November 1, 1972.

A ring, $\mathcal{F}_A$ = free f.g. A-modules $\subset \mathcal{P}_A$. The problem is to show that

$$Q(\mathcal{F}_A) \longrightarrow Q(\mathcal{P}_A)$$

is essentially the covering of the latter belonging to the K_0A set $K_0A/\mathbb{Z} \cong \operatorname{Coker}\{K_0\mathcal{F}_A \to K_0A\}$.

Lemma: $P \in \mathcal{P}_A$. Consider the category whose objects are surjections $F \twoheadrightarrow P$ with F in $\mathcal{F}_A$ and in which the arrows are triangles

$$\begin{tikzcd} F' \arrow[r] \arrow[dr] & F \arrow[d] \\ & P \end{tikzcd}$$

such that $F' \to F$ is surjective with kernel in $\mathcal{F}_A$. This category is contractible.

Proof: Write $P \oplus P_0 = F_0$.

$$\begin{tikzcd} (fa, c) & (a,b,c) \arrow[l, mapsto] & F \oplus P \oplus P_0 \arrow[dl, "g"'] \arrow[dr, "pr_1"] & \\ & F_0 = P \oplus P_0 \arrow[dr, "pr_1"'] & & F \arrow[dl, "f"] \\ & & P & \end{tikzcd}$$

Clearly $\operatorname{Ker} g = \operatorname{Ker} f \oplus P \cong F$

so contractible by the cone construction.

~~40~~ Problem: Show ~~$Q(P_A)$ ...~~

Q(free A modules) is a covering space of $Q(P_A)$, in fact the covering associated to the K_0A sets $\tilde{K}_0(A) = K(A)/\mathbb{Z}$

<u>November 5, 1972</u>

May's theorem: Let $p \mapsto (E_p \longrightarrow B_p)$ be a map of simplicial spaces with each B_p connected; let F_p be the h-fibre over the basepoint of B_p, (assumed to be the degeneracy of a given basepoint of B_0). Then $|F|$ is the h-fibre of $|E| \longrightarrow |B|$.

Proof. Put $\Omega_q X$ = space of maps $\Delta(q) \longrightarrow X$ carrying the 0-skeleton to the basepoint. I recall the heq

$$|\Omega_\cdot X| \xrightarrow{\sim} X$$

for X-connected; it arises from:

$$\begin{array}{ccccc} \cdots P_1X & \rightrightarrows & P_0X & \longrightarrow & X \\ \downarrow & & \downarrow & & \\ \cdots \Omega_1X & \rightrightarrows & pt & & \end{array}$$

where P_q = maps $\Delta(q+1) \longrightarrow X$ carrying $\{0,\dots,q\}$ to the basepoint. Vertically we have heq's; horizontally one has heq's

$$P_qX \longrightarrow (P_0X/X)^{q+1}$$

and the latter is contractible locally over X, hence globally by a Cech covering.

Now we let J_pE_q be the space of maps $\Delta(q) \longrightarrow E_q$ carrying the vertices into the fibre over the basepoint of B_q. Then

$$J_pE_q \xrightarrow{\sim} \Omega_pB_q \times F_q$$

by the CHT. Then

$$|J_p E_q|^h \sim E_q \qquad \text{as} \quad B_q \text{ is connected}$$

(proof analogous to above: Replace $J_p E_q$ by maps $\Delta(pH) \to E_q$ carrying all but last vertex to the fibres). On other hand

$$|J_p E_q|^v \sim |\Omega_p B_q|^v \times |F| \qquad \text{here use } |X \times Y| = |X| \times |Y|$$

so the squares

$$\begin{array}{ccc} |J_p E_q|^v & \longrightarrow & |J_{p'} E_q|^v \\ \downarrow & & \downarrow \\ |\Omega_p B_q|^v & \longrightarrow & |\Omega_{p'} B_q|^v \end{array}$$

will be homotopy cartesian with h-fibre $|F|$. Thus $\underset{|E|}{\underset{\shortparallel}{|J_p E_q|}}$ fibres over $\underset{(B)}{\underset{\shortparallel}{|\Omega_p B_q|}}$ with fibre $|F|$ by Segal's fibration thm.

__Analogous result.__ If X_p is a simplicial space with X_p connected for all p, then

$$|\Omega X_\cdot| = \Omega |X_\cdot|$$

In effect: Because realization commutes with products:

$$B|\Omega X_\cdot| = \big|\,|\Omega_q X_p|^h\big|^v = \big|\,|\Omega_q X|^v\big|^h \underset{X_p \text{ conn}}{\overset{\uparrow}{=}} |X| \qquad \text{and so } |\Omega X| \text{ group-like} \implies |\Omega X| = \Omega|X|.$$

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DEPARTMENT OF MATHEMATICS

November 7, 1972

Dear Peter,

I recently found a new proof of the group-completion theorem which seems to be much better than the others. Here it is:

Let M be a simplicial monoid and E a simplicial set on which M acts to the right. The category of such simplicial M-sets has the projective generators $\Delta(n)\times M$, $n\geq 0$, and so one has a "standard resolution" for E:

$$\cdots\ \coprod_{\substack{\Delta(n_0)\times M\to\Delta(n_1)\times M \\ \searrow\ E\ \swarrow}} \Delta(n_0)\times M \rightrightarrows \coprod_{\Delta(n)\times M\to E} \Delta(n)\times M \longrightarrow E$$

Precisely, let $\mathcal{C}$ be the full sub-category of simplicial M-sets over E of the form $\Delta(n)\times M\to E$, $n\geq 0$. Then ~~[illegible]~~ the ~~[illegible]~~ above gadget is the simplicial object, in the category of simplicial M-sets, ~~of chains on $\mathcal{C}$ with coefficients~~ which in degree p is

$$N_p\mathcal{C}\times_{Ob\,\mathcal{C}} F \qquad\qquad F = \coprod_{\substack{\Delta(n)\times M\to E \\ \in Ob\,\mathcal{C}}} \Delta(n)\times M$$

where $N_p\mathcal{C}$ is the set of p-simplices: $x_0\leftarrow x_1\leftarrow\cdots\leftarrow x_p$ in the nerve of $\mathcal{C}$, and the map $N_p\mathcal{C}\to Ob\,\mathcal{C}$ sends this simplex to x_p.

If I regard this simplicial object as a bisimplicial set, then in each vertical degree q it contracts to E_q. This is ~~[illegible]~~ standard triple theory: $p\mapsto (N_p\mathcal{C}\times_{Ob\,\mathcal{C}} F)_q$ ~~[illegible]~~ is the nerve of the category of arrows $\Delta(q)\times M \longrightarrow \Delta(n)\times M \longrightarrow E$ with $\Delta(q)\times M$ fixed, and this category is a disjoint union of categories with ~~final~~ initial objects, one for each element of E_q.

Now if h_* is any homology theory on simplicial sets, we can apply it to the simplicial object $N_p\mathcal{C}\times_{\mathrm{Ob}\,\mathcal{C}} F$ and get a Segal-style spectral sequence

$$E^2_{pq} = H_p\Big(N\mathcal{C},\ \begin{matrix}\Delta(n)\times M\\ \downarrow\\ E\end{matrix} \longmapsto h_q(\Delta(n)\times M)\Big) \Longrightarrow h_{p+q}\big(\mathrm{diag}_0(N\mathcal{C}\times_{\mathrm{Ob}\,\mathcal{C}} F)\big) \overset{\sim}{\downarrow} h_{p+q}(E)$$

~~[illegible]~~ Now the right multiplication of M on E induces an action of $\pi_0 M$ on this spectral sequence, so if $\pi_0 M$ is abelian, one can localize obtaining a spectral sequence

$$(*)\qquad E^2_{pq} = H_p\Big(N\mathcal{C},\ \begin{matrix}\Delta(n)\times M\\ \downarrow\\ E\end{matrix} \longmapsto h_q(M)[\pi_0 M^{-1}]\Big) \Longrightarrow h_{p+q}(E)[\pi_0 M^{-1}].$$

If in addition left multiplication by an element α of $\pi_0 M$ on $h_q(M)[\pi_0 M^{-1}]$ is an automorphism, e.g. if ~~[illegible]~~ left and right multiplication by α on $h_*(M)$ coincide, then ~~the coefficient system in the E^2 term is~~ it is easy to see that the functor $\begin{matrix}\Delta(n)\times M\\ \downarrow\\ E\end{matrix} \longmapsto h_q(M)[\pi_0 M^{-1}]$ carries all arrows into isomorphisms, i.e. it is a local coefficient system on $N\mathcal{C}$. Thus the E^2 term is a homotopy invariant of $N\mathcal{C}$ under these assumptions.

Now suppose M_q acts freely on E_q for each q and set $X_q = E_q/M_q$, so that $E_q \simeq X_q\times M_q$ as right M_q-sets. The simplicial object

$$E\times M^2 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; E\times M \rightrightarrows E$$

contracts to X_q in each vertical dimension q, hence we have a (weak) homotopy equivalence.

$$\mathrm{diag}(p \mapsto E\times M^p) \longrightarrow X.$$

We now show $N\mathcal{C}$ and X are (weak) homotopy equivalent. Consider the bisimplicial object in the category of simplicial sets

$$\begin{tikzcd} N_p\mathcal{C}\times_{\mathrm{Ob}\,\mathcal{C}} F\times M^q \arrow[r, "\text{horizontal}", "\text{augmentation}"'] \arrow[d, "\text{vertical augmentation}"] & E\times M^q \\ N_p\mathcal{C} & \end{tikzcd}$$

The horizontal augmentation will induce a homotopy equivalence on ~~the~~ diagonal simplicial sets, since $\mathrm{diag}(N\mathcal{C}\times_{\mathrm{Ob}\,\mathcal{C}} F)$ is homotopy equiv. to E. Vertically, we also have a homotopy equiv. ~~since the fibres~~ $q \mapsto \Delta(n)\times M\times M^q$ are contractible. Thus we get

$$N\mathcal{C} \xleftarrow{\text{h.eq.}} \mathrm{diag}(N_p\mathcal{C}\times_{\mathrm{Ob}\,\mathcal{C}} F\times M^q) \xrightarrow{\text{h.eq.}} \mathrm{diag}(E\times M^q) \xrightarrow{\text{h.eq.}} X$$

so the spectral sequence (*) can be written

$$(**)\qquad \boxed{E^2_{pq} = H_p(X, \mathcal{L}_q) \Longrightarrow h_{p+q}(E)[(\pi_0 M)^{-1}].}$$

~~where $\mathcal{L}_q$ is a local coefficient system on X.~~

Now I can prove the group-completion theorem. Let $BM = \mathrm{diag}(p\mapsto M^p)$, $EM = \mathrm{diag}(p \mapsto M^p\times M)$, and let $X \xrightarrow{f} BM$ be a Kan fibration with X contractible, Ω = fibre of f over basepoint, whence Ω has the homotopy type of ΩBM. Let

$E = X \times_{BM} PM$. Then E fibres over PM, which is contractible, with fibre Ω so

$$h_*(\Omega BM) \xrightarrow{\simeq} h_*(E).$$

But now the spectral sequence (**) can be applied to the inclusion of the fibre of E over the basepoint of X

$$\begin{tikzcd} M \arrow[r] \arrow[d] & E \arrow[d] \\ pt \arrow[r] & X \end{tikzcd}$$

giving

$$\begin{tikzcd} E^2_{pq} = H_p(pt, \mathcal{L}_q) \arrow[r, Rightarrow] \arrow[d, "\cong"] & h_{p+q}(M)[(\pi_0 M)^{-1}] \arrow[d, "\cong"] \\ E^2_{pq} = H_p(X, \mathcal{L}_q) \arrow[r, Rightarrow] & h_{p+q}(E)[\pi_0 M^{-1}] \end{tikzcd}.$$

because X is contractible. Finally one notes that right multiplication by $m \in M_0$ on E is a homotopy equivalence because of the cartesian square

$$\begin{tikzcd} E \arrow[r, "\cdot m"] \arrow[d] & E \arrow[d] \\ PM \arrow[r, "\cdot m"] & PM \end{tikzcd} \quad \leftarrow \text{h.eq. as } PM \sim pt.$$

so $h_*(E) \xrightarrow{\simeq} h_*(E)[\pi_0 M^{-1}]$.

Therefore

$$h_*(M)[\pi_0 M^{-1}] \simeq h_*(\Omega BM)$$

yielding the group-completion theorem

November 7, 1972

The classifying space of a simplicial monoid M.

Let $E \to B$ be a map of simplicial (right) M-sets with M acting trivially on B. Then we have the simplicial object $p \mapsto E \times M^p$ in simplicial sets over B. If $|E \times M^p| \to B$ is a heq ($|\;|$ denotes diagonal), we call E a h-torsor for M over B.

~~Example 1 Let $EM = |M^\mu \times M|$ over $BM = |M^\mu|$.~~
~~Then $|EM \times M^\mu| = |\mu \mapsto |\nu \mapsto M^\nu \times M| \times M^\mu| \xrightarrow{heq} |\mu \mapsto M^\mu| = BM$~~
~~so EM is an h-torsor over BM.~~

~~Example 2: Suppose $E_q \xleftarrow{\simeq} B_q \times M_q$ as right M_q-sets for each q. Then~~
~~$|E \times M^p| = |(p,q) \mapsto E_q \times M_q^p| \xrightarrow{heq} |$~~

Example 1: Suppose $E_q \simeq B_q \times M_q$ as right M_q-sets for each q. Then

$$|E \times M^p| \xrightarrow{heq} B$$

because $|p \mapsto E_q \times M_q^p| \xrightarrow{heq} B_q$ for each q.

Example 2: $EM = |M^p \times M|$, $BM = |M^p|$. Special case of preceding since $(EM)_p = M_p^p \times M_p$, $(BM)_p = M_p^p$.

Def. morphism of h-torsors.

Suppose E is an h-torsor for M over B. Then we have morphisms of M-torsors

$$\begin{tikzcd}
E \arrow[d] & {|E \times M^p \times M|} \arrow[l, "\alpha"'] \arrow[r] \arrow[d] & {|M^p \times M| = EM} \arrow[d] \\
B & {|E \times M^p|} \arrow[l, "heq"'] \arrow[r] & {|M^p| = BM}
\end{tikzcd}$$

<u>Lemma</u>: α heq (true for any M-space).

<u>Question</u>: Given $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$ and $f : B' \to B$, should $f^*(E) \to B'$ be an M-torsor? This is true for examples 1, 2.

We see from the above diagram that there is a well-defined $\chi_E \in [B, BM]$ associated to E.

Given
$$\begin{tikzcd} E' \arrow[r, "\tilde{f}"] \arrow[d] & E \arrow[d] \\ B' \arrow[r, "f"'] & B \end{tikzcd}$$
a morphism of torsors

we have $\chi_{E'} = \chi_E \circ f$. (Moreover $\tilde{f}$ a heq $\Rightarrow$ $|E' \times MP| \longrightarrow |E \times MP|$ heq $\Rightarrow$ f heq. (goes with defn.))

E contractible $\Longrightarrow$ B heq BM (precisely $\chi_E \cong$).

Call a $E \to B$ <u>universal</u> if E contractible.

<u>Question</u>: If $E \to B$ universal, in what sense is any other torsor induced from it?

<u>November 21, 1972:</u> Periodicity

Given an admissible $\mathcal{M}$ we have a spectrum

$$B_0(\mathcal{M}), \quad B_1(\mathcal{M}), \quad B_2(\mathcal{M})$$

with $B_1(\mathcal{M}) \cong Q(\mathcal{M})$. For example in the case of $k = \overline{\mathbb{F}}_p$, we have

	$B_0(k)$	$B_1(k)$	$B_2(k)$
0	$\mathbb{Z}$ $\quad 1$		
1	$\mathbb{Q}/\mathbb{Z}'$ $\quad q$	$\mathbb{Z}$ $\quad 1$	
2		$\mathbb{Q}/\mathbb{Z}'$ $\quad q$	$\mathbb{Z}$ $\quad 1$
3	$\mathbb{Q}/\mathbb{Z}'$ $\quad q^2$		$\mathbb{Q}/\mathbb{Z}'$ $\quad q$
4		$\mathbb{Q}/\mathbb{Z}'$ $\quad q^2$	
5	$\mathbb{Q}/\mathbb{Z}'$ $\quad q^3$		$\mathbb{Q}/\mathbb{Z}'$ $\quad q^2$

I have indicated the eigenvalues of Frobenius $x \mapsto x^q$.
Note the analogue for connected complex k^*:

$\mathbb{Z} \times BU$	U	BU	SU	BSU
$\mathbb{Z}$				
0	$\mathbb{Z}$			
$\mathbb{Z}$	0	$\mathbb{Z}$		
0	$\mathbb{Z}$	0	$\mathbb{Z}$	
$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$

~~One thing is clear: There is no Ψ^q on connected complex k^* which is compatible with Frobenius. In effect we would need an $f : BU \to BU$ which is additive such that $f = 1$ on $\pi_2 BU$, q on $\pi_4 BU$, etc~~

<u>Observe</u>: there is no stable operation on k^* compatible with Ψ^q on k^0. For if we have $f: BU \to BU$ such that

$$\begin{tikzcd} \mathbb{Z}\times BU \arrow[r, phantom, "\overset{\text{periodicity}}{\simeq}"] \arrow[d, "\Psi^q"'] & \Omega^2 BU \arrow[d, "\Omega^2(f)"] \\ \mathbb{Z}\times BU \arrow[r, phantom, "\overset{\text{periodicity}}{\simeq}"] & \Omega^2 BU \end{tikzcd}$$

commutes, then

$$\begin{tikzcd} K(X) \arrow[r, "\cdot\beta", "\sim"'] \arrow[d, "\Psi^q"'] & \tilde{K}(S^2X) \arrow[d, "f"] \\ K(X) \arrow[r, "\sim", "\cdot\beta"'] & \tilde{K}(S^2X) \end{tikzcd}$$

commutes, so

$$f(\beta\cdot x) = \beta\,\Psi^q(x)$$

$$f(x) = \beta\,\Psi^q(\beta^{-1}x) = \beta\beta^{-1}q^{-1}\,\Psi^q(x)$$

so

$$f(x) = \frac{1}{q}\Psi^q(x)$$

But $\frac{1}{q}\Psi^q$ is not an integral operation on $\tilde{K}$, e.g.

$$\frac{1}{q}\Psi^q(L-1) = \frac{1}{q}\left(L^q-1\right)$$

$$= \frac{1}{q}\left\{[1+(L-1)]^q - 1\right]$$

$$= \sum_{i=1}^{q-1}\frac{1}{q}\binom{q}{i}(L-1)^i + \frac{(L-1)^q}{q}$$

So trouble arises from the last term.

Conclude: It will not be possible to define Adams operations on the connected theory $K^*(X, m)$ in a general way so that they are compatible with suspension.

Suppose I work over $k = \overline{\mathbb{F}}_p$. Let $F = \overline{\mathbb{F}_p(T)}$. If various conjectures about curves over finite fields hold, then we will have

$$K_i k = K_i F \qquad i \geq 2, \; i = 0$$

$$K_1 F = F^\cdot$$

~~so~~

$B_0(F)$	$B_1(F)$	$B_2(F)$
$\mathbb{Z}$		
$F^\cdot$	$\mathbb{Z}$	
0	$F^\cdot$	$\mathbb{Z}$
$\mathbb{Q}/\mathbb{Z}'$	0	$F^\cdot$
0	$\mathbb{Q}/\mathbb{Z}'$	0

Perhaps I can hope to produce a map

$$B_2(F) \longrightarrow B_0(F)$$

$$K_{i-2}(F) \longrightarrow K_i(F)$$

which is the cap product with a canonical $\beta^{-1} \in$ $\pi_2(B_0(k)^\wedge) = \hat{\mathbb{Z}}'$.

1. Preliminaries on quasifibrations

We recall the definition and some properties of quasifibrations (see [3]).

1.1 Definition. Let E, B be topological spaces. A continuous map $p:E \to B$ *onto* B is a *quasifibration* (= q.f.) if

$$(1) \qquad p_*:\pi_i(E, p^{-1}(x), y) \cong \pi_i(B, x) \qquad \text{for all } x \in B,\ y \in p^{-1}(x), \text{ and } i \geq 0.$$

For $i = 0, 1$ this means that we have an isomorphism between sets with distinguished elements (see [3], 1.2). We define a group structure on $\pi_1(E, p^{-1}(x))$ by the requirement that (1) (for $i = 1$) should be an isomorphism of groups. E, p, B, $p^{-1}(x)$ in this order are the *total space*, the *projection*, the *base*, the *fibre* over x of the q.f.

As in the case of fibre bundles the isomorphisms (1) lead to the exact homotopy sequence of a q.f. (see [3], 1.4)

$$(2) \qquad \cdots \to \pi_{i+1}(B) \to \pi_i(p^{-1}(x)) \to \pi_i(E) \to \pi_i(B) \to \cdots.$$

If in a q.f. the base is arcwise connected, then any two fibres are of the same weak homotopy type (see [3], 1.10).

1.2 Definition. Let $p:E \to B$, $p':E' \to B'$ be q.f.s. A map $f:E \to E'$ is called *fibrewise* if there exists a (continuous) application $\bar{f}:B \to B'$ such that commutativity holds in

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "\bar{f}"'] & B'. \end{tikzcd}$$

We say $\bar{f}$ is *induced* by f or f *lies over* $\bar{f}$.

A fibrewise map induces a homomorphism of the exact homotopy sequence of p into that of p' (see [3], 1.8). If $p:E \to B$ is a continuous map, we call a subset $A \subset B$ *distinguished* with respect to p if $p_A:p^{-1}(A) \to A$, the restriction of p to $p^{-1}(A)$, is a q.f. Then we have the following criteria.

1.3 Lemma (see [3], 2.10). *Let $p:E \to B$ be a continuous map onto B, let $B' \subset B$ be a distinguished set, and put $E' = p^{-1}(B')$. Assume there is a "fibre-preserving" deformation of E into E', i.e., there are deformations*

$$D_t:E \to E, \qquad d_t:B \to B \qquad (t \in [0, 1])$$

with

$$D_0 = \mathrm{id}, \quad D_t(E') \subset E', \quad D_1(E) \subset E' \qquad (\mathrm{id} = \textit{identity map}),$$

$$d_0 = \mathrm{id}, \quad d_t(B') \subset B', \quad d_1(B) \subset B', \quad \textit{and} \quad pD_t = d_t p.$$

Assume further

$$D_{1*}:\pi_i(p^{-1}(x)) \cong \pi_i(p^{-1}(d_1(x))), \qquad \text{all } x \in B \text{ and } i \geqq 0.$$

Then B itself is distinguished, i.e., p is a q.f.

1.4 LEMMA ([3], 2.2). *Let $p:E \to B$ be a continuous map, and let U, $V \subset B$ be open sets. If U, V, and $U \cap V$ are distinguished with respect to p, then $U \cup V$ is distinguished.*

1.5 LEMMA ([3], 2.15). *Let $p:E \to B$ be a continuous map. Assume that B is the inductive limit of a sequence of subspaces $B_1 \subset B_2 \subset \cdots \subset B$, satisfying the first separation axiom (points are closed), and each B_ν is distinguished with respect to p. Then p is a q.f.*

2. The basic construction

Every q.f. $E \to B$ in which an $\mathfrak{H}$-space H operates (see Definition 2.2) is embedded in a q.f. $\hat{E} \to \hat{B}$ such that E is contractible to a point in $\hat{E}$. This is done by suitably attaching $CE \times H$ to E where CE is the cone over E.

2.1 DEFINITION (see [5], IV, 1). An *$\mathfrak{H}$-space* is a topological space H together with a continuous multiplication

$$H \times H \to H, \qquad (h, h') \to hh'$$

with two-sided unit e. We also require that the left translations

$$L_{h'}:H \to H, \qquad L_{h'}(h) = h'h$$

induce isomorphisms of all homotopy groups. (If H is arcwise connected, this follows from the existence of a unit.)

2.2 DEFINITION. Let $p:E \to B$ be a q.f., and H an $\mathfrak{H}$-space. An *operation* of H in this q.f. is a continuous map

$$\mu:E \times H \to E, \qquad \mu(y, h) = yh \qquad y \in E, \quad h \in H$$

such that

(1) $ye = y$

(2) $\mu(y \times H) \subset F_y = p^{-1}(p(y)) =$ fibre through y.

Define

(3) $$\mu_y:H \to F_y, \qquad \mu_y(h) = yh$$
$$\mu_{y*}:\pi_i(H) \cong \pi_i(F_y) \qquad \text{for all } y \text{ and all } i \geqq 0.$$

This is obviously a generalization of the notion of a principal bundle. The word "principal" is reserved, however, for the case of an associative $\mathfrak{H}$-space. Note that we do not require $y(hh') = (yh)h'$.

Given a q.f. $p:E \to B$ in which H operates we shall embed it in a q.f. $\hat{p}:\hat{E} \to \hat{B}$ such that the inclusion map $E \subset \hat{E}$ is nullhomotopic. Roughly