Q_n be a sequence of quadratics form on V. Then by selecting a subsequence we can arrange that the eigenvalues of the Q_n converge in $[0,\infty]$, and we can also arrange that the eigenspaces converge. Then if W is the 0 eigenspace and if W' is the ∞ eigenspace of the limit, we get a quadratic form on W'/W which is positive definite.

Perhaps it would be nice to fix a Q_0 ~~so~~ that any other form Q can be identified with a positive definite symmetric operator:

$$Q(x) = Q_0(Ax, x)$$

Then we take the limit of the operators A_n, i.e. we take the graph of A_n in $V \times V$ and take the limit.

Intrinsically Q_n determines a map $V \to V^*$ hence a subspace Γ_{Q_n} of $V \times V^*$

$$\Gamma_{Q_n} = \{(x,\lambda) \mid \forall y \quad \lambda(y) = B_n(y,x)\}$$

and we take the ~~[illegible]~~ subsequence so that these converge. The fact that B_n is symmetric can be expressed

$$\begin{array}{ccc} \Gamma_{Q_n} & \qquad & (\Gamma_{Q_n})^* \\ \cap & & \uparrow \\ V \times V^* & & V^* \times V \\ & & \uparrow \\ & & \Gamma_{Q_n}^{\perp} \end{array}$$

by saying that Γ_{Q_n} is isotropic for the symplectic bilinear form on $V \times V^*$:

$$\langle (x,\lambda),(x',\lambda') \rangle = \lambda(x') - \lambda'(x)$$

$$\cancel{} \qquad = B_n(x',x) - B_n(x,x').$$

And the fact that Q_n is positive definite means

$$(x,\lambda) \in \Gamma_{Q_n} \implies \lambda(x) \geq 0.$$

So this must be preserved in the limit. So maybe

$\overline{X(V)}$ = subspaces Γ of $V \times V^*$ which are maximal isotropic and ~~positive~~ non-negative

~~This is not quite correct because [illegible] the limit of [illegible] and after is a quadratic form (non negative) defined on a~~

May 23, 1972: vector bundles on curves.

Weil theorem: (X curve $/\mathbb{C}$) A bundle E is obtained from a representation of $\pi_1 X \iff$ each indecomposable factor of E has ~~[illegible]~~ degree 0.

Example: line bundles

$$\begin{array}{ccccccccc} H^1(X,\mathbb{Z}) & \to & H^1(X,\mathcal{O}) & \xrightarrow{e^{2\pi i}} & H^1(X,\mathcal{O}^*) & \to & H^2(X,\mathbb{Z}) & \to & H^2(X,\mathcal{O}) \\ \| & & \uparrow\wr \text{ any Kahler man.} & & \uparrow & & \uparrow\wr & & \uparrow \text{ not inj.} \\ H^1(X,\mathbb{Z}) & \to & H^1(X,\mathbb{R}) & \to & H^1(X,S^1) & \to & H^2(X,\mathbb{Z}) & \to & H^2(X,\mathbb{R}) \end{array}$$

shows that

$$\mathrm{Hom}(\pi_1 X, S^1) = H^1(X,S^1) = \mathrm{Ker}\{H^1(X,\mathcal{O}^*) \xrightarrow{\deg} H^2(X,\mathbb{R})\}$$

for any Kahler manifold. Thus given a line bundle of degree 0 on a curve there is a unique character $\chi : \pi_1 X \to S^1$ it is obtained from.

Any bundle obtained from a repn. of $\pi_1 X$ is flat, hence has torsion Chern classes.

A vector bundle E over a curve X is called stable (resp. semi-stable) if

$$\frac{\deg(W)}{\mathrm{rg}(W)} < \frac{\deg(E)}{\mathrm{rg}(E)} \qquad (\text{resp. } \leq)$$

for all subbundles $0 < W < E$. Since

$$\deg(\underline{\mathrm{Hom}}(W,E)) = \deg(W^* \otimes E) = rg(W)\deg(E) - \deg(W)rg(E)$$

it is equivalent to require that $\deg(\underline{\mathrm{Hom}}(W,E)) > 0$ (resp. ≥ 0) for all $0 < W < E$. This shows that E is stable $\Longleftrightarrow$ $E \otimes L$ is for any line bundle L.

Mumford shows that the set of stable bundles of rank $= r$ and deg [redacted] $= d$ forms a quasi-projective variety. Sheshadri + Narashiman show over $\mathbb{C}$ that semi-stable of degree 0 $\Longleftrightarrow$ comes from a unitary rep. of $\pi_1 X$, and they give a generalization for other degrees. Sheshadri shows the semi-stable bundles of rank r and degree 0 form a projective variety.

Stable bundles are indecomposable, in fact $H^0(\underline{\mathrm{Hom}}(E,E)) = k$ (at least if $\deg(E) = 0$). Sheshadri shows the degree 0 semi-stable bundles form an abelian category. In effect if one has a map $E \longrightarrow E'$ with kernel K, coimage E/K, image I, cokernel E'/I. Then

$$\deg(K) \leq 0 \qquad \deg(I) \leq 0$$

by semi-stab.

and

$$\deg(E/K) \leq \deg I$$

$(\Lambda^r(E/K) \subset \Lambda^r I)$

so difference is a line bundle with section

so

$$\overset{\shortparallel}{-\deg(K)}$$

so $\deg K = \deg I = 0$, and $E/K = I$.

June 13, 1972 <u>fibration problem</u>: when is $|\mathcal{C}_y|$ the h-fibre of $|\mathcal{C}| \to |\mathcal{C}'|$?

Let $f: \mathcal{C} \to \mathcal{C}'$ be fibred such that the base change functors are heq's. Then for any $y \in \mathcal{C}'$, the square

$$\begin{array}{ccc} \mathcal{C}_y & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow f \\ e & \longrightarrow & \mathcal{C}' \end{array}$$

gives rise upon geometric realization to a square

$$\begin{array}{ccc} |\mathcal{C}_y| & \longrightarrow & |\mathcal{C}| \\ \downarrow & & \downarrow \\ pt & \longrightarrow & |\mathcal{C}'| \end{array}$$

hence there is a canonical map

$$|\mathcal{C}_y| \longrightarrow \text{homotopy-fibre of } |f| \text{ at } y$$

which I want to show is a homotopy equivalence. Recall the homotopy-fibre is the fibre of the map p

(*) $$|\mathcal{C}| \xrightarrow{i} W \xrightarrow{p} |\mathcal{C}'|$$

where i is a heq and p is a fibration $\ni$ $|f| = pi$. (Usually we take $W = \{(z,\lambda) \mid z \in |\mathcal{C}| \text{ and } \lambda \text{ is a path starting from } z\}$.) Note that we can realize (*) by taking a factorization of Nf

$$N\mathcal{C} \xrightarrow{i} E \xrightarrow{p} N\mathcal{C}' \qquad N = \text{Nerve}$$

where p is a Kan fibration (minimal if desired) and i is a weq of simplicial sets; then $W = |E|$. So the problem becomes semi-simplicial: to show that NC_y is weq to the fibre of $E \longrightarrow NC'$ over y.

But this leads to the following problem, already encountered by Friedlander. Suppose that $F = p^{-1}\{y\}$. By cohomological means we can probably prove that

$$H^*(p^{-1}\{y\}, L) \xrightarrow{\sim} H^*(C_y, L)$$

for all local coeff. systems L on $\mathcal{C}$. The problem comes with the possible non-simply-connectedness of C_y. Thus one is lead to:

Question: If $f: \mathcal{C} \longrightarrow \mathcal{C}'$ is as above and $\mathcal{C}$ is 1-connected, then is each component of C_y simple?

June 19, 1972 — fibration problem

Suppose $f: X \longrightarrow Y$ is a morphism of simplicial sets. For any simplex y in Y, let X_y be defined by a cartesian square

$$\begin{tikzcd} X_y \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Delta(d) \arrow[r, "y"] & Y \end{tikzcd}$$

where $d = \dim(y)$. Then ~~[illegible]~~ to a map $y' \longrightarrow y$ (e.g. if y' is a face of y) we have associated a map

$$X_{y'} \longrightarrow X_y .$$

Assume that these maps are all heq's. Let

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "f"'] & & E \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

be a factorization with i a heq and p a fibration. I want to prove that

$$X_y \longrightarrow E_y$$

is a heq for all y. Can assume Y connected

Let F be a minimal complex with the homotopy type of X_y, and let $G = \underline{Aut}(F)$ be the simplicial group of its autos. I recall that because F is minimal, every heq $F \to F$ is an auto, and conversely.

For each 0-simplex $y \in Y_0$ choose a heq

$$\varphi_y : X_y \longrightarrow F$$

Let now $y \in Y_1$ with faces $d_0 y = v$, $d_1 y = v'$. Then we have heq's

$$\begin{tikzcd} X_v \arrow[rr, "\varphi_v"] \arrow[dd] & & F \arrow[d, dashed, shift left, "\theta_y"] \\ & X_{v'} \arrow[r, "\varphi_{v'}"'] \arrow[dl] & F \arrow[u, dashed, shift left] \\ X_y & & \end{tikzcd}$$

hence there is a dotted arrow θ rendering the diagram homotopy commutative. Then we can solve

$$\begin{tikzcd} X_v \amalg X_{v'} \arrow[r, "\varphi_v + \theta_y \varphi_{v'}"] \arrow[d, "cof"] & F \\ X_y \arrow[ur, dotted, "\varphi_y"'] & \end{tikzcd}$$

by the homotopy extension theorem. To continue this construction.

What I am really doing is constructing by induction a minimal fibration $E(n)$ over the n-skeleton $Y(n)$ of Y together with a homotopy equivalence

$$\begin{tikzcd} X(n) \arrow[rr] \arrow[dr] & & E(n) \arrow[dl] \\ & Y(n) & \end{tikzcd}$$

Consider the step from $n-1$ to n. Let y be a n-simplex, $\dot{y} : \Delta(n)^\circ \longrightarrow Y$ its boundary. Then

$$\begin{tikzcd} X_{\dot y} \arrow[rr] \arrow[dr, "f_{\dot y}"'] & & E(n-1)_{\dot y} \arrow[dl] \\ & \Delta(n)^\circ & \end{tikzcd}$$

is a minimal factorization of $f_{\dot y}$ (means horizontal arrow is a weq and $\swarrow$ is a minimal fibn). Let

$$\begin{tikzcd} X_y \arrow[rr] \arrow[dr, "f_y"'] & & Z_y \arrow[dl] \\ & \Delta(n) & \end{tikzcd}$$

be a minimal factorization of f_y. Then restricting to $\dot\Delta(n)$ gives a minimal fact. of $f_{\dot y}$. Since minimal facts. are unique up to (non-canonical) isom, we have an isom

$$Z_y | \dot\Delta(n) \simeq E(n-1)_{\dot y}$$

whence we can glue

$$\begin{tikzcd} \bigvee E(n-1)_{\dot y} \arrow[r] \arrow[d] & \bigvee Z_y \arrow[d] \\ E(n-1) \arrow[r] & E(n) \end{tikzcd}$$

$\bigvee$ being taken for each nd, n-simplex. Now it is necessary to show that $X(n) \longrightarrow E(n) \longrightarrow Y(n)$ is a minimal factorization of $f(n)$. The point is that the second map is a twisted cartesian product with fibre F, hence must be (?) a fibration. The first map will be a heq by the Whitehead criterion ultimately.

This should work and provide a minimal factorization

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"'] & E \arrow[d] \\ & Y \end{tikzcd}$$

of f. Since the induced maps $X_y \to E_y$ are heq's by construction.

~~The essential point of the above argument is why~~

Steps of the above argument.

1.

$$\begin{tikzcd} X(n-1) \arrow[r] \arrow[dr, "f_{n-1}"'] & E(n-1) \arrow[d, "p_{n-1}"] \\ & Y(n-1) \end{tikzcd}$$

has been constructed so that p_{n-1} is a minimal fibration and such that the map on each fibre is an heq. It follows that that the horizontal map restricted to any subcomplex of $Y(n-1)$ is a heq.

2. Let

$$\begin{tikzcd} X_y \arrow[r] \arrow[dr, "f_y"'] & Z_y \arrow[d] \\ & \Delta(n) \end{tikzcd}$$

be a minimal fact of f_y. Then for each $y' < y$ we have

$$\begin{tikzcd} X^{\#}_{y'} \arrow[r] \arrow[d, "heq"'] & Z_{y'} \arrow[d, "heq"] \\ X_y \arrow[r, "heq"'] & Z_y \end{tikzcd}$$

because $\begin{matrix} \Delta(s) \\ \downarrow \\ \Delta(n) \end{matrix}$ is whence it follows that the restriction of $X_y \to Z_y$ to any subcomplex of $\Delta(n)$ is an heq. Thus

$$Z_j \cong E(n-1)_j$$

and we can enlarge $E(n-1)$.

<u>Remark</u>: The above probably suffices for the needs of your paper. But the problem remains — suppose $f: \mathcal{C} \to \mathcal{C}'$ fibred and for all L on $\mathcal{C}$, $R^*f_*(L)$ is locally constant on $\mathcal{C}'$. Does it follow that $\mathcal{C}_y$ has the homotopy type of the h-fibre of $|f|$ over y?

June 14, 1972

> $f: C \to C'$ fibred $\ni$ $R^0 f_*$ (loc. const) $\subset$ loc. const. does not seem to imply that the base change functors are heq's.

Grothendieck's approach to the fundamental groupoid of a category C:

Consider the category of $F: C^\circ \longrightarrow$ Sets which transform all maps into isomorphisms. This category $\mathcal{L}$ is the full subcategory of locally constant objects in the topos $C^\wedge = \underline{\mathrm{Hom}}(C^\circ, \mathrm{Sets})$. Each object X in C determines a fibre functor

$$\begin{aligned} \mathcal{L} &\longrightarrow \mathrm{Sets} \\ F &\longmapsto F(X) \end{aligned}$$

and $\underline{\underline{\pi}} C$ can be identified with the dual of the full subcategory of $\underline{\mathrm{Hom}}(\mathcal{L}, \mathrm{Sets})$ consisting of the functors of the above form.

To see this we can define $\underline{\underline{\pi}} C$ by localizing C with respect to all its morphisms. Then

$$\mathcal{L} = (\underline{\underline{\pi}} C)^\wedge$$

and the inclusion of $\mathcal{L}$ in $C^\wedge$ can be viewed as the inverse image for the morphism of topoi

$$C^\wedge \longrightarrow \mathcal{L}$$

induced by the functor $C \longrightarrow \underline{\underline{\pi}} C$. Now points in $C^\wedge$ may be identified with $\mathrm{Pro}(C)$, and since $\underline{\underline{\pi}} C$ is a groupoid, $\mathrm{Pro}(\underline{\underline{\Pi}}\mathcal{C}) \cong \underline{\underline{\Pi}}\mathcal{C}$. Thus we can recover $\underline{\underline{\Pi}}\mathcal{C}$ as fibre functors on $\mathcal{L}$.

Let us now consider the functor

$$\mathcal{C} \xrightarrow{u} \underline{\underline{\Pi}}\mathcal{C}$$

and factor it in the standard way

$$\mathcal{C} \longrightarrow \tilde{\mathcal{C}} \longrightarrow \underline{\underline{\Pi}}\mathcal{C}$$

where an object of $\tilde{\mathcal{C}}$ is a triple $(x, y, u(x) \to y)$. Think of $\underline{\underline{\Pi}}\mathcal{C}$ as being the category of 1-conn. coverings of $\mathcal{C}$ and u as the functor assigning to x the pointed universal covering over x. Then we can view $\tilde{\mathcal{C}}$ as being the fibred cat. over $\mathcal{C}$ consisting of an $x \in \mathcal{C}$, a 1-conn. covering y, and a point in the fibre of y over x. Clearly $\tilde{\mathcal{C}}$ is equivalent to $\mathcal{C}$. (In general if we have a functor $\mathcal{C} \longrightarrow \mathcal{G}$ with $\mathcal{G}$ a groupoid, then the category of $(x, y, f(x) \to y)$ is equivalent to $\mathcal{C}$.)

On the other hand, the fibre of $\tilde{\mathcal{C}}$ over y is simply the 1-conn. covering category of $\mathcal{C}$ defined by y.

Now suppose that we are given a functor

$$f : \mathcal{C} \longrightarrow \mathcal{C}'.$$

~~Suppose that f is fibred and that the base change functors $\mathcal{C}_y \to \mathcal{C}'_{y'}$ are all heq's. We can c~~

We can consider the full subcategory $\mathcal{L}$ of $\mathcal{C}^\wedge$ consisting of those F which are locally in the image of f^*. What this means is that for every $X_0 \in \mathcal{C}$, there exists a $G : (\mathcal{C}'/fX_0)^\circ \longrightarrow$ Sets and an isomorphism

$$(*) \qquad F(X) \simeq G(fX)$$

of functors on $\mathcal{C}/X_0$. Clearly this implies that when $X \to X_0$ is $\ni$ $fX \xrightarrow{\sim} fX_0$, then $F(X) \xleftarrow{\sim} F(X_0)$. In particular F is locally constant on each fibre of f.

Conversely, suppose we are given F and X_0 and we want to construct G so that $(*)$ holds on $\mathcal{C}/X_0$. Assume f is fibred. Then given $Y \in \mathcal{C}'/fX_0$, say $u : Y \to fX_0$, put

$$G(Y) = F(u^* X_0).$$

Then because $(uv)^* = v^* u^*$ it follows that G is a well-defined functor on $\mathcal{C}'/fX_0$. If F is locally constant on each fibre of f, then given $X \in \mathcal{C}/X_0$ we have

$$\begin{tikzcd} X \arrow[rr, "\alpha"] \arrow[d] & & X_0 \\ u^*X_0 \arrow[urr] & & \end{tikzcd}$$

$$fX \xrightarrow{u = f(\alpha)} fX_0$$

and $\quad F(X) \xleftarrow{\sim} F(u^* X_0) = G(fX)$, (better notation:

$$F(X) \xleftarrow{\sim} F(fX \times_{fX_0} X_0) \quad)$$

showing F is locally in the image of f^*.

<u>Conclude</u>: If $f: \mathcal{C} \longrightarrow \mathcal{C}'$ is fibred, then $F \in \mathcal{C}^\wedge$ is locally in the image of f^* iff F is locally constant on each fibre of f.

So now let $\mathcal{L}$ be the full subcat of $\mathcal{C}^\wedge$ consisting of these functors. Let $\overline{\mathcal{C}}$ denote the category obtained by inverting all of the arrows in $\mathcal{C}$ which become isomorphisms in $\mathcal{C}'$, or equivalently inverting just the arrows in the fibres. Then we may identify

$$\mathcal{L} = \overline{\mathcal{C}}^\wedge$$

and the inclusion of $\mathcal{L}$ in $\mathcal{C}^\wedge$ is just the inverse image for the functor

$$\mathcal{C} \longrightarrow \overline{\mathcal{C}}$$

It is cleare that

$$\overline{\mathcal{C}} \longrightarrow \mathcal{C}'$$

is fibred and the fibre over y is

$$\overline{\mathcal{C}}_y = \underline{\pi}\, \mathcal{C}_y .$$

(Actually, this requires a good proof. Intuitively the base change functor

$$\mathcal{C}_y \longrightarrow \mathcal{C}_{y'}$$

will extend to a functor of groupoids

$$\underline{\pi}\, \mathcal{C}_y \longrightarrow \underline{\pi}\, \mathcal{C}_{y'}$$

and the resulting fibred category in groupoids will clearly be

June 15, 1972

$\overline{\mathcal{C}}$.)

We can think of objects of $\overline{\mathcal{C}}$ as the 1-connected coverings of the fibres of f. g assigns to $x \in \mathcal{C}$ the pointed 1-connected covering with basepoint over X.

~~Suppose from now on that all base change functors $\mathcal{C}_y \to \mathcal{C}_{y'}$ are heq's.~~

Suppose now that $f : \mathcal{C} \longrightarrow \mathcal{C}'$ is fibred and that

(*) for any local coeff system of sets (resp. grps, resp. ab. gps) L on $\mathcal{C}$, $R^q f_*(L)$ is locally constant for $q=0$ (resp. $q \leq 1$, resp. all q). Here $R^q f_*$ is computed using covariant functors so that

$$R^q f_*(L)_y = H^q(\mathcal{C}_y, L).$$

I want then to conclude that all the base change functors $\mathcal{C}_y \longrightarrow \mathcal{C}_{y'}$ are heq's. I can assume $\mathcal{C}'$ is 1-connected by pulling back to any 1-conn. covering, as this doesn't change the fibres.

Now for any set S we have that $f_*(S)$ is locally constant, hence for any y

$$\begin{array}{l} H^0(\mathcal{C}', f_*(S)) \xrightarrow{\sim} f_*(S)_y = H^0(\mathcal{C}_y, S) = \mathrm{Hom}(\pi_0 \mathcal{C}_y, S) \\ \quad \| \\ H^0(\mathcal{C}, S) = \mathrm{Hom}(\pi_0 \mathcal{C}, S) \end{array}$$

and so we conclude that $\pi_0 \mathcal{C}_y \xrightarrow{\sim} \pi_0 \mathcal{C}$ for all

y. ~~[illegible]~~

We can suppose C is connected since f is the sum of its restrictions to the components of C and since the $R^q f_*(L)$ decompose. Then C_y is also connected.

Let F be a covering of C, i.e. a local coefficient system on C and suppose F has a section over C_y. Since $f_* F$ is locally constant, hence constant

$$H^0(C_y, F) = (f_* F)_y \xleftarrow{\sim} H^0(C', f_* F) = H^0(C, F)$$

so the ~~[illegible]~~ section over C_y may be extended to all of C. This implies that

$$\pi_1(C_y, x) \twoheadrightarrow \pi_1(C, x)$$

for any x in C_y. (Take F to be the covering defined by the $\pi_1(C,x)$-set

$$\pi_1(C,x)/\operatorname{Im} \pi_1(C_y, x).)$$

Let $\tilde{C} \xrightarrow{p} C$ be the universal covering of C; it is fibred so the composite $\tilde{C} \xrightarrow{p} C \xrightarrow{f} C'$ is fibred. It is clear that $\tilde{C}_y = C_y \times_C \tilde{C}$ is the induced covering. Now if L is a local system on C, then

$$R^q (fp)_*(L) = R^q f_* (p_* L)$$

is locally constant. If we succeed in establishing that the base change $\tilde{C}_y \to \tilde{C}_{y'}$ is an heq, it will follow that $C_y \to C_{y'}$ is. (Our problem is that we have only

$$H^*(C_{y'}, L) \xrightarrow{\sim} H^*(C_y, L)$$

for L which come from $\mathcal{C}$ and not all local coeff. systems on $\mathcal{C}_{y}$.) Thus we may assume $\mathcal{C}$ is 1-connected.

So we are now in the following situation. $f: \mathcal{C} \to \mathcal{C}'$ is fibred, $\mathcal{C}, \mathcal{C}'$ are 1-connected, and for all local coeff systems L on $\mathcal{C}$, $R^q f_*(L)$ is loc. const.

Let $u: y' \longrightarrow y$ be an arrow in $\mathcal{C}'$ ~~[illegible]~~ and

$$u^* : \mathcal{C}_y \longrightarrow \mathcal{C}_{y'}$$

the base change functor. Let $x \in \mathcal{C}_y$ and consider

$$\pi_1(\mathcal{C}_y, x) \longrightarrow \pi_1(\mathcal{C}_{y'}, u^* x).$$

We have for any group G that $R^1 f_*(G)$ is locally constant hence

$$\begin{array}{ccc} H^1(\mathcal{C}_{y'}, G) & \xrightarrow{\sim} & H^1(\mathcal{C}_y, G) \\ \| & & \| \\ \mathrm{Hom}(\pi_1 \mathcal{C}_{y'}, G) & \xrightarrow{\sim} & \mathrm{Hom}(\pi_1 \mathcal{C}_y, G) \end{array}$$

where the Hom's are taken in the category of groups up to inner automorphisms. Since this holds for all G we can conclude that

$$\pi_1(\mathcal{C}_y, x) \xrightarrow{\sim} \pi_1(\mathcal{C}_{y'}, u^* x).$$

Suppose we now consider the factorization of f

$$\mathcal{C} \xrightarrow{g} \overline{\mathcal{C}} \xrightarrow{h} \mathcal{C}'$$

where $\overline{C}$ is obtained from C by inverting the maps in the fibres. Then $\overline{C}$ is fibred with $\overline{C}_y = \amalg\, C_y$. By what we have just shown the base changes $\overline{C}_y \to \overline{C}_{y'}$ are equivalences of groupoids.

Let x_0 be a basepoint of C and set
$$G = \pi_1(C_{fx_0}, x_0)$$
For each $y \in C'$ choose a principal G-bundle $P_y \to C_y$ which is a universal covering. This is possible as we have shown $\pi_1 C_y \simeq G$. For each map $u: y' \to y$ in C there exists at least one covering map θ_u

$$\begin{tikzcd}
P_y \arrow[r, "\theta_u"] \arrow[d] & P_{y'} \arrow[d] \\
C_y \arrow[r, "u^*"] & C_{y'}
\end{tikzcd}$$

compatible with the action of G. Any two choices for θ_u differ by right multiplication by an element of the center Z of G. (In effect we only have to check this for autos. Any auto. θ of P_y is determined by its effect on the fibre over the basepoint and hence is a map $P_{y,x} \to P_{y,x}$ compatible with left mult. by $\pi_1(C_y, x)$ and also right mult. by G. The only maps $G \to G$ which commute with both left & right mult. are mult. by elements of Z.) Therefore we obtain a compatible family of covering maps.

$$\begin{tikzcd} P_y/Z \arrow[r] \arrow[d] & P_{y'}/Z \arrow[d] \\ C_y \arrow[r] & C_{y'} \end{tikzcd}$$

whence we obtain a principal covering of C with group G/Z. Since $\pi_1 C = 0$, it follows this covering is trivial, so restricting to the fibre C_{fx_0} we see that $G/Z = 0$. Thus G is abelian. So we conclude

$\pi_1 C_y$ is <u>abelian</u>.

<u>Remark:</u> $\overline{C} \longrightarrow C'$ is a gerb for the group $G = \pi_1 C_y$. It is non-trivial, otherwise we would able to find a coherent system of P_y and hence construct a non-trivial covering of C.

Now we have reached the following problem. Consider the map $C \xrightarrow{g} \overline{C}$ whose fibres are essentially the universal coverings of the fibres of f. Given a map

$$\begin{tikzcd} P_y \arrow[r] \arrow[d] & P_{y'} \arrow[d] \\ C_y \arrow[r] & C_{y'} \end{tikzcd}$$

in $\overline{C}$, we know that $H^*(C_{y'}, A) \xrightarrow{\sim} H^*(C_y, A)$ for all abelian groups A, but we don't know this for all G-modules, $G = \pi_1 C_y$. Thus for example we have

$$\begin{tikzcd}
0 \arrow[r] & H^2(G,A) \arrow[r] \arrow[d, "\wr"] & H^2(\mathcal{C}_{y'},A) \arrow[r] \arrow[d, "\wr"] & H^2(P_{y'},A)^G \arrow[r] \arrow[d] & H^3(G,A) \arrow[r] \arrow[d, "\wr"] & H^3(\mathcal{C}_{y'},A) \arrow[d, "\wr"] \\
0 \arrow[r] & H^2(G,A) \arrow[r] & H^2(\mathcal{C}_y,A) \arrow[r] & H^2(P_y,A)^G \arrow[r] & H^3(G,A) \arrow[r] & H^3(\mathcal{C}_y,A)
\end{tikzcd}$$

which shows that

$$(\pi_2 P_{y'})_G \xrightarrow{\sim} (\pi_2 P_y)_G .$$

~~Thus for example it appears that it~~

I don't see how to get anything better. Thus

$$\mathcal{C} \xrightarrow{g} \bar{\mathcal{C}}$$

$Rg_*(A)$ is some complex of covariant functors on $\bar{\mathcal{C}}$ such that when pushed down to $\mathcal{C}'$, it becomes locally constant.

?

June 17, 1972

Let $X \xrightarrow{f} Y$ be a finite radicial surjective morphism of noetherian schemes. Then

$$f_* : \mathrm{Modf}(X) \longrightarrow \mathrm{Modf}(Y)$$

is exact and preserves dimension of the support, hence it induces a map of spectral sequences

$$\begin{tikzcd}
E^1_{pq} = \bigoplus\limits_{\dim(x)=p} K_{p+q}(k(x)) \arrow[r, Rightarrow] \arrow[d] & G_{p+q}(X) \arrow[d] \\
E^1_{pq} = \bigoplus\limits_{\dim(y)=p} K_{p+q}(k(y)) \arrow[r, Rightarrow] & G_{p+q}(Y).
\end{tikzcd}$$

Recall that f is a universal homeomorphism and in particular is a homeomorphism of X and Y. For ~~any~~ each $y \in Y$, $\exists!\ x \in X$ with $f(x) = y$ and the extension $k(y) \longrightarrow k(x)$ is finite and purely inseparable.

Consider now a purely insep. field extension E/F of degree p^a, and let $g : F \longrightarrow E$ denote the inclusion. Then for any E-module V we have

$$f^* f_* V = E \otimes_F V = (E \otimes_F E) \otimes_E V$$

But we may filter $E \otimes_F E$ by powers of the augmentation ideal ~~thereby filtering~~ I; I^n/I^{n+1} is an E-module.

Thus on the level of K-groups
$$f^*f_* \alpha = \left[\bigoplus I^n/I^{n+1}\right]\cdot\alpha$$
$$= [E:F]\,\alpha .$$
Also we have $f_*f^*\alpha = [E:F]\,\alpha$ in general. Thus we see that f^* is essentially an inverse to f_* once p gets inverted so
$$K_*(E)[p^{-1}] \xrightarrow{\sim} K_*(F)[p^{-1}] .$$

So we conclude:

<u>Proposition</u>: If $f: X \to Y$ is finite radicial surjective and if ℓ is a prime number invertible on Y, then
$$f_* : G_*(X)\otimes \mathbb{Z}_{(\ell)} \xrightarrow{\sim} G_*(Y)\otimes\mathbb{Z}_{(\ell)} .$$

<u>Cor</u>: If Y is of char. 0, then $G_*(X) \xrightarrow{\sim} G_*(Y)$.

__Conjecture:__ $\mathcal{C}$ category

$P \longrightarrow B$ $\mathcal{C}$-torsor $\ni$ fibres of $P \longrightarrow Ob\,\mathcal{C}$ are contractible.

X nice space (e.g. CW complex).

Then

$$[X, B] = \pi_0 \underline{\mathrm{Tors}}(X, \mathcal{C})$$

more generally

$$\underline{\mathrm{Hom}}(X, B) = B\,\underline{\underline{\mathrm{Tors}}}(X, \mathcal{C})$$

category

June 21, 1972

Suppose we have a cube

$$\begin{tikzcd}
 & Y' \arrow[rr] \arrow[dd] & & Z' \arrow[dd] \\
W' \arrow[ur] \arrow[rr] \arrow[dd] & & X' \arrow[ur] \arrow[dd] & \\
 & Y \arrow[rr] & & Z \\
W \arrow[ur] \arrow[rr] & & X \arrow[ur] &
\end{tikzcd}$$

such that (i) top and bottom are homotopy-cocartesian
(ii) front and left side are homotopy-cartesian.

Then the back and the right side are homotopy-cartesian.

~~Proof. Can assume all objects are Kan cxs. (apply Ex^∞) and that all vertical arrows are fibrations (since factorization is functorial).~~ ~~Recall that a minimal factorization $X \xrightarrow{i} Z \xrightarrow{p} Y$ of a fibration f is unique up to canonical isom. because i is a fibration with contractible fibre (see paper: Geom. real. of Kan fibn is a Serre fibn)~~

~~Proof.~~ Assume this for the moment, and suppose that we have a map $X \longrightarrow Y$ of simplicial spaces such that for every simplicial operation $\Delta(p) \longrightarrow \Delta(q)$ we have a h-cartesian square

$$\begin{tikzcd} X_q \arrow[r] \arrow[d] & X_p \arrow[d] \\ Y_q \arrow[r] & Y_p \end{tikzcd}$$

I want to show that

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & |X| \arrow[d] \\ Y_0 \arrow[r] & |Y| \end{tikzcd}$$

is homotopy-cartesian.

~~The idea is to inductively construct~~ $\cancel{|X| \to |Y|}$
~~Denote by~~ $\cancel{X^{(n)}, Y^{(n)}}$ ~~the n-skeleton of~~ $\cancel{X, Y}$. ~~Suppose we know that~~

Philosophy of simplicial sets - especially their skeletal description.

June 22, 1972. (32 years old)

Given categories and functors

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "f"'] & & E \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

such that (i) f, p are fibred [redacted] and all base change functors are heq's, (ii) i heq. ~~To~~ prove $\forall y$

$$X_y \longrightarrow E_y$$

is a heq.

Can assume Y 1-conn.

Can assume X, E connected, whence fibres X_y, E_y are connected. [redacted]

Can assume X, E 1-connected. In effect if $\tilde{X} \to X$ is a 1-connected covering, then $\tilde{X}_{y'} \to \tilde{X}_y$ is the universal covering of $\tilde{X}_{y'} \to X_y$, hence is a heq. And if we show $\tilde{X}_y \longrightarrow E_y$ is an heq for all y it will follow that $X_y \longrightarrow E_y$ is also.

Now we know [redacted] from previous work that the fundamental group of the fibre X_y is abelian and fits into an exact sequence

$$[redacted]\ H_2X \longrightarrow H_2Y \longrightarrow \pi_1 X_y \longrightarrow 0$$

Thus $X_y \to E_y$ induces an iso. of π_1. Next we can factor f and p:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f'"'] & E \arrow[d, "p'"] \\ \bar{X} \arrow[r, "\bar{\imath}"] \arrow[dr, "k"'] & \bar{E} \arrow[d, "q"] \\ & Y \end{tikzcd}$$

by inverting morphisms along the fibres. We can think of an object of $\bar{X}$ as being a 1-connected covering of a fibre of f. Then $\bar{\imath}$ is an equivalence of categories, because both k, q are fibred in groupoids with equivalent fibres. Finally we note that when we make f' fibred in the standard way, then the base change

$$X_z \longrightarrow X_{z'}$$

associated to a map $z' \longrightarrow z$ in $\bar{X}$ is essentially the map of coverings

$$\begin{tikzcd} X_z \arrow[r] \arrow[d] & X_{z'} \arrow[d] \\ X_y \arrow[r] & X_{y'} \end{tikzcd}$$

so $X_z \to X_{z'}$ is a heq. Similarly for p'. Thus up to equivalence we can write the above in the form

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "f'"'] & & E \arrow[dl, "p'"] \\ & Z \arrow[d, "k"] & \\ & Y & \end{tikzcd}$$

where f' and p' satisfy the same conditions. Thus we reduce to the case where the fibres of X and of E are 1-connected. At this point we need only show that $X_y \longrightarrow E_y$ induces an isom. on integral cohomology.

Now one can use either the Zeeman theorem, or argue with D_{lc} as you did before to show that $Rf_*(L) \longrightarrow Rp_*(L)$ must be an isom. in $D_{lc}(Y)$.

June 25, 1972: p-adic skew-fields

K locally compact sfield
μ Haar measure for K^+
Then put for $a \in K$

$$\operatorname{mod}(a) = \frac{\mu(aX)}{\mu(X)}$$

for any measurable $X \ni \quad 0 < \mu(X) < \infty$, eg. X compact nbd. of 0.

$$\operatorname{mod}(ab) = \operatorname{mod}(a) \cdot \operatorname{mod}(b)$$

~~mod: $K \to \mathbb{R}^+$ [illegible]~~

Observe that

$$\operatorname{mod}(x) = |x| \quad \text{on } \mathbb{R}$$
$$\operatorname{mod}(x) = |x|^2 \quad \text{on } \mathbb{C}$$
$$\operatorname{mod}(x) = |x|_p \quad \text{on } \mathbb{Q}_p$$

i.e. one gets the good absolute values for the product formula. In general, if V is a vector space of dim n over K, then

$$\operatorname{mod}_V(a) = \operatorname{mod}_K(a)^n \qquad a \in K.$$

Now suppose K totally disconnected, and let $R = \{x \in K \mid \operatorname{mod}(x) \leq 1\}$ be the maximal compact subring, let $P = \{x \mid \operatorname{mod}(x) < 1\}$ be the maximal ideal. Then R/P is a finite field, say it has $q = p^f$ elements. If π is an element of R of max module, clearly $P = R\pi = \pi R$ has index q so

$$\operatorname{mod}(\pi) = \frac{1}{q}.$$

Now define $\omega : R \to R$ by
$$\omega(x) = \lim_n x^{q^n}.$$
Clearly $\omega(x) = 0 \iff x \in P$. If $x \in R^\times = R - P$, then $x^{q-1} \equiv 1 \ (P)$.

Now if $u \equiv 1 \ (P^r)$ then
$$u^p = (1 + (u-1))^p = 1 + p(u-1) + \binom{p}{2}(u-1)^2 + \cdots + (u-1)^p$$
$$\in 1 + pP^r + P^{rp} \subset 1 + P^{r+1}.$$
Thus
$$x^{(q-1)q^s} \in P^{s+1}$$
i.e. $x^{q^s} \equiv x^{q^{s+1}} \ (P^{s+1})$ and so $\omega(x)$ is well defined.

Clearly
$$\omega(x)^q = \omega(x)$$
whence
$$\mathrm{Im}(\omega) = \{x \in R^\times \mid x^q = x\}.$$

Clearly $\mathrm{Im}(\omega)$ is the subset of all elements of $R^\times$ of order prime to p.

<u>Question</u>: Let A be a d.v.r. quotient field K, let D be a sfield fin. central $/K$. Is $\{x \in D \mid x \text{ integral over } A\}$ a subring of D? (It would then be the maximal order in D.)

<u>Note</u>: Exact sequence

$$1 \longrightarrow (1+P) \longrightarrow R^\times \longrightarrow (R/P)^\times \longrightarrow 1$$

$(1+P)$: pro-p group; $(R/P)^\times$: cyclic order $q-1$

shows that $R^\times$ and $(R/P)^\times$ have same mod ℓ cohomology $\ell \neq p$. This suggests that if L is an unramified splitting field for K, then

$$GL_n(L) \longrightarrow GL_n(K)$$

should induce an isomorphism on mod ℓ cohomology. Try to prove via the building.

<u>Question</u>: What are the forms of GL_n over local and global fields? More precisely, suppose Λ is a semi-simple algebra over a field K, what is the algebraic group

$$R \longmapsto (\Lambda \otimes_K R)^* \ ?$$

June 27, 1972   homotopy theory of cats

1. $\mathcal{C}$ small category, $\mathcal{L}$ = full subcategory of $\mathcal{C}^\wedge$ consisting of locally constant F, i.e. $F(u)$ iso. for all u in $\mathcal{C}$. Let $\pi\mathcal{C}$ = localization of $\mathcal{C}$ wrt all arrows. Then

$$\mathcal{L} \simeq (\pi\mathcal{C})^\wedge$$

More precisely, this equivalence is given by γ^*, where $\gamma : \mathcal{C} \to \pi\mathcal{C}$ is the canonical functor.

We have adjoint functors

$$\begin{tikzcd} \mathcal{C}^\wedge \arrow[r, shift left=2, "\gamma_!"] \arrow[r, shift right=2, "\gamma_*"'] & \mathcal{L} \arrow[l, "\gamma^*"'] \end{tikzcd}$$

which signifies for any $F \in \mathcal{C}^\wedge$, there exist universal arrows

$$F \longrightarrow \gamma_! F$$

$$\gamma_* F \longrightarrow F$$

to and from a locally constant sheaf. Formulas

$$(\gamma_! F)(x) = H_0(P_x, F)$$

$$(\gamma_* F)(x) = H^0(P_x, F)$$

where $P_x \longrightarrow \mathcal{C}$ is the pointed universal covering with basepoint over $x \in \mathcal{C}$.

I want to make this more precise, so it is necessary to understand $\pi\mathcal{C}$. Given $x \in \mathcal{C}$, it determines a functor

$$L \longmapsto L(x)$$

from $\mathcal{L}$ to Sets which is left exact and commutes with ind limits, hence is a point in $\mathcal{L}$, which one knows is a pro-object in $\pi\mathcal{C}$, hence representable as $\pi\mathcal{C}$ is a groupoid. Thus we have that $L \mapsto L(x)$ is representable:

$$L(x) = \mathrm{Hom}_{\mathcal{L}}(P_x, L)$$

If $x \to y$, then $L(y) \to L(x)$, hence $P_x \to P_y$ so we obtain a functor

$$\mathcal{C} \longrightarrow \mathcal{L}$$

We may identify $\pi\mathcal{C}$ with the full sub-category of P_x in $\mathcal{L}$, and γ with the functor $x \mapsto P_x$.

Now

$$(\gamma_* F)(y) = \varprojlim_{\gamma x \to y} F(x) = H^0(\gamma/y, F)$$

where γ/y is the fibred category over $\mathcal{C}$ defined by the "universal covering" y. It should be mentioned that any L is essentially equivalent to a bifibred category with discrete fibres over $\mathcal{C}$. Thus P_x can be interpreted as "the" universal pointed covering with basepoint over x. Thus γ/y equivalent to $y\backslash\gamma$.

$$(\gamma_! F)(y) = \varinjlim_{y \to \gamma x} F(x) = H_0(y\backslash\gamma, F)$$

$$= H_0(\gamma/y, F).$$

Summary: coverings of C, locally constant sheaves in $C^\wedge$

universal covering P_x, $x \in C$.

fundamental groupoid πC.

adjoint functors

$$C^\wedge \underset{\gamma_*}{\overset{\gamma_!}{\rightleftarrows}} \mathcal{L}$$

2. Let $D(C)$ = derived cat of $C^\wedge$ and $D_{lc}(C)$ the full subcat consisting of complexes E such that $\mathcal{H}^q E \in \mathcal{L}$ for all q. To show that the inclusion functor

$$D_{lc}(C) \xrightarrow{i} D(C)$$

admits left and right adjoints defined at least as follows:

$$D(C)^- \xrightarrow{i_!} D_{lc}(C)^-$$

$$D(C)^+ \xrightarrow{i_*} D_{lc}(C)^+$$

We consider the case of i_*. Given $F^{\cdot} \in D(C)^+$

we wish to find $E^\bullet \in D_{lc}(\mathcal{C})^+$ together with a map $E^\bullet \to F^\bullet$ such that

$$\mathrm{Hom}(L^\bullet, E^\bullet) \xrightarrow{\sim} \mathrm{Hom}(L^\bullet, F^\bullet)$$

for all $L^\bullet \in D_{lc}(\mathcal{C})$.

a) Reduction to case $L^\bullet = L[n]$ with $L \in \mathcal{L}, n \in \mathbb{N}$:

If $\mathcal{H}^q(L^\bullet) = 0 \quad q > m, \quad \mathcal{H}^q(E^\bullet) = 0 \quad q \le m$

then $\mathrm{Hom}(L^\bullet, E^\bullet) = 0$. Thus can assume $L^\bullet \in D_{lc}(\mathcal{C})^+$ by using triangle

$$L^\bullet_{\le m} \longrightarrow L^\bullet \longrightarrow L^\bullet_{> m}.$$

Milnor exact sequence

$$0 \to R^1\varprojlim_n \mathrm{Hom}^{-1}(L^\bullet_{\le n}, E^\bullet) \to \mathrm{Hom}(L^\bullet, E^\bullet) \longrightarrow \varprojlim_n \mathrm{Hom}(L^\bullet_{\le n}, E^\bullet) \longrightarrow 0$$

+ five lemma reduce us to case $L^\bullet \in D_{lc}(\mathcal{C})^b$.
Now use Postnikov decompose of $L^\bullet$ into $\mathcal{H}^n(L^\bullet)[-n]$.

b) Inductive construction of $E^\bullet_{\le n} \in D_{lc}(\mathcal{C})^b$ + map $E^\bullet_{\le n} \to F^\bullet$ such that

$$\mathrm{Hom}^q(L, E^\bullet_{\le n}) \longrightarrow \mathrm{Hom}^q(L, F^\bullet) \qquad L = L[0]$$

for all $q < n$, and $\hookrightarrow$ for $q = n$. For $n \ll 0$ take $E^\bullet_{\le n} = 0$. Assuming $E^\bullet_{\le n-1}$ exists form triangle

$$K^\bullet \longrightarrow E^\bullet_{\le n-1} \longrightarrow F^\bullet$$

whence

~~[illegible]~~

$$\cdots \xrightarrow{\delta} \mathrm{Hom}^q(L\text{[redacted]}, K^\cdot) \longrightarrow \mathrm{Hom}^q(L\text{[redacted]}, E^\cdot_{\le n-1}) \longrightarrow \mathrm{Hom}^q(L\text{[redacted]}, F^\cdot)$$

so

$$\mathrm{Hom}^q(L\text{[redacted]}, K^\cdot) = 0 \qquad q \le n-1.$$

Thus

$$L \longmapsto \mathrm{Hom}^n(L\text{[redacted]}, K^\cdot)$$

is left exact, and as it sends sums to products it is representable:

$$(*) \qquad \mathrm{Hom}_{\mathcal{L}_{ab}}(L, L_n) = \mathrm{Hom}^n(L\text{[redacted]}, K^\cdot).$$

In fact

$$L_n(x) = \mathrm{Hom}_{\mathcal{L}_{ab}}(\mathbb{Z}P_x, L_n) = \mathrm{Hom}^n(\mathbb{Z}P_x\text{[redacted]}, K^\cdot)$$

is an explicit formula for L_n.

~~Meaning of (*): $\exists$ canon. elt. $\chi_n : L_n[-n] \to K^\cdot$ such that~~

$$\begin{tikzcd} & L[n] \arrow[dl, dashed, "\exists !"'] \arrow[d] \\ L_n[-n] \arrow[r] & K^\cdot \end{tikzcd}$$

~~(note $\mathrm{Hom}^\cdot(L[-n], L'[-n]) = \mathrm{Hom}_{\mathcal{L}_{ab}}(L, L')$).~~

~~Thus we have a map $L_n[n] \to E^\cdot$ such that~~

$(*) \Longrightarrow \exists$ canon. map $L_n[-n] \longrightarrow K^\cdot$ such that

(1) $\mathrm{Hom}_{\mathcal{L}_{ab}}(L, L_n) \xrightarrow{\sim} \mathrm{Hom}^n_{D(\mathcal{C})}(L, L[-n]) \xrightarrow{\sim} \mathrm{Hom}^n_{D(\mathcal{C})}(L, K^\cdot)$.

Consider the morphism of coh. functors of $L \in \mathcal{L}$

$$\mathrm{Hom}^{n+q}_{D(\mathcal{C})}(L, L_n[-n]) \longrightarrow \mathrm{Hom}^{n+q}_{D(\mathcal{C})}(L, K^\cdot)$$

induced by this map. We know it is an isom for $q=0$. Moreover

$$\mathrm{Hom}^{n+1}_{D(\mathcal{C})}(L, L'[-n]) = \mathrm{Hom}^1_{D(\mathcal{C})}(L, L')$$

$$= \mathrm{Ext}^1_{\mathcal{C}^\wedge_{ab}}(L, L') \cong \mathrm{Ext}^1_{\mathcal{L}_{ab}}(L, L')$$

because any extension of locally const. functors is locally constant; thus for $q=1$ the left functor is effaceable so we conclude

(2) $$\mathrm{Hom}^{n+1}_{D(\mathcal{C})}(L, L_n[-n]) \hookrightarrow \mathrm{Hom}^{n+1}_{D(\mathcal{C})}(L, K^\cdot)$$

by general facts about coh. functors.

Let us now form cones on the maps $L_n[-n] \to K^\cdot$, $K^\cdot \to E^\cdot$ and their composite, getting triangles

$$\begin{tikzcd}
L_n[n] \arrow[r] \arrow[d, equal] & K^\cdot \arrow[r] \arrow[d] & C^\cdot \arrow[d] \\
L_n[n] \arrow[r] & E^\cdot_{\leq n-1} \arrow[r] \arrow[d] & E^\cdot_{\leq n} \arrow[d] \\
& F^\cdot \arrow[r, equal] & F^\cdot
\end{tikzcd}$$

by octahedral axiom. From (1) and (2) we get

$$\mathrm{Hom}^q(L, C^\circ) = 0 \qquad q \leq n$$

so

$$\mathrm{Hom}^q(L, E^\circ_{\leq n}) \longrightarrow \mathrm{Hom}^q(L, F^\circ)$$

isom. $q < n$
inj. $q = n$

completing the induction.

c) Take $E^\circ = \varinjlim E_{\leq n}$. Milnor ex. seq $\Rightarrow \exists$ $E^\circ \to F^\circ$ and now done.

YOGA: Pretend there is a functor $i : \mathcal{C} \to \mathcal{C}'$ such that i^* is equivalent to the inclusion of $D_{lc}(\mathcal{C})$ in $D(\mathcal{C})$. (Such a cat. $\mathcal{C}'$ does not usually exist; but perhaps it exists as a ∞-category). Then $i_* : D(\mathcal{C}) \longrightarrow D_{lc}(\mathcal{C})$ is just the functor constructed above

3. Non-abelian variations on the preceding.

Given a functor $f : C \longrightarrow C'$ and a sheaf of groups G on C let

$$R^{\leq 1} f_*(G)$$

be the sheaf in groupoids over C' whose stalk at y is

$$R^{\leq 1} f_*(G)_y = \underline{\mathrm{Tors}}(f/y, G).$$

More generally, this definition makes sense if G is replaced by a sheaf in groupoids $\mathcal{G}$ on C. (sheaf in groupoids = fibred category whose fibres are groupoids.)

<u>Question</u>: Let $\mathcal{G}$ be a fibred category in groupoids over C and assume that $u^* : \mathcal{G}_y \longrightarrow \mathcal{G}_{y'}$ is an equivalence of categories forall arrows u in C. It is equivalent that $\mathcal{G}$ also be cofibred over C. (In effect then for x over y, x' over y'

$$\mathrm{Hom}(u_* x', x)_{id_y} \simeq \mathrm{Hom}(x', x)_u \simeq \mathrm{Hom}(x', u^* x)_{id_{y'}}$$

so

$$\mathcal{G}_{y'} \underset{u^*}{\overset{u_*}{\rightleftarrows}} \mathcal{G}_y$$

are adjoint functors, necessarily equivalences as the categories are groupoids).

The question is whether $\mathcal{G}$ is equivalent to the inverse image of a $\mathcal{G}'$ over πC.

Answer: NO. Example: Suppose that C is 1-connected with $H^2(C,A) \neq 0$ for some abelian group A. Taking a non-zero element of $H^2(C,A)$, it classifies a gerb $\mathcal{G}$, which is a fibred category in connected groupoids, each fibre being equivalent to A. Thus it is bifibred. (Moral: A gerb whose lien is locally constant will be bifibred.) But πC is equivalent to the punctual category, hence $\mathcal{G}$ can't come from πC.

~~Question: Let G be a sheaf of groupoid over C, and let $\gamma : C \longrightarrow \pi C$ be the canonical functor. Is there an equivalence~~

$$\cancel{\underline{\mathrm{Hom}}(\mathcal{G}, \gamma^* R^{\leq 1}\gamma_*(G)) = \underline{\mathrm{Hom}}(\mathcal{G}, G)}$$

~~for all bifibred $\mathcal{G}$?~~

Question: Let $\mathcal{G}$ be a fibred cat. in groupoids over C and let $\mathcal{L}$ be a bifibred cat. in groupoids. If $\gamma : C \longrightarrow \pi C$ is the canon. functor, then $R^{\leq 1}\gamma_*(\mathcal{G})$ should be a bifibred cat in groupoids over πC and there should be a canon. functor

$$\gamma^* R^{\leq 1}\gamma_*(\mathcal{G}) \longrightarrow \mathcal{G}$$

The question is whether this induces an equivalence

$$\underline{\mathrm{Hom}}(\mathcal{L}, \gamma^* R^{\leq 1}\gamma_*(\mathcal{G})) \longrightarrow \underline{\mathrm{Hom}}(\mathcal{L}, \mathcal{G}).$$