April 2, 1972. Resolution problem

$\mathcal{M}(r) =$ modules of proj dim. $\leq r$ and their isos.
$S\mathcal{M}(r) =$ cat. with same objects but in which an $M' \longrightarrow M$ is a pair (M_0, ξ) with M_0 a submodule of M such that $M/M_0 \in \mathcal{M}(r)$, and where $\xi : M_0 \twoheadrightarrow M$.
$\mathcal{M}(r-1)$, $S\mathcal{M}(r-1)$ defined analogously; their objects denoted by P, Q, P', Q', etc.

Theorem: ~~The inclusion functor~~ The categories

$$S\mathcal{M}(r-1) \text{ and } S\mathcal{M}(r)$$

~~is a~~ are homotopy equivalent.

Scheme of the demonstration: Given $P \in \mathcal{M}(r-1)$ $M \in \mathcal{M}(r)$, let $C_{P,M}$ be the category whose objects are surjections

$$Q \twoheadrightarrow P \times M$$

with $Q \in \mathcal{M}(r-1)$ and whose morphisms are isomorphisms over $P \times M$. Given arrows

$$\alpha_1 : \begin{array}{c} P_0 \subset P \\ \downarrow\!\!\!\downarrow \\ P' \end{array} \qquad\qquad \alpha_2 : \begin{array}{c} M_0 \subset M \\ \downarrow\!\!\!\downarrow \\ M' \end{array}$$

in $S\mathcal{M}(r-1)$ and $S\mathcal{M}(r)$ respectively, we have a base change functor

$$(*) \qquad (\alpha_1, \alpha_2)^* : C_{P,M} \longrightarrow C_{P',M'}$$

sending

$$\begin{pmatrix} Q \\ \downarrow\!\!\!\downarrow \\ P \times M \end{pmatrix} \longmapsto \left((P_0 \times M_0) \times_{(P \times M)} Q \twoheadrightarrow P' \times M' \right).$$

This is well-defined because from the cartesian square

$$\begin{tikzcd} (P_0 \times M_0) \times_{(P\times M)} Q \arrow[r, "\alpha"] \arrow[d] & Q \arrow[d] \\ P_0 \times M_0 & \subset \quad P \times M \end{tikzcd}$$

one sees that $\operatorname{Cok}(\alpha) \simeq P/P_0 \times M/M_0 \in \mathcal{M}(r)$, and since $Q \in \mathcal{M}(r-1)$ it follows that

$$(P_0 \times M_0) \times_{(P\times M)} Q \in \mathcal{M}(r-1).$$

(Recall if you have

$$0 \longrightarrow Q' \longrightarrow \underset{\mathcal{M}(r-1)}{\underset{\cap}{Q}} \longrightarrow \underset{\mathcal{M}(r)}{\underset{\cap}{Q/Q'}} \longrightarrow 0$$

then $Q' \in \mathcal{M}(r-1)$.).

Interpreting $(P_0 \times M_0) \times_{(P\times M)} Q$ as the submodule of Q which is the inverse image of $P_0 \times M_0$ considered as a subobject of $P \times M$, one sees that

$$P, M \longmapsto \mathcal{C}_{P,M}$$

is a functor

$$[\mathcal{S}\mathcal{M}(r-1) \times \mathcal{S}\mathcal{M}(r)]^{\circ} \longrightarrow \mathcal{C}at$$

hence we can form a <u>scinded fibred category</u> $\mathcal{C}$ over $\mathcal{S}\mathcal{M}(r-1) \times \mathcal{S}\mathcal{M}(r)$. Thus the objects of $\mathcal{C}$ are diagrams

$$\begin{array}{c} Q \\ \downarrow \\ P \times M \end{array} \qquad \begin{array}{l} Q, P \in \mathcal{M}(r-1) \\ M \in \mathcal{M}(r) \end{array}$$

and an arrow from $\begin{matrix} Q' \\ \downarrow \\ P'\times M' \end{matrix}$ to $\begin{matrix} Q \\ \downarrow \\ P\times M \end{matrix}$ is

a diagram

$$\begin{tikzcd}
Q' \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & Q \arrow[d] \\
P_0\times M_0 \arrow[d] & P\times M \arrow[l, hook'] \\
P'\times M' &
\end{tikzcd}$$

such that the square is cartesian, and the composite vertical arrow is the given one for Q'.

~~Prop 1. $\mathfrak{C}$ is a fibred cat.~~

Clearly the functors

$$\mathfrak{C} \longrightarrow \mathcal{SM}(r-1) \qquad\qquad \begin{pmatrix} Q \\ \downarrow \\ P\times M \end{pmatrix} \longrightarrow P$$

$$\mathfrak{C} \longrightarrow \mathcal{SM}(r) \qquad\qquad\qquad\qquad\qquad \longrightarrow M$$

are fibrant (fibrant functors are closed under composition, and the projections $pr_i : \mathcal{C}_1\times\mathcal{C}_2 \to \mathcal{C}_i$ are fibrant.) An direct way of seeing this is to argue as follows. Given $M \in \mathcal{SM}(r)$ let $\mathfrak{C}_M$ be the fibre of $\mathfrak{C}$ over M, i.e. the category of $Q \twoheadrightarrow P\times M$ with arrows

$$\begin{tikzcd}
Q' \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & Q \arrow[d] \\
P_0\times M \arrow[d] & P\times M \arrow[l, phantom, "<"] \\
P'\times M &
\end{tikzcd} \qquad\qquad P/P_0 \in \mathfrak{M}(r-1)$$

Then observe that there is a functor

$$M \longmapsto \mathcal{C}_M$$
$$\mathcal{SM}(r)^\circ \longrightarrow \mathcal{C}at$$

<u>Proposition</u>: (i) The category $\mathcal{C}_M$ is contractible
(ii) ——— $\mathcal{C}_P$ is ———.
(for any $P \in \mathcal{SM}(r-1)$, $M \in \mathcal{SM}(r)$).

<u>Proof of (i)</u>: Let $\mathcal{K}_M$ be the category of surjections $Q \twoheadrightarrow M$ with $Q \in \mathcal{M}(r-1)$ whose morphisms

$$(Q' \twoheadrightarrow M) \longrightarrow (Q \twoheadrightarrow M)$$

are injections $\alpha : Q' \longrightarrow Q$ over M such that $\mathrm{Cok}(\alpha) \in \mathcal{M}(r-1)$.

<u>Lemma 1</u>: $\mathcal{K}_M$ is contractible.

<u>Proof</u>: "Cone construction" Let $Q_0 \to M$ be a fixed object (note: We use here the fact that $\mathcal{K}_M \neq \phi$). Then we have functors from $\mathcal{K}_M$ to itself

$$\begin{tikzcd}
 & (Q \to M) \arrow[d] \\
(Q \to M) \arrow[ur, mapsto, "id"] \arrow[r, mapsto] \arrow[dr, mapsto] & (Q \oplus Q_0 \to M) \\
 & (Q_0 \to M) \arrow[u]
\end{tikzcd}$$

where the vertical arrows are natural transformations.

Define the functor

$$f : \mathcal{C}_M \longrightarrow \mathcal{K}_M$$

$$(Q \twoheadrightarrow P \times M) \longmapsto (Q \twoheadrightarrow M)$$

<u>Lemma 2</u>: f <u>is</u> <u>cofibrant</u> with <u>contractible</u> fibres.

<u>Proof</u>: The fibre $f^{-1}\{Q\}$ has for objects pairs (P, ξ) where $P \in \mathcal{M}(r-1)$, & $\xi : Q \longrightarrow P$ is such that $Q \twoheadrightarrow P \times M$.

$$\mathrm{Hom}_{f^{-1}\{Q\}}\left(\begin{matrix} Q \\ \downarrow \\ P' \end{matrix} , \begin{matrix} Q \\ \downarrow \\ P \end{matrix} \right) = \mathrm{Hom}_{Q\backslash}(P, P')$$

which has one element if $P' \ll P$ as a quotient of Q and is empty otherwise. Thus $f^{-1}\{Q\}$ is equivalence to the ordered set of quotients P of Q such that $Q \twoheadrightarrow P \times M$. As this ordered set has a least element 0, the category $f^{-1}\{Q\}$ is contractible.

On the other hand given $\alpha : Q' \longrightarrow Q$ over M with $\mathrm{Cok}(\alpha) \in \mathcal{M}(r-1)$ define

$$\alpha_* : f^{-1}\{Q'\} \longrightarrow f^{-1}\{Q\}$$

by

$$\begin{pmatrix} Q' \\ \downarrow \\ P' \end{pmatrix} \longmapsto \begin{pmatrix} Q \\ \downarrow \\ Q/\alpha(\mathrm{Ker}\, Q' \to P') \end{pmatrix}$$

well-defined: $\alpha(\mathrm{Ker}\, Q' \to P') \twoheadrightarrow M$.

$$0 \longrightarrow \underset{\in \mathcal{M}(r-1)}{P'} \longrightarrow Q/\alpha(\mathrm{Ker}\, Q' \to P') \longrightarrow \underset{\in \mathcal{M}(r-1)}{Q/\alpha Q'} \longrightarrow 0$$

$$\operatorname{Hom}_{f^{-1}\{Q\}}\left(\alpha_*(Q' \to P'), (Q \to P)\right) \cong \begin{cases} \{\phi\} & \text{if } \alpha \operatorname{Ker}(Q' \to P') \supset \operatorname{Ker}(Q \to P) \\ \phi & \text{otherwise} \end{cases}$$

On the other hand, $\exists$ diagram of the form

$$\begin{tikzcd} Q' \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & Q \arrow[d] \\ P_0 \arrow[d] \arrow[r, phantom, "\subset"] & P \\ P' & \end{tikzcd}$$

iff $\alpha\{\operatorname{Ker}(Q' \to P')\} \supset \operatorname{Ker}(Q \to P)$.

<u>Better method.</u> Given $\alpha : Q' \hookrightarrow Q$ in $\mathcal{K}_M$ compute ~~that there is at most one map~~ ~~$(Q' \to P') \to (Q \to P)$ lying over α, and in fact there is one~~ the maps in C_M lying over α:

$$\operatorname{Hom}_{C_M}\left(\begin{matrix} Q' \\ \downarrow \\ P' \end{matrix}, \begin{matrix} Q \\ \downarrow \\ P \end{matrix}\right)_\alpha = \begin{cases} \text{one element} & \alpha \operatorname{Ker}(Q' \to P') \supset \operatorname{Ker}(Q \to P) \\ \phi & \text{otherwise} \end{cases}$$

$$= \operatorname{Hom}_{C_M}\left(\begin{matrix} Q \\ \downarrow \\ Q/\alpha \operatorname{Ker}(Q' \to P') \end{matrix}, \begin{matrix} Q \\ \downarrow \\ P \end{matrix}\right)_{id_Q}.$$

Proof of (ii): Let $\mathcal{K}'_P$ be the category of surjections $Q \twoheadrightarrow P$ with $Q \in \mathcal{M}(r-1)$ whose arrows are injections α $Q' \hookrightarrow Q$ over P such that $\mathrm{Cok}(\alpha) \in \mathcal{M}(r)$.

Lemma 1': $\mathcal{K}'_P$ is contractible.

$$\begin{tikzcd}
Q \arrow[r, "in_1"] \arrow[dr] & Q \oplus Q_0 \arrow[d] & Q_0 \arrow[l, "in_2"'] \arrow[dl] \\
& P &
\end{tikzcd}$$

Lemma 2': The functor $C_P \longrightarrow \mathcal{K}'_P$ sending $Q \twoheadrightarrow P \times M$ to $Q \twoheadrightarrow M$ is cofibrant with contractible fibres.

Proof: Given $\alpha : Q' \hookrightarrow Q$ over P with $\mathrm{Coker}(\alpha) \in \mathcal{M}(r)$ and given $Q' \twoheadrightarrow M'$ in C_P define

$$\alpha_*(Q' \twoheadrightarrow M') = \left(Q \longrightarrow Q/\alpha \operatorname{Ker}(Q' \to M')\right)$$

$$\begin{tikzcd}
Q' \arrow[r, hook, "\alpha"] \arrow[d] & Q \arrow[d] & \\
M' \arrow[r, hook] \arrow[d, phantom, "\in" sloped] & Q/\alpha \operatorname{Ker}(Q' \to M') \arrow[d, phantom, "\in" sloped] & \mathrm{IS} \\
\mathcal{M}(r) & \mathcal{M}(r) & \mathcal{M}(r)
\end{tikzcd}$$

so α_* is well-defined. Next compute

$$\operatorname{Hom}_{\mathcal{C}_P}\left(\begin{matrix} Q' \\ \downarrow \\ M' \end{matrix}, \begin{matrix} Q \\ \downarrow \\ M \end{matrix}\right)_{\alpha} \cong \begin{cases} \{\phi\} & \text{if } \alpha\,\mathrm{Ker}(Q' \to M') \supset \mathrm{Ker}(Q \to M) \\ \phi & \text{otherwise.} \end{cases}$$

$$\cong \operatorname{Hom}_{\mathcal{C}_P}\left(\alpha_*\begin{pmatrix} Q' \\ \downarrow \\ M' \end{pmatrix}, \begin{matrix} Q \\ \downarrow \\ M \end{matrix}\right)_{id_Q}.$$

Thus it is cofibred. The fibre over Q is the category of $Q \to M$ such that $M \in \mathfrak{M}(r)$ and $Q \twoheadrightarrow P \times M$, where M' maps to M iff $M' \ll M$ as quotient objects. This category has the initial object 0.

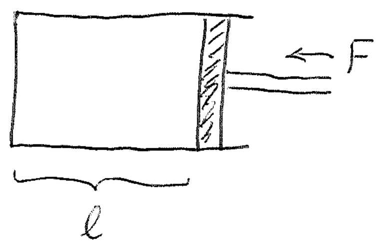
At this stage we have homotopy equivalences

$$\begin{tikzcd} & \mathcal{C} \arrow[dl] \arrow[dr] & \\ S\mathfrak{M}(r-1) & & S\mathfrak{M}(r) \end{tikzcd}$$

and the problem remains to show the homotopy equivalence between $S\mathfrak{M}(r-1)$ and $S\mathfrak{M}(r)$ thus obtained also comes from the inclusion functor.

April 15, 1972. education in statistical mechanics

Ideal gas.



area of piston $= A$ $cm.^2$

force $= F$ $gr\ cm/sec^2$

pressure $= P = F/A$ $gr/cm\ sec^2$

The gas is made up of many particles. Suppose we consider one particle; ~~let~~ m be its mass, and v the magnitude of the x-component of its velocity. It hits the piston once every $2l/v$ sec. imparting momentum $2mv$ each collision. Total momentum/sec imparted to piston by this particle is

$$2mv \cdot \frac{v}{2l} = \frac{mv^2}{l}$$

Recall that (general formula)

$$\int_a^b F\,dt = \int_a^b m\frac{dv}{dt} = [mv]_a^b$$

$$= \text{change in momentum.}$$

Thus force of the gas on the piston is

$$F = \sum_i \frac{m_i v_i^2}{l} \qquad \text{sum over particles}$$

$$= \frac{2}{l} E_x$$

where E_x is the x-part of the kinetic energy. For symmetry reasons $E_x = \frac{1}{3}E$ so we get

$$PA\ell = \frac{2}{3} E$$

or

$$\boxed{PV = \frac{2}{3} E}$$

$P =$ pressure
$V =$ volume
$E =$ kinetic energy of gas

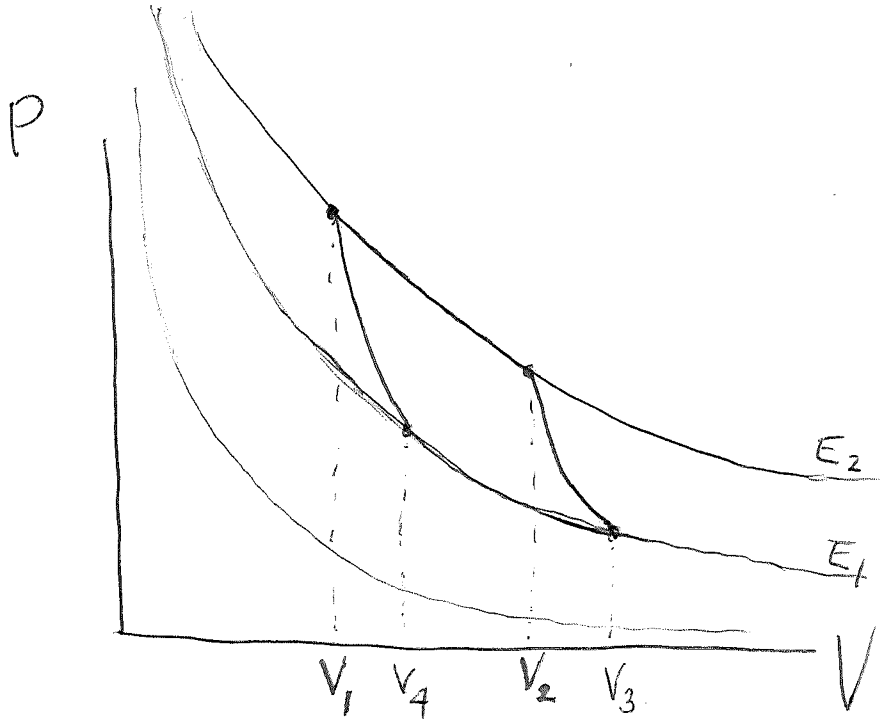
<u>Carnot cycle</u>:

Stage 1: (isothermal reversible expansion!) Heat the gas and allow the piston to expand in such a way that the internal energy doesn't change. Thus we transform a quantity q_2 of heat into work at constant internal energy E_2.

<u>Stage 2</u>: (adiabatic rev. expansion). Allow gas to expand so that internal energy goes down from E_2 to E_1, and work done is $E_2 - E_1$.

<u>Stage 3</u>: (isothermal rev. compression). Volume decreases without changing internal energy. Cool gas.

<u>Stage 4</u>: (adiabatic rev. compression). Piston moves in so that internal energy goes from E_1 to E_2. Work done by gas $= -(E_2 - E_1)$.



1) $PV = \frac{2}{3} E_2$ V goes from V_1 to V_2

2) work done by gas is PdV always.

want

$$PdV + dE = 0$$

i.e.

$$PdV + \frac{3}{2} d(PV) = 0$$

$$\frac{5}{2} PdV + \frac{3}{2} VdP = 0$$

$$5 \frac{dV}{V} + 3 \frac{dP}{P} = 0$$

$$V^5 P^3 = \text{Constant}$$

$$\boxed{P = (\text{Constant})\, V^{-5/3} \qquad \text{adiabatic}}$$

Digression: The quantities P and V describe the state of the gas. Other quantities such as E = internal energy, T = temperature are functions of P, V which are independent variables.

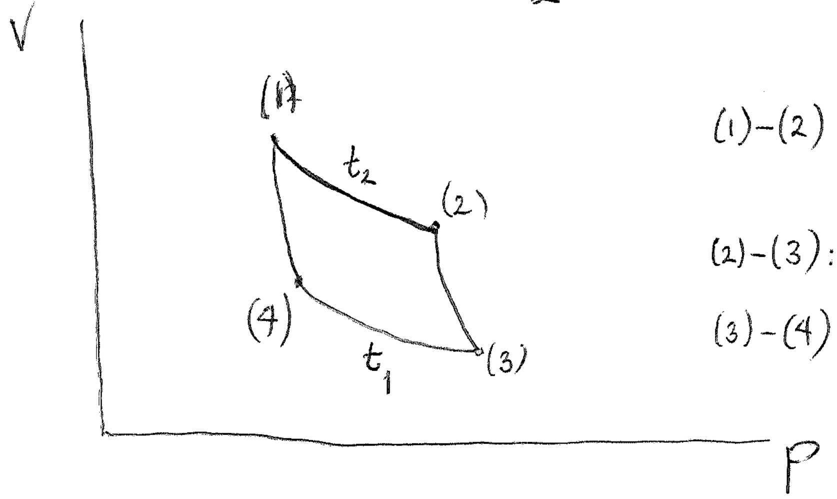
If a change dP, dV is produced in the gas, then PdV is the work done by the gas, so

$$PdV + dE$$

is the heat added to the gas during the change. This differential is misleadingly denoted dq.

Carnot cycles make sense for non-ideal gases, where it will not be true that E and T are proportional. I review the theory:

Run a Carnot cycle between two temperatures t_1 and t_2:



(1)–(2): heat q_2 added, work w_1 done

(2)–(3): work w_2 done

(3)–(4): heat $-q_1$ added, work $-w_3$ done

(4)–(1): work $-w_4$ done

$$q_2 - q_1 = w_1 + w_2 - w_3 - w_4$$

The efficiency of the engine =

$$\frac{\text{work done}}{\text{heat used up}} = \frac{q_2 - q_1}{q_2}$$

Claim this is the same for any other engine working between the same 2 temperatures by the Second Law of Therm. Granted this

$$\frac{q_2 - q_1}{q_2} = f(t_1, t_2)$$

or

$$\frac{q_1}{q_2} = \bar{f}(t_1, t_2) = 1 - f(t_1, t_2)$$

so

$$\bar{f}(t_1, t_3) = \frac{q_1}{q_2} \cdot \frac{q_2}{q_3} = \bar{f}(t_1, t_2)\bar{f}(t_2, t_3)$$

i.e.

$$\bar{f}(t_1, t_2) = \frac{F(t_1)}{F(t_2)}$$

$F(t)$ unique up to a constant

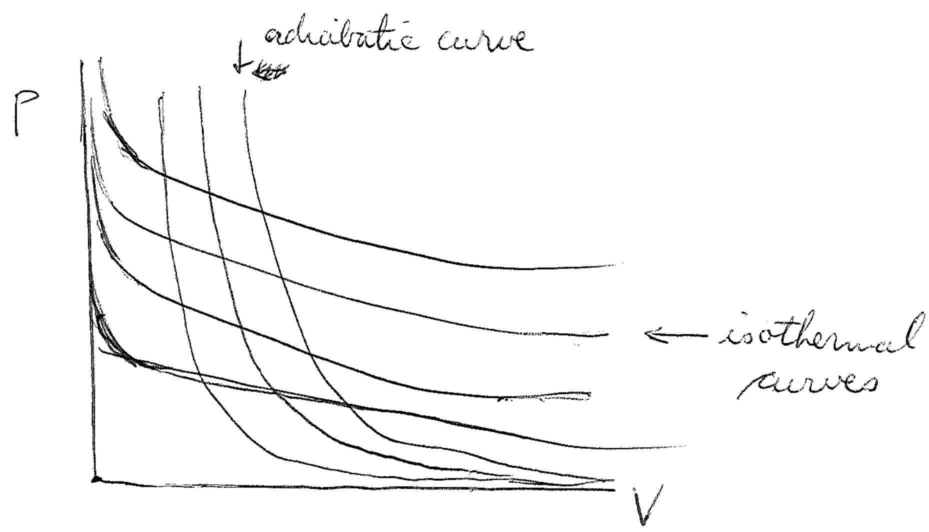
So take $F(t) = T$ Kelvin temperature function so that

$$\frac{q_1}{q_2} = \frac{T_1}{T_2}.$$

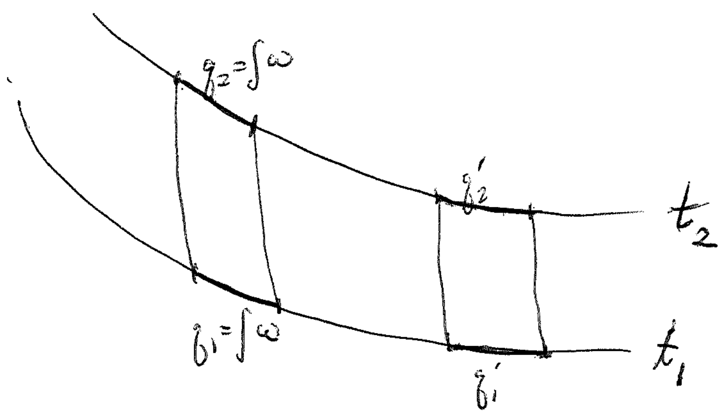
Meaning: Let $\omega = PdV + dE$, so that

$$\int_C \omega$$

is the amount of heat that has to be added to the system as it traverses the curve C. The curves on which ω vanishes are the adiabatic curves.



The point of the 2nd law is that given two isothermal curves



the ratio of the measures on these curves, with respect to the correspondence induced by the adiabatic curves, is constant $= T_2/T_1$, where T = Kelvin temperature. Thus

$$\frac{\omega}{T} = dS$$

is an exact differential. S is called the entropy.

Back to ideal gas:

$$\omega = P\,dV + dE = P\,dV + \frac{3}{2}\,d(PV)$$

$$= \frac{5}{2}P\,dV + \frac{3}{2}V\,dP = (PV)\,d\left(\frac{5}{2}\ln V + \frac{3}{2}\ln P\right)$$

Actually T is not determined by property of being an integrating factor for ω (can be changed by any function of S). So put in other input:

$$PV = RT \qquad\qquad E = \frac{3}{2}RT$$

whence

$$S = \frac{R}{2}\ln(P^3V^5)$$

(Observe: T does not seem to come out of the kinetic theory of the ideal gas. ~~[illegible]~~ It seems necessary to give one isothermal curve in order to determine T.)

April 16, 1972

Suppose I have a classical mechanical system described by a Riemannian manifold X with a potential function V. I wish to do statistical mechanics corresponding to this system.

A state for the classical system is a point in the cotangent bundle $M = T^*_X$, whose evolution in time is described by the Hamiltonian vector field associated to the Hamiltonian

$$H = E + V$$

Example: The simple harmonic oscillator with

$$H = \tfrac{1}{2}(p^2 + q^2)$$

$q : X \xrightarrow{\sim} \mathbb{R}$ and $p(adq) = a$. Then the equations of motion are

$$\dot{p} = \{H, p\} = -q$$

$$\dot{q} = \{H, q\} = p$$

In general

$$\dot{f} = \{H, f\} = \sum_i \frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial f}{\partial p_i}$$

A state for the statistical system should be a probablity measure on the Hamiltonian manifold M. Thus a point of M gives rise to the Dirac measure, and n-points to the average of their Dirac measures. It is clear how probability measures change in time, in fact, if dV is the canonical volume on M, then any measure is of the form

$$fdV \qquad\qquad f \text{ a "distribution"}$$

whose time derivative is $\{H,f\}dV$.

The basic idea: Take a large number N of points in M and let them run almost independently of each other. "Almost independently" means that there is enough interaction to transfer energy between the particles, but not enough to include in the calculations. Then after "large" time the system reaches "equilibrium". The equilibrium state is described by the Maxwell-Boltzmann-Gibbs measure on M:

$$\frac{e^{-H/kT}\,dV}{Z}$$

$$Z = \int_M e^{-H/kT}\,dV$$

Observe T is something new that enters after we understand what equilibrium means.

To make this work, one would want to take a large number N of independent systems, described by the Hamiltonian manifold M^N and introduce "interaction" which might be a perturbation of the Hamiltonian. Another possibility would be to perturb the closed 2 form on M^N. Then one would want to take a limit to get the equilibrium distribution.

Examples. Consider the simple harmonic oscillator and take N identical systems. The Hamiltonian is

$$H = \sum_{i=1}^{N} \tfrac{1}{2}(p_i^2+q_i^2)$$

Now suppose the total energy of the system is E. Suppose $f(q,p)$ is a function on M and I am interested in the average value of f for the N-identical systems, given that the total energy is E. Thus I want to evaluate

$$\int_{H=E} \frac{1}{N}\sum_{1}^{N} f(q_i,p_i)$$

the integral being taken over the hypersurface $H=E$ with respect to the natural prob. measure induced by $dV = \prod dq_i\, dp_i$ on that hypersurface.

Start with the Liouville measure $\omega = \prod dq_i\, dp_i$

and write it

$$\omega = dH \wedge \eta$$

so that η is determined up to $f\,dH$, hence η has a well-defined restriction to any of the surfaces $H =$ constant, and

$$\int_{M^N} F\,\omega = \int_0^\infty dE \int_{H=E} F\eta$$

As a start take $F = f(H)$ so that

$$\int_{M^N} f(H)\,\omega = \int_0^\infty f(E)\,v(E)\,dE$$

where

$$\cancel{v(E) = \int_{H=E} \eta = \text{volume of}}$$

$$v(E)\,dE = H_*(\omega) \quad = \text{volume of phase space between } H=E \text{ and } H=E+dE.$$

In the example

$$\int_{M^N} e^{-sH}\,\omega = \prod_1^N \int_{\mathbb{R}^2} e^{-s(p^2+q^2)/2}\,dq\,dp$$

and

$$\int_{\mathbb{R}^2} e^{-sr^2/2}\,r\,dr\,d\theta = \frac{2\pi}{-s}\int_0^\infty e^{-sr^2/2}(-sr\,dr)$$

$$= \frac{2\pi}{-s}\left[e^{-sr^2/2}\right]_0^\infty = \frac{2\pi}{s}$$

Thus $$\int_0^\infty e^{-sE}\, v(E)\,dE = \left(\frac{2\pi}{s}\right)^N$$

Recall that

$$\Gamma(k) = \int_0^\infty e^{-t}\, t^{k-1}\,dt$$

$$\Gamma(k) = \int_0^\infty e^{-st}\,(st)^{k-1}\, s\,dt \qquad s>0$$

$$\frac{\Gamma(k)}{s^k} = \int_0^\infty e^{-sE}\, E^{k-1}\,dE$$

$$\frac{(2\pi)^N}{s^N} = \int_0^\infty e^{-sE}\, \frac{(2\pi)^N E^{N-1}}{(N-1)!}\,dE$$

and so

$$v(E)\,dE = \frac{(2\pi)^N E^{N-1}}{(N-1)!}$$

Now I want to evaluate

$$\frac{1}{N}\int_{H=E} \sum_{i=1}^{n} f(q_i,p_i)\,\eta = \frac{1}{N}\sum \int_{H=E} f(q_i,p_i)\,\eta$$

$$= \int_{H=E} f(q_1,p_1)\,\eta$$

by symmetry considerations. Thus for each E I get a measure on the Hamiltonian manifold M, the direct image of the measure on $H=E$ induced by Liouville with respect to the projection $pr_1 : M^N \longrightarrow M$

In the example

$$\int_0^\infty e^{-sE}\, dE \int_{H=E} (pr_1^* f)\, \eta \;=\; \int_{M^N} e^{-sH}\, pr_1^* f\ \omega$$

$$= \int_{\mathbb{R}^2} e^{-s(p_1^2+q_1^2)/2} f(q_1,p_1)\, dq_1\, dp_1 \cdot \prod_1^{N-1} \int_{\mathbb{R}^2} e^{-s(p^2+q^2)/2}\, dp\, dq$$

$$= \left(\frac{2\pi}{s}\right)^{N-1} \int_{\mathbb{R}^2} e^{-s(q^2+p^2)/2} f(q,p)\, dq\, dp$$

Now take

$$f(q,p)\, dq\, dp \;=\; \delta \text{ measure at } (q_0,p_0)$$

and this becomes

$$\left(\frac{2\pi}{s}\right)^{N-1} e^{-s(q_0^2+p_0^2)/2}$$

$$= \int_0^\infty e^{-sE}\, (?)\, dE$$

To evaluate (?) recall that

$$\cancel{\int_0 e^{-sE}\, \delta(E-k)\, dE = e^{-sk}} \qquad \cancel{k>0}$$

$$\int_0^\infty e^{-sE} E^n \, dE = \frac{n!}{s^{n+1}}$$

$$\int_0^\infty e^{-s(E+k)} E^n \, dE = \frac{n! \, e^{-sk}}{s^{n+1}}$$

$$\parallel$$

$$\int_k^\infty e^{-sE} (E-k)^n \, dE = \int_0^\infty e^{-sE} \begin{Bmatrix} 0 & E<k \\ (E-k)^n & E \geq k \end{Bmatrix} dE$$

Therefore

$$(?) = (2\pi)^{N-1} \begin{cases} 0 & E < (q_0^2+p_0^2)/2 \\ \dfrac{\left(E - (q_0^2+p_0^2)/2\right)^{N-2}}{(N-2)!} & E \geq (q_0^2+p_0^2)/2 \end{cases}$$

Thus

$$\boxed{\cancel{(pr_1)_* \eta|_{H=E}}}$$

$$(pr_1|_{H=E})_*(\eta) = (2\pi)^{N-1} \frac{\left(E - \frac{1}{2}(q^2+p^2)\right)^{N-2}}{(N-2)!} \operatorname{Heav}\left(E - \tfrac{1}{2}(p^2+q^2)\right) \cdot dq\,dp$$

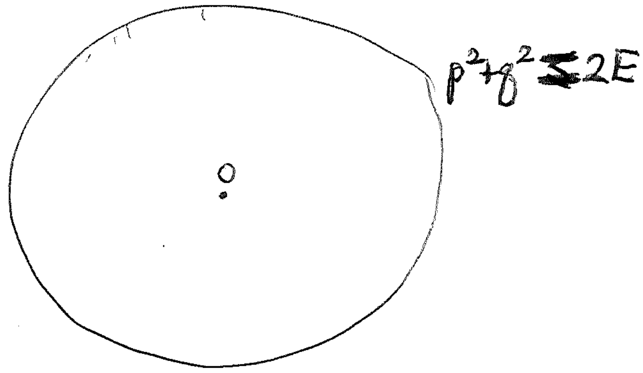
Now I want to divide this by the measure of the hypersurface $H=E$, which is

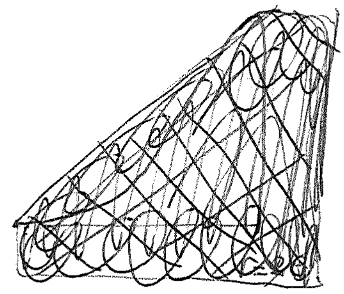
$$\frac{(2\pi)^N E^{N-1}}{(N-1)!}$$

by previous calculations. Thus I get

$$\frac{N-1}{2\pi} \frac{\left(E - \frac{1}{2}(q^2+p^2)\right)^{N-2}}{E^{N-1}} \, \text{Heav}\left(E - \tfrac{1}{2}(q^2+p^2)\right) \, dq\, dp$$

This is a probability measure in the plane





concentrated inside the circle $p^2+q^2 = 2E$. Rewrite

$$\frac{N-1}{2\pi E} \left(1 - \frac{r^2}{2E}\right)^{N-2} \text{Heav}\left(1 - \frac{r^2}{2E}\right) \; r\,dr\,d\theta$$

Note: As $N \to \infty$, this converges to the δ-function at the origin, because for $r > 0$ the exponential factor will cancel the $N-1$. This could have been seen a priori – any single particle has average energy $E/N \to 0$ as $N \to \infty$.

So try a different limit. Let $E, N \to \infty$ so that $E/N = \lambda$, so that the average energy of a particle is λ. Then look what happens to the above distribution in the limit:

$$\lim_{N\to\infty} \frac{N-1}{2\pi\lambda N}\left(1-\frac{r^2}{2\lambda N}\right)^{N-2} \mathrm{Heav}\left(1-\frac{r^2}{2\lambda N}\right) r\,dr\,d\theta$$

$$= \frac{1}{2\pi\lambda} e^{-r^2/2\lambda}\, r\,dr\,d\theta \qquad !$$

It should be possible to derive this in general.
Thus suppose I single out the first variable and put

$$\omega = \omega_1 \cdot \omega' \qquad (\omega_1 = dq_1\, dp_1 \text{ in example})$$
$$H = H_1 + H'$$

$$\omega' = dH'\,\eta'$$
$$\omega_1 = dH_1\,\eta_1$$

so that
$$\omega = \omega_1\,\omega' = dH_1\,\eta_1\, dH'\,\eta'$$
$$= \pm d(H_1+H')\,\underbrace{dH'\,\eta_1\,\eta'}_{\pm\eta}$$

Thus

$$\int_{H=E} f_1\,\eta = \pm\int_{H_1+H'=E} f_1\, dH'\,\eta_1\,\eta'$$

$$= \int_{0\le E_1\le E} \left(\int_{H_1=E_1} f_1\,\eta_1\right)\left(\int_{H'=E-E_1}\eta'\right) dE_1$$

so recall $E = N\lambda$ where $N \longrightarrow \infty$, and we wish to compute what happens to

$$\frac{\displaystyle\int_{H=E} f_1 \eta}{\displaystyle\int_{H=E} \eta} = \int_0^E dE_1 \left(\int_{H_1=E_1} f_1 \eta_1 \right) \left(\frac{\displaystyle\int_{H'=E-E_1} \eta'}{\displaystyle\int_{H=E} \eta} \right)$$

In the example considered before

$$\int_{H=E} \eta = \frac{(2\pi)^N E^{N-1}}{(N-1)!}$$

$$\int_{H'=E-E_1} \eta' = \frac{(2\pi)^{N-1} (E-E_1)^{N-2}}{(N-2)!}$$

and the ratio is

$$\frac{1}{2\pi} \frac{(N-1)}{E} \left(1 - \frac{E_1}{E}\right)^{N-2}$$

$$= \frac{1}{2\pi} \left(\frac{N-1}{N\lambda}\right) \left(1 - \frac{E_1}{\lambda N}\right)^{N-2} \longmapsto \frac{1}{2\pi\lambda} e^{-E_1/\lambda}$$

The ratio I am after is the probability that the first particle has its energy between E_1 and $E_1 + dE_1$ given that the total energy is E.

In general we could ask the following: Given N identical independent variables $X_1, \ldots, X_N$, what is the distribution of X_1 given that $\sum X_i = E$?

Suppose we have random variables X, Y giving a prob. measure $\mu(x,y)\,dx\,dy$ on the plane. Fixing $x = x_0$ we get the conditional probability distribution

$$\frac{\mu(x_0, y)\,dy}{\int_{-\infty}^{\infty} \mu(x_0,y)\,dy}$$

Its characteristic function is

$$\frac{\int_{-\infty}^{\infty} e^{ity}\,\mu(x_0,y)\,dy}{\int_{-\infty}^{\infty} \mu(x_0,y)\,dy} = \frac{\phi(t)}{\phi(0)}$$

where

$$\phi(t) = \int_{-\infty}^{\infty} e^{ity}\,\mu(x_0,y)\,dy$$

Now take r.v. $X_1, \dots, X_N$, identical + independent and put

$$X = X_1 + \dots + X_N$$
$$Y = X_1$$

so that the char. function of μ is

$$\iint e^{isx+ity}\,\mu(x,y)\,dx\,dy = \int \cdots \int e^{is(X_1+\dots+X_N)+it(X_1)}\,d\mu_1 \cdots d\mu_N$$

$$= \varphi(s+t)\,\varphi(t)^{N-1} \qquad \text{where}$$

$$\varphi(s) = \int e^{isX} d\mu$$

To get $\psi(t)$ use the inverse Fourier transform

$$\psi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx_0}\, \varphi(s+t)\varphi(t)^{N-1}\, ds$$

The the characteristic function of X_1 subject to the condition that $X_1 + \cdots + X_N = E$ is

$$\frac{\psi(t)}{\psi(0)} = \frac{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isE}\, \varphi(s+t)\, \varphi(t)^{N-1}\, ds}{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isE}\, \varphi(s)\, ds}$$

I have a slightly different problem because the measure on phase space is not a prob. measure. But nevertheless let

$$x = H_1 + \cdots + H_N$$
$$y = H_1$$

and let $\mu(x,y)\,dx\,dy$ be the image measure so that

$$\iint e^{isx+ity}\, \mu(x,y)\,dx\,dy = \int_{M^N} e^{is(H_1+\cdots+H_N)} e^{itH_1}\, dL$$

$$= \varphi(s+t)\,\varphi(t)^{N-1} \quad \text{where}$$

$$\varphi(s) = \int_M e^{isH} dL \qquad \mathrm{Im}(s) > 0$$

What I want is the prob. measure

$$\frac{\mu(E,y)\,dy}{\int_{-\infty}^{\infty} \mu(E,y)\,dy}$$

~~[illegible]~~ and I should be able to get this by Fourier inversion

$$\mu(E,y) = \frac{1}{(2\pi)^2} \iint e^{-isE-ity} \varphi(s+t)\varphi(t)^{N-1} ds\,dt$$

Put $E = N\lambda$?

Gibbs point of view:

Take a large number N of copies of the system with energy $E = N\lambda$, $\lambda =$ average energy per particle. Average dynamical quantities over phase space. Thus if f is a function on the phase space M of the system we consider

$$\int_{\sum H_i = E} f(pr_1)$$

which gives the expected value of f for the first particle (hence any particle). Claim is that as $N \longrightarrow \infty$ this gives [redacted] us the Gibbs distribution

$$\frac{e^{-H/\lambda}\, d(\text{Liouville})}{\int e^{-H/\lambda}\, d\text{Liouville}} .$$

Possible ideas to use:

1. Replacing an integral over a simplex $\sum x_i = 1$ by some sort of characteristic function
2. Replacing $n!$ by Γ
3. Entropy and how it arises from the Γ replacement for factorials.

April 19, 1972.

Candidate for the exact sequence K-theory:

Consider the groupoid of v.b. over a scheme S and denote it $\mathcal{M}$. Form the free monoid:

$$\mathcal{C} = \coprod_{n \geq 0} \mathcal{M}^n$$

~~and consider the functor $\mathrm{Filt}: \mathcal{C} \longrightarrow \mathrm{Sets}$~~

$$\mathrm{Filt}(E_1, \dots, E_n) = \mathrm{Filt}(E_1) \times \mathrm{Filt}(E_2) \times \dots \times \mathrm{Filt}(E_n)$$

~~where an element of $\mathrm{Filt}(E)$ is a filtration of E by sub-vector-bundles. Next define~~

$$\mathcal{C}_{/F} \longrightarrow \mathcal{C}$$

$$\{E_i, \Phi_i\} \longmapsto gr\{E_i, \Phi_i\}.$$

~~Messy: What I am trying to describe is~~

Let $\mathcal{C}'$ be the category with the same objects as $\mathcal{C}$, but in which an arrow

$$(E_1', \dots, E_m') \longleftarrow (E_1, \dots, E_n)$$

consists of filtrations

$$0 = E_{10} \subset E_{11} \subset \cdots \subset E_{1j_1} = E_1$$

$$\cdots\cdots$$

$$0 = E_{n0} \subset E_{n1} \subset \cdots \subset E_{nj_n} = E_n$$

and isomorphisms

$$E_{ij}/E_{i,j-1} \simeq E'_a \qquad \begin{array}{l} 1 \le i \le n \\ 0 < j \le j_i \end{array}$$

where $\quad a = j_1 + \cdots + j_{i-1} + j.$

$$m = j_1 + \cdots + j_n$$

Thus there is a functor

$$\mathcal{C} \longrightarrow \mathcal{C}'$$

and $\mathcal{C}'$ is ~~the~~ analogous to an étale topological category with object space $\mathcal{C}$.

My idea is that the category $\mathcal{C}'$ should be the monoid generated by vector bundles subject to the relations generated by exact sequences. Group-completing $\mathcal{C}'$ might yield the desired space giving the K-theory of S. Hopefully one can compute the homology of $\mathcal{C}'$.

Try $\mathbb{P}_1$ over a field k. Consider line bundles. The non-degenerate part is simply

$$\coprod_{n \in \mathbb{Z}} k^* .$$

Now consider rank 2 bundles. Then $\mathcal{C}'$ has **two** kinds of non-degenerate ~~[illegible]~~ objects:

vector bundles of rank 2 $\qquad E \simeq \mathcal{O}(n) \oplus \mathcal{O}(m) \qquad n \geq m$

pairs of line bundles $\qquad (L_1, L_2) \simeq (\mathcal{O}(n), \mathcal{O}(m)) \qquad n, m \in \mathbb{Z}$.

morphisms

$$\mathrm{Aut}\,(\mathcal{O}(n) \oplus \mathcal{O}(m)) = \begin{cases} GL_2(k) & n = m \\ k^* \times k^* \times \Gamma(\mathcal{O}(n-m)) & n > m \end{cases}$$

$$\mathrm{Aut}\,(\mathcal{O}(n), \mathcal{O}(m)) = k^* \times k^* \qquad \text{all } n, m$$

We must also worry about filtered bundles

$$0 \longrightarrow L' \hookrightarrow E \twoheadrightarrow L'' \longrightarrow 0$$

and their isos. Given E the possible L' are points of $\mathbb{P}_1(E \otimes_{\mathcal{O}} K)$. Take $E = \mathcal{O}(n) \oplus \mathcal{O}(m)$ $\quad n > m$.
There are three kinds:

1) $\qquad \mathcal{O}(n) \oplus \mathcal{O}(m) \twoheadrightarrow \mathcal{O}(q) \qquad q > n$

Here $\mathcal{O}(m) \longrightarrow \mathcal{O}(n) \oplus \mathcal{O}(m) \longrightarrow \mathcal{O}(q)$ is non-zero as $\mathcal{O}(n)$ cannot map onto $\mathcal{O}(q)$. Thus the kernel of $\twoheadrightarrow \mathcal{O}(q)$ is generically complementary to $\mathcal{O}(m)$, hence stabilizer will be $\underline{k^*}$.

2) $\qquad \mathcal{O}(n) \oplus \mathcal{O}(m) \twoheadrightarrow \mathcal{O}(n) \qquad$ Stabilizer $\mathrm{Aut}\,(\mathcal{O}(n) \oplus \mathcal{O}(m))$

3) $\qquad \mathcal{O}(n) \oplus \mathcal{O}(m) \twoheadrightarrow \mathcal{O}(m) \qquad$ Stabilizer $k^* \times k^*$.