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March 1, 1972

Infinitesimal form of cocycle:

Consider first the problem of integrating a vector field $v = a(x)\frac{d}{dx}$ on $\mathbb{R}$ (with compact support; i.e. $a(x) = 0$ for $|x|$ large) to a flow

$$t \longmapsto F_t : \mathbb{R} \longrightarrow \mathbb{R}.$$

Thus for x fixed, the tangent vector to the path $t \mapsto F_t(x)$ should be the vector field v at $F_t(x)$:

$$\frac{d}{dt} f(F_t(x)) = a(F_t(x)) \frac{df}{dx}(F_t(x))$$

for all functions f on the line. Taking $f = \mathrm{id}$

$$\begin{cases} \frac{d}{dt} F_t(x) = a(F_t(x)) \\ F_0(x) = x \end{cases}$$

this separates, so reduces to an integration.

This differential equation determines F_t. (Check that F is a flow: $F_{t+s} = F_t \circ F_s$. But

$$\frac{d}{dt} F_{t+s}(x) = \frac{d}{d(t+s)} F_{t+s}(x)$$

$$= a(F_{t+s}(x))$$

and $$\frac{d}{dt} F_t(F_s(x)) = a(F_t(F_s(x)))$$

so have two solutions of the initial value problem

$$\frac{d}{dt} U_t = a(U_t)$$
$$U_0 = F_s(x)$$

hence they must be equal.)

So let's now solve the D.E. to the second order

$$F_t(x) = x + t \cdot \frac{d}{dt} F_t(x)\Big|_{t=0} + \frac{t^2}{2} \frac{d^2}{dt^2} F_t(x)\Big|_{t=0}$$

$$\frac{d^2}{dt^2} F_t(x) = \frac{d}{dt} a(F_t(x)) = a'(F_t(x)) \frac{dF_t(x)}{dt}$$
$$= a'(F_t(x))\, a(F_t(x)) = (aa')(F_t(x))$$

Thus

$$F_t(x) = x + t\,a(x) + \frac{t^2}{2}(aa')(x) + O(t^3)$$
$$= e^{ta\frac{d}{dx}} x = \sum_{n\geq 0} \frac{t^n}{n!}\left(a\frac{d}{dx}\right)^n \cdot x$$

Now consider the cocycle on page 17 with

$$g_{vu}(x) = x + ta(x) + \frac{t^2}{2}(aa')(x) + O(t^3)$$
$$g_{wu}(x) = x + tb(x) + \frac{t^2}{2}(bb')(x) + O(t^3)$$

and determine the leading terms as $t \to 0$.

$$\frac{\log(1+u)}{u} = \frac{u - \frac{u^2}{2} + \frac{u^3}{3}}{u} = 1 - \frac{u}{2} + O(u^2)$$

$$g'_{vu}(x) = 1 + t a'(x) + \frac{t^2}{2}(aa')'(x) + \cdots$$
$$g'_{wu}(x) = 1 + t b'(x) + \frac{t^2}{2}(bb')'(x) + \cdots$$
$$g''_{vu}(x) = t a''(x) + \frac{t^2}{2}(aa')''(x) + \cdots$$
$$g''_{wu}(x) = t b''(x) + \frac{t^2}{2}(bb')''(x) + \cdots$$

$$\frac{\log g'_{vu}}{g'_{vu} - 1} = 1 - \frac{1}{2}[ta'] + O(t^2)$$

$$\frac{\log g'_{wu}}{g'_{wu} - 1} = 1 - \frac{1}{2}[tb'] + O(t^2)$$

$$g'_{wu} - g'_{vu} = t(b' - a') + O(t^2)$$

Thus the leading term of the expression in brackets at bottom of page 17 is

$$\frac{-\frac{1}{2}ta' + \frac{1}{2}tb'}{t(b'-a')} = \frac{1}{2}$$

and

$$\begin{vmatrix} g'_{vu} - 1 & g'_{wu} - 1 \\ g''_{vu} & g''_{wu} \end{vmatrix} = \begin{vmatrix} ta' & tb' \\ ta'' & tb'' \end{vmatrix}$$

$$= t^2(a'b'' - a''b') + O(t^3)$$

Thus we get the infinitesimal cocycle

$$\lambda\left(a\frac{d}{dx}, b\frac{d}{dx}\right) = \frac{1}{2}\int_{-\infty}^{\infty}(a'b''-a''b')\,dx$$

$$= \int_{-\infty}^{\infty} a'b''\,dx$$

clearly skew-symmetric by integration by parts.

Check this is a cocycle on the Lie algebra of vector fields on R with compact support: This means it satisfies the Jacobi identity

$$\lambda([X,Y],Z) + \lambda([Y,Z],X) + \lambda([Z,X],Y) = 0 \quad ?$$

$$X = a\frac{d}{dx}, \quad Y = b\frac{d}{dx}, \quad Z = c\frac{d}{dx}$$

$$[X,Y] = (ab'-ba')\frac{d}{dx}$$

$$\lambda([X,Y],Z) = \int (ab'-ba')'c''\,dx = \int (ab''c''-ba''c'')\,dx$$

$$\lambda([Y,Z],X) = \int (bc''a''-cb''a'')\,dx$$

$$\lambda([Z,X],Y) = \int (ca''b''-ac''b'')\,dx$$

so it is indeed a 2-cocycle.

(This is essentially the Gelfand-Fuchs 2-cocycle for vector fields on S^1. They claim that $\int \begin{vmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{vmatrix} dx$ is a 3-cocycle.)

March 2, 1972

I want to understand the Lie algebra extension defined by this cocycle. Thus if $\mathfrak{g}$ is the Lie alg. of vector fields on $\mathbb{R}$ with compact support, the extension is $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}$ with bracket

$$[X+\alpha, Y+\beta] = [X,Y] + \lambda(X,Y).$$

Hence to realize $\tilde{\mathfrak{g}}$ concretely what we want to do is associate to each $X \in \mathfrak{g}$ an operator $A(X)$ satisfying the commutation relations

$$[A(X), A(Y)] = A([X,Y]) + \lambda(X,Y)$$

$\lambda(X,Y)$ being viewed as a scalar operator.

Idea: Let M be a symplectic manifold, ω the canonical closed non-degenerate 2-form. Then

$$\theta(X)\omega = 0 \iff d\,i(X)\omega = 0.$$

so there is a 1-1 correspondence between Hamiltonian vector fields on M and closed 1-forms. In particular to each f on M belongs X_f $\ni$ $i(X_f)\omega = df$. The Poisson bracket of two functions f, g is defined by

$$\{f,g\} = X_f g = i(X_f)dg = i(X_f)i(X_g)\omega.$$

Then

$$d\{f,g\} = d\,i(X_f)dg = \theta(X_f)dg = \theta(X_f)i(X_g)\omega$$
$$= i([X_f, X_g])\omega.$$

so that $f \longmapsto X_f$ is a Lie homomorphism. Thus we get a Lie algebra extension

$$0 \longrightarrow \mathbb{R} \longrightarrow \left\{\begin{matrix}\text{functions} \\ \text{under} \\ \text{Poisson bracket}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{Hamiltonian} \\ \text{vector fields}\end{matrix}\right\} \longrightarrow 0$$

(these are sheaves of Lie algebras). I might hope to induce the extension $\tilde{\mathfrak{g}}$ above by making $\mathfrak{g}$ act as Hamiltonian vector fields on some symplectic manifold. The obvious candidate, $M =$ cotangent bundle of $\mathbb{R}$ doesn't work. In effect if

$$v = a(x)\frac{d}{dx}$$

is a vector field on $\mathbb{R}$, then the induced vector field $\tilde{v}$ on the cotangent bundle can be shown to be

$$\cancel{X_{ap} = a'p\frac{\partial}{\partial q} + a\frac{\partial}{\partial p}\Big)\, dp\,dq}$$

the Hamiltonian vector field X_{ay} where $y : M \to \mathbb{R}$ sends $a\,dx$ to a. Formulas:

$$\text{canonical 1-form on } M \quad = \quad y\,dx$$

$$\text{2-form on } M \quad \omega = dy\,dx$$

$$\tilde{v} = a\frac{\partial}{\partial x} - ya'\frac{\partial}{\partial y}$$

$$i(\tilde{v})\,\omega = +a\,dy + ya'\,dx = d(ay)$$

Thus in this example $v \longmapsto \tilde{v}$ lifts to the Lie algebra of functions.

Further possibilities:

- Is extension

$$0 \longrightarrow \mathbb{R} \longrightarrow \text{functions} \longrightarrow \underset{\text{v.f.}}{\text{Hamilt.}} \longrightarrow 0$$

non-trivial on the formal level (formal power series at a point)? Probably, otherwise the extension would be trivial for a canonical reason. In fact {Hamiltonian vector fields} is probably a perfect Lie algebra and so this extension would have to split canonically if it split.

In fact there is a distinguished linear ~~local~~ section. Assign to v the unique f with $X_f = v$ such that $f(0)=0$. Then the cocycle is going to be given by

$$v, w \longmapsto [\tilde{v}, \tilde{w}] - \widetilde{[v,w]}.$$

Review of Kostant's theory: Let a Lie algebra $\mathfrak{g}$ act on a symplectic manifold (M,ω). Then to each $x \in \mathfrak{g}$ ~~[illegible]~~ we have a Hamiltonian vector field v_x on M. Exact sequence of Lie algs.

$$0 \longrightarrow \mathbb{R} \longrightarrow \underset{\text{under } \{\ \}}{\text{functions on } M} \longrightarrow \underset{[\ ,\ ]}{\text{Hamilt v.f}} \longrightarrow H^1(M,\mathbb{R}) \to 0$$

where $H^1(M,\mathbb{R})$ is abelian. In effect if X, Y are Hamiltonian v.f. we can local solve for f, g $\ni$

$$i(X)\,\omega = df \qquad i(Y)\,\omega = dg$$

and then $\{f,g\} \overset{= i(X)i(Y)\omega}{}$ is a well-defined global function on X such that $i([X,Y])\,\omega = d\{f,g\}$.

Thus the last map above vanishes on brackets. Now suppose then that $H^1(\mathfrak{g},\mathbb{R}) = 0$, i.e. that $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$. Then $v_x = X_{f_x}$ where f_x is a function unique up to constants. If in addition $H^2(\mathfrak{g},\mathbb{R}) = 0$, then the central extension obtained by pull-back

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r] \arrow[d, equal] & \tilde{\mathfrak{g}} \arrow[r] \arrow[d] & \mathfrak{g} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r] & \text{functions under } \{,\} \arrow[r] & \text{exact Ham. v.f.} \arrow[r] & 0
\end{tikzcd}$$

will be trivial, hence we ~~[illegible]~~ obtain a Lie homom.

$$\mathfrak{g} \longrightarrow \underset{M \text{ under } \{\ \}}{\text{functions on}}$$

By duality this gives us an *equivariant* map

$$M \longrightarrow \mathfrak{g}^*.$$

<u>Conclude</u>: If $H^1(\mathfrak{g}, \mathbb{R}) = H^2(\mathfrak{g}, \mathbb{R}) = 0$, then there exists a canonical map $M \longrightarrow \mathfrak{g}^*$ for any symplectic $\mathfrak{g}$-manifold.

In particular any homogeneous symplectic manifold (for the Lie group belonging to $\mathfrak{g}$) must cover an [redacted] orbit of $\mathfrak{g}$ in $\mathfrak{g}^*$ in [redacted] a canonical way. But more is true in this case: Thus suppose M homogeneous and that $m \in M$, and let $\mathfrak{g}_m$ be the stabilizer, i.e. $x \in \mathfrak{g}_m \Leftrightarrow v_x(m) = 0$. Let [redacted] λ be the elt of $\mathfrak{g}^*$ determined by m, i.e.

$$\lambda(x) = f_x(m)$$

and $\mathfrak{g}_\lambda$ the stabilizer of λ:

$$x \in \mathfrak{g}_\lambda \iff \lambda([x,y]) = 0 \quad \forall y \in \mathfrak{g}.$$

Now

$$\lambda([x,y]) = f_{[x,y]}(m) = \{f_x, f_y\}(m)$$

$$= \big(i(v_x)\,i(v_y)\,\omega\big)(m)$$

and by assumption M is homogeneous so $x \mapsto v_x(m)$ from $\mathfrak{g} \to T_M(m)$ is surjective. Thus if $x \in \mathfrak{g}_\lambda$ we have $i(v_x)\,\omega(m) = 0$, so $v_x(m) = 0$ and $x \in \mathfrak{g}_m$. So $\mathfrak{g}_\lambda \subset \mathfrak{g}_m$, and as the other inclusion is clear, we have $\mathfrak{g}_m = \mathfrak{g}_\lambda$. Thus $\mathfrak{g}/\mathfrak{g}_\lambda \xrightarrow{\sim} T_M(m)$ and so one has that M is a covering space of the orbit of λ.

The idea behind reviewing Kostant theory was to construct the representation mentioned on page 23 by the following scheme. Thus suppose $\mathfrak{g}$ perfect and form the universal central extension $\mathfrak{g}_{univ}$ of $\mathfrak{g}$ by $H_2(\mathfrak{g})$.

Given a 2-cocycle on $\mathfrak{g}$, it determines an element of $(\mathfrak{g}_{univ})^*$, on which $\mathfrak{g}$ operates, so we can look at the orbit O, which is a symplectic homogeneous $\mathfrak{g}$-manifold. Quantizing O as in Kostant's theory should lead to the desired representation.

Example. Take $\mathfrak{g}$ ~~to~~ be abelian and suppose the 2-cocycle λ is a non-degenerate bilinear form. Then $O = \mathfrak{g}$ acting as translations and quantization here means we construct the Heisenberg representation:

$$[A(x), A(y)] = \lambda(x,y).id$$

This roughly signifies that to produce a representation when $\mathfrak{g}$ = vector fields on $\mathbb{R}$ with compact support, we will need an infinite dimensional O, i.e. second quantization?

March 3, 1972: $SL_2\mathbb{R}$

Consider $SL_2\mathbb{R}$ as a discrete group. We propose to define an element of $H^2(SL_2\mathbb{R}, \mathbb{R})$. Let [redacted] Z be the upper half plane $\{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. Then $SL_2\mathbb{R}$ acts continuously on Z by [redacted]

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az+b}{cz+d}$$

and the stabilizer of i is

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \qquad a^2+b^2=1$$

which is SO_2 [redacted]. Thus

$$Z = SL_2(\mathbb{R})/SO_2 .$$

Z is the symmetric space belonging to $SL_2(\mathbb{R})$; SO_2 is the maximal compact subgroup. Since Z is contractible $SO_2 \longrightarrow SL_2\mathbb{R}$ is a homotopy equivalence, hence the universal covering of $SL_2\mathbb{R}$ as a top. group is contractible.

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{SL_2\mathbb{R}}^{top} \longrightarrow SL_2\mathbb{R} \longrightarrow 1$$

Let ω denote an invariant volume form on Z, e.g.

$$\omega = \frac{dx\,dy}{y^2} = \frac{\frac{i}{2}\,dz\,d\bar{z}}{y^2}$$

(Proof of invariance

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \omega = \frac{\frac{i}{2}\, d\left(\frac{az+b}{cz+d}\right) \cdot d\left(\frac{a\bar z+b}{c\bar z+d}\right)}{\operatorname{Im}\left(\frac{az+b}{cz+d}\right)^2}$$

$$\operatorname{Im}\left(\frac{az+b}{cz+d}\right) = \frac{1}{2i}\left(\frac{az+b}{cz+d} - \frac{a\bar z+b}{c\bar z+d}\right)$$

$$= \frac{1}{2i}\,\frac{1}{|cz+d|^2}\Big[\cancel{ac z\bar z} + azd + bc\bar z + \cancel{bd} - \cancel{ac z\bar z} - bcz - ad\bar z - \cancel{bd}\Big]$$

$$= (ad-bc)\,\frac{\operatorname{Im}(z)}{|cz+d|^2}$$

$$d\left(\frac{az+b}{cz+d}\right) = \frac{(cz+d)a - (az+b)c}{(cz+d)^2}\,dz = (ad-bc)\frac{dz}{(cz+d)^2}$$

$$\therefore \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \omega = \frac{(ad-bc)^2\,\frac{\frac{i}{2}dz\,d\bar z}{|cz+d|^4}}{(ad-bc)^2\,\frac{y^2}{|cz+d|^4}} = \frac{dx\,dy}{y^2} = \omega$$

Observe that ω is in fact invariant under $(GL_2\mathbb{R})^+$ (+ signifies det >0) so that UHP is preserved).).

Now to define an element of $H^2(SL_2\mathbb{R}^{disc.}, \mathbb{R})$ it will suffice to define a (characteristic) class in $H^2_{DR}(X)$ for any principal $SL_2\mathbb{R}^{disc.}$-bundle P over a manifold X.

But form the fibre bundle

$$E = P \times^{SL_2\mathbb{R}} Z \longrightarrow X$$

with contractible fibres $\simeq Z$. On this fibre bundle we can define a closed 2-form $\tilde{\omega}$ by pulling ω up to $P \times Z$ via the projection and then descending. More precisely, if U is an open set of X over which P is trivial, then $E|U \simeq U \times Z$ has form $\tilde{\omega}_U = pr_2^*(\omega)$, $pr_2 : U \times Z \to Z$. The transition functions being loc. constant map $U \cap V \longrightarrow SL_2(\mathbb{R})$ we have $\tilde{\omega}_U = \tilde{\omega}_V$ on $U \cap V$, because ω is invariant. Pulling $\tilde{\omega}$ back by a section of E gives a well-defined element of $H^2_{DR}(X)$.

To show this element is non-trivial, let Γ be a discrete subgroup of $SL_2\mathbb{R}$ which is torsion-free and has compact quotient, so that the quotient manifold $\Gamma\backslash Z$ exists and is compact. Then take $X = \Gamma\backslash Z$

$$
\begin{tikzcd}
P = Z \times^{\Gamma} SL_2\mathbb{R} \arrow[d, "(pr_1)"] \\
X = \Gamma\backslash Z
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
P \times^{SL_2\mathbb{R}} Z = Z \times^{\Gamma} Z \arrow[d, "(pr_1)"] \\
X \quad \Gamma\backslash Z
\end{tikzcd}
$$

Observe that ω descends to a closed 2 form $\bar{\omega}$ on $\Gamma\backslash Z$ and that in this case $\tilde{\omega}$ on $Z \times^{\Gamma} Z$ is the pull-back of ω by the map $Z \times^{\Gamma} Z \longrightarrow \Gamma\backslash Z$ induced by (pr_2). Here there is a diagonal section $\Gamma\backslash Z \to Z \times^{\Gamma} Z$, and so we see that the form we get on X is just $\bar{\omega}$, which is the induced volume form on $\Gamma\backslash Z$. In particular since $\Gamma\backslash Z$ is compact this cohomology class is non-trivial.

Construction of such Γ: $\Gamma\backslash Z$ must be a closed Riemann surface (of genus >1, as its universal covering is Z.) Conversely given a closed Riemann surface X of genus >1, its universal covering is analytically isomorphic to Z. Since $PSL_2(\mathbb{R})$ is the group of analytic isos. of Z, it follows that we have a homom. $\pi_1 X \to PSL_2\mathbb{R}$, well-defined up to inner autos., which is injective. Thus:

Discrete torsion-free subgroups of $PSL_2\mathbb{R}$ with compact quotient are the same thing as uniformized closed Riemann surfaces.

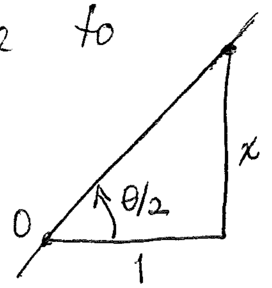
Now to lift $\pi_1 X$ up into $SL_2(\mathbb{R})$ is possible when an obstruction in $H^2(X, \mathbb{Z}/2)$ vanishes (maybe same as putting a spinor structure on X?) In any case by Poincare duality we can kill such a class by passing to any non-trivial double covering, so passing to any subgroup of $\pi_1 X$ of index 2 we get a $\Gamma \subset SL_2\mathbb{R}$.

Observe that Z being a Riemann ~~manifold~~ surface of constant negative curvature, its volume form is a neg. constant times its Gauss-Bonnet form, so consequently for any $\Gamma \subset SL_2\mathbb{R}$ as above

$$\int_{\Gamma\backslash Z} \bar{\omega} = (\text{constant})\ \underset{\substack{\parallel \\ 2-2g}}{\chi(\Gamma\backslash Z)}$$

which ~~means~~ indicates that the class in $H^2(PSL_2(\mathbb{R}), \mathbb{R})$ comes from an integral class, probably the class of the universal covering extension.

Action of $PSL_2\mathbb{R}$ on $P_1(\mathbb{R}) = S^1$. Identify $P_1(\mathbb{R})$ with S^1 the unit circle by sending a line in the plane to 2· the angle it makes with the x-axis.



i.e.

$$x = \tan\left(\frac{\theta}{2}\right) \qquad -\pi < \theta < \pi$$

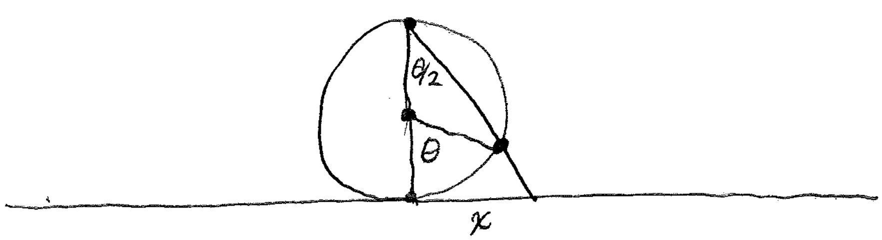
if the line contains $(1,x)$. Another way of viewing this transformation:

$$z \longmapsto \frac{1+iz}{1-iz}$$

maps $\mathbb{Z}$ holomorphically onto $|z|<1$ and the real axis onto the circle:

$$\frac{1+ix}{1-ix} = e^{i\theta}$$

Also stereographic projection:



Recall that sl_2 has basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

satisfying:

$$[H,X] = 2X$$
$$[H,Y] = -2Y$$
$$[X,Y] = H$$

$$\exp(tH) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

acts on $P_1(\mathbb{R})$ by $\exp(tH)(x) = e^{2t}x$

so the vector field on $P_1(\mathbb{R})$ is

$$f \longmapsto \frac{d}{dt} f(e^{2t}x)\Big|_{t=0} = f'(x)\,2x = \left\langle 2x\frac{d}{dx}, df \right\rangle(x)$$

i.e.

$$v_H = 2x\frac{d}{dx}$$

which becomes

$$2\tan\left(\frac{\theta}{2}\right)\left(\frac{dx}{d\theta}\right)^{-1}\frac{d}{d\theta} = \frac{2\tan\left(\frac{\theta}{2}\right)}{\frac{1}{2}\sec^2\left(\frac{\theta}{2}\right)}\frac{d}{d\theta} = 4\sin\frac{\theta}{2}\cos\frac{\theta}{2}\frac{d}{d\theta}$$

so

$$\boxed{v_H = 2\sin\theta\frac{d}{d\theta}}.$$

$$(\exp tX)(x) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}(x) = x+t$$

$$v_X(x) = \frac{d}{dx} = 2\cos^2\frac{\theta}{2}\frac{d}{d\theta}$$

$$\boxed{v_X = (\cos\theta + 1)\frac{d}{d\theta}}$$

(check v_X on P_1 vanishes at $x=\infty$ which is $\theta=\pi$).

$$(\exp tY)(x) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}(x) = \frac{x}{tx+1} = \frac{1}{\frac{1}{x}+t}$$

$$v_Y = \frac{d}{dt}\left(\frac{x}{tx+1}\right)\Big|_{t=0} \cdot \frac{d}{dx} = -x^2\frac{d}{dx} = -2\tan^2\frac{\theta}{2}\cos^2\frac{\theta}{2}\frac{d}{d\theta}$$

$$= -2\sin^2\frac{\theta}{2}\,\frac{d}{d\theta}$$

$$\boxed{v_Y = (\cos\theta - 1)\frac{d}{d\theta}}$$

(Unfortunately signs are off, because

$$[v_X, v_Y] = \left[\frac{d}{dx}, -x^2\frac{d}{dx}\right] = -2x\frac{d}{dx}$$

$$[v_H, v_X] = \left[2x\frac{d}{dx}, \frac{d}{dx}\right] = -2\frac{d}{dx}.$$

The reason for this comes from general incompatibility of conventions adopted at the beginning. Thus we want the Lie group G to act on X to the left and we want to define vector field v_H assoc. to $H \in \mathfrak{g}$ by formula

$$\frac{d}{dt} f(e^{tH}x)\Big|_{t=0} = (Hf)(x)$$

so that we have "Taylor formula"

$$f(e^{tH}x) = \sum_n \frac{t^n}{n!}(H^n f)(x) \overset{\text{defn.}}{=} (e^{tH}f)(x).$$

Unfortunately this will force us to set

$$(gf)(x) = f(gx)$$

making G act to the right on functions. So the only consistent thing to do (from the category viewpoint) is to define

$$(Hf)(x) = \frac{d}{dt}\left(e^{tH}f\right)(x)\Big|_{t=0}$$
$$= \frac{d}{dt} f\left(e^{-tH}x\right)\Big|_{t=0}$$

and accept

$$f\left(e^{-tH}x\right) = \sum \frac{t^n}{n!}\left(H^n f\right)(x)$$

which is an ugly version of Taylor's formula. Thus on $\mathbb{R}$ we have to accept the formula

$$e^{-tD}x = x + t$$

so that

$$\left(e^{tD}f\right)(x) = f(x+t).$$

This seems unpleasant, but there is a real problem: We can't preserve all of:

(i) bracket of vector fields $\sim$ commutator of inf. flows

(ii) bracket of vector fields $=$ commutator of operators on the functions.

(iii) formula $\quad \frac{d}{dt} f\left(e^{tH}x\right)\Big|_{t=0} = (Hf)(x)$

Perhaps the thing to do is to define

$$(v_H f)(x) = -\frac{d}{dt} f\left(e^{tH}x\right)\Big|_{t=0}$$

Thus we have

$$g \longmapsto (gf)(x) = f(g^{-1}x)$$
$$H \longmapsto (v_H f)(x) = -\langle Hx, df\rangle.$$

Then v will be a Lie homomorphism. So in the present situation we must put

$$v_H = -2\sin\theta \frac{d}{d\theta}$$

$$v_X = -(1+\cos\theta)\frac{d}{d\theta}$$

$$v_Y = -(\cos\theta - 1)\frac{d}{d\theta} .$$

A slightly better basis is perhaps

$$\frac{d}{d\theta}, \ \cos\theta \cdot \frac{d}{d\theta}, \ \sin\theta \cdot \frac{d}{d\theta}$$

Ideas for future work: Construct explicitly the representations of the universal covering of $PSL_2(\mathbb{R})$ and try to see if these can be extended to the group of orient. pres. diffeomorphisms of S^1. *

Explicit realization of $\widetilde{PSL_2(\mathbb{R})}$? (means orient. preserving)

Borel subgroup B of SDiff(S^1) might be subgroup fixing $x=\infty$. Observe have character $B \to \mathbb{R}^+$ obtained from derivative at $x=\infty$. Can one induce characters of B up to representations of SDiff(S^1)? Bruhat decomposition of SDiff(S^1) and the construction of its universal central extension by Moore-Matsumoto methods.

(Thurston points out that SDiff(S^1) has at least 2 real classes of dim 2 — the Euler class and the interesting (Godbillon-V.) class. Thus even if we construct $\widetilde{PSL_2(\mathbb{R})}$ and extend to SDiff(S^1), we don't get the interesting class.)

March 3, 1971. Milnor model for BG

A review of general nonsense which should be well understood:

Milnor model for BG: Let G be a group (sans topologie pour fixer les idées). The principal bundle PG over Milnor's BG is the infinite join

$$\bigcup_n G^{*n}$$

whose points are linear combinations $\sum_{i\geq 0} t_i g_i$ where $t_i \geq 0$, $\sum t_i = 1$, and almost all the t_i are zero. Thus the Milnor PG is a simplicial complex with vertices $\mathbb{N} \times G$ and where a set of distinct vertices

$$(i_1, g_1), \ldots, (i_m, g_m)$$

forms a simplex iff the i_j are distinct.

~~[illegible] BG is the quotient of this by the action of G. Since G acts freely on the simplices of PG, the quotient is a simplicial complex. One sees the set of vertices of BG is [illegible]~~

The Milnor BG is the quotient of this by the action of G. Unfortunately the simplicial complex structure of PG does not induce one on BG, because the **G-orbit** of a sequence $(g_1, \ldots, g_m)$ in G^m

is not determined by the m-tuples of G-orbits of the vertices. Nevertheless we ought to be able to describe it as the realization of the singular complex (or nerve) of a category, so Segal claims.

<u>Fact</u>: Let K be a simplicial complex endowed with an ordering on its vertices such that each simplex is linearly ordered. Then to K is associated a semi-simplicial set $\gamma(K)$, its singular complex :

$$\gamma(K)_p = \operatorname{Hom}(\Delta(p), K) \qquad \text{ordering-preserving}$$

or equivalently $\gamma(K)$ is the nerve of K viewed as a category. Then the canonical map

$$|\gamma(K)| \longrightarrow |K|$$

is a homeomorphism. (This is clear set-theoretically because a point in $|\gamma(K)|$ is of the form $\sum_{i=0}^{p} t_i k_i$, all $t_i > 0$, $(k_0, \ldots, k_p)$ a non-deg. simplex of K. The same is true for the geom. realization of $|K|$.

Apply this fact to PG whose vertices $\mathbb{N} \times G$ are ordered via the natural order on $\mathbb{N}$. Thus PG is the realization of the nerve of the category whose objects are pairs (i, g) and where

$$\operatorname{Hom}((i,g),(i',g')) = \begin{cases} \phi & i > i' \text{ or } i = i' \text{ and } g \neq g' \\ \{id\} & i = i' \quad g = g' \\ \{\phi\} & i < i' \end{cases}$$

Since G acts freely on the objects, hence also the arrows of this category, we see that BG is the realization of the category whose set of objects is $\mathbb{N}$ and

$$\mathrm{Hom}(i,i') = \begin{cases} \phi & \text{if } i > i' \\ \{id\} & \text{if } i = i' \\ G & \text{if } i < i' . \end{cases}$$

(Must check this against Segal's paper eventually.)

As a check we observe that geometrically we get in the realization of the nerve of this category one q-simplex for each collection

$$i_0 \overset{g_{01}}{<} i_1 \overset{g_{12}}{<} \cdots \overset{g_{q-1,q}}{<} i_q$$

and that the same is true for the Milnor BG, namely to the simplex $((i_0,g_0), \dots, (i_q,g_q))$ in PG goes the simplex

$$i_0 \overset{g_0 g_1^{-1}}{<} i_1 \overset{g_1 g_2^{-1}}{<} \cdots\cdots \overset{g_{q-1} g_q^{-1}}{<} i_q$$

Now this construction makes sense for a monoid M, hence we have a Milnor BM. Moreover there is a canonical map

$$BM \xrightarrow{p} B\mathbb{N} = \bigcup_{n \geq 0} \Delta(n)$$

obtained by mapping M to 1. Let now X be a compact space, and let $f: X \to BM$ be a map. The map $pf: X \longrightarrow BN$ is the same thing as a family of cont. functions $\rho_i : X \longrightarrow [0,1]$ $i \geq 0$, almost all zero such that

$$\sum_{i \geq 0} \rho_i = 1$$

Such a partition of unity determines an open covering U_i of X by $U_i = \rho^{-1}(0,1]$. ~~If σ is a finite subset of $\mathbb{N}$ such that~~

$$U_\sigma = \bigcap_{i \in \sigma} U_i \neq \phi$$

~~then for x in U_σ, $f(x)$ is a point of BM of the form~~

$$i_0 \underset{m^x_{01}}{<} i_1 \underset{m^x_{12}}{<} \cdots\cdots \underset{m^x_{q-1,q}}{<} i_q \qquad \sigma = \{i_0, \ldots, i_q\}$$

~~and $m^x_{j-1,j}$ is a locally constant function of $x \in U_\sigma$, because the simplices over U_σ are topologically disjoint.~~

Let $x \in X$ and let $\sigma = \{i_0, \ldots, i_q\}$ be the subset of $\mathbb{N}$ such that $i \in \sigma \Leftrightarrow \rho_i(x) > 0$. Then $f(x)$ is a point of BM of the form

$$i_0 \overset{m^x_{01}}{<} i_1 \overset{m^x_{12}}{<} i_2 < \cdots < i_q$$

with $m^x_{j_0-1,j} \in M$. ~~[illegible]~~ ~~[illegible]~~ If $i,j \in \sigma$ and $i<j$, let $m_{ij}(x)$ be the product of the various m's between i and j. I claim then that $m_{ij} : U_i \cap U_j \longrightarrow M$ is a locally constant function. The only thing to show is that if x specializes to a point y in a such a way that certain of the $\rho_k(x)$, $k \in \sigma - \{i,j\}$ go to zero, and if $\rho_i(y)$ and $\rho_j(y) > 0$, then $m_{ij}(x) \longrightarrow m_{ij}(y)$. But this is clear from the topology on BM.

<u>Conclusion:</u> A map $f : X \longrightarrow BM$ is the same as a partition of unity

$$\sum_{i \geq 0} \rho_i(x) = 1 \qquad \rho_i : X \longrightarrow [0,1]$$

together with locally constant maps

$$m_{ij} : U_i \cap U_j \longrightarrow M \qquad i<j$$

satisfying the cocycle condition

$$m_{ij} m_{jk} = m_{ik} \quad \text{on } U_i \cap U_j \cap U_k \quad \text{if } i<j<k$$

(Here $U_i = \rho_i^{-1}(0,1]$.)

As a check suppose given this data, and try to construct f. Then given [redacted] x define $f(x)$ in the way you must namely if $\sigma = \{i_0, \ldots, i_k\}$ are the indices $\ni$ $U_i \ni x$, then

$$f(x) = \overbrace{\text{point of the simplex}}\; i_0 \overset{m_{i_0 i_1}}{<} i_1 < \cdots\cdots \overset{m_{i_{k-1} i_k}}{<} i_k \quad \text{with coordinates } (\rho_i(x)).$$

Now you want to check the continuity of f, which somehow seems messy (?)

<u>Example 1:</u> Let K be a simplicial complex, ~~with an ordering on its~~ and let K' be its barycentric subdivision. A simplex of K' is a sequence of simplices $\sigma_1 < \cdots\cdots < \sigma_m$ in K. Thus K' has a natural ordering and its vertices form a category, namely, the category of simplices of K with inclusion maps.

$|K|$ has a natural covering by stars of vertices (= open sets U_v of points whose v th coordinate is > 0). Lubkin forms the family of finite intersections, thus obtaining the open stars of simplices (= open sets U_σ consisting of the points whose coordinates at each v in σ are > 0). Then the category of these open sets is the same as the category of simplices of K, $Cat(K)$.

Let Nerve $Cat(K)$ be the nerve of $Cat(K)$; it is the semi-simplicial set whose (non-degenerate) i-simplices are chains of proper inclusions of length $i+1$. Thus an i-simplex of Nerve $Cat(K)$ is the same as an i-simplex of K'. Consequently

$$\mathrm{Real}[\text{Nerve } Cat(K)] = |K'|$$

<u>Summary:</u> Given a simplicial complex K, it determines a category, namely, it's ordered set of simplices, and the geom. real. of the nerve of this category is the real. of K'.

<u>Example 2</u>: Given a category $\mathcal{C}$, denote by $\mathbb{N}\cdot\mathcal{C}$ the category whose objects are pairs (n, X) $n \in \mathbb{N}$, $X \in \mathrm{Ob}\,\mathcal{C}$ in which the morphisms are given by

$$\mathrm{Hom}(n, X; n', X') = \begin{cases} \phi & \text{if } n > n' \\ & \text{or if } n = n' \text{ and } X \neq X' \\ \{id\} & \text{if } n = n' \text{ and } X = X' \\ \mathrm{Hom}_{\mathcal{C}}(X, X') & \text{if } n < n', \end{cases}$$

with evident composition. Then there are two obvious functors

$$\begin{tikzcd} & \mathbb{N}\cdot\mathcal{C} \arrow[dl] \arrow[dr] & \\ \mathbb{N} & & \mathcal{C} \end{tikzcd}$$

inducing maps of Real.(Nerve ?). The functor $\mathbb{N}\cdot\mathcal{C} \to \mathcal{C}$ is surely going to induce a homotopy equivalence on nerve realizations.

How Real(Nerve $\mathcal{C}$) looks: A typical point might be written

$$t_0 X_0,\ f_{01},\ t_1 X_1,\ f_{12},\ \ldots\ldots,\ t_q X_q \qquad \begin{matrix} t_i > 0 \\ \sum t_i = 1 \end{matrix}$$

This belongs to the simplex

$$X_0 \xrightarrow{f_{01}} X_1 \xrightarrow{f_{12}} X_2 \longrightarrow \cdots \longrightarrow X_q$$

One makes identifications as follows

faces: If $t_i \to 0$ one deletes $t_i X_i$ and composes $f_{i-1,i}$ and $f_{i,i+1}$.

degeneracies: If $f_{i,i+1} = id$ one deletes X_{i+1} and adds t_i and t_{i+1}.

The ~~...~~ space $Real(Nerve(\mathbb{N}\cdot\mathcal{C}))$ has points corresponding to sequences

$$t_{i_0} X_{i_0}, f_{i_0 i_1}, t_{i_1} X_{i_1}, f_{i_1 i_2}, \ldots\ldots, t_{i_q} X_{i_q}$$

where $i_0 < i_1 < \cdots < i_q$ are in $\mathbb{N}$ and $t_{i_j} > 0$, $\sum t_{i_j} = 1$. A map of a (compact, say) space X into $Real(Nerve(\mathbb{N}\cdot\mathcal{C}))$ is therefore the same as a partition of unity

$$\sum_{i \in \mathbb{N}} \rho_i = 1$$

and for each $U_i = \rho_i^{-1}(0,1]$ a continuous map $X_i : U_i \to Ob\,\mathcal{C}$ and for each $i<j$ a continuous map $f_{ij} : U_i \cap U_j \to Ar\,\mathcal{C}$ such that the cocycle condition holds.

Problem: Is there a reasonable way to think about maps of X into $Real(Nerve\,\mathcal{C})$?

<u>March 4, 1972:</u> Group-completion theorem.

Let M be a topological monoid, to fix the ideas. I want to understand its "group-completion" ΩBM.

The **basic construction**: Let M act to the right on $M\times M$ by the rule $(m_1, m_2)m = (m_1 m, m_2 m)$. Thus $M\times M$ is a right M space and so we obtain a topological category $(M\times M/\Delta M)$ whose nerve $\mathrm{Nerv}(M\times M/\Delta M)$ is the simplicial space:

$$(M\times M)\times M\times M \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; (M\times M)\times M \;\underset{s\,=\,\text{action}}{\overset{t\,=\,pr_1}{\rightrightarrows}}\; M\times M$$

This will be the candidate for ΩBM. To obtain a map of $\mathrm{Nerv}(M\times M/\Delta M)$ to ΩBM, we produce a "fibre space" over BM with $\mathrm{Nerv}(M\times M/\Delta M)$ as fibre whose total space is contractible.

Let M act to the left on $M\times M$ by the rule $m(m_1, m_2) = (mm_1, m_2)$. This commutes with the right action, hence M acts on $\mathrm{Nerv}(M\times M/\Delta M)$ and we can form a simplicial topological category $(M\backslash \mathrm{Nerv}(M\times M/\Delta M))$ whose nerve

$$\mathrm{Nerv}(M\backslash \mathrm{Nerv}(M\times M/\Delta M))$$

will be a bisimplicial space

$$(p,q) \longmapsto M^p \times (M\times M)\times M^q$$

It is clear that the "vertical augmentation"

$$\mathrm{Nerv}(M\backslash \mathrm{Nerv}(M\times M/\Delta M)) \longrightarrow \mathrm{Nerv}(M\backslash e)$$

has fibres $\simeq$ $\mathrm{Nerv}(M\times M/\Delta M)$.
To obtain contractibility consider the map

$$\mathrm{Nerv}\,(M\backslash\,\mathrm{Nerv}\,(M\times M/\Delta M)) \longrightarrow [redacted]\ \mathrm{Nerv}\,(M/M) [redacted]$$

$$m \qquad m_1, m_2 \quad m' \qquad \longmapsto \qquad [redacted] \qquad m_2, m'$$

[redacted], more precisely the map

$$M^p \times (M\times M)\times M^q \longmapsto M\times M^q$$

given by projection on the last two factors. The fibres of this map are $\approx$ $\mathrm{Nerv}(M\backslash M)$ which is contractible and the [redacted] base is contractible, so the "total" space is contractible.

[redacted]

We now get from the above considerations a "map"

$$\mathrm{Nerv}\,(M\times M/\Delta M) \longrightarrow \Omega BM.$$

For [redacted] this to be a homotopy equivalence, [redacted] it is necessary and sufficient that [redacted] the action of any element of M [redacted] produce a homotopy self-equivalence of $\mathrm{Nerv}\,(M\times M/\Delta M)$.

March 5, 1971 Integrating classifying topoes.

Let $\mathcal{T}$ be a topos, let S be an object of $\mathcal{T}$, and let G be a group in $\mathcal{T}/S$. By BG, I means the topos $(\mathcal{T}/S)_G$ consisting of objects M of $\mathcal{T}$ over S with an action of G:

$$\begin{tikzcd} G \times_S M \arrow[rr] \arrow[dr] & & M \arrow[dl] \\ & S & \end{tikzcd}$$

By $\underline{\Gamma}(S, BG)$ I mean the stack over $\mathcal{T}$ obtained by "integrating" BG over S. Thus ~~[illegible]~~G is a pre-stack over $\mathcal{T}/S$; to each $U \to S$ we have the category associated to the group $\mathrm{Hom}_{/S}(U, G)$. One enlarges this in the standard way to a stack; to each $U \to S$ one ~~[illegible]~~ associates the category of G_U-torsors. Now one takes the ~~[illegible]~~ direct image ~~[illegible]~~ of this stack relative to the map $S \to e$; one obtains the stack associating to U in $\mathcal{T}$, the category of $G_{S\times U}$-torsors over $S \times U$. This last stack is $\underline{\Gamma}(S, BG)$.

It is clear that there is a canonical map

$$B\underline{\Gamma}(S, G) \longrightarrow \underline{\Gamma}(S, BG)$$

obtained by ~~[illegible]~~ viewing the ~~[illegible]~~ group $\underline{\Gamma}(S,G)$

in $\mathcal{T}$ as a prestack. It is clear that this map is the full subcategory consisting of the trivial torsors.

March 5, 1972:

I want now to understand the homotopy type of New-$(M\times M/\Delta M)$ in the case of K-theory. Thus M will now be replaced by the category $\mathcal{a}$ of f.g. proj. R-modules and their isomorphisms.

For New-$(M\times M/\Delta M)$ I take the category $\mathcal{C}$ cofibred over Δ^0 whose fibre $\mathcal{C}_n$ is the category $\mathcal{a}^{n+2} = (\mathcal{a}^2)\times \mathcal{a}^n$. An arrow from $(V_0^{\pm}, V_1, \ldots, V_n)$ to $(W_0^{\pm}, W_1, \ldots, W_m)$ lying over a monotone map $[m] \xrightarrow{\varphi} [n]$ is a collection of isomorphisms

$$W_j^{\pm} \xleftarrow{\sim} \bigoplus_{\varphi(j-1) < i \leq \varphi j} V_i^{\pm} \qquad (V_i^+ = V_i^- \quad \text{for } i > 0).$$

Thus the source operator $\mathcal{C}_1 \longrightarrow \mathcal{C}_0$ (cobase change wrt the last vertex $[0] \xrightarrow{\varphi} [1]$, $\varphi(0) = 1$ is

$$(V_0^{\pm}, V_1) \longmapsto V_0^{\pm} \oplus V_1$$

while the target operator $\varphi(0)=0$ is

$$(V_0^{\pm}, V_1) \longmapsto V_0^{\pm} .$$

Now an important thing to note is that the source operator is ~~...~~ faithful, hence the (pseudo-) simplicial category

$$\cdots \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; \mathcal{C}_2 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \mathcal{C}_1 \;\substack{\longrightarrow\\\longrightarrow}\; \mathcal{C}_0$$

is essentially the nerve of a category object $\mathcal{C}$ in ~~...~~ (Cat) with $\mathrm{Ob}\,\mathcal{C} \cong \mathcal{C}_0$ and $\mathrm{Ar}\,\mathcal{C} = \mathcal{C}_1$ étale over $\mathrm{Ob}\,\mathcal{C}$. Then I know a nice topos of sheaves to consider.

More precisely, let $\mathcal{C}_1$ denote the cofibred category over $\mathcal{C}_0 = \mathcal{a} \times \mathcal{a}$ defined by the functor

$$F(E^\pm) = \left\{ \text{splittings } E^\pm \simeq F^\pm \oplus P^\pm \text{ together with an isom } P^+ \simeq P^- \right\}.$$

(a splitting is simply a projection operator. Thus I ask for projection operators $\pi^\pm$ together with an isom. $\operatorname{Im}\pi^+ \simeq \operatorname{Im}\pi^-$).

Thus an object of $\mathcal{C}_1$ consists of a pair E^+, E^- together with splittings

$$E^\pm = \operatorname{Im}\pi^\pm \oplus \operatorname{Ker}\pi^\pm$$

and an isomorphism $\alpha : \operatorname{Im}\pi^+ \simeq \operatorname{Im}\pi^-$. Define

$$s : \mathcal{C}_1 \longrightarrow \mathcal{C}_0$$

to be the structural map, and

$$t : \mathcal{C}_1 \longrightarrow \mathcal{C}_0$$

$$t(E^\pm, \pi^\pm, \alpha) = (\operatorname{Ker}\pi^\pm).$$

Finally it is clear how to define composition

$$\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \longrightarrow \mathcal{C}_1 .$$

Thus we have a category object in (Cat) whose nerve is equivalent to the pseudo-simplicial category $\mathcal{C}_*$ defined above.

Now intuition [redacted] from Mather's theorem tells me that it is natural to consider the category of local coefficient systems on $\mathcal{C}_*$. Such a thing consists of a sheaf F_0 over $\mathcal{C}_0$ together with an action of $\mathcal{C}_1$. ~~Now F_0 will be a discrete fibred and cofibred category over $\mathcal{C}_0$, and we must lift F_0 to $\mathcal{C}_1$ via source~~

~~$\mathcal{C}_1 \times_{\mathcal{C}_0} F_0$~~

~~in order that it be etale over $\mathcal{C}_1$.~~ Think of F_0 as a bifibred category over $\mathcal{C}_0$ with discrete fibres. From the sheaf theory it is natural to ask for a right action of $\mathcal{C}_1$

$$(*) \qquad F_0 \times_{\mathcal{C}_0} \mathcal{C}_1 \longrightarrow F_0 .$$

This means given $E^{\pm} = V^{\pm} \oplus P$, we want a map

$$(**) \qquad F(V^{\pm}) \longrightarrow F(E^{\pm}) .$$

More precisely $F_0 \times_{\mathcal{C}_0} \mathcal{C}_1$ has for its objects $(f, V^{\pm} \oplus P^{\pm} = E^{\pm})$ with $f \in F(V^{\pm})$ and morphisms are isomorphisms. Thus the action $(*)$ [redacted] being a functor means that $(**)$ is equivariant for $\mathrm{Aut}(V^{\pm})$. The [redacted] identity axiom for a functor implies that the map $(**)$ must [redacted] reduce to the given action of $\mathrm{Aut}(V^{\pm})$ on $F(V^{\pm})$.

Conclude that [redacted] the natural category of sheaves on $\mathcal{C}_*$ is the topos of covariant functors on the category with objects pairs $V^{\pm}$ and morphisms

$$V^{\pm} \longrightarrow E^{\pm}$$

to consists of two complemented inclusions $V^{\pm} \oplus P^{\pm} \xrightarrow{\sim} E^{\pm}$

together with an isomorphism $P^+ \simeq P^-$.

So now let $\mathcal{B}$ be this category. I want to show that it has the good properties. The first thing is to show it gives rise to an H-space. But we have direct sum

$$\begin{tikzcd} V^\pm, W^\pm \arrow[r, mapsto] \arrow[d] & V^\pm \oplus W^\pm \arrow[d] \\ V'^\pm, W'^\pm & V'^\pm \oplus W'^\pm \\ \Delta Q \quad \Delta R \arrow[u, hook] & \Delta Q \oplus \Delta R \arrow[u, phantom, "\cup"] \end{tikzcd}$$

which is associative and commutative up to isomorphism. Moreover, $(0,0)$ behaves as a unit for this operation. Thus the realization of $\mathcal{B}$ is an H-space. In fact, it an invertible H-space because given any object $V^\pm$ of $\mathcal{B}$, its inverse is (V^-, V^+). That's because

$$(V^-, V^+) \oplus (V^+, V^-) = (V^- \oplus V^+, V^+ \oplus V^-) \hookleftarrow (0,0).$$

~~Why $\mathcal{B} = \Omega BB\mathfrak{a}$: [redacted]~~

~~[redacted] By $BB\mathfrak{a}$ we mean the analogue of $\mathrm{Nerv}(M)$, i.e. the (pseudo-)simplicial category in degree n [redacted] $= \mathfrak{a}^n$. [redacted] Using the evident $\mathfrak{a}$-action on $\mathcal{B}$ we can form a pseudo-simplicial category [redacted] $\mathrm{Nerv}(B\ddot{\mathfrak{a}}^{M} \backslash \mathcal{B})$; (here $\mathfrak{a}$ acts by $V \oplus (V^\pm) = (V+V^+, V^-)$. An important point is again the fact that the map~~

~~$\mathcal{B}$~~

Why $\mathcal{B} = \Omega BB\mathcal{A}$. Let $\mathcal{A}$ act on $\mathcal{B}$ by the rule $V\cdot(V^+, V^-) = (V\oplus V^+, V^-)$ and form the pseudo-simplicial category $\mathrm{Nerv}(\mathcal{A}\backslash\mathcal{B})$:

$$\begin{matrix} \to \\ \to \\ \to \end{matrix} \quad \mathcal{A}\times\mathcal{B} \rightrightarrows \mathcal{B}.$$

Observe that the action map $\mathcal{A}\times\mathcal{B}\longrightarrow\mathcal{B}$ is "étale" i.e. $\mathcal{A}\times\mathcal{B}$ is equivalent to the cofibred category over $\mathcal{B}$ defined by the functor

$$F(V^+, V^-) = \{\text{splittings of } V^+\}.$$

(The way to check these things is to note that an object of $\mathcal{A}^n\times\mathcal{B}$ is a collection

$$(V_1, \dots, V_n, V^\pm)$$

and the first vertex map

$$(V_1, \dots, V_n, V^\pm) \longmapsto (V_1\oplus\cdots\oplus V_n\oplus V^+, V^-)$$

allows one to identify $\mathcal{A}^n\times\mathcal{B}$ with an object of $\mathcal{B}$ together with an $(n+1)$-fold splitting of the first space. It follows then that $\mathrm{Nerv}(\mathcal{A}\backslash\mathcal{B})$ is homotopy equivalent to the category with objects $V^\pm$ and in which a map $V^\pm \longrightarrow W^\pm$ consists of splittings

$$V^+\oplus P\oplus Q^+ \xrightarrow{\sim} W^+$$

$$V^-\oplus Q^- \xrightarrow{\sim} W^-$$

together with an isomorphism $Q^+ \simeq Q^-$. To show the last category, call it $\mathcal{L}$, is contractible. Since

~~Why $B \simeq \Omega BB$? Let a act on B by $(V^+, V^-) \mapsto (V^+ \oplus V, V^-)$ and let Nerv$(B\backslash a)$ be the~~

a acts invertibly on B.

Nerv$(a\backslash B) \longrightarrow$ Nerv(a) is a quasi-fibration with fibres $\sim B$, so this will establish ΩNerv$(a) \sim B$.

To show $\mathcal{L}$ contractible, project: $V^{\pm} \longmapsto V^-$. This provides a functor from $\mathcal{L}$ to the category $\mathcal{I}$ of complemented inclusions (reduced version of $B(a/a)$. Then

$$\mathcal{L} \longrightarrow \mathcal{I}$$

is cofibred. The fiber over W^- is the category $\mathcal{I}$ $(V^+ \oplus P \xrightarrow{\sim} W^+)$, hence is contractible as it has an initial element. Thus $\mathcal{L}$ is contractible.

Now I have to understand the homology of B. The idea will be to consider the functor

$$\text{Nerv}(a \times a/a) \longrightarrow B$$

given by last vertex:

$$(V_0^{\pm}, V_1, \dots, V_n) \longmapsto V_0^{\pm} \oplus \Delta V_1 \oplus \cdots \oplus \Delta V_n$$

This functor is a homotopy equivalence as mentioned before. Why?

This is the nerve of a category in $B^\wedge$.

<u>Claim</u>: Let C be a topological category with etale source map, and $C^\wedge$ the associated category of sheaves. Then for computing cohomology in $C^\wedge$, I have found useful the resolution

$$\mathrm{Ar}_2 C \rightrightarrows \mathrm{Ar}\, C \xrightarrow{s} \mathrm{Ob}\, C \qquad (= e \text{ in } C^\wedge)$$

where $\mathrm{Ar}\, C$ acts on the right. This is the nerve of a category $\mathcal{T}$ in $C^\wedge$ with

$$\mathrm{Ob}(\mathcal{T}) = \mathrm{Ar}\, C$$
$$\mathrm{Ar}(\mathcal{T}) = \mathrm{Ar}_2 C$$

etc. The classifying topos of $(C^\wedge, \mathcal{T})$ is $C^\wedge$ itself.

In effect we already know that $C^\wedge_{/\mathrm{Ar} C} \simeq (\mathrm{Ob}\, C)^\wedge$. Thus a diagram

$$\begin{tikzcd}
\cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & F \times_{\mathrm{Ar} C} \mathrm{Ar}_2 C \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & F \arrow[d] \\
\cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathrm{Ar}_2 C \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Ar}\, C
\end{tikzcd}$$

in $C^\wedge$ will be equivalent to a diagram

$$\begin{tikzcd}
\cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & F' \times_{\mathrm{Ob} C} \mathrm{Ar}\, C \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & F' \arrow[d] \\
\cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathrm{Ar}\, C \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Ob}\, C
\end{tikzcd}$$

so its pretty clear. (The origin of this question arose because I thought source etale top. categories had to be treated differently from categories in topoi.)

March 6, 1972: Mumford's conjecture again

Let V be a representation of a group G, G being discretes. Assume G perfect and no mod p cohomology where V is of characteristic p. Now consider the bigraded ring

$$H^*(G, SV)$$

Think of this as $H^*(X, \mathcal{O}_X)$, where X is a ringed topoi of char p. Thus it has Steenrod operations with a Bockstein operations of degree 1, and $\mathcal{P}^0$ operation induced by ~~[illegible]~~ Frobenius.

new idea for proof that category $\mathcal{J}(R)$ of pairs (V^+, V^-) with diag. action is $BGL(R)^+$ (see p. 4 and 7)

March 6, 1972:

Let R be a ring and $\mathcal{J} = \mathcal{J}(R)$ the category of finitely generated projective R-modules with complemented injections for morphisms (i.e. a map $P \to P'$ in $\mathcal{J}(R)$ consists of a pair of R-module maps $\varepsilon : P \to P'$, $\pi : P' \to P$ such that $\pi\varepsilon = id_P$). I propose to determine the category $\mathrm{Ind}(\mathcal{J}(R))$.

So let I be a filtering category and

$$\begin{aligned} I &\longrightarrow \mathcal{J} \\ i &\longmapsto P_i \end{aligned}$$

a functor. Set

$$P = \varinjlim P_i$$

and for each i let the maps

$$P_i \underset{\varepsilon_i}{\overset{\pi_i}{\rightleftarrows}} P$$

be defined by taking the limit over the category $i \backslash I$ of the maps

$$P_i \underset{\varepsilon_u}{\overset{\pi_u}{\rightleftarrows}} P_j . \qquad u : i \to j$$

Then $\pi_i \varepsilon_i = id_{P_i}$, so $E_i = \varepsilon_i \pi_i$ is an idempotent in $\mathrm{End}(P)$. Suppose now that $u : i \to j$. Then we clearly have

$$\begin{tikzcd}
P_i \arrow[r, shift right, "\varepsilon_u"'] \arrow[rr, bend right, "\varepsilon_i"'] & P_j \arrow[l, shift right, "\pi_u"'] \arrow[r, shift right, "\varepsilon_j"'] & P \arrow[l, shift right, "\pi_j"'] \arrow[ll, bend right, "\pi_i"']
\end{tikzcd}$$

so
$$E_i E_j = \varepsilon_i \pi_i \varepsilon_j \pi_j = \varepsilon_i \pi_u \cancel{\pi_j \varepsilon_j} \pi_j = E_i$$

$$E_j E_i = \varepsilon_j \pi_j \varepsilon_i \pi_i = \varepsilon_j \cancel{\pi_j \varepsilon_j} \varepsilon_u \pi_i = E_i$$

So that $E_i \leq E_j$ in the usual sense of projectors.

Thus
$$i \longmapsto E_i \qquad \left[\begin{array}{l} E \leq F \iff EF = FE = E \\ \qquad \iff \left\{\begin{array}{l} \operatorname{Im} E \subset \operatorname{Im} F \\ \operatorname{Ker} E \supset \operatorname{Ker} F \end{array}\right\} \end{array}\right.$$

is a map from I to the ordered set of projectors in P. Its image $\bar{I}$ will be a directed set and the functor $I \longrightarrow \bar{I}$ will be cofinal.

Therefore to any ind-object in $\mathcal{I}$, we can associate an R-module P together with a set $\mathcal{E}$ of projectors in P satisfying

i) $\mathcal{E}$ directed

ii) $\forall\, p \in P$, $\exists\, E$ with $p \in \operatorname{Im}(E)$

iii) $\forall\, E \in \mathcal{E}$, $\operatorname{Im}(E)$ is a f.g. projective R-module.

The functor represented by the ind-object is
$$Q \longmapsto \left\{ Q \underset{i}{\overset{\pi}{\leftrightarrows}} P \;\middle|\; i\pi \leq \text{some member of } \mathcal{E} \right\}.$$

Proposition: $\operatorname{Ind}(\mathcal{I})$ is equivalent to the following category:

Objects: An R-module P endowed with a set $\mathcal{E}$ of projectors which is directed, exhaustive, and hereditary, and such that $\forall E \in \mathcal{E}$, $\operatorname{Im}(E)$ is fin. gen. projective.

Arrows: $(P, \mathcal{E}) \longrightarrow (P', \mathcal{E}')$ consists of $P \underset{\varepsilon}{\overset{\pi}{\leftrightarrows}} P'$ such that $E \longmapsto \varepsilon E \pi$ carries $\mathcal{E}$ into $\mathcal{E}'$.

Proof: Let $\mathcal{I}'$ be the category ~~[illegible]~~ just described. We have a functor $\mathrm{Ind}(\mathcal{I}) \longrightarrow \mathcal{I}'$, and similarly ~~[illegible]~~ a functor in the opposite direction. Observe that compositions are the same.

Example to show that we can have $P \xrightarrow{\sim} P'$ but not $\mathcal{E} = \mathcal{E}'$. Take R to be a field to simplify and consider $\mathcal{E}'$ to be all projection operators on $V = k^{(\mathbb{N})}$. The point is that given E_1', E_2' we ~~[illegible]~~ can find a subspace of finite-codimension $Q \subset \mathrm{Ker}\, E_1' \cap \mathrm{Ker}\, E_2'$ and such that $Q \cap \{\mathrm{Im}(E_1') + \mathrm{Im}(E_2')\} = 0$. Then extending the sum of the images to a complement for Q and letting E' be the resulting projector we have

$$\left.\begin{array}{l} \mathrm{Ker}\, E' \subset \mathrm{Ker}\, E_i' \\ \mathrm{Im}\, E' \supset \mathrm{Im}\, E_i' \end{array}\right\} \Longrightarrow E_i' < E_i$$

Thus the set of all projectors works. So we can take $\mathcal{E}$ to be the projectors on subspaces corresp. to finite $S \subset \mathbb{N}$.

Question: Is P necessarily projective?

Let S denote the monoid of isomorphism classes of fin. gen. proj. R-modules. Let (S/S) be the category obtained by letting S act on itself by addition. Then we have an evident functor

$$f : \mathcal{J} \longrightarrow (S/S)$$

which sends P to its iso. class $cl(P) \in S$, and a morphism $P \underset{\varepsilon}{\overset{\pi}{\rightleftarrows}} P'$ to the morphism $(cl(P), cl(\operatorname{Ker}\pi))$.

<u>Proposition</u>: f is acyclic, i.e. for all $F : (S/S) \to \mathcal{A}b$ we have

$$H_*(\mathcal{J}, f^*F) \xrightarrow{\sim} H_*(S/S, F).$$

<u>Proof</u>. It suffices to show that for each object s in S/S the category of arrows $s \longrightarrow cl(P)$ is contractible; i.e. the category of pairs (P, t) with $s + t = cl(P)$. But observe we have a functor

$$(P,t), (P',t') \longmapsto (P \oplus P', s+t+t')$$

and natural transfs.

$$(P,t) \longrightarrow (P \oplus P', s+t+t') \longleftarrow (P',t')$$

so the contractibility follows from.

<u>Lemma</u>: Let $\mathcal{C}$ be a category with a functor

$$\mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$
$$(X, Y) \longmapsto X + Y$$

together with natural transfs.

$$X \xrightarrow{i_1} X + Y \xleftarrow{i_2} Y.$$

If $\mathcal{C}$ is non-empty it is contractible.

Proof: The functor $X \mapsto X_0$ is joined to the functor $X \mapsto X + X_0$ by the natural transf. i_2. Similar the latter is joined to the identity by i_1.

Better to observe simply that if $cl(P_0) = s$ then there are natural transformations

$$\begin{tikzcd}
(P,t) \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, mapsto] & (P,t) \arrow[d] \\
(P_0, 0) \arrow[r] & (P \oplus P_0, t+s)
\end{tikzcd}$$

joining the identity to the constant functor.

Corollary: $$H_*(S, f^*F) = \begin{cases} 0 & * > 0 \\ \varinjlim\limits_{(S/S)} F(s) & * = 0 \end{cases}$$

Proof: The category S/S is filtering, hence the inductive limit functor is exact.

Proposition: Let B denote the category of pairs (V^+, V^-) with morphism $f : (V^+, V^-) \longrightarrow (W^+, V^+)$ defined to be a complemented inclusion together with an isomorphism $\mathrm{Ker}(\pi^+) \simeq \mathrm{Ker}(\pi^-)$. Then

$$H_*(\mathcal{B}, \mathbb{Z}) \xleftarrow{\sim} H_*(\mathcal{Q})[(\pi_0 \mathcal{Q})^{-1}]$$

Proof: The projection $B \xrightarrow{p} \mathcal{I}$ is cofibred with fibre over V equivalent to $\mathcal{Q}$, hence we have a spectral sequence

$$H_*(B,\mathbb{Z}) \Longleftarrow H_*(\mathcal{I}, L_*p_!(\mathbb{Z})) = E^2$$

$$Lp_!(\mathbb{Z})_V \simeq H_*(\mathcal{Q},\mathbb{Z})$$

Thus $Lp_!(\mathbb{Z})$ is the functor on $\mathcal{I}$ which sends V to

$$H_*(\mathcal{Q}) = \bigoplus_{s\in S} H_*(\mathrm{Aut}(P_s))$$

and which sends a map $V \overset{\pi}{\underset{i}{\leftrightarrows}} V'$ into multiplication by $cl(\mathrm{Ker}\,\pi) \in \pi_0\mathcal{Q}$. Applying the preceding we have $E^2_{pq}=0$ for $q>0$, and the spectral sequence degenerates yielding the desired result.

Idea: Go back to the category $\mathcal{J}$ of pairs (V^+, V^-) and assume that there exists a stable range functions, namely a function $n(i)$ so that

$$H_j(GL_n) \xrightarrow{\sim} H_j(GL_{n+1})$$

for all $j \leq i$, $n(i) \leq n$. Then I see by considering the projection $(V^+, V^-) \longrightarrow V^+$ that the component of $\mathcal{J}$ with $\dim V^- = \dim V^+ + m$ will clearly have the right homotopy type in a ranges. The point is that the local coefficient system

$$V^+ \longmapsto H_j(GL_{\dim V^+ + m}, \mathcal{L})$$

$\mathcal{L}$ some K-module

is locally constant, hence its homology over $\mathcal{J}$ is trivial as $\mathcal{J}$ is contractible. Now the next point is to use the equivalence of the components.

Scheme for new proof of computation of $H_*(\mathcal{J})$: Idea is to consider projection $(V^+, V^-) \longrightarrow V^+$ giving spectral sequence

$$E^2_{pq} = L_p \varinjlim_{\mathcal{J}} (V^- \mapsto H_*(\mathrm{Aut}\, V^-)) \Longrightarrow H_*(\mathcal{J})$$

Now multiply by S = Iso classes, and localize. It doesn't affect $H_*(\mathcal{J})$ and it makes the local coeff. system invertible. Then the spectral sequence degenerates by contractibility of $\mathcal{J}$.

D. Quillen

<u>March 9, 1972</u>

Thursten described Eilenberg-Maclane cohomology in the following way. Let G be a Lie group, say. Then an E-M class for G with coeff. $\mathbb{R}$ "is" a class for the underlying discrete group which "varies continuously". For example given a family of homomorphisms

$$\varphi_s : \Gamma \longrightarrow G$$

parameterized by $s \in S$, then $\varphi_s^*(\alpha) \in H^*(\Gamma, \mathbb{R})$ should be continuous in s for any E-M class α of G.

~~We can view the family as a map~~

$$\cancel{S \times \Gamma \longrightarrow S \times G}$$

~~of top. groups over S, where $S\times\Gamma$ is discrete over S.~~

One should also allow the group Γ to vary over S. (?)

We can consider ~~a fibre bundle~~ a proper smooth map $X \longrightarrow S$ and a principal G-bundle $P \longrightarrow X$ which is stratified with respect to S. Then if we give a section ε of X/S and a trivialization $\varepsilon^* P \xleftarrow{\sim} X \times G$ of P over the section, we obtain a family of homomorphisms

$$\varphi_s : \pi_1(X_s, \varepsilon(s)) \longrightarrow G$$

as described above. Thus an E-M class α of G should give a section of the DR cohomology of X/S:

$$s \longmapsto \varphi_s^*(\alpha) \in H^*(\pi_1(X_s, \varepsilon(s)), \mathbb{R}) \longrightarrow H^*(X_s, \mathbb{R})$$

<u>Question</u>: Can we identify an EM class α of G with a characteristic class for stratified principal G-bundles over foliated manifolds, that is, a char. class Θ which assigns to a foliated manifold $(X, S \subset T_X)$ and a principal G-bundle P stratified wrt S a class

$$\Theta(P, X, S) \in H^*(X, \cancel{\Omega_{X/S}}\ \Lambda^{\cdot} S^*)?$$

~~where~~ $\cancel{\Omega_{X/S}: \Lambda^0 S^* \to \Lambda^1 S^* \to \Lambda^2 S^* \cdots}$

<u>Example</u>: G Lie group, say connected, $Z = G/K$ its associated symmetric space. Then given a principal G-bundle $P \to X$ stratified with respect to a foliation S, we form $P \times^G Z = P/K$. Let ω be a left invariant differential form on Z; the complex of these is $(\Lambda^{\cdot}(\mathfrak{g}/\mathfrak{k})^*)^K$ with suitable differential, ~~In fact one has an embedding~~

$$\cancel{\Lambda^{\cdot}(\mathfrak{g}/\mathfrak{k})^* \hookrightarrow \Lambda^{\cdot}(\mathfrak{g})^* \quad (\text{lift back via } G \to G/K)}$$

~~with image the forms ω ∋ $i(k)\omega = 0$. Now~~

~~choose a G-invariant connection on P extending the given S-connection. (This is possible because such an invariant connection on S is a section of an affine space bundle over X. More precisely consider the fibre bdl.~~

the differential being induced by the embedding

$$\left(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k})^*\right)^K \hookrightarrow \Lambda^\bullet \mathfrak{g}^*. \qquad (\text{map } G \to G/K).$$

Then if $\pi : P \times^G Z \to X$ is the projection, we can associate to ω a form along the fibres of π obtaining a map of complexes

$$\left(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k})^*\right)^K \longrightarrow \Gamma(P \times^G Z, \Lambda^\bullet T_\pi^*)$$

Now the point is that because P is stratified with respect to the foliation S, this map lifts to a map

$$(*) \qquad \left(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k})^*\right)^K \longrightarrow \Gamma(P \times^G Z, \Lambda^\bullet \mathcal{U}^*)$$

where $\mathcal{U} \subset T_{P \times^G Z}$ is the subbundle spanned by the lift of S provided by the S-connection and T_π, that is

$$0 \longrightarrow \mathcal{U} \longrightarrow T_{P \times^G Z} \longrightarrow \pi^* Q \longrightarrow 0$$

$$0 \longrightarrow T_\pi \longrightarrow \mathcal{U} \longrightarrow \pi^* S \longrightarrow 0$$

(connection provides a splitting of latter sequence, hence a lifting of cxs: $\Lambda^\bullet T_\pi^* \longrightarrow \Lambda^\bullet \mathcal{U}^*$.) Another version: locally $\exists$ quotient $\bar{X}$ by foliation, and P comes from $\bar{P}$ over $\bar{X}$. Then we have

$$\left(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k})^*\right)^K \longrightarrow \Gamma\left(\bar{P} \times^G Z, \Lambda^\bullet T_{\bar{\pi}}^*\right) \longrightarrow \Gamma(P \times^G Z, \Lambda^\bullet \mathcal{U}^*).$$

and being canonical this gives a global map $(*)$

Now given a section $s : X \longrightarrow P \times^G Z$ we have a map

$$\Gamma(P \times^G Z, \Lambda^\bullet \mathcal{U}^*) \longrightarrow \Gamma(X, \Lambda^\bullet S^*)$$

pull-back of forms. We see therefore that to any reduction of P to K we have associated a map of complexes

$$(+) \qquad [\Lambda^\cdot(\mathfrak{g}/\mathfrak{k})^*]^K \longrightarrow \Gamma(X, \Lambda^\cdot S^*)$$

hence a well-defined map in cohomology. (by usual argument)

$$H^*(\mathfrak{g}, \mathfrak{k}) \longrightarrow H^*(X, \Lambda^\cdot S^*).$$

I'd like a converse to this example. Suppose we have a characteristic class for G-torsors stratified with respect to foliations. Do we get a class in $H^*(\mathfrak{g}, \mathfrak{k})$?

Simpler description: Given P, X, S, $s: X \to P/K$ as above. Given a tangent vector $v \in S(x)$ it can be lifted in two ways, via s and via the connection. The difference then is tangent to the fibre, so we get

$$\begin{array}{ccc} S(x) & \longrightarrow & T_{\pi}(s(x)) \\ & & \| \\ & & q^{-1}[s(x)] \times^K \mathfrak{g}/\mathfrak{k} \end{array}$$

where $q: P \to P/K$.

This gives the map $(+)$ above, I guess.

<u>Example:</u> Go back to the case of $\pi : Y \to X$ foliated $\mathbb{R}$-bundle flat at the ends. Then if we choose a fibre coordinate $z : Y \to \mathbb{R}$ such that $z = f_u$ at the "ends" of Y_u, we have a way of mapping tangent vectors on X into vector fields with compact support on $\mathbb{R}$, hence we can pull back the Gelfand-Fuchs cocycles. In the notation used before (except z replaces t)

$$\omega = dz + \sum a_i dx_i \qquad x_1, \ldots, x_m \text{ local coords on } X.$$

Then $\frac{\partial}{\partial x_i}$ lifts to

$$-a_i \frac{\partial}{\partial z} + \frac{\partial}{\partial x_i} \qquad \text{via foliation}$$

$$\frac{\partial}{\partial x_i} \qquad \text{via } z$$

and the difference is

$$\left(\frac{\partial}{\partial x_i}\right)_x \longmapsto a_i(x,z)\frac{\partial}{\partial z} \qquad \text{vector field on } \mathbb{R}$$

so the 2-form is

$$\left(\frac{\partial}{\partial x_i}\right)_x, \left(\frac{\partial}{\partial x_j}\right)_x \longmapsto \lambda\left(a_i^{(x,z)}\frac{\partial}{\partial z}, a_j^{(x,z)}\frac{\partial}{\partial z}\right)$$

$$= \int_{-\infty}^{\infty} \frac{\partial^2 a_i}{\partial z}(x,z)\frac{\partial a_j}{\partial z}(x,z)\, dz$$

~~the~~ i.e. we get our previous formula

$$\pi_*(\theta \cdot d\theta) = \sum_{i,j} \int_{-\infty}^{\infty} \frac{\partial^2 a_i}{\partial z^2}(x,z) \frac{\partial a_j}{\partial z}(x,z)\, dz \quad dx_i\, dx_j$$

(up to sign).

March 11, 1972:

Fix a space J. Consider the following topological category $\mathcal{J}(J)$. Its objects are pairs of finite sets $S^+ \to J$, $S^- \to J$ over J. An arrow $(S^+, S^-) \longrightarrow (T^+, T^-)$ consists of a pair of injections

$$S^+ \hookrightarrow T^+$$
$$S^- \hookrightarrow T^-$$

over J together with an isomorphism $T^+ - S^+ \xrightarrow{\sim} T^- - S^-$ over J.

$$\mathrm{Ob}\, \mathcal{J}(J) = \coprod_{\substack{S^+, S^- \text{ finite} \\ \text{sets}}} J^{S^+} \times J^{S^-}$$

$$\mathrm{Ar}\, \mathcal{J}(J) = \coprod_{\substack{S^+ \hookrightarrow T^+ \\ S^- \hookrightarrow T^- \\ T^+ - S^+ \xrightarrow{\sim} T^- - S^-}} J^{T^+} \underset{J^{T^+ - S^+} = J^{T^- - S^-}}{\times} J^{T^-}$$

Note that the target map is not etale. Thus the simplicial category

$$\begin{array}{c}\longrightarrow \\ \longrightarrow \\ \longrightarrow\end{array} (M \times M) \times M \rightrightarrows M \times M$$

$$M = \coprod_n (X^n, \Sigma_n)$$

is not of the type you studied before ie.

$$\underset{\alpha \quad \beta}{(X^p \times X^q)} \times \underset{\gamma}{X^r} \longrightarrow X^{p+r} \times X^{q+r}$$
$$\longmapsto (\alpha\gamma, \beta\gamma)$$

is <u>not</u> etale.

Next suppose J is a space with basepoint. Consider pairs of finite sets S^+, S^- plus a map $\xi : S^+ \to J$. An arrow $(S^+, S^-, \xi) \longrightarrow (T^+, T^-, \eta)$ consists of

$$S^+ \hookrightarrow T^+ \quad \text{over } J$$

$$S^- \hookrightarrow T^-$$

$$T^+ - S^+ \xrightarrow{\sim} T^- - S^-$$

such that $\eta(T^+ - S^+) =$ the basepoint.

$$Ob = \coprod_{S^+, S^-} J^{S^+}$$

$$Ar = \coprod_{\substack{S^+ \hookrightarrow T^+ \\ S^- \hookrightarrow T^- \\ T^+ - S^+ \xrightarrow{\sim} T^- - S^-}} J^{S^+}$$

Thus source is etale here. $S \mapsto J^S$ functor covariant for $S \hookrightarrow T$ if we use the basepoint. This shouldn't work either.

Conclude: In order to work $\coprod_n P\Sigma_n \times^{\Sigma_n} J^n$ into your setup it is necessary to make the diagonal maps $J \longrightarrow J^k$

etale.

Lang's theorem again

Let σ be an endomorphism of a group G. Given a G-torsor P let

$$\sigma_* P = P \times^{G} \overset{\sigma}{\to} G$$

and consider the category of (P, α) where $\alpha : \sigma_* P \xrightarrow{\sim} P$, i.e. $\alpha : P \longrightarrow P$ satisfies $\alpha(pg) = \alpha(p)\sigma(g)$. Define

$$(*) \qquad (G^\sigma\text{-torsors}) \longrightarrow \text{cat. of } (P, \alpha)$$

$$Q \longmapsto Q \times^{G^\sigma} G \quad \text{with} \quad \alpha(qx) = q\sigma(x).$$

<u>Claim:</u> $(*)$ is ~~an equivalence of categories~~ fully-faithful.
It is an equivalence of categories $\Leftrightarrow$

$$G/G^\sigma \xrightarrow{\sim} G$$
$$gG^\sigma \longmapsto g(\sigma g)^{-1}.$$

<u>Proof.</u> Given $\theta : Q \times^{G^\sigma} G \longrightarrow Q' \times^{G^\sigma} G$ compatible with α, α' choose $q \in Q$, and let $\theta(q) = q' g_0$. Then

$$q' \sigma g_0 = \alpha' \theta(q) = \theta \alpha(q) = \theta q = q' g_0$$

so $\sigma g_0 = g_0$ and $q' g_0 \in Q'$, so fully faithful.
Given (P, α), choose p, whence $\alpha(p) = pg$; let $g = g_1(\sigma g_1)^{-1}$ whence $\alpha(pg_1) = pg\sigma g_1 = pg_1$ and hence $P = Q \times^{G^\sigma} G$ where $Q = \{p \mid \alpha(p) = p\}$. Thus equivalence if $G/G^\sigma = G$.
Conversely given $g_0 \in G$ and define $\alpha : G \to G$ by $\alpha(g) = g_0\sigma(g)$. If an equivalence, $\exists\, g_1$ with $\alpha(g_1) = g_1$, i.e.

$$g_0 = g_1(\sigma g_1)^{-1}.$$

March 12, 1972. ~~[illegible]~~ stable splitting theorem

$\mathcal{A}$ category of f.g. proj. R-modules and isos.

$\mathcal{J}$ ——————————————— and split injections

$\mathcal{J}$ cat. of pairs (V^+, V^-) an arrow $(V^+, V^-) \longrightarrow (W^+, W^-)$ consists of split injections $(i^\pm, Q^\pm) : V^\pm \longrightarrow W^\pm$ together with $Q^+ \xrightarrow{\sim} Q^-$.

Fix V in $\mathcal{A}$. Let $\mathcal{A}_V$ be the category of injections $V \hookrightarrow E$ with E/V in $\mathcal{A}$, arrows being isoms. under V. Then we have a direct sum operation

$$E, E' \longmapsto E \overset{V}{+} E'$$

which is associative, commutative, and unitary. ~~Note~~ ~~Assume that $\mathcal{A}_V$ is not an additive category~~ Note that the category of injections $V \hookrightarrow E$ is not an additive category. Fibred over additive category $\mathcal{P}_R$.

Form $\mathcal{J}_V$ in analogy with $\mathcal{J}$. Its objects are the same as those of $\mathcal{A}_V$. A morphism $E \longrightarrow E'$ in $\mathcal{J}_V$ consists of an injection $i : E \hookrightarrow E'$ under V together with a submodule Q of E' containing V such that

$$E \overset{V}{+} Q \xrightarrow{\sim} E'.$$

Composition is clear.

NO $\rightarrow$ Claim that

$$\begin{aligned} \mathcal{J}_V &\longrightarrow \mathcal{J} \\ E &\longmapsto E/V \end{aligned}$$

is essentially cofibrant since the fibre over W is $\underline{\text{Ext}}(W, V)$

which is a groupoid (in fact a Picard category), we must show every arrow in $\mathcal{J}_V$ is cocartesian. So suppose given two arrows

~~$E \overset{\vee}{+} Q' \xrightarrow{\sim} E'$~~

~~$E \overset{\vee}{+} Q'' \xrightarrow{\sim} E''$~~

~~lying over an arrow $E/V \oplus Q \longrightarrow E'/V$. Thus we have~~

$$\begin{tikzcd}[row sep=small] & & E' \arrow[dr] & & \\ 0 \arrow[r] & V \arrow[ur] \arrow[dr] & & E'/V \arrow[r] & 0 \\ & & E'' \arrow[ur] & & \end{tikzcd}$$

$$E \overset{i_1}{\hookrightarrow} E_1 \supset Q_1$$

$$E \overset{i_2}{\hookrightarrow} E_2 \supset Q_2$$

lying over the same arrow in $\mathcal{J}$. This means we have an isomorphism

$$(*) \qquad E_1/V \simeq E_2/V$$

compatible with i_1, i_2 and Q_1, Q_2. Then I want to show there is a unique isomorphism $E_1 \longrightarrow E_2$ compatible with i_1, i_2, Q_1, Q_2 and $(*)$. But given ?

What can be done in the above is to define $\mathcal{J}'_V$ with same objects as $\mathcal{J}_V$ but with $E \longrightarrow E'$ defined to be a split injection from E to E', under V of course. Then $\mathcal{J}'_V$

is cofibred over $\mathcal{I}$.

The point: ~~Let G act on E. Then~~
Consider characteristic classes of representations in $\mathcal{A}_V$. Then any stable char. class extends to the Grothendieck group of representations, in which we have the identity $E = \bar{E}$. The point is that stability $\Rightarrow$ invertibility. What does this amount to?

Suppose we have a stable class. My idea is to send $E \longmapsto E$ from $\mathcal{A}_V \longrightarrow \mathcal{A}$. Then stable classes in $\mathcal{A}$ will induce stable classes for $\mathcal{A}_V$. The point is that if G acts on E, then G acts on $E+Q$ which is isomorphic as a G-module to $E+(Q/V)$.
Thus the value of a stable class on E and on $E \overset{V}{+} Q$ will be the same.

Question: Can you see geometrically why stability implies invertibility? Thus in the preceding we first want to understand why in $\mathcal{J}_V$ we will have a homotopy

$$\begin{tikzcd} \mathrm{Aut}(E) \arrow[rr] \arrow[d] & \arrow[d, phantom, "\Downarrow"] & \mathcal{J}V \\ \mathrm{Aut}(\bar{E}) \arrow[rru] & & \end{tikzcd}$$

This is easy. We have arrows in $\mathcal{J}V$
$$(E,F) \longrightarrow (E \overset{V}{+} E,\ F \overset{V}{+} E) \simeq (\bar{E} \overset{V}{+} E,\ F \overset{V}{+} E) \leftarrow (\bar{E}, F).$$

~~Thus if G acts on E, then we have the functor~~

$$\cancel{\begin{array}{ccc} G & \longrightarrow & \mathcal{J}_V \\ e & \longmapsto & (E,V) \\ g & \longmapsto & (g, id) \end{array}}$$

~~and the natural transformation~~

$$\cancel{e \quad (E,V) \longrightarrow (E \overset{\vee}{+} E, V \overset{\vee}{+} E)}$$

Thus if G acts on E, we have the following diagram of G-objects in $\mathcal{J}_V$

$$(E,F) \xrightarrow[\substack{\text{with}\\ \text{complement}\\ E,E}]{in_1, in_1} (E \overset{\vee}{+} E, F \overset{\vee}{+} E) \simeq (\bar{E} \overset{\vee}{+} E, F \overset{\vee}{+} E) \xleftarrow[\substack{\text{comp.}\\ E,E}]{(in_1, in_1)} (\bar{E}, F)$$

where G acts trivially on F. Thus the functors from G to $\mathcal{J}_V$ given by the G-objects (E,F), $(\bar{E},F)$ are related by natural transformations.

This argument is the geometric reason why E and $\bar{E}$ become equivalent in $\mathcal{J}_V$. Now to see why E and $\bar{E}$ become equivalent in $\mathcal{J}$.

Cohomologically what seems to happen is that we compute

$$H_*(\mathcal{J}_V^{\Delta}) = \varinjlim_{E} H_*(\mathrm{Aut}(E)).$$

In other words for each extension E we have its group of autos and for each map $E \to E'$ in $\mathcal{J}_V$ we have a well-defined homomorphism

$$\mathrm{Aut}(E) \longrightarrow \mathrm{Aut}(E')$$

except that more is true, namely the functor

$$E \longrightarrow H_*(\mathrm{Aut}(E))$$

depends only on the iso. class of E, and the arrow

$$H_*(\mathrm{Aut}(E)) \longrightarrow H_*(\mathrm{Aut}(E'))$$

depends only on the iso. class of the complement. Thus we have a filtered inductive limit.

March 13, 1972

Suppose over J we form the cofibred category belonging to the functor $V \longmapsto \mathrm{Aut}(V)$ to sets. Thus we consider pairs (V,θ) where $\theta \in \mathrm{Aut}(V)$ and an arrow $(V,\theta) \longrightarrow (V',\theta')$ is a split injection $V \longrightarrow V'$ such that $\theta \mapsto \theta'$. To the pair (V,θ), associate the diagram

$$\begin{tikzcd}
(0,0) \arrow[dr, "{(\alpha,\alpha,\theta)}"] & \\
& (V,V) \\
(0,0) \arrow[ur, "{(\alpha,\alpha,id)}"'] &
\end{tikzcd}$$

in J. We see that if $(V,\theta) \xrightarrow{V \underset{i}{\overset{\pi}{\rightleftarrows}} V'} (V',\theta')$ is an arrow, then

$$\begin{tikzcd}
(0,0) \arrow[dr, "\theta"] \arrow[drr, bend left, "\theta'"] & & \\
& (V,V) \arrow[r] & (V',V') \\
(0,0) \arrow[ur, "id"'] \arrow[urr, bend right, "id"'] & &
\end{tikzcd}$$

commutes. It therefore is clear that the category of (V,θ) is equivalent to the category of paths in J of the form

$$\begin{tikzcd}
(0,0) \arrow[dr] & \\
& \cdot \\
(0,0) \arrow[ur] &
\end{tikzcd}$$

Question: I have already seen that the category $A\backslash\mathcal{I}$ of objects of $\mathcal{I}$ under a given object is contractible, because of the existence of an initial object. But suppose you give several objects $A_1, \dots, A_n$ and consider the category of $n+1$-tuples $(V, u_1, \dots, u_n)$ where $u_i : A_i \longrightarrow V$. Is this category contractible? For example consider the category of diagrams

$$\begin{tikzcd}[row sep=small] A_1 \arrow[dr] & \\ & B \\ A_2 \arrow[ur] & \end{tikzcd}$$

Is this contractible? NO

Heuristic argument: We know $\mathcal{I}$ is contractible, hence so must be the space of paths joining A_1 to A_2 in the realization of the nerve of $\mathcal{I}$. But because of the relative sums, such paths should be replaceable by 2-stage paths.

$$\begin{tikzcd} A_1 \arrow[r] & B & A_2 \arrow[l] \\ A_1 & 0 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & A_2 \\ A_1 \arrow[r] & A_1 \oplus A_2 & A_2 \arrow[l] \end{tikzcd}$$

Why this doesn't work: Assign to $\bullet$ $\begin{smallmatrix} A_1 \to B \\ A_2 \to B \end{smallmatrix}$ the kernel of the map $A_1 + A_2 \longrightarrow B$, and observe the kernel does not change for maps $B \longrightarrow B'$ as they are injective. So the category is not connected.

Given (V^+, V^-) in $\mathcal{J}$ we consider the category of arrows

$$\begin{tikzcd}[row sep=small] (V^+,V^-) \arrow[dr] & \\ & (W^+,W^-) \\ (0,0) \arrow[ur] & \end{tikzcd}$$

i.e. of stable isomorphisms of V^+ and V^-. Then $\mathrm{Aut}(V^+,V^-) = \mathrm{Aut}(V^+) \times \mathrm{Aut}(V^-)$ acts on this category. Observe that the category is not connected. In effect

$$\begin{tikzcd} V^+ \arrow[r, hook] & W^+ \arrow[d, phantom, "\wr"] \\ V^- \arrow[r, hook] & W^- \end{tikzcd}$$

so we can intersect V^+ and V^-, and the dimension of the intersection I at least

$$\begin{tikzcd}[row sep=small] & & V^+ \arrow[dr] & & \\ 0 \arrow[r] & I \arrow[ur] \arrow[dr] & & V^+ + V^- \arrow[r] & W \\ & & V^- \arrow[ur] & & \end{tikzcd}$$

is an invariant.

<u>Question</u>: Can we modify the above category so that its components are K_1 at least in the stable range?

March 15, 1972: Review of power operations and the symmetric gp.

~~Recall that power operations lead to ma~~

$$\cancel{H^{2i}(X) \xrightarrow{ext.} H^{2ni}_{\Sigma_n}(X^n) \xrightarrow{\Delta} H^{2ni}_{\Sigma_n}(X)}$$

I shall be interested in power operations in mod p cohomology. Begin with the external power operation

$$H^{2i}(X) \longrightarrow H^{2ni}_{\Sigma_n}(X^n)$$

$$H^{2i+1}(X) \longrightarrow H^{n(2i+1)}_{\Sigma_n}(X^n, \mathbb{F}_p^{\,sign})$$

I don't know how to think of this yet, so from now on treat $p=2$. Then we have power operation (external)

$$P_n^{ext}: H^{\cdot}(X) \longrightarrow H^{\cdot}_{\Sigma_n}(X^n)$$

with the following properties

$$P_n^{ext}(x+y) = \sum_{i+j=n} \operatorname{ind}_{\Sigma_i \times \Sigma_j}^{\Sigma_n} \left(P_i^{ext} x \boxtimes P_j^{ext} y\right)$$

$$P_n^{ext}(xy) = P_n^{ext}x \cdot P_n^{ext}y$$

$$P_n^{ext}(0) = \begin{cases} 1 & n=0 \\ 0 & n>0. \end{cases}$$

Here if $H \subset G$ is a subgroup of finite index, and if X is any G-space, then

$$\operatorname{ind} : H_H(X) \longrightarrow H_G(X)$$

is the trace map for the covering

$$\begin{array}{ccc} PH \times^H X & \longrightarrow & PG \times^G X \\ \downarrow & \text{cart.} & \downarrow \\ BH & \longrightarrow & BG \end{array}.$$

It is natural for maps of G-spaces, so

$$\Delta^* \operatorname{ind}_{\Sigma_n}^{\Sigma_i \times \Sigma_j} \left(P_i^{ext} x \boxtimes P_j^{ext} y \right) = \operatorname{ind}_{\Sigma_n}^{\Sigma_i \times \Sigma_j} \left(P_i x \boxtimes P_j y \right)$$

where

$$P_i x = \Delta P_i^{ext}(x) \qquad \Delta : X \longrightarrow X^i$$

is the internal Steenrod operation.

Now I want to ~~consider~~ arrange the family of P_n^{ext} in a coherent way. So let me consider the functor

$$F(X) = \left\{ (\alpha_n) \in \prod_{n \geq 0} H_{\Sigma_n}(X^n) \;\middle|\; \begin{array}{l} \forall i+j=n \quad \operatorname{res}_{\Sigma_i \times \Sigma_j}^{\Sigma_n} \alpha_n = \alpha_i \boxtimes \alpha_j \\ \alpha_0 = 1 \end{array} \right\}$$

<u>Proposition:</u> $F(X)$ is a ring with

$$(\alpha+\beta)_n = \sum_{i+j=n} \operatorname{ind}_{\Sigma_n}^{\Sigma_i \times \Sigma_j} \alpha_i \boxtimes \beta_j$$

$$(\alpha\beta)_n = \alpha_n \beta_n$$

Proof: ~~[illegible]~~ Must show that addition is well-defined. Need to know

$$\mathrm{res}^{\Sigma_n}_{\Sigma_i \times \Sigma_j} \ \mathrm{ind}^{\Sigma_a \times \Sigma_b}_{\Sigma_n}$$

For this use the double coset formula

$$\mathrm{res}^G_K \ \mathrm{ind}^H_G = \sum_{KxH} \mathrm{ind}^{K \cap xHx^{-1}}_K \ \mathrm{res}^H_{K \cap xHx^{-1}}$$

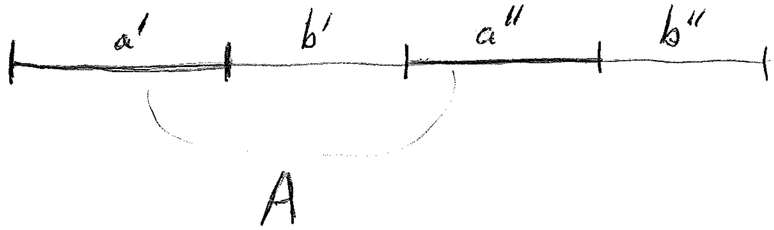
$$\begin{tikzcd} G/K \times G/H \arrow[r] \arrow[d] & G/H \arrow[d] \\ G/K \arrow[r] & G \end{tikzcd}$$

Now $\Sigma_n / \Sigma_a \times \Sigma_b$ = subsets of $\{1, \ldots, n\} = S$ of card a

$\cong$ (a, b) shuffles.

Thus given $A \subset S$ of order a there is a nice shuffle permutation σ_A sending $\{1, \ldots, a\}$ to A in order. Now we want to look at the orbits of $\Sigma_i \times \Sigma_j$ on $\Sigma_n / \Sigma_a \times \Sigma_b$. So what's important is $A \cap \{1, \ldots, i\}$ so given a double coset we get a decomp.

$$\begin{aligned} a &= a' + a'' \\ b &= b' + b'' \\ a' + b' &= i \\ a'' + b'' &= j. \end{aligned}$$



~~[illegible]~~ The double cosets are thus in 1-1 correspondence with $0 \le a' \le a$. For our coset representative take the element $\tau_{a'}$ ~~interchanging the~~ sending

$$[a'+1, \dots, a] \longmapsto [a'+b', \dots, a+b']$$
$$[a+1, \dots, a+b'] \longmapsto [a'+1, \dots, a'+b']$$

and fixing the rest. The stabilizer of A is clearly

$$\begin{array}{ccc} \Sigma_{a'} \times \Sigma_{b'} & \times & \Sigma_{a''} \times \Sigma_{b''} \longrightarrow \Sigma_a \times \Sigma_b \\ \cap & & \cap \\ \Sigma_i & \times & \Sigma_j \end{array}$$

and so it is clear that

$$\boxed{\begin{array}{c} \operatorname{res}^{\Sigma_n}_{\Sigma_i \times \Sigma_j} \operatorname{ind}^{\Sigma_a \times \Sigma_b}_{\Sigma_n} = \sum\limits_{0 \le a' \le a} \operatorname{ind}^{\Sigma_{a'} \times \Sigma_{b'} \times \Sigma_{a''} \times \Sigma_{b''}}_{\Sigma_i \times \Sigma_j} \operatorname{res}^{\Sigma_a \times \Sigma_b}_{\Sigma_{a'} \times \Sigma_{b'} \times \Sigma_{a''} \times \Sigma_{b''}} \\ \text{where} \quad \begin{array}{l} a+b=n \\ i+j=n \end{array} \quad \begin{array}{l} a = a'+a'' \\ b = b'+b'' \end{array} \quad \begin{array}{l} i = a'+b' \\ j = a''+b'' \end{array} \end{array}}$$

Starting with $\alpha_a \boxtimes \alpha_b$

$$\operatorname{res}^{\Sigma_n}_{\Sigma_i \times \Sigma_j} \operatorname{ind}^{\Sigma_a \times \Sigma_b}_{\Sigma_n} \alpha_a \boxtimes \alpha_b = \sum_{0 \le a' \le a} \operatorname{ind}^{\Sigma_{a'} \times \Sigma_{b'} \times \Sigma_{a''} \times \Sigma_{b''}}_{\Sigma_i \times \Sigma_j} \alpha_{a'} \boxtimes \alpha_{b'} \boxtimes \alpha_{a''} \boxtimes \alpha_{b''}$$

$$\| $$

$$\left(\operatorname{ind}^{\Sigma_{a'} \times \Sigma_{b'}}_{\Sigma_i} \alpha_{a'} \boxtimes \alpha_{b'} \right) \boxtimes \left(\operatorname{ind}^{\Sigma_{a''} \times \Sigma_{b''}}_{\Sigma_j} \alpha_{a''} \boxtimes \alpha_{b''} \right)$$

so addition is well-defined.

Associativity and commutativity of addition clear. The fact that $-\alpha$ exists is clear by recursion

$$(\alpha+\beta)_n = \alpha_n + \mathrm{ind}\ \alpha_1 \boxtimes \beta_{n-1} + \cdots + \beta_n .$$

Distributivity clear from

$$[\gamma(\alpha+\beta)]_n = \gamma_n \Big[\sum_{i+j=n} \mathrm{ind}_{\Sigma_n}^{\Sigma_i\times\Sigma_j}\ \alpha_i \boxtimes \beta_j\Big]$$

$$= \sum_{i+j=n} \mathrm{ind}_{\Sigma_n}^{\Sigma_i\times\Sigma_j}\Big[\underbrace{\Big(\mathrm{res}^{\Sigma_n}_{\Sigma_i\times\Sigma_j}\ \gamma_n\Big)\cdot \alpha_i \boxtimes \beta_j}_{\displaystyle (\gamma_i\alpha_i)\boxtimes(\gamma_j\beta_j)}\Big] .$$

~~Basic conclusion~~

Variants. Let R be a graded anti-comm. $\mathbb{F}_p$-algebra and consider

$$F(R) = \Big\{ (\alpha_n) \in \prod_{n\geq 0} H^0(B\Sigma_n, R)\ \Big|\ \begin{matrix} \forall i+j=n & \alpha_0 = 1 \\ & \mathrm{res}^{\Sigma_n}_{\Sigma_i\times\Sigma_j}\ \alpha_n = \alpha_i \boxtimes \alpha_j \end{matrix} \Big\}$$

This is also a ring. But such an α is the same as a ring homomorphism

$$\Gamma = \bigoplus_{n\geq 0} H_*(B\Sigma_n) \longrightarrow R.$$

Thus

$$F(R) = \mathrm{Hom}_{\text{rings}}(\Gamma, R)$$

is a ring so F is an affine ring scheme.

Question: I also have an interpretation of $F(R)$ as exp. characteristic classes for representations of groups on finite sets with coefficients in R. The standard operation on such exp. classes

$$(\theta\cdot\theta')(E) = \theta(E)\,\theta'(E)$$

corresponds to the multiplication in $F(R)$. What is the interpretation of the addition of $F(R)$?

The point seems to be this. Given an n-fold covering $E \to X$, consider the induced covering of i-fold subsets of fibres of E, call it $Y_i \xrightarrow{u_i} X$. Then $u_i^*(E) = E_i' + E_i''$ so we can form $\theta(E_i')\,\theta'(E_i'')$ and then

$$(\theta+\theta')(E) = \sum_{i\geq 0} u_{i*}\left[\theta(E_i')\,\theta'(E_i'')\right].$$

Proof that $(\theta+\theta')$ is an exponential characteristic class: Set $A_i(E)$ for the i-fold subsets of the fibres of E. Then ~~[illegible]~~

$$\Lambda_i(E+F) = \coprod_{j=0}^{i} A_j(E) \times_X A_{j-i}(F)$$

source étale cat. with ob = G-sets
a model for $BGL(R)^+$

March 15, 1972

I have seen already the usefulness of ~~[illegible]~~ topological categories $\mathcal{C}$ in which the source map is etale. ~~[illegible]~~ Consider now the analogue in which the space $Ob\,\mathcal{C}$ is replaced by the topos of G-sets.

Thus $Ob\,\mathcal{C}$ is the topos of G-sets, and $Ar\,\mathcal{C}$ is the topos of G-sets over a G-set S

$$Ar\,\mathcal{C} = (G\text{set})_{/S}$$

and the source map is the localization arrow

$$(G\text{sets})_{/S} \longrightarrow (G\text{-sets})$$

The target map must be a similar morphism of topoi, hence must be given by a G-torsor P over S in the category of G-sets. Thus P is a $(G\times G^0)$-set which is free for the action of the second factor. The composition arrow will be given by a map

$$P \times^G P \longrightarrow P$$

of $G\times G^0$ sets, ~~[illegible]~~ which is associative in the evident sense. The point is that once s,t are given we can consider sheaves over $Ob\,\mathcal{C}$ on which $Ar\,\mathcal{C}$ acts

$$\begin{array}{c} F \\ \downarrow \\ Ob\,\mathcal{C} \end{array} \qquad \begin{tikzcd} F \times_{Ob\,\mathcal{C}} Ar\,\mathcal{C} \arrow[rr] \arrow[dr] & & F \arrow[dl] \\ & Ob\,\mathcal{C} & \end{tikzcd}$$

i.e. we give a map

$$\begin{array}{c} t^*F \longrightarrow s^*F \\ \shortparallel \\ P\times^G F \longrightarrow (P/G)\times F \end{array}$$

Thus the action will be a map $P \times^G F \longrightarrow F$ of G-sets. Associativity will be expressed in the obvious way. Next the identity which must be a section of S, i.e. a fixed point $\varepsilon \in S$. Then the axiom $t\varepsilon = id_{pt}$ must be expressed by giving a trivialization of the the torsor P over ε, i.e. we give a point $e \in P$. Then it is more or less clear that the identity axioms may be expressed as saying that e is an identity for the multiplication of P.

Thus P is an associative monoid. The group G acts to the left and right of P, freely on the right. I claim that $g \mapsto ge = eg$ is an inj. homomorphism from G to P and that the left and right action by G is just multiplication with respect to the monoid structure of P. Indeed let $\mu : P \times P \longrightarrow P$ is the mult. of P. (Note that the homom. $g \mapsto g'$ defined by

$$ge = eg'$$

is required to be the identity of G, so $ge = eg$.) Then

$$\mu(x, ge) = \mu(xg, e) = xg$$
$$\mu(eg, x) = \mu(e, gx) = gx$$

so the claim is clear.

Summarizing

<u>Proposition:</u> Let G be a group and let [redacted] P be a monoid containing G such that right mult. $p, g \longmapsto pg$ is a free G-action. Then

$$\mathrm{Ob}\,\mathcal{C} = (G\text{-sets})$$

$$\mathrm{Ar}\,\mathcal{C} = (G\text{-sets})_{/(P/G)}$$

$$\begin{tikzcd}
(G\text{-sets})_{/P\times^G P/G} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & (G\text{-sets})_{/P/G} \arrow[r, shift left, "s^*F = P/G\times F"] \arrow[r, shift right, "t^*F = P\times^G F"'] & (G\text{-sets}) \ni F
\end{tikzcd}$$

constitutes the analogue of an etale topological category with $\mathrm{Ob}\text{-}\mathcal{C} =$ G-sets. Every such is obtained in this way.

<u>Examples:</u>

~~1) Suppose G is a subgroup of a group P. Then we have to consider~~ [redacted] ~~P as a G-torsor (right action) in the category of G-sets (left action). Such a P is determined up to isomorphism by~~

1) Let G be the diffeomorphisms of $[0,1] = I$ with support in the interior, and let P be the monoid of smooth injections

$$[0,1] \xrightarrow{\sim} [0,t] \subset [0,1]$$

where the first diffeomorphism $= x$ [redacted] near $x = 0$. Then

G is a subgroup of P. Moreover G acts freely on P to the right.

2) Suppose B is a subgroup of a group G. Then with the notation change ($G \mapsto B$, $P \mapsto G$), we ~~[illegible]~~ can consider G as a $B \times B^\circ$-set P. Then we consider the B-set $S = G/B$ and break it into B-orbits

$$G/B = \coprod_{w \in W} BwB/B$$

where w runs over double coset representatives. For each w, we have stabilizer of coset wB

$$B \cap wBw^{-1},$$

and two injections of it into B. Thus one way of describing of P as a $B \times B^\circ$-set is by choosing representatives $\{wB, w \in W\}$ for the B-orbits on G/B and using the stabilizers and the homom. $B \cap wBw^{-1} \longrightarrow B$ defined by the B-torsor over ~~[illegible]~~ the point wB. To reconstruct G I also need what data?

$$\begin{array}{ccc} G \times_B G & \longrightarrow & G = \coprod_w BwB \\ \| & & \\ \coprod_{w_1, w_2} Bw_1B \times_B Bw_2B & & \\ \| & & \\ Bw_1Bw_2B & & \end{array}$$

$$G = \coprod_{w} B \overset{i_w \swarrow\, B \cap wBw^{-1} \,\searrow j_w}{\times} B \qquad ?$$

<u>Example:</u> Consider the following category. Its objects are pairs (S, V) where S is a finite linearly ordered set, V is a f.d. vector space over k. An arrow $(S, V) \longrightarrow (S', V')$ consists of a monotone injection $S \longrightarrow S'$, a split injection $V \oplus Q \xrightarrow{\sim} V'$, and an isom. of $S' - S$ with a basis of Q. We consider the full subcategory with $\dim V - \operatorname{card} S = m$. It is clear the category is equivalent to the full subcategory with objects $(\{1, \dots, n\}, k^{m+n})$.

It is the cofibred category belonging to the functor

$$\langle n \rangle = \{1, \dots, n\} \longmapsto GL_{m+n}(k) = \operatorname{Aut}(k[\langle m \rangle \cup \langle n \rangle])$$

where given a map $\langle n \rangle \longrightarrow \langle n' \rangle$ one considers the induced map $GL_{m+n} \longrightarrow GL_{m+n'}$ doing the appropriate thing on the last coordinates.

(Should be careful. Clear that fibre over $\langle n \rangle$ equivalent to $\operatorname{Aut}(k^{m+n})$ with the understanding that this be $\emptyset$ if $m < n$. Now given k^{m+n} $k^{m+n'}$

$$\langle n \rangle \xrightarrow{u} \langle n' \rangle$$

we get split injection $k^{m+n} \oplus Q \xrightarrow{\sim} k^{m+n'}$ together with an isomorphism of the complement of u with a basis for Q. When $m \geq 0$ there is a canonical such arrow working on the last n-coordinates, so the given arrow $(\langle n \rangle, k^{m+n}) \longrightarrow (\langle n' \rangle, k^{m+n'})$ is uniquely expressible as the product of the canonical arrow and an auto. of k^{m+n}. When $m<0$, there is no canonical such arrow, however we could look at all subspaces of $k[n>]$ of codim $-m$ if we wanted to.)

Anyway let's worry only about $m \geq 0$. Then we have

$$GL_m \longrightarrow GL_{m+1} \rightrightarrows GL_{m+2} \; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \; GL_{m+3} \cdots\cdots$$

$$0 \longrightarrow 1 \rightrightarrows 2 \; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \; 3 \cdots\cdots$$

What is the fundamental group of the category

$$GL_m \longrightarrow GL_{m+1} \rightrightarrows GL_{m+2} \quad ?$$

Assuming m is in a stable range, the homology of the fibres will be constant over the base which is the category

$$0 \longrightarrow 1 \rightrightarrows 2$$

which has an initial object, hence is contractible.

<u>Prop.</u> Let $i \mapsto G_i$ be a functor from I to groups. ~~If I is connected, then the fundamental group~~ And let $\mathcal{G}$ be the associated cofibred category over I. Assume I has an initial object i_0. Then

$$\pi_1(\mathcal{G}, i_0) \xleftarrow{\sim} \varinjlim_{i \in I} G_i$$

the limit inductive being taken in the category of groups.

Proof. Let F be a local coefficient system on $\mathcal{G}$. Then for each i, $F(i)$ is a G_i set, and for each $i \xrightarrow{u} i'$, $F(u): F(i) \xrightarrow{\sim} F(i')$ is compatible with the map $G_i \longrightarrow G_{i'}$ induced by u.

Suppose S is a $\varinjlim G_i$ set. Then set $F(i) = S$ for all i and let G_i act on $F(i)$ as it should. Then we get a local coefficient system on $\mathcal{G}$ with $F(u) = id_S$ for all arrows u in I. This defines a homomorphism

$$\pi_1(\mathcal{G}, i_0) \longrightarrow \varinjlim G_i$$

in general.

Now in general we have a map

$$\varinjlim_{i_0 \to i} G_i \longrightarrow \pi_1(\mathcal{G}, i_0)$$

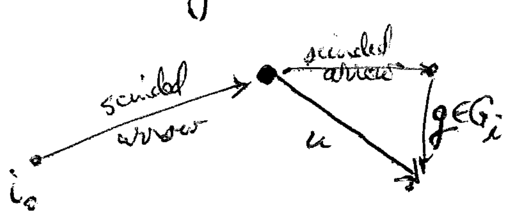
for any object i_0. This assigns to F the set

$$\varinjlim_{i_0 \to i} F(i) \xleftarrow{\sim} F(i_0).$$

The composition

$$\varinjlim_{i_0 \to i} G_i \longrightarrow \pi_1(\mathcal{G}, i_0) \longrightarrow \varinjlim_{i} G_i$$

is evidently the obvious one, so it is the identity when i_0 is an initial object. Finally it is clear that any arrow u in $\mathcal{G}$



comes from an element of G_i for some i. QED.

So we see that it is necessary to consider the inductive limit of

$$GL_m \longrightarrow GL_{m+1} \rightrightarrows GL_{m+2}$$

$$\bullet \longrightarrow 1 \rightrightarrows 2$$

We must identify the two images of an element of GL_{m+1} in GL_{m+2}. The two embeddings are conjugate by the matrix

$$\gamma : \left(\begin{array}{c|cc} I_m & & \\ \hline & & +1 \\ & +1 & \end{array}\right)$$

whose centralizer is small, consequently lots of commutators $\alpha\gamma\alpha^{-1}\gamma^{-1}$ become zero in the inductive limit. Now with $m \geq 1$, we have to divide out by the normal subgroup of $SL_3(\mathbb{Z})$ generated by the difference of a transposition and any conjugate. Thus have to kill all of $SL_3(\mathbb{Z})$. So with $m \geq 1$ the fundamental group problably is GL_{m+2} modulo the normal subgroup generated by elementary matrices.

March 17, 1972

Consider the local field situation $K, A, \mathfrak{m}, k$. If V is a finite-dimensional vector space over K, let $X(V)$ be ~~its~~ its building. Thus $X(V)$ is the simplicial complex whose i-simplices are chains of lattices

$$L_0 < \cdots < L_i$$

such that $\mathfrak{m} L_i \subset L_0$. I know that $X(V)$ is contractible, hence ~~...~~ the cohomology of $\mathrm{Aut}(V)$ should be accessible through the stabilizers of the simplices of $X(V)$.

What I ~~...~~ want to do is consider the complex of chains on X:

$$0 \to C_{n-1}(X) \to \cdots \to C_1(X) \longrightarrow C_0(X) \longrightarrow \mathbb{Z}$$

$$C_i(X) = \mathbb{Z}[X_i]$$

where X_i is the set of i-simplices. Given a simplex

$$\sigma : \quad L_0 < \cdots < L_i$$

let

$$\underline{b}(\sigma) = (b_0, b_1, \ldots, b_i)$$

$$b_0 = \dim_k L_0 / \mathfrak{m} L_i$$

$$b_j = \dim L_j / L_{j-1} \qquad 1 \leq j \leq i.$$

so that $\underline{b}(\sigma)$ is a sequence satisfying

$$(*) \qquad \begin{cases} b_0 \geq 0, \; b_1 > 0, \ldots, b_i > 0 \\ \sum_0^i b_j = n. \end{cases}$$

It is easy to see that $\underline{b}(\sigma) = \underline{b}(\sigma') \iff \sigma$ and σ' are conjugate under $G = \mathrm{Aut}(V)$. Thus the G-orbits on X_i are in 1-1 correspondence with sequences (*), so

$$X_i = \coprod_{\sigma \in S_i} G/G_\sigma$$

where S_i is a set of representatives for the orbits, so

$$C_i(X) = \bigoplus_{\sigma \in S_i} \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_\sigma]} \mathbb{Z}.$$

~~Now I want a spectral sequence~~

$$\cancel{E^1_{p*} = H_*}$$

Now if I regard G as a discrete group, then since the complex of chains $C_\cdot(X)$ is a resolution of $\mathbb{Z}$, I have a spectral sequence

$$E^1_{p*} = H_*(G, C_p(X)) \Longrightarrow H_*(G, \mathbb{Z})$$

$$\| $$

$$\bigoplus_{\sigma \in S_p} H_*(G_\sigma, \mathbb{Z})$$

and similarly one in cohomology, using the cochain complex. I recall having trouble with the topological situation.

~~Now suppose (I consider~~

Consider G as a topological category. (More invariantly, consider the top. cat. $\mathcal{G}$ whose objects are f.d. v.s. $/K$ of dim n (discrete top.) and whose morphisms are isomorphisms, $\mathrm{Isom}(V,W)$ being endowed with its customary topology.) We then can consider (G,X) as a cofibred and fibred category over G. Specifically the category whose objects are pairs (V,σ) where σ is a simplex of $X(V)$, and where a morphism $(V,\sigma) \longrightarrow (W,\tau)$ consists of an isom $\alpha : V \xrightarrow{\sim} W$ such that ~~[illegible]~~ $\alpha(\sigma) \leq \tau$. Evident topology.

Now consider the morphism

$$(G,X) \xrightarrow{\quad f \quad} G .$$

The fibres are contractible. Here I have an idea that f is an essential morphism of topoi when one considers gross topoi.

On the other hand we have a functor

$$(G,X) \longrightarrow \Delta^{+}_{\leq n} \left\{ \begin{array}{l} \text{full subcat. of } \Delta \text{ consisting of} \\ [1], [2], \ldots, [n], \text{ and injective} \\ \text{monotone maps.} \end{array} \right.$$

This functor is fibred. So we should be able to conclude that cohomology $H^*(G, \mathbb{F}_\ell)$ (E-M coh.) is abutment of a spectral sequence with

$$E_1^{pq} = \prod_{\sigma \in S_p} H^q(G_\sigma, \mathbb{F}_\ell) .$$

Alternative approach. X simplicial set (allow degenerate simplices). G acts on X, so can form (X,G) simplicial category, and $\mathrm{Nerv}(X,G)$ which is a bisimplicial space. Now given a top. abelian group A, we obtain a double complex of abelian groups

$$C^*(\mathrm{Nerv}(X,G), A) = \mathrm{Map}(G^q \times X_p, A).$$

Now for fixed p we have the cochains with values in A of the top. category (G, X_p) which is equivalent to $\coprod_{\sigma \in S_p^+} (G_p, e)$, so we get the desired spectral sequence — even degenerate simplices

$$E_1^{pq} = \bigoplus_{\sigma \in S_p^+} H^q(G_\sigma, A) \Longrightarrow H^{p+q}(C^*(\mathrm{Nerv}(X,G), A))$$

In the other direction

$$\mathrm{Map}(G^q \times X_p, A) = \prod_{X_p} \mathrm{Map}(G^q, A)$$

so

$$C^*(\mathrm{Nerv}(X,G), A) = C(X, C(\mathrm{Nerv}\, G, A))$$

so we get contractibility.

<u>Conclude</u>: The Eilenberg-Maclane cohomology of $GL_n(K)$ with coefficients in any top. abelian group A can be reduced to that of the stabilizers of the simplices of the building.

Applications:

(1) Take $A = \mathbb{Q}_p$. According to Lazard

$$\varinjlim_{U} H^*(U, \mathbb{Q}_p) = H^*(\mathfrak{g}, \mathbb{Q}_p)$$

where U runs over the open subsets of $GL_n(A)$. The point now is that if U is normal in V, then

$$H^*(U, \mathbb{Q}_p)^{V/U} \xleftarrow{\sim} H^*(V, \mathbb{Q}_p)$$

so taking the limit over U and using the fact that

? $\underline{GL_n K \text{ acts trivially on } H^*(\mathfrak{g}, \mathbb{Q}_p)}$ we see that for all σ in the building

$$H^*(G_\sigma, \mathbb{Q}_p) \xrightarrow{\sim} H^*(\mathfrak{g}, \mathbb{Q}_p).$$

Consequently the E_1 term is the chains on the orbit space X/G. Thus in the SL_n case done as this orbit is a simplex. In the case of $GL_n K$, I have to consider the simplicial set whose i-simplices are sequences

$$b_0, b_1, \ldots\ldots, b_i \geq 0$$

with $\sum b_i = n$ where the faces and deg. ops. are

$$d_j(b_0, \ldots, b_i) = \begin{cases} (b_0 + b_1, b_2, \ldots, b_i) & j = 0 \\ (b_0, \ldots\ldots, b_{i-1} + b_i) & j = i-1 \\ (b_0 + b_i, b_1, \ldots\ldots, b_{i-1}) & j = i \end{cases}$$

and

$$s_j(b_0,\dots,b_i) = (b_0,\dots,b_j,0,b_{j+1},\dots,b_i) \qquad 0\le j\le i$$

We compute the homology as follows. Let $R=\mathbb{Z}[T]$ be the group ring of $\mathbb{N}$. Then we start with the standard resolution

$$\cdots\ R\otimes R\otimes R \rightrightarrows R\otimes R$$

$$d_j(b_0\otimes\cdots\otimes b_{n+1}) = b_0\otimes\cdots\otimes b_j b_{j+1}\otimes\cdots\otimes b_{n+1} \qquad 0\le j\le n$$

of R as an $R\otimes R^{\circ}$-module. (let $[b_0,\dots,b_n]=b_0\otimes\cdots\otimes b_n$)

$$(r\otimes s)[b_0,\dots,b_{n+1}] = [rb_0,\dots,b_{n+1}s].$$

The complex we are looking at is obtained by tensoring with R over $R\otimes R$, ~~Thus I wish to look~~ because

$$R\otimes_{R\otimes R}(R\otimes R^n\otimes R) = R\otimes R^n$$

$$\begin{array}{ccc}
x\otimes[b_0,\dots,b_{n+1}] & \longmapsto & b_0 x b_{n+1}\otimes[b_1,\dots,b_n] \\
\downarrow d_n & & \downarrow d_n \\
x\otimes[b_0,\dots,b_n b_{n+1}] & \longmapsto & b_0 x b_n b_{n+1}\otimes[b_1,\dots,b_{n-1}]
\end{array}$$

OKAY. Thus the homology of the complex under consideration is

$$\mathrm{Tor}_p^{R\otimes R}(R,R) = \begin{cases} R & p=0 \\ R\cdot\overline{T} & p=1 \\ 0 & p>1 \end{cases}$$

Thus it seems that $X/G \sim S^1$; (have to take the degree n part of the preceding, and this contributes for $n \geq 1$.)

<u>Conclude:</u> E-M cohomology of $GL_n(K)$, $[K:\mathbb{Q}_p]<\infty$ with coefficients in $\mathbb{Q}_p$ is

$$\cong H^*(S^1, \mathbb{Q}_p) \otimes H^*(\mathfrak{gl}_n(K), \mathbb{Q}_p)$$

$$\cong \Lambda\left[K[1], K[3], \ldots, K[2n-1]\right].$$

(2) Take $A = \mathbb{F}_\ell$, where ℓ prime to p. Here we know that

$$H^*(G_\sigma, \mathbb{F}_\ell)$$

is finite-dimensional in each dimension, in fact,

$$H^*(G_\sigma) = H^*(GL_{b_0}(k)) \otimes H^*(GL_{b_1}(k)) \otimes \cdots \otimes H^*(GL_{b_n}(k))$$

if $\underline{b}(\sigma) = (b_0, \ldots, b_n)$. It will be preferable to work with homology defined as the dual of cohomology. Then

$$E^1_{p*} = \bigoplus_{\sigma\in\tilde{S}_p} H_*(G_\sigma) = \bigoplus_{\substack{(b_0,\ldots,b_p)\\ \sum b_i = n}} \bigotimes_i H_*(GL_{b_i}(k))$$

$$= R \otimes R^p \qquad \text{where} \qquad R = \bigoplus_{n\geq 0} H_*(GL_n k)$$

Again it should be possible to identify

$$E^2_{p*} = \mathrm{Tor}_p^{R\otimes R}(R, R)$$

But I know

$$R = \cancel{\mathbb{F}_\ell[\xi, \xi_1, \xi_2, \dots, \eta_1, \eta_2, \dots]}$$
$$P[\varepsilon, \xi_1, \xi_2, \dots] \otimes \Lambda[\eta_1, \eta_2, \dots]$$

Thus

$$\mathrm{Tor}_*^{R\otimes R}(R, R) = \mathrm{Tor}^{R_1\otimes R_1}(R_1, R_1) \otimes \mathrm{Tor}^{R_2\otimes R_2}(R_2, R_2)$$

$$\cong \begin{array}{lcl} P[\varepsilon, \xi_1, \xi_2, \dots] & \otimes & \Lambda[\eta_1, \eta_2, \dots] \\ \otimes \Lambda[\bar{\varepsilon}, \bar{\xi}_1, \bar{\xi}_2, \dots] & \otimes & \Gamma[\bar{\eta}_1, \bar{\eta}_2, \dots] \end{array}$$

March 19, 1972

Problem: To prove the homotopy equivalence of the following two [redacted] categories:

(i) $\mathcal{J}$. This [redacted] I think of as being the simplest version of $B(M\times M, \Delta M)$ where $M = B$(f.g. proj R-modules + isos.). Its objects are pairs (V^+, V^-) of f.g. proj. R-modules. An arrow $(V^+, V^-) \longrightarrow (W^+, W^-)$ consists of a pair of split injections and an isomorphism of the complements.

(ii) Simple version of $B(M\times N, N)$ where $N = B$(finite sets). Call it $\mathcal{J}_0$. Its objects are pairs (V, S) where V is a fgp module and S is a finite set. An arrow consists of a split-injection $V \to V'$, + injection $S \to S'$, + isom. of [redacted] $V'-V$ with $R(S'-S)$.

Idea of a solution. Let $M\times N$ act on $M\times M\times N$ by

$$w(v^+, v^-, s)t = (w+v^+, w+v^- + t, s+t)$$
$$\cancel{(v^+, v^-, s)(w,t) = (v^+ + w + t, v^- + w, s+t)}$$

and call the result $B(M\backslash M\times M\times N/N)$. Then we have projections

$$\begin{tikzcd}
 & B(M\backslash M\times M\times N/N) \arrow[dl, "\text{fibre } B(M\times N/N)"'] \arrow[dr, "\text{fibre } B(M\backslash M\times M)"] & \\
B(M|M) & & B(N,N)
\end{tikzcd}$$

and the point is that the projections are quasi-fibrations with contractible bases.

So I wish to consider the category $\mathcal{J}_1$ whose objects are triples (V^+, V^-, S) and in which an arrow $(V^+, V^-, S) \longrightarrow (W^+, W^-, S')$ consists of

$$V^+ \overset{\pi^+}{\underset{\longrightarrow}{\longleftarrow}} W^+$$

$$V^- \overset{\pi^-}{\underset{\longrightarrow}{\longleftarrow}} W^- \qquad \operatorname{Ker} \pi^- \overset{\sim}{\longleftarrow} \operatorname{Ker} \pi^+ \oplus R[S-S']$$

$$S \hookrightarrow S'$$

(intuitively $\quad W^+ = V^+ + Q, \quad W^- = V^- + Q + R[T], \quad S' = S + T$)

Consider now the projection $(V^+, V^-, S) \longrightarrow S$ from $\mathcal{J}_1$ to $\mathcal{J}_0$ (= finite sets and injections). The fibre over S consists of pairs (W^+, W^-) with usual maps, hence the fibre is equivalent to $\mathcal{J}$. Now given $S \hookrightarrow S'$ and an object (V^+, V^-) over S I want to consider the arrows

$$(V^+, V^-, S) \xrightarrow{\ \alpha\ } (W^+, W^-, S')$$

lying over $S \hookrightarrow S'$. Such an α consists of

$$V^+ \rightarrow W^+ \hookleftarrow Q^+$$

$$V^- \longrightarrow W^- \hookleftarrow Q^-$$

$$Q^- \overset{\sim}{\longleftarrow} Q^+ \oplus R[S'-S].$$

Thus if I consider $(V^+, V^- \oplus R[S'-S], S')$ and canonical arrow $(V^+, V^-, S') \nearrow$, $\exists$ unique map in $\mathcal{J}$

$$(V^+, V^- \oplus R[S'-S]) \longrightarrow (W^+, W^-)$$

yielding α. Thus $\mathcal{J}_1 \longrightarrow \mathcal{J}_0$ is cofibred,

~~initial objects~~ and further, the cobase change functors

$$(V^+, V^-, S) \longmapsto (V^+, V^- \oplus R[S'-S])$$

are homotopy equivalences as we have seen before. Thus since $\mathcal{I}_0$ is contractible, the inclusion of a fibre $\mathcal{J} \to \mathcal{J}_1$ is a homotopy equivalence. Here I use

<u>Lemma</u>: Let $\mathcal{E} \to \mathcal{B}$ be cofibred and suppose the cobase change functors between the fibres are homotopy equivalences (e.g. if bifibred). If $\mathcal{B}$ is contractible, then for any b, $\mathcal{E}_b \hookrightarrow \mathcal{E}$ is a homotopy equivalence.

In the situation just considered one can contract $\mathcal{J}_1$ to the fiber over the **initial object** ϕ of $\mathcal{I}_0$ as follows. Start with the identity functor

$$(V^+, V^-, S) \longmapsto (V^+, V^-, S)$$

and the functors

$$\begin{tikzcd}
(V^+, V^-, S) \arrow[r, mapsto] & (V^+, V^-, S) \arrow[d] \\
(V^+, V^-, S) \arrow[r, mapsto] & (R[S]\oplus V^+, R[S]\oplus V^-, S) \\
(V^+, V^-, S) \arrow[r, mapsto] & (R[S]\oplus V^+, V^-, \phi) \arrow[u]
\end{tikzcd}$$

The vertical arrows are natural transformations giving the deformation.

On the other hand we have the projection from $\mathcal{J}_1$ to $\mathcal{I}$ sending (V^+, V^-, S) to V^+. This should again be cofibred associated to the pseudo-functor

$$V^+ \longmapsto \mathcal{J}_0$$

~~[illegible]~~

$V^+ \oplus Q = W^+$ goes to arrow $(V^+, V^-, S) \longrightarrow (W^+, V^- \oplus Q, S)$.

Thus $\mathcal{J}_1$ is cofibred over $\mathcal{I}$ with fibres $\mathcal{J}_0$, $\mathcal{a}$ acting by sum on the first factor. Now ~~suppose that~~ I know that the action is invertible so again by the lemma it should follow that the inclusion of a fibre $\mathcal{J}_0 \hookrightarrow \mathcal{J}_1$ is a homotopy equivalence.

March 20, 1972. Return to buildings.

Consider the (additive) category a of vector bundles over $\mathbb{P}_1$ over a field k. According to Grothendieck-Hilbert any such bundle $\mathcal{E}$ is a direct sum of the line bundles $\mathcal{O}(n)$. (How this may be proved: One associates to $\mathcal{E}$ the graded module

$$\Gamma_*(\mathcal{E}) = \bigoplus_n \Gamma(\mathbb{P}_1, \mathcal{E}(n))$$

over $\bigoplus_n \Gamma(\mathbb{P}_1, \mathcal{O}(n)) = k[T_0, T_1]$. To prove $\Gamma_*(\mathcal{E})$ is a free $\Gamma_*(\mathcal{O})$-module, proceed as follows. Choose a trivialization of $\mathcal{E}$ off $T_0 = 0$, which is possible as $k[z]$, $z = T_1/T_0$, is a P.I.D. Thus we know that

$$M = \varinjlim_n \Gamma(\mathbb{P}_1, \mathcal{E}(n)) \qquad \text{multiplication by } T_0$$

is a free $k[z]$-module. Now $\mathcal{E}$ is completely determined by M and by the stalk at $z=\infty$ which is a $\mathcal{O}_\infty$-lattice in

$$L \subset k(z) \otimes_{k[z]} M$$

where $\mathcal{O}_\infty = k[z^{-1}]$ localized at ideal $z^{-1}k[z^{-1}]$.

Moreover

$$\Gamma(\mathbb{P}, \mathcal{E}) = L \cap M$$

$$\Gamma(\mathbb{P}, \mathcal{E}(n)) = z^n L \cap M$$

with T_0 acting as the inclusion of $z^n L \cap M$ in

$z^{n+1}L\cap M$ and T_1 as multiplication by z. Now T_0 is regular as far as $\Gamma_*(\mathcal{E})$ is concerned so what we must show is that z is injective on

$$\Gamma_*(\mathcal{E})/T_0\Gamma_*(\mathcal{E}) = \bigoplus_n \frac{z^nL\cap M}{z^{n-1}L\cap M} \qquad n\in\mathbb{Z}.$$

But if $\omega\in z^nL\cap M$ and $z\omega\in z^nL\cap M$, then

$$z\omega = z^n l \qquad l\in L$$

$$\text{so}\quad \omega = z^{n-1}l \in z^{n-1}L\cap M$$

which proves what we want.)

Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}_1$ as above. Then we know

$$\mathcal{E} = \bigoplus_{i=1}^{r} \mathcal{O}(a_i) \qquad r=\operatorname{rank}\mathcal{E}$$

where $a_1\geq a_2\geq \cdots \geq a_r$ (are uniquely determined by Krull-Schmidt.) I claim there is a canonical filtration on $\mathcal{E}$.

$$F_p\mathcal{E} = \bigoplus_{a_i\geq p}\mathcal{O}(a_i)$$

decreasing filtration

Thus we can filter the category $\mathcal{A}$ by saying $\mathcal{E}\in F_p\mathcal{A}$
$\Leftrightarrow$ $\mathcal{E}$ is a direct sum of $\mathcal{O}(n)$ with $n\geq p$
$\Leftrightarrow$ $\operatorname{Hom}(\mathcal{E},\mathcal{O}(p-1))=0$.

Then for any $\mathcal{E}$ it has a largest subobject $F_p\mathcal{E}$

which is in $F_p \mathcal{A}$. Can characterize $F_p \mathcal{E}$ as being generated by the images of all the maps from $\mathcal{O}(p)$ to $\mathcal{E}$.

~~Now by construction we know that the filtration~~

$$\cancel{F_{p+1}\mathcal{E} \subset F_p \mathcal{E} \subset \cdots}$$

~~splits. Therefore let us consider the direct sum K-theory of $\mathcal{A}$.~~

$$\cancel{H_*(\mathrm{Is}\,\mathcal{A}) = \bigoplus}$$

Thus we can describe any $\mathcal{E}$ up to isomorphism by the numbers

$$\nu_p = \dim F_p \mathcal{E} / F_{p+1} \mathcal{E}$$

i.e. the number of times the indecomposable $\mathcal{O}(p)$ occurs in $\mathcal{E}$. The direct sum Grothendieck group is

$$K_0(\mathcal{A}, \oplus) = \mathbb{Z}[X, X^{-1}] \qquad X = \mathrm{cl}(\mathcal{O}(1)).$$

Next consider the K-theory. Given a repn. of G on $\mathcal{E}$ it must preserve the filtration which splits forgetting the action. Thus for any invertible exp. charac. class for repns in $\mathcal{A}$ we have

$$\theta(\mathcal{E}) = \prod \theta(\mathrm{gr}_p \mathcal{E}).$$

by the stable splitting theorem. Therefore it is clear (use $\mathrm{Hom}(\mathcal{O}(p), \mathcal{O}(p)) = k$) that <u>the direct sum K-theory</u>

associated to $\mathcal{A}$ is the direct sum of copies of the K-theory of k, one for each $\mathcal{O}(p)$, $p \in \mathbb{Z}$.

I want to ~~[illegible]~~ understand the K-theory of $k[z]$. So I consider the category of finitely generated projective $k[z]$-modules M and their isomorphisms, and later the direct sum operation. ~~[illegible] I replace M by the category whose objects~~ ~~To M I associate the building which is the simplicial complex~~

Let $K = k(z)$, $A = k[z^{-1}]_{\mathfrak{m}}$, $\mathfrak{m} = z^{-1}k[z^{-1}]$. To a f.g. proj. (i.e. torsion free) $k[z]$-module M, I associate the building $X(M)$ of ~~[illegible]~~ A-lattices L in $K \otimes_{k[z]} M$. Thus $X(M)$ is a simplicial complex whose vertices represent the extension of M to a ~~[illegible]~~ vector bundle on $\mathbb{P}_1$. To a lattice L we have $\mathcal{E}_L$ as on page 1.

$X = X(M)$ being contractible it furnishes a spectral sequence relating the cohomology

$$E_1^{pq} = \bigoplus^p(X/\Gamma, \mathcal{H}^q) \Longrightarrow H^*_\Gamma$$

of Γ and its stabilizers. The first thing I want to do is determine the orbit of Γ on the various simplices and their stabilizers.

<u>Claim:</u> Two lattices L, L' are Γ-conjugate $\Longleftrightarrow$ the sheaves $\mathcal{E}_L$ and $\mathcal{E}_{L'}$ are isomorphic. Moreover

Γ_L is the group of automorphisms of $\mathcal{E}_L$.

<u>Proof:</u> This is clear because a map $\mathcal{E} \to \mathcal{E}'$ ~~is determined by what it does to ... and~~ may be identified with a map $M=\Gamma(\mathbb{A}^1,\mathcal{E}) \longrightarrow M'=\Gamma(\mathbb{A}^1,\mathcal{E}')$ carrying L into L'.

It might be better to note that the category of L in $X(M)$ with morphisms: $\gamma L \subset L'$ is equivalent to the category of $\mathcal{E}$ on $\mathbb{P}^1$ of rank r whose morphisms: $\mathcal{E} \to \mathcal{E}'$ are injections which are isos. on $\mathbb{A}^1$.

I should review now my earlier ideas on the homotopy axiom.

Given M as above, the idea was to consider L which are sufficiently positive so that $V = L \cap M$ is an "involutive" k-subspace of M, i.e. V generates M and $z^{-1}V \overset{\text{defn.}}{=} \{m \in M \mid zm \in V\} \subset V$, whence it follows that

$$0 \to k[z] \otimes z^{-1}V \longrightarrow k[z] \otimes_k V \longrightarrow M \longrightarrow 0$$

is exact. Observe that if $V \subset V'$ are both involutive, then

$$z: \quad V'/V \xrightarrow{\ \sim\ } V'+zV'/V+zV \qquad (?)$$

This should roughly mean that $\mathcal{E}_L$ is of filtration ≥ 0 i.e. contains copies of $\mathcal{O}(n)$, $n \geq 0$.

March 22, 1972 (Carl is [redacted]7 today).

The situation: k field. We are interested in K-theory of $k[z]$ and of the projective line $\mathbb{P}^1$ over k.

$$K = k(z) \quad \text{function field}$$

$$A = \text{valuation ring at } z=\infty$$

$$= k[z^{-1}]_{\mathscr{Y}}, \quad \mathscr{Y} = z^{-1}k[z^{-1}].$$

Then a vector bundle $\mathcal{E}$ over $\mathbb{P}^1$ is the same as a free f.t. $k[z]$-module M together with an A-lattice L in $K \otimes_{k[z]} M$. In fact

$$M = \Gamma(\mathbb{A}^1, \mathcal{E}) \qquad [redacted]$$

$$L = \mathcal{E}_\infty \quad \text{stalk at } \infty.$$

[redacted] (The correspondence $\mathcal{E} \Longleftrightarrow (M, L, L \hookrightarrow [redacted] K \otimes M)$ is essentially [redacted] a special case of the one used [redacted] by Artin to describe sheaves in the situation

$$Y \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U$$

$$\mathcal{E}$$

$$\mathcal{E} \longleftrightarrow (j^*\mathcal{E}, i^*\mathcal{E}, \overset{i^*\mathcal{E}}{[redacted]} \longrightarrow i^*j_*j^*\mathcal{E})$$

Remarks:

The square

[redacted]

$$\begin{array}{ccc} \mathrm{Vect}(\mathbb{P}^1) & \longrightarrow & \mathrm{Vect}(A) \\ \downarrow & & \downarrow \\ \mathrm{Vect}(\mathbb{A}^1) & \longrightarrow & \mathrm{Vect}(K) \end{array}$$

is 2-cartesian,

and, at least conjecturally, it gives rise to a Mayer-Vietoris sequence. Somehow therefore the arrows $\longrightarrow^{\downarrow}$ are "transversal". A basic problem is to define a suitable notion of tranversality.

Idea for proving $K_*(k) \xrightarrow{\sim} K_*(k[z])$:

The point is to show surjectivity. Thus suppose a free f.t. $k[z]$-module M. We then consider the extensions $\mathcal{E} = (M, L)$ of M to a vector bundle over $\mathbb{P}^1$ such that $\mathcal{E} \geq 0$, meaning that it is generated by its global sections (hence that it is isomorphic to $\oplus \mathcal{O}(n_i)$ with $n_i \geq 0$). To such an $\mathcal{E}$ we associate the pair of k-modules of f.t.

$$\left(\Gamma(\mathcal{E}), \Gamma(\mathcal{E}(-1))\right)$$

If $\mathcal{E} \subset \mathcal{E}'$, (i.e. $L \subset L'$) then from

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}' \longrightarrow \underset{\text{support at } \infty}{\mathcal{F}_1} \longrightarrow 0$$

we obtain exact sequences

$$0 \longrightarrow \Gamma\mathcal{E} \longrightarrow \Gamma\mathcal{E}' \longrightarrow \Gamma\mathcal{F} \longrightarrow 0$$

$$0 \longrightarrow \Gamma\mathcal{E}(-1) \longrightarrow \Gamma\mathcal{E}'(-1) \longrightarrow \Gamma\mathcal{F}(-1) \longrightarrow 0$$

because $\mathcal{E} \geq 0 \Rightarrow H^1(\mathcal{E}) = H^1(\mathcal{E}(-1)) = 0$. ~~[illegible]~~
[redacted] But multiplication by t_1 (recall $z = \frac{t_1}{t_0}$ and $t_0 = 0$ at ∞) defines an isomorphism

$$\Gamma\mathcal{F} \xrightarrow{\sim} \Gamma\mathcal{F}(-1).$$

Thus to the inclusion $\mathcal{E} \hookrightarrow \mathcal{E}'$ belongs an inclusion of pairs

$$(\Gamma\mathcal{E}, \Gamma\mathcal{E}(-1)) \longrightarrow (\Gamma\mathcal{E}', \Gamma\mathcal{E}'(-1))$$

together with a trivialization of the cokernel. It's clear this is compatible with composition of injections.

March 23, 1972

Conjecture: Let k be a finite field of characteristic p, and let X be the building of proper subspaces of a fdvs. V over k, $G = \mathrm{Aut}(V)$. Then

$$H^*_G(X; \mathbb{F}_p) \xleftarrow{\sim} H^*_G(pt; \mathbb{F}_p)$$

Meaning: We know that there is a spectral sequence

$$E_2^{pq} = H^p(G, H^q(X)) \Longrightarrow H^{p+q}_G(X)$$

(mod p coefficients) and that

$$H^0(X) = \mathbb{F}_p$$

$$H^{r-1}(X) = \mathrm{Hom}(\text{Steinberg representation}, \mathbb{F}_p)$$

$$H^i(X) = 0 \qquad i \neq 0, r-1.$$

Thus the conjecture asserts that

$$H^*(G, \mathrm{Hom}(\text{Steinberg}, \mathbb{F}_p)) = 0$$

~~Note that if B is a Borel subgr~~

~~res^G_P~~

Recall that if P is a Sylow subgroup of G, then

(*) $\quad \operatorname{res}^G_P(\text{Steinberg}) = \mathbb{Z}[P]$ $\qquad$ not quite correct in case $r = 2$ $\qquad \bar{\mathbb{Z}}[P]$

hence

$$H^*(G, \operatorname{Hom}(\text{Steinberg}, \mathbb{F}_p))$$

$$\Big\downarrow \operatorname{res}$$

$$H^*(P, \operatorname{Map}(P, \mathbb{F}_p)) = \begin{cases} 0 & * > 0 \\ \mathbb{F}_p & * = 0. \end{cases}$$

So what is required to prove the conjecture is to show that the $\mathbb{F}_p$ is not there, for example that it is not fixed by the Borel subgroup.

Example: $r = 2$. Here one knows that P is abelian with normalizer B, so

$$H^*(G, M) \xrightarrow{\sim} H^*(B, M) = H^*(P, M)^{\overset{H}{\overset{\shortparallel}{B/P}}}$$

r all G-modules M which are p-primary. Thus

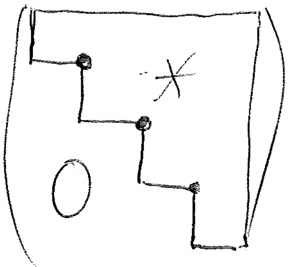
$$H^*_G(X) = H^*_G(G/B) = H^*(B)$$

l the conjecture is clear.

~~To prove the conjecture, suppose we can~~
~~lish the following improvement of (*).~~

<u>March 24, 1972.</u> buildings

Set $G = GL_n k$, k a finite field of characteristic p. Let B be the standard Borel subgroup:



The dots denote the **simple** roots associated to B. These are

$$t = (t_i) \longmapsto t_{i+1}/t_i$$

for $1 \leq i < n$.

Better notation: Let σ denote a simplex of the building of G. Thus σ is a flag

$$0 < V_1 < \cdots < V_p < k^n$$

in k^n. Let P_σ denote the corresponding parabolic subgroup. One knows then that every subgroup $P_\sigma \subset K \subset G$ is of the form $K = P_\tau$ for a unique $\tau \subset \sigma$, $(G = P_\phi)$.

~~Conjecture: Let M be any parabolic subgroup of G~~

<u>Conjecture.</u> Let M be any $\mathbb{Z}[p^{-1}, G]$-module. Then for any σ (including ϕ), the complex

$$\bigoplus_{\substack{\sigma\subset\tau\subset S\\ \mathrm{card}(\tau-\sigma)=2}} H_*(P_\tau, M) \rightrightarrows \bigoplus_{\substack{\sigma\subset\tau\subset S\\ \mathrm{card}(\tau-\sigma)=1}} H_*(P_\tau, M) \longrightarrow H_*(P_\sigma, M) \longrightarrow 0$$

is acyclic. (Here S is a fixed max. flag containing σ).

Remarks:

1) maybe M should be a complex of modules

2) the geometric significance of the above complex: Given σ consider all conjugate flags and all possible refinements of this. This forms a simplicial category whose category of non-deg. p simplices consists of flags

$$W_1 < V_1 < \cdot < W_i < V_q < \cdot\, \cdot W_p\ ?$$

3) induction. Given $P_\sigma \supset P_S = B$, all subgps between P_σ and B contain the radical (unipotent) N_σ of P_σ, and hence

$$H_*(P_\tau, M) = H_*\left(P_\tau/N_\sigma,\ \mathbb{Z} \overset{L}{\otimes}_{\mathbb{Z}[N_\sigma]} M\right)$$

Example: rank 1. $G = GL_2(k)$ acts on $G/B = \mathbb{P}^1$. Since B contains a Sylow p-subgroups one have

$$H^*(G,M) \longrightarrow H^*(B,M) \Longrightarrow H^*_B(G/B, M)$$

for any $\mathbb{Z}[p^{-1}][G]$ module. But there are two orbits

$$G = eB \amalg BsB$$

for B on G/B and the second orbit is free. Thus one has immediately that the second orbit doesn't count for positive dimensional cohomology, i.e.

$$H^+(G,M) \xrightarrow{\sim} H^+(B,M).$$

In dimension zero all one knows from the exact sequence is that

$$H^0(G,M) = \{m \in H^0(B,M) \mid sm = m\}.$$

But this shows the conjecture is wrong because

$$M = \mathrm{Map}(B\backslash G, A) \qquad A \text{ a } \mathbb{Z}[p^{-1}]\text{-module}$$

$$H^0(G,M) = \mathrm{Map}(B\backslash G/G, A) \cong A$$

$$H^0(B,M) = \mathrm{Map}(B\backslash G/B, A) \cong A \oplus A$$

The problem was with the generalized Grün theorem which will not be valid for G-modules.

Thus if the Sylow p-subgroup P of G is abelian

$$H^*(G,A) \xrightarrow{\sim} H^*(P,A)^N = H^*(N,A)$$

$$N = \mathrm{Norm}(P).$$

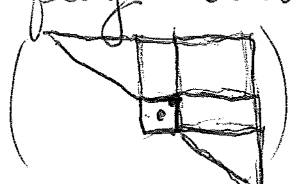
for any $\boxed{\text{trivial}}$ $\mathbb{Z}[p^{-1}, G]$-module A, but not for non-trivial ones. In effect given

$$P \cap xPx^{-1} \underset{j_x}{\overset{i_x}{\rightrightarrows}} P$$

one knows as P is abelian that $\exists\, y \in N$ carrying i_x into j_x, but one needs to worry about the effect of y on A. Review the argument: P being abelian P and xPx^{-1} are both Sylow subgroups of the centralizer of $P \cap xPx^{-1}$ in G, so $\exists\, z$ in centralizer $\ni$ $zPz^{-1} = xPx^{-1}$. Then $y = z^{-1}x$ normalizes P and and $y^{-1}(i_x)y = j_x$. So to make things work with a non-trivial G module M, it is necessary to know that z can be chosen to centralize M.

So modify the conjecture by putting $M = \mathbb{F}_p$.

<u>Example of rank 1.</u> Let $\sigma \subset S$ be the complement of a single element. Thus P_σ is the stablizer of a flag with a single jump of 1 dimension

P_σ :  $0 < V^1 < \cdots < V^p < V^{p+2} < \cdots < V^n = V$

Is $H^*(B) \xrightarrow{\sim} H^*(P_\sigma)$? Again there are two double cosets

$$P_\sigma = B \cup BsB \qquad s \text{ omitted reflection}$$

and

$$B \cap sBs = N_\sigma \quad \text{the unipotent rad. of } P_\sigma.$$

$$1 \to N_\sigma \longrightarrow P_s \longrightarrow\!\!\!\!\to \operatorname{Aut}(V^{p+2}/V^p) \to 1$$

Choose

$$\begin{tikzcd}[row sep=small] & W \arrow[dr, phantom, "\subset"] & \\ V^p \arrow[ur, phantom, "\subset"] \arrow[dr, phantom, "\subset"] & & V^{p+2} \\ & W' \arrow[ur, phantom, "\subset"] & \end{tikzcd}$$

and let B stabilize W, and let s interchange W and W'. The question is whether the homomorphisms

$$N \underset{\text{conj. by } s}{\overset{\text{incl.}}{\rightrightarrows}} B$$

induce the same homomorphism from $H^*(B)$ to $H^*(N)$.

Example: In dimension 3

$$\begin{pmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \begin{matrix} \alpha \\ \beta \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \gamma \end{pmatrix} \qquad\qquad \begin{pmatrix} 1 & 0 & \beta \\ 0 & 1 & \alpha \\ 0 & 0 & \gamma \end{pmatrix}$$

$$N \xrightarrow{\quad x \longmapsto sxs^{-1} \quad} B$$

This is certainly not conjugate in B to the standard embedding of N in B, and it clearly does not induce the identity on H^1 when $k = \mathbb{F}_2$.

Thus the conjecture should be modified to

<u>Conjecture:</u> Let $G = \text{Aut}(V)$ ~~[illegible]~~ and let σ be a maximal flag ~~[illegible]~~ in V, that is, a top-dimensional simplex of the building. Here $V \simeq k^n$; k finite of char. p. Then with coefficients in $\mathbb{F}_p$, the complex

$$(*) \qquad \rightrightarrows \bigoplus_{\substack{\tau\subset\sigma \\ \mathrm{card}(\tau)=2}} H_*(P_\tau) \rightrightarrows \bigoplus_{\substack{\tau\subset\sigma \\ \mathrm{card}\,\tau=1}} H_*(P_\tau) \longrightarrow H_*(G) \longrightarrow 0$$

is exact.

Let $\tilde{C}$ be the complex of mod p chains on the suspension of X:

$$0 \longrightarrow C_{n-1}(X) \longrightarrow \cdots\cdots \longrightarrow C_1(X) \longrightarrow C_0(X) \longrightarrow \mathbb{F}_p \longrightarrow 0$$

where X is the building. The homology of this complex is concentrated in degree n and is the Steinberg representation of $G \otimes \mathbb{F}_p$. I claim

$$(**) \qquad H_*(G, \tilde{C}_p) \cong \bigoplus_{\substack{\tau\subset\sigma \\ \mathrm{card}(\tau)=p}} H_*(P_\tau)$$

In effect

$$\tilde{C}_p(X) = C_{p-1}(X) = \mathbb{F}_p\Big[\coprod_{\substack{\mathrm{card}(\tau)=p \\ \tau\subset\sigma}} G/P_\tau\Big]$$

so this is clear. (The point is that each simplex in the building is conjugate to a unique $\tau \subset \sigma$. Pursuing this, one sees that the differential in (*) is induced by the differential in $\tilde{C}_\cdot$.)

Thus I see that (*) is the E^1 term of the spectral sequence

$$E^1_{pq} = H_q(G, \tilde{C}_p) \Longrightarrow H_{p+q}(G, \text{Steinberg} \otimes \mathbb{F}_p)$$

so what the conjecture signifies is that the E^2 term is zero. I am going to try to prove this by induction on n, having seen already that it is true for $n = 2$.

Let L be a line in V and let P be the parabolic subgroup normalizing L. Then

$$H_*(G, M) \xrightarrow{\text{ind}} H_*(P, M) \xrightarrow{\text{res}} H_*(G, M)$$

is multiplication by $[G:P]$ = no. of lines $= \frac{q^n - 1}{q - 1} \equiv 1 \ (p)$, so that if we knew triviality over P, we would have it over G. So we wish to consider the complex

$$p \longmapsto H_q(P, \tilde{C}_p).$$

Review now how one inductively determines the homotopy type of X. Let $\mathcal{H}_L$ be the set of hyperplanes in V complementary to L, and $Y \subset X$ the full subcomplex consisting of all vertices not in $\mathcal{H}_L$. Then Y is contractible ($W \mapsto W + L$ retracts Y into the closed star of the vertex L.) Observe that this retraction gives an equivariant for P map

$$Y \times \Delta(1) \longrightarrow \text{Closed star}_L$$

and that C is a functor from simplicial complexes to chain complexes transforming simplicial homotopies to chain homotopies. Thus Y is P-equivariantly contractible, whence the complex

$$p \longmapsto H_q(P, C_p(Y))$$

~~Also~~ will contract to $H_q(P, \mathbb{F}_\ell)$ for $p=0$. A better way of putting it is to say that the complex $\tilde{C}(Y)$, defined in analogy with $\tilde{C}(X)$, is equivariantly homotopic to zero. Thus

$$p \longmapsto H_q(P, \tilde{C}_p(Y))$$

is acyclic for all q.

But we have clearly an exact sequence

$$(*) \qquad 0 \longrightarrow \tilde{C}(Y) \longrightarrow \tilde{C}(X) \longrightarrow \underset{\tilde{C}(X,Y)}{\overset{\shortparallel}{C(X,Y)[1]}} \longrightarrow 0.$$

Claim

$$C(X,Y) = \bigoplus_{W \in \mathcal{H}_L} \tilde{C}(X_W).$$

In effect $C_p(X,Y)$ has as basis all $p+1$ simplices σ containing some W, necessarily unique. So σ:

$$0 < V_1 < \cdots < V_p < W,$$

~~[illegible]~~ determines a simplex of some X_W or possibly the empty simplex.

~~[illegible]~~ Claim for each q, p

$$(*) \qquad 0 \longrightarrow H_q(P, \tilde{C}_p(Y)) \longrightarrow H_q(P, \tilde{C}_p(X)) \longrightarrow H_q(P, \tilde{C}_p(X,Y)) \to 0$$

is exact. Because as a P-module $\tilde{C}_p(X)$ is the free abelian group generated by a P-set, while $\tilde{C}_p(Y)$ is generated by a P-subset. Now ~~consider~~ choosing a W and letting P_W be its stabilizer we have

$$H_q(P, \tilde{C}_p(X,Y)) = H_q(P_W, \tilde{C}_p(X_W)).$$

Now $P_W = k^* \times \mathrm{Aut}(W)$, k^* acting trivially. By induction

$$H_*(p \mapsto H_q(\mathrm{Aut}(W), \tilde{C}_p(X_W))) = 0 \qquad (\text{assuming } n \geq 3)$$

so we see using the long exact sequence in homology associated to $(*)$ (considered as an exact seq. of cxs. p varying, q fixed) that

$$H_*(p \mapsto H_q(P, \tilde{C}_p(X))) = 0$$

concluding the induction. $\therefore$ Conjecture on page 6 is proved.

<u>Remark</u>: Observe for $n = 1$ that the conjecture is not true. Thus the complex is $H_*(G) = \mathbb{F}_p$ in degree $p = 0$. ~~[illegible]~~ Note that

$$H_*(B, \mathbb{F}_p \otimes \text{Steinberg}) = \mathbb{F}_p \qquad n \geq 2$$

$$H_*(G, \mathbb{F}_p \otimes \text{Steinberg}) = 0 \qquad n \geq 2$$

follows from the preceding proof.

Questions

1. Can the preceding be generalized to any Tits system with finite Weyl group?

Suppose given G, B, N, S. Don't see the analogue of the line.

2. $H^*(G, \text{Steinberg} \otimes \mathbb{F}_\ell) = ?$ $\qquad \ell \neq p$.

In mod ℓ cohomology the complex $H_*(G, \tilde{C}_\cdot(X))$ takes the form

$$\rightrightarrows\!\!\to \bigoplus_{0<i<j<n} H_*(GL_i \times GL_{j-i} \times GL_{n-j}) \rightrightarrows \bigoplus_{0<i<n} H_*(GL_i \times GL_{n-i}) \longrightarrow H_*(GL_n) \longrightarrow 0$$

Recall that if R is an augmented algebra over k, then $T(\bar{R}[1])$ with differential defined to be

$$d[r_1, \dots, r_n] = \sum_{i=1}^{n-1} (-1)^i [r_1, \dots, r_i r_{i+1}, \dots, r_n]$$

is the bar resolution for calculating

$$\mathrm{Tor}_*^R(k, k).$$

Do not confuse the map $H_*(M) = \bigoplus_{n\geq 0} H_*(GL_n) \longrightarrow k$ sending GL_n to 0 for $n>0$ with the map induced by the map $M \longrightarrow pt$.

Recall the exact sequence model for BM:

$$(*) \qquad \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix} \coprod_{a,b\geq 0} BG_{a,b} \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix} \coprod_{a\geq 0} BG_a \rightrightarrows pt.$$

If we filter this by calling $BG_{a_1,a_2\cdots a_p}$ of filtration n if $a_1+a_2+\cdots+a_p \leq n$, then the n-th graded complex is

$$\cdots\cdots \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix} \begin{matrix} pt \\ \amalg \\ \coprod_{a+b=n} BG_{a,b}\end{matrix} \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix} \begin{matrix} pt \\ \amalg \\ BG_n\end{matrix} \rightrightarrows pt$$

It appears that the normalized homology is the complex

$$\longrightarrow \bigoplus_{\substack{a+b=n\\ a,b>0}} H_*(G_{a,b}) \longrightarrow H_*(G_n) \longrightarrow 0$$

which is the thing obtained before, i.e. the complex we showed was acyclic for $n \geq 2$. Thus it appears that

$\mathrm{Filt}_n(*)$ is acyclic mod p

except for the trivial copy of $B\mathbb{Z} = S^1$.

March 25, 1972

K local field (of char. 0 & residue char. p. To understand the mod p cohomology of $GL_2(K)$ which hopefully is nice.

I consider the building $X(K^2)$ with its natural $G = GL_2(K)$ action. This gives rise to a category cofibred over Δ^0 whose fibre over $[p]$ is the category of filtered A-modules

$$L_0 \subset L_1 \subset \cdots \subset L_p,$$

where each L_i is free of rank 2 and $\pi L_p \subset L_0$, and their isomorphisms. The non-degenerate objects are as follows

$p=0$	L_0	free A-module of rank 2	—	$GL_2 A$
$p=1$	$L_0 < L_1$	with $\dim L_1/L_0 = 1$	—	Iwahori subgp $\begin{pmatrix} * & * \\ \equiv 0 & * \end{pmatrix}$
$p=1$	$L_0 < L_1$	with $L_0 = \pi L_1$	—	$GL_2 A$
$p=2$	$L_0 < L_1 < L_2$		—	Iwahori subgrp.

Set $U = GL_2 A$, $U' =$ Iwahori subgroup. Then the category, or properly its non-degenerate part takes the form

$$\begin{tikzcd}
 & U \arrow[dr, shift left, "d_0 = d_1"] \arrow[dr, shift right] & \\
U' \arrow[ur, "d_1 = \mathrm{in}"] \arrow[dr, shift left, "d_0 = \mathrm{id}"] \arrow[dr, shift right, "d_2 = \text{conj by } \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}"'] & & U \\
 & U' \arrow[ur, shift left] \arrow[ur, shift right] &
\end{tikzcd}$$

To understand this a bit better, let us compute these homomorphisms. The idea is that we take for the last vertex the lattice $Ae_1 + Ae_2$ and identify the stabilizer of this with matrices.

basepoint lattice $\Lambda = Ae_1 + Ae_2$ stabilizer $U = \begin{pmatrix} * & * \\ * & * \end{pmatrix}$

codim 1 lattice $\Lambda' = Ae_1 + A\pi e_2$ stabilizer $U' = \begin{pmatrix} * & * \\ \equiv 0 & * \end{pmatrix}$

Then all the faces by the last are inclusions on the stabilizers. Next want

d_1 from $\Lambda' < \Lambda$ to Λ'

d_2 from $\pi\Lambda < \Lambda' < \Lambda$ to $\pi\Lambda < \Lambda'$

Thus we have to choose an isomorphism

$$\begin{array}{ccc} \varphi: \quad \Lambda & \xrightarrow{\sim} & \Lambda' \\ \cup\!\shortmid & & \\ \Lambda' & \xrightarrow{\sim} & \pi\Lambda \end{array}$$

so

$$\varphi(e_1) = \pi e_2 \qquad\qquad \varphi : \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}$$
$$\varphi(e_2) = e_1$$

Then want the induced map

$$\mathrm{Aut}(\Lambda' < \Lambda) = U' \longrightarrow \mathrm{Aut}(\Lambda') \xrightarrow{\theta \mapsto \varphi^{-1}\theta\varphi} \underset{\substack{\shortparallel \\ U}}{\mathrm{Aut}}(\Lambda)$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto \begin{pmatrix} 0 & \pi^{-1} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & \pi^{-1} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \beta\pi & \alpha \\ \delta\pi & \gamma \end{pmatrix} = \begin{pmatrix} \delta & \pi^{-1}\gamma \\ \pi\beta & \alpha \end{pmatrix}$$

Conclude that the category realizing $GL_2(K)$ is

$$\begin{tikzcd}
 & \mathcal{U} \arrow[dr, shift left, "d_0=id"] \arrow[dr, shift right, "d_1=id"'] & \\
\mathcal{U}' \arrow[ur, "d_0=i"] \arrow[dr, shift left, "d_0=id"] \arrow[dr, shift right, "d_2=h"'] & & \mathcal{U} \\
 & \mathcal{U}' \arrow[ur, shift left, "d_0=i"] \arrow[ur, shift right, "d_1=ih"'] &
\end{tikzcd}$$

where $\mathcal{U} = GL_2 A$

$\mathcal{U}' = \begin{pmatrix} * & * \\ \equiv 0 & * \end{pmatrix}$ subgroup of $\mathcal{U}$

$i : \mathcal{U}' \longrightarrow \mathcal{U}$ inclusion

$h\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \delta & \pi^{-1}\gamma \\ \pi\beta & \alpha \end{pmatrix}$ $\qquad$ h order 2

March 26, 1972

model for $GL_2(k)$:

$$\begin{array}{ccc} \begin{pmatrix} * & * \\ & 1 \end{pmatrix} & \longrightarrow & GL_2(k) \\ \downarrow & & \\ GL_1(k) & & (?) \end{array}$$

~~If k is finite then the vertical map~~
The vertical map induces isoms. on mod l cohomology and $\mathbb{Q}$ cohomology, $l \neq \mathrm{char}(k)$. Thus in these case $GL_2(k) \longrightarrow (?)$ ~~in~~duces isos. on cohomology. When k is finite of characteristic p, the horizontal arrow induces isoms on mod p coh. Thus $(?)$ has trivial mod p homology.

~~The clear deficit to~~
To generalize to higher dimensions. The idea I think is to look for what one might call

$$\mathrm{Filt}_n\ BGL(k)^+$$

which should be generated by vector spaces of dimension $\leq n$. Thus

$$\mathrm{Filt}_n(BU) = BU_n$$

and

$$H^*(\mathrm{Filt}_n(BU), \mathrm{Filt}_{n-1}(BU)) \simeq H^*(BU_n, BU_{n-1})$$
$$= \tilde{H}^*(MU_n)$$

From my earlier work on stability I have a good idea as to what $\mathrm{gr}_n\, BGL(R)^+$ might be. Its homology should be

$$H_*(GL_n k, \tilde{C}(X(k^n)))$$

where $X(k^n)$ is the unimodular vector complex of k^n.

Problem with the preceding model: A d.v.r. in K, $[K:\mathbb{Q}_p]<\infty$ where $\mu_p \subset A$. Then if I try the preceding to modify $GL_2(A)$, I get the wrong spectrum.

Try instead

$$\begin{tikzcd} GL_{1,1}(A) \arrow[r] \arrow[d] & GL_2(A) \arrow[d] \\ GL_1(A)^2 \arrow[r] & (?) \end{tikzcd}$$

I know that

$$\begin{tikzcd} \operatorname{Spec} H^*(GL_{1,1}A) \arrow[r] \arrow[d] & \operatorname{Spec} H^*(GL_2 A) \arrow[d] \\ \operatorname{Spec} H^*((GL_1 A)^2) \arrow[r] & \operatorname{Spec} H^*(?) \end{tikzcd}$$

will be a pushout diagram. The point is that any elementary abelian p-subgroup of $GL_2 A$ comes from $GL_{1,1}A$, and the different ones of rank 2 in $GL_{1,1}A$ all get identified in $(GL_1 A)^2$.

March 29, 1972. Compactification of the Buildings

A d.v.r. quotient field K. Note: The building of a K-vector space V is the same as for the $\hat{K}$ vector space $\hat{V} = \hat{K} \otimes_K V$, so we will suppose A complete.

I propose to compactify the building $Im(V)$. So let L_α be a directed system of lattices. Assuming the residue field k is finite we can, for each interval $L \subset L'$ in the building, arrange, by selecting a suitable subsystem, that

$$L + L_\alpha \cap L' = (L + L_\alpha) \cap L'$$

stabilizes. Call it $E_{LL'}$, and note that

$$(1) \qquad L + E_{L_1 L'} = E_{LL'} \qquad\qquad L_1 \subset L \subset L'$$

$$(2) \qquad L' \cap E_{LL''} = E_{LL'} \qquad\qquad L \subset L' \subset L''$$

Set

$$E_{L'} = \varprojlim_{L \subset L'} E_{LL'} = \bigcap_{L \subset L'} E_{LL'}$$

and note that by completeness of A (pass to limit in (1))

$$(1)' \qquad L + E_{L'} = E_{LL'}.$$

Also by passage to limit in (2)

$$(2)' \qquad L' \cap E_{L''} = E_{L'} \qquad\qquad L' \subset L''$$

Now set
$$E = \bigcup_{L''} E_{L''}$$
so that
$$E_{L'} = L' \cap E$$
and so
$$E_{LL'} = L + L' \cap E = (L+E) \cap L' \qquad L \subset L'$$
Thus have proved.

<u>Lemma:</u> Given any directed system of A-submodules E_α of V, there exists a subsystem which converges to a submodule E of V in the sense that $\forall L \subset L'$ (lattices)
$$L + E \cap L' = (L+E) \cap L'$$
$$= L + E_\alpha \cap L'$$
for all suff. large α.

What this amounts to is that
$$\begin{pmatrix}\text{A-submodules} \\ \text{of } V\end{pmatrix} \xrightarrow{\sim} \varprojlim_{[L\,L']} \begin{pmatrix}\text{A-submodules} \\ \text{of } L'/L\end{pmatrix}$$
and this must be due to the fact that A is compact.

So we consider the set $Y(V)$ of A submodules of V as the vertices of the compactification. A simplex will be a chain $E_0 < E_1 < \cdots < E_n$ of submodules such that $\pi E_n \subset E_0$. It will be necessary to add a topology.

Note that for any E we have a canonical exact sequence

$$0 \longrightarrow E_{div} \longrightarrow E \longrightarrow \hat{E} \longrightarrow 0$$

$$\hat{E} = \varprojlim E/\pi^n E$$

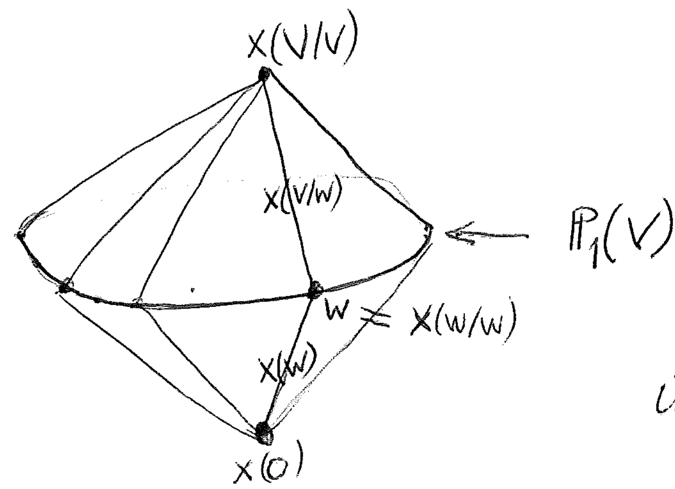
$$E_{div} = \bigcap \pi^n E$$

where $\hat{E} \simeq A^r$ and $E_{div} \simeq K^s$. Thus given a chain $E_0 < E_1 < \cdots < E_n$ we have $E_0 \supset \pi E_n \supset (E_n)_{div}$ and so the simplex comes from a simplex of the building of $\cancel{V}/E_{n,div}$. Thus without topology

$E_n \otimes K$

$$\boxed{Y(V) = \coprod_{W \subset W' \subset V} X(W'/W)}$$

Picture for $\dim V = 2$:



interior is $X(V)$

~~Conjecture:~~ Define a topology on the realization of $Y(V)$ as follows. A point of the realization $|Y(V)|$ is a pair (σ, z) where σ is a simplex and z is a point in the geometric simplex with vertices σ, i.e. if $\sigma = (E_0, \dots, E_m)$, then $z = \sum t_i E_i$. It's clear what is meant by a nbd of σ - we give an interval $L \subset L'$ in V and consider all simplices τ whose image in the interval is the same as σ. Example: suppose $\tau = L_0 < \dots < L_m$ is in the interior. Then take $L = \pi L_0$ and $L' = \pi^{-1} L_m$. If then $E_0 < \dots < E_n$ is in this neighborhood we must have

$$\{\pi L_0 + E_i \cap \pi^{-1} L_m\} = \{L_i\}$$

$$\pi L_0 + E_0 \cap \pi^{-1} L_m = L_0 \implies E_0 \cap \pi^{-1} L_m = L_0$$

$$\text{(by Nakayama's lemma)} \implies L_0 \subset E_0$$

$$\text{and } E_0 \cap \pi^{-1} L_0 = L_0$$

$$\implies E_0 = L_0 \quad (V/L_0$$

$= \text{injective hull of } \pi^{-1} L_0 / L_0).$

Example: Let $E \simeq A^s$ in V. Choose L so that $L \cap E \subset \pi E$ and $L' \supset E$. Then if E' is in the L, L' neighborhood, we have

$$L + E' \cap L' = L + E \cap L'$$

$$= L + E$$

$$\frac{E' \cap L'}{E' \cap L} = \frac{L + E' \cap L'}{L} = \frac{L+E}{E} = \frac{E}{L \cap E} \quad ?$$

The topology is then defined by saying that a nbd of (σ, z) consists of (τ, w) where τ is in an (L, L') nbd. of σ and where the image w is in the image of a nbd. of z.

Conjecture: 1) The above definition makes sense and makes $|Y(V)|$ into a compact space.

2) $|Y(V)|$ is homeomorphic to the suspension of the Borel–Serre compactification of the building associated to $SL(V)$, V being given a volume.

The argument on page 4 uses

Lemma: If L is a lattice in V, E a submodule, and if

$$\pi L + \pi^{-1} L \cap E = L$$

then $E = L$.

Proof: Nakyama $\Rightarrow$ $\pi^{-1} L \cap E = L$, so $L \subset E$. But $\pi^{-1}L/L$ is the socle of V/L, so $E/L \cap \pi^{-1}L/L = 0 \Rightarrow E/L = 0$.

Corollary: Assume E generates V so that we can find an L in E such that $L \twoheadrightarrow \hat{E}$. Let E' be in the $(\pi L, \pi^{-1} L)$ nbd. of E, i.e.

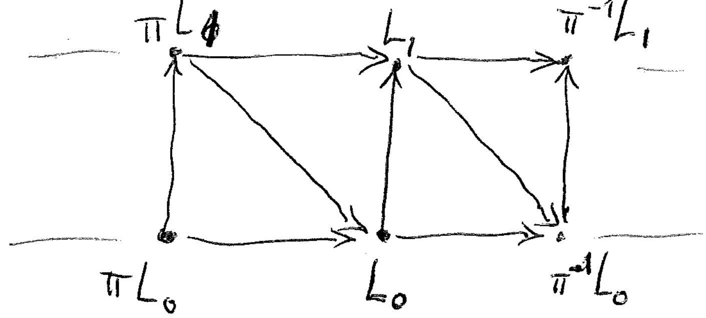
$$\pi L + E' \cap \pi^{-1} L = \pi L + E \cap \pi^{-1} L$$

Then $E' \subset E$ and $\hat{E}' \twoheadrightarrow E$

Why ~~is~~ the Bruhat-Tits building of V is contractible. Recall that if $X(V)$ is my building and $Y(V)$ theirs there is a map

$$X(V) \xrightarrow{f} Y(V)$$

Vertices of $Y(V)$ are homothety classes of $L \subset V$, i.e. $L \sim \lambda L'$ $\lambda \in K^*$. A ~~simplex~~ subset of $Y(V)$ is a simplex $\Leftrightarrow$ it is the image of a simplex of $X(V)$. Thus the map f is simplicial and $Y(V)$ is contractible provided all of the fibres are. But a typical fibre looks like



so it is clear all the fibres are contractible.

It is also clear from this, granted the Borel-Serre thm. that the coh. with comp. supports of X is the suspension of that of Y.

April 29, 1972 — Compactification of the building

$A, K, k, \mathfrak{m}$ usual d.v.r. situation.
V a K-module f.t. $X(V)$ its building. Then

$$X(V) = \bigcup_n X(\pi^{-n}\Lambda/\pi^n\Lambda)$$

where Λ is a fixed lattice in V and $X(\pi^{-n}\Lambda/\pi^n\Lambda)$ is the subcomplex of $X(V)$ whose vertices lie between $\pi^n\Lambda$ and $\pi^{-n}\Lambda$. More generally have $X(\Lambda_1/\Lambda_0)$ when $\Lambda_0 \subset \Lambda_1$ are lattices in V.

Suppose the layer $\Lambda_0' \subset \Lambda_1'$ is contained in the layer $\Lambda_0 \subset \Lambda_1$, i.e.

$$\Lambda_0 \subset \Lambda_0' \subset \Lambda_1' \subset \Lambda_1$$

Then there is a retraction

$$X(\Lambda_1/\Lambda_0) \longrightarrow X(\Lambda_1'/\Lambda_0')$$

$$L \longmapsto (L \cap \Lambda_1') + \Lambda_0' = (L + \Lambda_0') \cap \Lambda_1'$$

Better, this is a retraction of $X(V)$ to $X(\Lambda_1'/\Lambda_0')$.

Using these retractions we obtain an inverse system of simplicial complexes

$$\longrightarrow X(\pi^{-n}\Lambda/\pi^n\Lambda) \longrightarrow X(\pi^{-n+1}\Lambda/\pi^{n-1}\Lambda) \longrightarrow \cdots$$

and we can take the inverse limit

$$\overline{X}(V) = \varprojlim_n X(\pi^{-n}\Lambda/\pi^n\Lambda).$$

Note that when k is finite, $\bar{X}(V)$ is a compact space, since $X(\pi^{-n}\Lambda/\pi^n\Lambda)$ is a finite simplicial complex.

Fix n and let L belong to $X(\pi^{-n-1}\Lambda/\pi^{n+1}\Lambda)$ but not $X(\pi^{-n}\Lambda/\pi^n\Lambda)$. Then there are two cases - either $L \not\subset \pi^{-n}\Lambda$ or $L \not\supset \pi^n\Lambda$ and both can occur simultaneously, e.g. $L = \langle \pi^{-n-1}e_1, \pi^{n+1}e_2 \rangle$ where $\Lambda = \langle e_1, e_2 \rangle$. But suppose L is immediately joinable to the subcomplex $X(\pi^{-n}\Lambda/\pi^n\Lambda)$; suppose $\{L, L_0\}$ is a one simplex, then either

ⓐ $L_0 \subset L \subset \pi^{-1}L_0 \implies \pi^{-n}\Lambda \subset L \subset \pi^{-n-1}\Lambda$

ⓑ $\pi L_0 \subset L \subset L_0 \implies \pi^{n+1}\Lambda \subset L \subset \pi^{-n}\Lambda$

and these cases can't occur at the same time. Perhaps this can be used to compute the cohomology of $X(V)$ with supports in $X(\pi^{-n}\Lambda/\pi^n\Lambda)$?

I want to identify the space $\bar{X}(V)$. Suppose that $x = (x_n) \in \bar{X}(V)$ with $x_n \in X(\pi^{-n}\Lambda/\pi^n\Lambda)$. Then the dimension $d(x_n)$ of the open simplex containing x_n is a bounded monotone function of n, hence it stabilizes: $d = d_n$ for n large. Then

$$x_n = \sum_{i=0}^{d} t_i L_{i,n} \qquad L_{0n} < L_{1n} < \cdots < L_{dn}$$

for uniquely determined $L_{i,n}$ in the layer $\pi^n\Lambda \subset \pi^{-n}\Lambda$. Assuming A, K complete, we know that

$$\lim_n L_{in} = E_i$$

where E_i is an A-submodule of V, and that

$$(*) \qquad E_0 < \cdots < E_d \qquad\qquad \pi E_d \subset E_0 .$$

Thus we see that set-theoretically $\bar{X}(V)$ is identifiable with the simplicial complex whose simplices are sequences of the form $(*)$. Thus set-theoretically at least

$$\bar{X}(V) = \bigcup_{0 \subset W_0 \subset W_1 \subset V} X(W_1/W_0)$$

by our previous work (uses the exact sequence

$$0 \longrightarrow E_{div} \longrightarrow E \longrightarrow \hat{E} \longrightarrow 0.)$$

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Why the last vertex functor

$$\mathrm{Nerv}(\mathcal{C}) \xrightarrow{f} \mathcal{C}$$

$$(X_p \leftarrow \cdots \leftarrow X_0) \longmapsto X_0$$

is a homotopy equivalence. In general given a functor $f : \mathcal{C} \longrightarrow \mathcal{C}'$, consider

$$(x, y) \longmapsto \mathrm{Hom}_{\mathcal{C}'}(y, f(x))$$

$$\mathcal{C} \times (\mathcal{C}')^{\circ} \longrightarrow \mathit{Sets}$$

and form the category M_f cofibred over $\mathcal{C} \times (\mathcal{C}')^{\circ}$ associated this functor. Thus we have

$$\begin{tikzcd} & M_f \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ \mathcal{C} & & \mathcal{C}' \end{tikzcd} \qquad (x, y, \xi : y \to f(x))$$

where pr_1 is cofibred, and pr_2 is fibred. In addition define a section of pr_1

$$\mathcal{C} \xrightarrow{s} M_f$$

$$x \qquad (x, f(x), id : f(x) \to f(x))$$

such that $pr_2 \cdot s = f$. Since

$$\mathrm{Hom}_{M_f}\big((x, y, \xi : y \to f(x)), (u, f(u), id : f(u) \to f(u))\big) = \mathrm{Hom}_{\mathcal{C}}(x, u)$$

it follows we have adjoint functors

$$\mathcal{C} \underset{s}{\overset{pr_1}{\leftrightarrows}} M_f$$

so that pr_1 and s are homotopy equivalences. Thus to prove f is a h.eq., it suffices to show that each of the categories [redacted] $y|\mathcal{C}$ consisting of $(x, \xi : y \to f(x))$ is contractible.

Now return to the last vertex functor

$$New(\mathcal{C}) \xrightarrow{f} \mathcal{C}$$

Then given y in $\mathcal{C}$ we [redacted] consider the category of arrows $y \to f(x)$ i.e. the simplicial set [redacted] whose p-simplices are:

$$y \to x_0 \to \cdots\cdots \to x_p .$$

Anyhow we still need to show contractibility of this.

The idea will be to consider the functors

$$\begin{array}{lcl}
(y \to x_0 \to \cdots \to x_p) & \longmapsto & (y \to x_0 \to \cdots\cdots \to x_p) \\
 & & \quad\uparrow d_0 \quad \text{nat. transf.} \\
 & \longmapsto & (y \to y \to x_0 \to \cdots\cdots \to x_p) \\
 & & \quad\downarrow d_1^p \quad [redacted] \\
 & \longmapsto & (y \to y)
\end{array}$$

which actually do provide the desired contraction

So now let $\mathcal{M}$ be the category of coh. sheaves on a noetherian scheme X and their isomorphisms. Let $\mathcal{R}$ be the cofibred category over Δ° whose fibre at p is the category of filtered objects of $\mathcal{M}$ of length p up to isom.

$$0 \subset M_1 \subset \cdots\cdots \subset M_p .$$

Let $\mathcal{S}$ be the category with $Ob(\mathcal{S}) = Ob(\mathcal{M})$, in which an arrow $M' \longrightarrow M$ is an isom. of M'' with a subquotient of M, i.e.:

$$\begin{tikzcd} F \arrow[d, two heads] & \subset M' \\ M'' , & \end{tikzcd}$$

Then we obtain a functor

$$\mathcal{R} \xrightarrow{\quad f \quad} \mathcal{S}$$

$$(0 \subset M_1 \subset \cdots \subset M_p) \longmapsto M_p$$

which I want to show is a homotopy equivalence. So I fix an object V of $\mathcal{S}$ and consider the category of arrows $y \to f(x)$. Thus I wish to consider diagrams

$$\begin{tikzcd} F \arrow[d, two heads] & \subset V \\ 0 \subset M_1 \subset \cdots\cdots \subset M_p . & \end{tikzcd}$$

as objects of the category. A morphism in $\mathcal{R}$

from $0 \subset M_1 \subset \cdots \subset M_p$ to $0 \subset N_1 \subset \cdots \subset N_q$ consists of $\varphi : [q] \longrightarrow [p]$ and an isomorphism

$$\begin{array}{ccc} \xi : M_{\varphi(q)}/M_{\varphi(0)} & \xrightarrow{\sim} & N_q \\ \cup & & \cup \end{array}$$

which induces: $M_{\varphi(i)}/M_{\varphi(0)} \longrightarrow N_i$ for $0 \leq i \leq q$.

Now we are given
$$\begin{array}{c} F \subset V \\ \downarrow \\ M_p \end{array} \quad \text{and} \quad \begin{array}{c} E \subset V \\ \downarrow \\ N_q \end{array}$$

so if (φ, ξ) carries the former to the latter then

$$\begin{array}{ccc} E & \hookrightarrow & F \\ \downarrow & & \downarrow \\ M_{\varphi(q)} & \hookrightarrow & M_p \end{array} \qquad \text{must be cartesian}$$

and $E \longrightarrow M_{\varphi(q)} \xrightarrow{\xi} N_q$ must be the given map from $E \twoheadrightarrow N_q$. Observe that ξ is uniquely determined. So what I am trying to say is that this $y \to f(x)$ category is cofibred over Δ° with discrete fibres, in fact the category is equivalent to the category belonging to the simplicial set whose p-simplices are chains of subobjects

$$M_0 \subset M_1 \subset \cdots \subset M_p \subset V$$

This is the nerve of the category of all $M \hookrightarrow V$ which is a category with a final object and hence is contractible.

To be very careful, given

$$(*)\qquad \begin{array}{ccc} & & F \hookrightarrow V \\ & & \downarrow \\ 0 \subset M_1 \subset & \cdots & \subset M_p \end{array}$$

send it into

$$0 \times_{M_p} F \subset M_1 \times_{M_p} F \subset \cdots \subset M_p \times_{M_p} F \subset V.$$

This is a functor of cofibred cats $/\Delta^0$. In the opposite direction send

$$M_0 \subset M_1 \subset \cdots \subset M_p \subset V$$

to

$$\begin{array}{ccc} & & M_p \subset V \\ & & \downarrow \\ 0 \subset M_1/M_0 \subset & \cdots & \subset M_p/M_0 . \end{array}$$

(The point: The fibre category of $y \to f(x)$ is cofibred over $\mathcal{R}$ hence also over Δ^0. And there is a unique isomorphism from any diagram $(*)$ to any other. Thus the fibres category is a simplicial set.) I have proved:

<u>Proposition</u>: $\mathcal{R}$ = the cat cof$/\Delta^0$ of $0 \subset M_1 \subset \cdots \subset M_p$
$\mathcal{S}$ = cat of M where $M \longrightarrow M'$ is $\begin{array}{c} F \subset M \\ \downarrow \\ M' \end{array}$

Then
$$f : \mathcal{R} \longrightarrow \mathcal{S}$$
$$(0 \subset M_1 \subset \cdots \subset M_p) \mapsto M_p$$
is a homotopy equivalence.