January 2, 1972: $GL_n(K)$, K a local field

Let K be a local field with residue field k finite ~~of ...~~ with $q = p^\nu$ elements. I wish to compute $H^*(GL_n(K), \mathbb{F}_\ell)$ where $\ell \neq p$. This is cohomology defined using continuous cochains.

Let X be the building belonging to $GL_n(K)$. It is the simplicial complex whose simplices are chains $L_0 < \cdots < L_q$ of lattices in K^n such that $\pi L_q \subset L_0$. I know X is contractible, hence the cochain complex ~~of~~ X

$$\cdots \; 0 \longrightarrow C^0(X, \mathbb{F}_\ell) \longrightarrow \cdots\cdots \longrightarrow C^n(X, \mathbb{F}_\ell) \longrightarrow 0 \cdots$$

is a resolution of $\mathbb{F}_\ell$ by discrete abelian groups on which G acts continuously. $G = GL_n(K)$

If

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is an exact sequence of discrete abelian groups with continuous G-operation, then one obtains an exact sequence of complexes of continuous cochains

$$0 \longrightarrow C^\cdot(G, M') \longrightarrow C^\cdot(G, M) \longrightarrow C^\cdot(G, M'') \longrightarrow 0$$

and hence a long exact sequence of cohomology. From this it follows by decomposing the complex above into short exact sequences that there is a spectral sequence

$$E_1^{st} = H^t(G, C^s(X, \mathbb{F}_\ell)) \Longrightarrow H^{s+t}(G, \mathbb{F}_\ell).$$

Now

$$C^s(X, \mathbb{F}_\ell) = Map(X_s, \mathbb{F}_\ell)$$

where X_s is the set of s-simplices of X. Given an s-simplex

$$\sigma : \quad L_0 < \cdots < L_s$$

let $\underline{b}(\sigma) = (b_0, b_1, \ldots, b_s)$ be the sequence of non-negative integers defined by

$$b_j = \dim_k L_j / L_{j-1} \qquad j = 1, \ldots, s$$

$$b_0 = \dim_k L_0 / \pi L_s$$

Then $b_j > 0$ for $j \geq 1$. It is clear that if σ and σ' are conjugate under G, then $\underline{b}(\sigma) = \underline{b}(\sigma')$. Conversely if $\underline{b}(\sigma) = \underline{b}(\sigma')$ we claim that σ and σ' are conjugate. Indeed if

$$\sigma' : \quad L_0' < \cdots < L_s'$$

then we can find an elt of G carrying L_s' to L_s (recall a lattice L in K^n is a free $\mathcal{O}$-module of rank n, hence choosing a basis one gets a $g \in G \ni gL = \mathcal{O}^n$.) The stabilizer of L_s maps onto $\mathrm{Aut}(L_s/\pi L_s)$ ($GL_n(\mathcal{O}) \longrightarrow GL_n(k)$ is surjective because $\mathcal{O}$ is a local ring.), and L_j is the inverse image of $L_j/\pi L_s \subset L_s/\pi L_s$. Using the fact that over a field the general linear group acts transitively on the set of flags with the same jumps in dimensions we see that $g \in G$ can be found so that $gL'_s = L_s$ and $g(L'_j/\pi L'_s) = L_j/\pi L_s$, hence $g \cdot \sigma' = \sigma$.

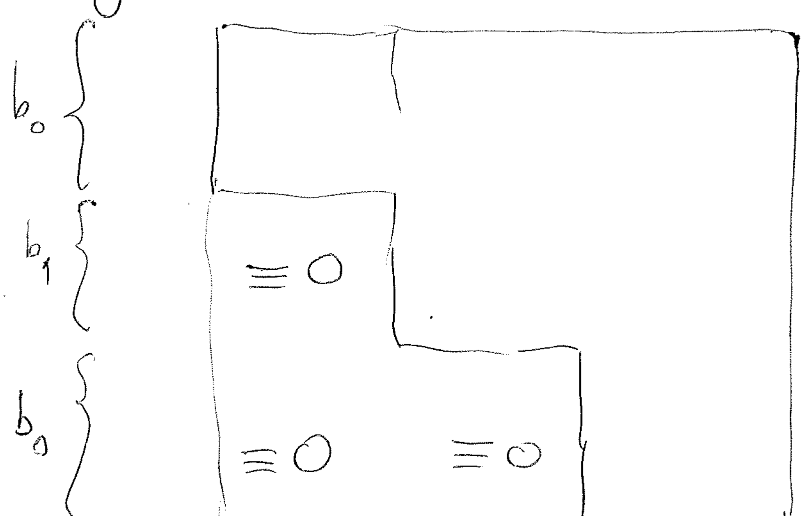
Given ~~a~~ sequence $\underline{b} = (b_0, b_1, \ldots, b_s)$ with $b_i \in \mathbb{N}$, $b_j > 0$, for $j > 0$, and $\sum b_j = n$, let $\sigma_{\underline{b}}$ be the following s-simplex of X. Let $e_1, \ldots, e_n$ be the standard basis of K^n. Then $\sigma_{\underline{b}}$ consists of the lattices ~~spanned by the following sets of vectors~~

$(\sigma_{\underline{b}})_0$ spanned by $e_1, \ldots, e_{b_0}, \pi e_{b_0+1}, \ldots, \pi e_n$

$(\sigma_{\underline{b}})_j$ " " $e_1, \ldots, e_{b_0+\cdots+b_j}, \pi e_{b_0+\cdots+b_j+1}, \ldots, \pi e_n$

$(\sigma_{\underline{b}})_s$ " " $e_1, \qquad\qquad, e_n$

where π generates the maximal ideal of $\mathcal{O}$.
The stabilizer of $\sigma_{\underline{b}}$ is the subgroup of $GL_n(\mathcal{O})$ described by the picture



Denote this subgroup by $G_{\underline{b}}$. It is open in G.
As we have seen

(?) **Lemma**: Let $P \xrightarrow{h} Q$ be a map of simplicial objects in X, and assume all of its stalk maps

$$P_x \longrightarrow Q_x$$

are weq's. Then

$$H^*((\Delta/Q)^\wedge; F) \xrightarrow{\sim} H^*((\Delta/P)^\wedge; F)$$

for any abelian sheaf F on X (probably also for any local system on Q, i.e. a simplicial sheaf F over Q such that all squares

$$\begin{tikzcd} F_n \arrow[r] \arrow[d] & F_m \arrow[d] \\ Q_n \arrow[r] & Q_m \end{tikzcd}$$

are cartesian.)

Obtain from lemma

$$H^q\left(\left(\Delta/f'^{-1}(V\times\Delta(d))\right)^{\wedge}, g'^{*}F\right)$$

$$= H^q\left(\Delta/f^{-1}V\times\Delta(d), F\right)$$

so reduce to case where

$$Q = f^{*}P.$$

In this case clear by localization. i.e. pulling to open R of $(\Delta/Y)^{\wedge}$.

The real point is that no matter what object W of $(\Delta/Q)^{\wedge}$ you look at you have

$$H^q\left((\Delta/f'^{*}W)^{\wedge}, F\right) \xleftarrow{\sim} H^q\left((\Delta/f^{*}W)^{\wedge}, F\right)$$

by the lemma, reducing to the case $P = f^{*}W$ whence the base change follows by localization

If the lemma holds, then let there be ~~given~~ a map $f : X \to Y$, a simplicial object Q over Y (resp. P over X) and a map $P \to f^*Q$ which is a stalk-wise weq.

Claim in the square

$$\begin{tikzcd} (\Delta/P)^\wedge \arrow[r, "g'"] \arrow[d, "f'"] & X \; F \arrow[d, "f"] \\ (\Delta/Q)^\wedge \arrow[r, "g"] & Y \end{tikzcd}$$

I have base change

$$\boxed{g^* R^{\delta} f_*(F) \xrightarrow{\sim} R^{\delta} f'_*(g'^* F).}$$

[illegible]

~~Special case: $P = X$, $Q = Y$. Then f'_* sends a simplicial $\{G_n\}$ in X to $\{f_* G_n\}$, and same for derived functors~~

$$R^{\delta} f_*(G)_n = R^{\delta} f_*(G_n)$$

~~(since any injective in $C^\wedge$ is injective over $\mathrm{Ob}\, C$.)~~

~~hence have base change if $G_n = F$ all n.~~

__Corollary:__ Assume P, Q ~~[illegible]~~ acyclic over X, Y resp. Then Leray spectral sequence of (f, F) isomorphic to that Leray spec. sequence of $(f', g'^{*}F)$ for square

$$\begin{tikzcd} (\Delta/P)^{\wedge} \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ (\Delta/Q)^{\wedge} \arrow[r, "g"'] & Y \end{tikzcd}$$

Proof: i) $R^{+}g'_{*}\, g'^{*}F = 0$ & $F \to g'_{*}g'^{*}F$
because $P \to X$ acyclic.

ii) $R^{+}g_{*}\left(R^{b}f'_{*}(g'^{*}F)\right) = R^{+}g_{*}\left(g^{*}R^{b}f_{*}(F)\right)$

by above base change fmla (uses $P \to f^{*}Q$ acyclic) and latter zero as $Q \to Y$ acyclic.

$GL_n K$ K local field in back

January 4, 1972:

Formulas: Let G be a topological group acting continuously on a space X. Then one has a top. category (G,X) whose nerve is

$$\text{Nerv}(G,X): \quad G^2\times X \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; G\times X \;\substack{\longrightarrow\\ \longrightarrow}\; X$$

where I recall that to get the simplicial operations one thinks of the arrows as running to the left, so

$$(g,x) \quad : \quad gx \xleftarrow{\;g\;} x$$

$$(g_1,g_2,x) \quad : \quad g_1g_2x \xleftarrow{\;g_1\;} g_2x \xleftarrow{\;g_2\;} x$$

and hence

$$d_0(g,x) = x$$
$$d_1(g,x) = gx$$

On the other hand if G acts to the right of Y we get a top. category (Y,G) with nerve

$$\text{Nerv}(Y,G): \quad Y\times G^2 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; Y\times G \;\substack{\longrightarrow\\ \longrightarrow}\; Y.$$

Set $\text{Nerv}(G) = \text{Nerv}(pt, G) = \text{Nerv}(G, pt)$.

Let M be an abelian group on which G operates in a continuous fashion w.r.t. the discrete topology on M. Then $\text{Nerv}(M,G)^*$ is a simplicial sheaf over $\text{Nerv}(G)$ which is special, i.e. all squares

*(make G act on M to the right by $mg = g^{-1}m$)

$$(*)\qquad \begin{array}{ccc} M\times G^q & \xrightarrow{\ {}^t\varphi\ } & M\times G^p \\ \downarrow & & \downarrow \\ G^q & \xrightarrow[{}^t\varphi^*]{} & G^p \end{array}$$

for any simplicial operator $\varphi : [p] \to [q]$, are cartesian. Thus $\mathrm{Nerv}(M, G)$ gives rise to a cosimplicial sheaf over $\mathrm{Nerv}(G)$, and hence to a cosimplicial abelian group of sections (taken dimension-wise):

$$C^r(G, M) = \text{sections of } M\times G^r \longrightarrow G^r$$
$$= \mathrm{Map}(G^r, M).$$

The coface operator $\delta_j : C^r(G,M) \longrightarrow C^{r+1}(G,M)$ is defined by letting $(\delta_j f)$ be the section of $M\times G^{r+1} \longrightarrow G^{r+1}$ compatible with f and the cartesian square $(*)$ with ${}^t\varphi = d_j$. One calculates:

$$(\delta_j f)(g_1, \ldots, g_{r+1}) = \begin{cases} g_1 f(g_2, \ldots, g_{r+1}) & j=0 \\ f(\ldots, g_j g_{j+1}, \ldots) & 0<j\leq r \\ f(g_1, \ldots, g_r) & j = r+1 \end{cases}$$

so that we do get the usual continuous cochain complex $C^\cdot(G, M)$ with

$$\delta = \sum (-1)^j \delta_j .$$

(<u>Remark</u>: This discussion is unconvincing w.r.t. naturality. I need a yoga which would explain all of this from a general viewpoint; perhaps the classifying topoi is needed.)

Let U be a subgroup of G, whence we have a morphism of topological categories

(1) $$(U, e) \longrightarrow (G, G/U),$$

and hence a morphism of simplicial spaces

Nero(1) $$\mathrm{Nero}(U) \longrightarrow \mathrm{Nero}(G, G/U).$$

Suppose now that the projection $G \to G/U$ has a continuous section $s : G/U \to G$. Then we can define a morphism of topological categories

We assume $s(U) = 1_G$

(2)
$$\begin{aligned} (G, G/U) &\longrightarrow (U, e) \\ x &\longmapsto e \\ y \xleftarrow{g} x &\longmapsto s(y)^{-1} g\, s(x) \end{aligned}$$

such that $(2)\circ(1) = id$. The composition $(1)\circ(2)$ is isomorphic to the identity via the natural transf. which associates to $x \in G/U$ the map $x \xleftarrow{s(x)} e$. Indeed $y \xleftarrow{g} x$ is sent into $e \xleftarrow{s(y)^{-1} g\, s(x)} e$ by $(1)\circ(2)$ and we have the comm. square

$$\begin{tikzcd} x \arrow[d, "g"'] & e \arrow[l, "s(x)"'] \arrow[d, "s(y)^{-1} g\, s(x)"] \\ y & e \arrow[l, "s(y)"] \end{tikzcd}$$

Consequently we have that

$$\mathrm{Nero}(2) \circ \mathrm{Nero}(1) = id_{\mathrm{Nero}(U)}$$

Nerv(1) ∘ Nerv(2) homotopic to $\mathrm{id}_{\mathrm{Nerv}(\mathcal{G},\, \mathcal{G}/\mathcal{U})}$

(The precise idea here is that a natural transf. θ of functors between topological categories (more generally, categorical objects in a category) induces a simplicial homotopy on the nerves as follows.

Recall that a simplicial homotopy $h : X \times I \longrightarrow Y$ consists of maps

$$h_n : X_n \times I_n \longrightarrow Y_n$$

compatible with simplicial operations. As I_n has simplices $\underbrace{0 \cdots 0}_{0 \cdots j} \, 1 \cdots 1 \atop \cdots n$ $\quad j = 0, \dots, n+1$

hence h_n consists of a family of maps

$$h_n^j : X_n \longrightarrow Y_n \qquad j = 1, \dots, n \qquad \begin{cases} h_n^0 = f \\ h_n^{n+1} = g. \end{cases}$$

such that certain identities with faces + degeneracies hold. In the present situation, we are given two functors $f, g : \mathcal{C} \longrightarrow \mathcal{C}'$ and a natural transformation θ and the associated map

$$\mathrm{Nerv}(\mathcal{C}) \times I \longrightarrow \mathrm{Nerv}(\mathcal{C}')$$

will send

$$h_n^j(X_0 \longleftarrow X_1 \longleftarrow \cdots \longleftarrow X_n) = f(X_n) \longleftarrow \cdots f(X_{j-1}) \xleftarrow{\text{constructed using } \theta} g(X_j) \longleftarrow \cdots \longleftarrow g(X_n)$$

).

Returning to page 2, let X be a G-space and let F be a sheaf over X with compatible G-action. Then we have a cochain complex $C^\bullet(X,G;F)$ with

$$C^r(X,G;F) = \mathrm{Map}_X(X\times G^r;F)$$

and with

$$(\delta_j f)(x,g_1,\dots,g_r) = \begin{cases} g_1 f(g_1^{-1}x, g_2,\dots,g_r) & j=0 \\ f(x,g_1,\dots,g_jg_{j+1},\dots,g_r) & 0<j<r \\ f(x,g_1,\dots,g_{r-1}) & j=r \end{cases}$$

This may be interpreted as the cosimplicial abelian group of dimension-wise sections of the special sheaf

$$\mathrm{Nerv}(F,G) \longrightarrow \mathrm{Nerv}(X,G)$$

where G acts on the right via $x\cdot g = g^{-1}x$ and similarly for F.

How to think: In nature, sheaves are contravariant animals. Thus if G were a monoid acting on X, the natural thing to consider would be a sheaf on X with a right G-action, i.e. maps $F_{gx} \longrightarrow F_x$ $\quad m \mapsto mg$. Thus the appropriate cochain complex would be

$$C^r(G,X;F) = \mathrm{Map}_X(G^r\times X, F)$$

$$(\delta_r f)(g_1,\dots,g_r,x) = f(g_1,\dots,g_r x)g_r .$$

Similarly if C is a category and F is a contravariant functor on $\mathcal{C}$, we have the cochain complex

$$C^n(\mathcal{C}; F) = \prod_{x_0 \leftarrow \cdots \leftarrow x_n} F(x_n)$$

It is important here <u>not</u> to think of F as giving rise to a simplicial object over $\mathrm{Nerv}(\mathbb{C})$.

<u>Situation of interest</u>: Let U be a subgroup of the topological group G such that $G \to G/U$ has a continuous section. Let A be an abelian group. Then the map of topological cats.

$$\mathrm{Nerv}(U, e) \longrightarrow \mathrm{Nerv}(G, G/U)$$

induces a map of cochain complexes

$$C^*(\mathrm{Nerv}(G, G/U); A) \longrightarrow C^*(\mathrm{Nerv}(U); A)$$

and the point is that this last map is a homotopy equivalence, and in particular a quasi-isom.

More generally if A is a U-module, then

January 23, 1972

Wu formulas for $Sq^i(w_j)$:

$E \mapsto Sq_s\, w_t(E)$ is an exponential char. class with

$$Sq_s\, w_t(L) = Sq_s(1+tx) = 1 + t(x+sx^2)$$
$$= 1 + tx + tsx^2 \qquad x = w_1(L).$$

Thus

$$Sq_s\, w_t(E) = w_{\lambda_1}(E)\, w_{\lambda_2}(E)$$

where $\quad \lambda_1 + \lambda_2 = t \qquad \lambda_1\lambda_2 = ts.$

$$Sq_s\, w_t = \sum_{i,j\geq 0} \lambda_1^i w_i \lambda_2^j w_j$$
$$= \sum_{i<j} (\lambda_1\lambda_2)^i \left(\lambda_1^{j-i} + \lambda_2^{j-i}\right) w_i w_j \;+ \sum_i (ts)^i w_i^2$$
$$= \sum_{i,k} (ts)^i \left(\lambda_1^k + \lambda_2^k\right) w_i w_{i+k} \;+ \sum (ts)^i w_i^2$$

But

$$\sum_{k\geq 0} \left(\lambda_1^k + \lambda_2^k\right) z^k = \frac{1}{1+\lambda_1 z} + \frac{1}{1+\lambda_2 z} = \frac{(\lambda_1+\lambda_2)z}{(\quad X \quad)}$$

$$= \frac{tz}{1+tz+tsz^2} = tz \sum_{a\geq 0} (tz + tsz^2)^a$$

$$\cancel{= tz \sum_{\nu\leq j} \binom{j}{\nu} s^{\nu} z^{j+\nu} t^j = \sum_{\nu\leq j} \binom{j}{\nu} t^{j+1} s^{\nu} z^{j+\nu+1}}$$

$$= \sum' (tz)^{a+1} (1+sz)^a = \sum_{b \le a}' \binom{a}{b} t^{a+1} s^b z^{a+1+b}$$

$$= \sum_{0 \le b < a}' \binom{a-1}{b} t^a s^b z^{a+b}$$

so $$(\lambda_1^k + \lambda_2^k) = \sum_{\substack{a+b=k \\ 0 \le b < a}}' \binom{a-1}{b} t^a s^b$$

$$Sq_s(w_t) = \sum_{i,a,b}' \binom{a-1}{b} t^{a+i} s^{b+i} w_i w_{i+a+b} + \sum (ts)^i w_i^2$$

$$= \sum_{\substack{i,a,b \\ 0 \le i \le b < a}}' \binom{a-1-i}{b-i} t^a s^b w_i w_{a+b-i} + \sum (ts)^i w_i^2$$

$$\boxed{Sq_b(w_a) = \sum_{0 \le i \le b}' \binom{a-1-i}{b-i} w_i w_{a+b-i} \quad \text{if } b<a}$$

$$Sq_{a-1}(w_a) = \sum_{0 \le i < a}' w_i w_{2a-1-i}$$

The point is that

$$Sq_{a-1}\Big(\sum_{i_1 < \cdots < i_a}' x_{i_1} \cdots x_{i_a}\Big) = \sum_{\substack{i_1 < \cdots < i_b \\ 1 \le b \le a}} x_{i_1}^2 \cdots \widehat{x_{i_b}^2} \cdots x_{i_a}^2 \, x_{i_b}$$

hence in $H^*(F\Phi^\theta)$ exceptional case we have

$$e_a^2 = \sum_{0 \le i \le a-1}' c_i c_{2a-1-i}$$

Relative to K-theory of $\mathbb{F}_q$.

I originally wanted to identify this K-theory with the fixpoints of Ψ^q on ordinary K-theory, ~~[illegible]~~ in all respects.

1. λ-ring structure

$$K(X;\mathbb{F}_q) \xrightarrow{\sim} [X, \mathbb{Z}\times \cancel{B}F\Psi^q]$$

is to be an isom of λ-rings.

2. interpretation of extension and restriction of scalars

All these identifications were to proceed from

$$[BU^n, \Omega BU] = 0$$

$$[BU^n, BU] = \mathrm{Hom}(\tilde{K}^n, \tilde{K})_{\text{compact spaces}}$$

things you can say about $F\Phi^q$.
λ-ring structure
part of a cohomology theory
l-primary part depends only
l-adic subgroup generated by q.
anything else doesn't follow.

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decision §9. Cohomology of $E\Phi^q$.

BU,

$$[BU, BU] = \varprojlim [Gr_{nn}, BU]$$

conclusion is that there is a unique element of $[BU, BU]$ ~~coinciding with~~ inducing Φ^q on $\tilde{K}(X)$ X a finite complex.

~~Suppose now~~ Given Φ^q let $E\Phi^q$ be the space whose points are pairs (x, λ) where x is a point of BU and λ is a path joining x to $\Phi^q(x)$. Then $\exists$

$$\begin{tikzcd} E\Phi^q \arrow[r, "t"] \arrow[d, "t"'] & BU \arrow[d, "\Delta"] \\ BU \arrow[r, "\Gamma"'] \arrow[ur, phantom, "H \Nearrow"] & BU \times BU \end{tikzcd}$$

$\Gamma = (id, \Phi^q)$.

where $t(x, \lambda) = x$ and H is the homotopy sending $((x, \lambda), r)$ to $(x, \lambda(r))$.

§9. Cohomology of $E\Psi^q$.

How much do you intend to do about the space $E\Psi^q$? First you must understand exacrtly what ~~ha~~ should be said, then you shoud ~~say~~ decide whether to spend the time saying it.

BU has endos. Ψ^q additive for each q in Z and composition is kike mult.

BU^* (completion profinite) has what endos? Reasobale to think $\mathbb{Z}^*$ undermult. e.g. BU^* = product of BU_ℓ for all primes ℓ, and there are no cross endos, and clearly $[BU_p, BU_p] = [BU,BU]_p$ by Artin-Mazur so done. Thus the endos. of BU_p should be known.

It's really enough to handle the case of the completion.afterward. Thus. So introduce the space $E\Psi^q$ and prove that $[X, E\Psi^q]$ is a -ring without identity, profinite. This being so we have a product decomposition

$$[X, E\Psi^q] = \prod_p [X, E\Psi^q]_p$$

where $[X, E\Psi^q]_p$ is a p-adic -ring in the sense of Atiyah-Tall. Now the next thing is the ~~determination~~ proof that the ℓ-primary component of depends only on the subgroup generated by q in $\mathbb{Z}_\ell^*$. Once you have established this it is enough to worry about the case where q is prime by Dirichlets theorem.

BU^* has auotos coming from the Galois group of $\mathbb{Q}$. If you take the m-th cyclo tomic extension of $\mathbb{Q}$, its Galois group is $(\mathbb{Z}/m)^*$, so the Galois group is $\mathbb{Z}^{c*}$ units in the completion. Thus one sees that Galois gives all possible additive autos of BU^c. Therefore also BU_p has $\mathbb{Z}_p^*$ for its group of autos. Also interesting is to see what in fact happens for ~~the cohomology of~~ BU_2 with its screwy behavior.

So the problem at the moment is to show that the extension of scalars map $E\Psi^q \quad E\Psi^{q^d}$ is a mod ℓ isomorphism if q and q^d generate the same sub-group of $\mathbb{Z}_\ell^*$. involves checkin on ly that $q^{di}-1/q^i-1$ is an ℓ-adic unit.

~~[illegible]~~ g, g^d generate same subgp. of $\mathbb{Z}_\ell^*$.

ℓ odd $\Longrightarrow$ $\mathbb{Z}_\ell^* \simeq \mu_{\ell-1} \times (1+\ell\mathbb{Z}_\ell)^\times$

$\exists c \ni g^{cd} \equiv g \mod \ell-1$

$$\begin{tikzcd}
E\Psi^g \arrow[r] \arrow[d, "\mathrm{ext}"] & BU \arrow[r, "\psi^g-1"] \arrow[d, "1"] & BU \arrow[d, "\frac{\psi^{g^d}-1}{\psi^g-1} = \sum_{a=0}^{d-1}\psi^{g^a}"] \\
E\Psi^{g^d} \arrow[r] & BU \arrow[r, "\psi^{g^d}-1"] & BU
\end{tikzcd}$$

assume r least $\ni g^r - 1 \equiv 0 \quad (\ell)$.

~~[illegible]~~

$i \not\equiv 0 \quad (\mathrm{mod}\ r)$ $\quad g^i - 1$ unit
$\quad g^{di} - 1$ unit

so $\dfrac{g^{di}-1}{g^i-1}$ unit.

$i \equiv 0 \quad (\mathrm{mod}\ r)$, then we know

$\dfrac{g^{di}-1}{d(g^i-1)}$ is ℓ-adic unit

and we know d prime to ℓ because otherwise ℓ-comp. wouldn't gen.

subgp gen. by $g = g_r g_s$ is $\langle g_r \rangle \langle g_s \rangle$

d is unit mod order of g_r.
$d \not\equiv 0 \ (\ell)$

$\langle g_s^d \rangle = \langle g_s^{[illegible]} \rangle \quad d$

~~Outline of~~ part of K-theory paper dealing with $\mathbb{F}_q$.

Let k be a finite field with $q = p^d$ elements and let $\bar{k}$ be an algebraic closure of k. The Galois group $\mathrm{Gal}(\bar{k}/k)$ is the free prof. gp. generated by ~~Let σ be~~ the Frobenius automorphism:

$\sigma x = x^q$. Then ~~σ induces an automorphism of $K_*(\bar{k})$ which we again denote by σ~~ the effect of σ on $K_*(\bar{k})$, according to [ref. to Part II] ~~it~~ coincides with the Adams operation Ψ^q. ~~on the K-groups~~

<u>Thm. 1:</u> For $i \geq 1$, we have isomorphism

$$K_{2i}(\bar{k}) = 0$$

$$K_{2i-1}(\bar{k}) \simeq \mathbb{Q}/\mathbb{Z}[p^{-1}]$$

and σ acts on $K_{2i-1}(\bar{k})$ by multiplying by q^i.

<u>Thm. 2:</u> The inclusion of k in $\bar{k}$ induces an isomorphism of $K_*(k)$ with the subgroup of $K_*(\bar{k})$ invariant under $\mathrm{Gal}(\bar{k}/k)$. In particular,

$$K_{2i}(k) = 0$$

$$K_{2i-1}(k) \simeq \mathbb{Z}/(q^i - 1)\mathbb{Z}$$

for $i \geq 1$.

Corollaries dealing with subfields $k' \subset k$:

$$K_*(k_1) \xrightarrow{\sim} K_*(k_2)^{\mathrm{Gal}(k_2/k_1)}$$

$[k_2 : k_1] < \infty \Rightarrow$ norm given by norm on invariants.
norm surjective

[you should be able to say that these follow from given action Σ on $\mathbb{Q}/\mathbb{Z}[p^{-1}]$].

§2. Proof of thm 1: ~~or better determination of~~ $\cancel{BGL(\bar{k})^+}$ ~~(following idea of Sullivan.)~~

Recall from other paper the Brauer map

$$\phi : BGL(\bar{k}) \longrightarrow BU$$

+ following facts.

1. ϕ induces isoms. on $H^*(\ , \mathbb{F}_\ell)$ all $\ell \neq p$
2. $H^*(BGL(\bar{k}), \mathbb{F}_p) = \mathbb{F}_p$.
3. ~~[illegible]~~

Using this, we will determine homotopy type of $BGL(\bar{k})^+$ following idea of D. Sullivan.

Construct

$$F \longrightarrow BU[p^{-1}] \longrightarrow BU_{\mathbb{Q}} \cong \prod_{m \geq 1} K(\mathbb{Q}, 2m)$$

Because $BGL(\bar{k})$ has trivial rational coh., $\exists !$ dotted arrow (up to $\sim$) ψ

$$(*) \quad \begin{tikzcd} BGL(\bar{k}) \arrow[r, dashed, "\psi"] \arrow[d, "\phi"] & F \arrow[d, "j"] \\ BU \arrow[r, "i"] & BU[p^{-1}] \end{tikzcd}$$

~~Now i isoms on $H^*(\ \mathbb{F}_\ell)$ $\ell \neq p$~~

~~j isoms on $H^*(\ \mathbb{F}_\ell)$ all ℓ~~

~~[illegible]~~

~~and $H^*(BU[p^{-1}], \mathbb{F}_p) = 0$. $\therefore$ $i\phi$ isos. finite coeffs.~~

~~same for j.~~

$$\left\{\begin{array}{l} ① \ \cancel{a} + \\ i \ \text{ induces isom on } H^*(\ , \mathbb{F}_\ell) \qquad \ell \neq p \\ H^*(BU[p^{-1}], \mathbb{F}_p) = 0 \ + ② \end{array}\right.$$

$\Rightarrow$ $i\phi$ induces isom. on $H^*(\ , \mathbb{F}_\ell)$ all ℓ.

+ j isoms ~~all~~ $\ell \neq p$

$\Rightarrow$ ψ induces isom $H^*(\ , \mathbb{F}_\ell)$ all ~~exc.~~ ℓ

But $H_*(BGL(\bar{k}), \mathbb{Q}) = H_*(F, \mathbb{Q}) = 0$

so ψ induces isoms with all coeffs.

$\Rightarrow$ $BGL(\bar{k})^+ \longrightarrow F$ heq.

proving first part of thm. 1.

from uniqueness of dotted arrow one sees

$\psi\sigma = \Psi^q \phi$.

Proof of thm 2:

Recall from other paper $\exists!\ \phi' \ni$

$$\begin{array}{ccc} BGL(k) & \xrightarrow{\phi'} & F\Psi^q \\ \downarrow & & \downarrow \\ BGL(\bar{k}) & \xrightarrow{\phi} & BU \end{array},$$

and the fact

REF $\boxed{\phi' \text{ induces isomorphisms on all kinds of coh.}}$

$\Rightarrow$ $BGL(k)^+ \longrightarrow F\Psi^q$ heq.

Rational coh. of $F\Psi^q$ trivial, get unique dotted arrow

$$(**) \qquad \begin{array}{ccc} F\Psi^q & \dashrightarrow & F \\ \downarrow & & \downarrow \\ BU & \longrightarrow & BU[p^{-1}] \end{array}$$

and similarly one gets commutative square

$$\begin{tikzcd}
BGL(k) \arrow[r, "\phi'"] \arrow[d] & F\Phi^{\delta} \arrow[d] \\
BGL(\bar{k}) \arrow[r, "\psi"] & F
\end{tikzcd}$$

in which both maps lead to heq.

As dotted arrow in $(**)$ may be obtained by map of fibrations

$$\begin{tikzcd}
F\Phi^{\delta} \arrow[r] \arrow[d, dotted] & BU \arrow[r, "\Phi^{\delta}-1"] \arrow[d, "1"] & BU \arrow[d, "(\Phi^{\delta}-1)^{-1}"] \\
F \arrow[r] & BU[p^{-1}] \arrow[r] & BU_{\mathbb{Q}}
\end{tikzcd}$$

one computes $\pi_*(F\Phi^{\delta}) \hookrightarrow \pi_*(F)$ ~~done~~

with thm. 2.

$A_n \subset \Sigma_n$

~~does~~ a transposition act as outer autom.

Yes $\quad \sigma x \sigma^{-1} = y x y^{-1}$ $\qquad$ σ transp.

$y^{-1}\sigma \in$ center of $A_n \Longrightarrow y^{-1}\sigma = id.$

$$A_n \longrightarrow \Sigma_n \longrightarrow \mathbb{Z}_2$$

$$H^*(\Sigma_n) \longleftarrow H^*(\mathbb{Z}_2)$$

not a n.z.d.
hence MESS.

A_5 $\quad$ 5 OKAY.

3 ✓

2 ✓

$SL_3(\mathbb{F}_2)$

$\underline{SL_2(\mathbb{F}_7)}$

$SL_2(\mathbb{F}_5)$ $\qquad 5 \cdot 24 = 120$

$r=2$

$3 \mid 25-1$

$$H^*(SL_2(\mathbb{F}_5), \mathbb{Z}/3) \xleftarrow{\sim} H^*(PSL_2(\mathbb{F}_5), \mathbb{Z}/3)$$

$$\mathbb{Z}/3[x_3, y_4] \qquad\qquad H^*(A_5, \mathbb{Z}/3)$$

$$H^*(SL_2(\mathbb{F}_5), \mathbb{Z}/5) \xleftarrow{\sim} H^*(A_5, \mathbb{Z}/5) \qquad 5 \mid 16-1$$

$$= H^*(SL_2(\mathbb{F}_4), \mathbb{Z}/5)$$

$$= \mathbb{Z}/5[x_3, y_4]$$

$GL_2(\mathbb{F}_5)$. $\quad$ 3?

$\mathbb{Z}/2 \,\tilde{\times}\, (\mathbb{Z}/4\mathbb{Z})^2$

$$H^*(PSL_2(\mathbb{F}_5), \mathbb{Z}/2)$$

$$H^*(SL_2(\mathbb{F}_5), \mathbb{Z}/2) = \mathbb{Z}/2[x_3, y_4]$$

$PSL_2(\mathbb{F}_5)$

$SL_2(\mathbb{F}_q)$ $\quad q = p^d$

here p, q odd

$\frac{p-1}{2}$ OKAY for $p \geq 5$

degree

$d \cdot \left(\frac{p-1}{2}\right)$

bad degree

$SL_2(\mathbb{F}_4)$

$4 \cdot 15 = 60$

$q=5 \quad d=1 \quad p=5$ bad

$q=9 \quad d=2 \quad p=3$ bad

$\Downarrow$

maybe $SL_2(\mathbb{F}_9)$ has a non-trivial H_2

$SL_2(\mathbb{F}_5)$ ——— H_2

non-perfect $SL_2(\mathbb{F}_3)$ $\quad 3 \cdot \frac{8}{2} = 12$ ✓

$\Downarrow$

$PSL_2(\mathbb{F}_9) = 9 \cdot \frac{80}{2} = 360$ / central extension mod 3. ?

$\boxed{PSL_2(\mathbb{F}_5)} = 5 \cdot \frac{24}{2} = \boxed{60}$ $\quad A_5$ does have cent. ext ~~[illegible]~~ mod 2

$2/4$..

$\mathbb{F}_5$ mod 5 $\quad$ good case. gen. 3, 4. $\quad$ no.

$\mathbb{F}_5^*$ $\quad \lambda^2$ $\quad$ 3 $\quad$ rule this out $\quad A_5 \quad \mathbb{Z}_5$

4

$SL_2(\mathbb{F}_2, 4, 8)$ ✓ ✓ | d.

SL_3 . . ✓ . . . ✓

$SL_3(\mathbb{F}_2) = PSL_2(\mathbb{F}_7)$ nice central ext. <u>order</u> 2 at least

---

A_5 has central $\mathbb{Z}_2$-extension.

$A_5 = PSL_2(\mathbb{F}_5) = G$

$PSL_2(\mathbb{F}_9)$ order 360 $\simeq A_6$

$\therefore SL_2(\mathbb{F}_9)$ probably 1-conn.

$SL_2(\mathbb{F}_5)$ also prob. 1-conn.

$SL_2(\mathbb{F}_4)$ $4 \cdot 15 = 60$

NO.

$SL_3(\mathbb{F}_2) = PSL_2(\mathbb{F}_7)$ <u>NO</u>.

$GL_4(\mathbb{F}_2) \simeq A_8$

begins

dim 2.

$SL_n(\mathbb{F}_q)$ ~~1-conn. except~~

0-connected except $n = 2$, $q = 2, 3$

1-connected except $n = 2$, $q = 4$

3, $q = 2$

Part II: §4.

It seems convenient to introduce the 2-category whose objects are pointed topological spaces, whose morphisms are continuous basepoint-preserving maps, and whose set of ~~two~~ 2-morphisms from f to g, where f,g are maps from X to Y, are the homotopy classes of homotopies joining f to g through basepoint-preserving maps.

Section 4. The mod ℓ cohomology of $E\Psi^q$. Recall that the space $E\Psi^q$ fits into a cartesian square

$$\begin{tikzcd} E\Psi^q \arrow[r] \arrow[d, "i"'] & BU^I \arrow[d, "(e_0,e_1)"] \\ BU \arrow[r, "\Gamma = (id,\Psi^q)"'] & BU\times BU \end{tikzcd}$$

where e_0,e_1 are the endpoint maps. In particular the square

$$(\ \ )\qquad \begin{tikzcd} E\Psi^q \arrow[r, "i"] \arrow[d, "i"'] & BU \arrow[d, "\Delta"] \\ BU \arrow[r, "\Gamma"'] & BU\times BU \end{tikzcd}$$

comes with a canonical homotopy from $\Gamma . i$ to $\Delta . i$, so we can ~~apply~~ define the operation

$$\Phi : (I\cap J/\,IJ)^{2a} \longrightarrow (\mathrm{Coker}\ i^*)^{2a-1} = H^{2a-1}(B\Psi^q)$$

where I is the kernel of Γ^* and J is the kernel of Δ^*. Define elements

$$(\ \ )\qquad \begin{aligned} c'_{jr} &= i^*(c_{jr}) \in H^{2jr}(E\Psi^q) && j\geq 1 && c'_0 = 1 \\ c''_{jr} &\quad H^{2jr-1}(E\Psi^q) && '' && c''_0 = 0 \\ &= \Phi(c_{jr}\otimes 1 - 1\otimes c_{jr}) \end{aligned}$$

where $c_i \in H^{2i}(BU)$ denotes the i-th universal Chern class. (Here you should point out that by the definition of r and the formula $(\Psi^q)^*(c_i) = q^i c_i$, the elements c_{jr} are invariant under the action of Ψ^q hence $c_{jr}\otimes 1 - 1\otimes c_{jr} \in I\cap J$.)

Let X be a space and let $x \in E\Psi^q(X)$. Then we define the elements

$$c'_{jr}(x) \in H^{2rj}(X)$$
$$c''_{jr}(x) \in H^{2jr-1}(X)$$

by pulling back the classes ( ) under the map $x\colon X \to E\Psi^q$.

Theorem 1: (Properties of the classes c'_{jr} and c''_{jr})

1) (Product formula) $c'_{jr}(x+y) = \sum_{a+b=j} c'_{ar}(x)c'_{br}(y)$

$$c''_{jr}(x+y) = \sum_{a+b=j} \left(c'_{ar}(x)\,c''_{br}(y) + c''_{br}(x)\,c'_{ar}(y) \right)$$

Or if $H^*(X)[\varepsilon]$ is the ring obtained by adjoining an element ε to $H^*(X)$ such that $\varepsilon^2=0$ and $\varepsilon u = (-1)^{\deg u} u\varepsilon$, then $c(x+y) = c(x)c(y)$ where

$$c(x) = \Sigma\, c'_{jr}(x) + c''_{jr}(y)\varepsilon$$

2) (Normalization) Let $\chi : C \longrightarrow S^1$ be a character of a cyclic group C of order dividing q^r-1, let E be the representation of C

$$E = \bigoplus_{1}^{r} \chi^{q^a}$$

and denote by $\alpha : BC \longrightarrow E\Psi^q$ the map obtained from E and the unique isomorphism $\Psi^q(E) \cong E$ (Unique because C is a finite group).

Let $u \in H^2(BC)$ be the first Chern class mod ℓ of the character χ and let $v \in H^1(BC)$ be the class of the homomorphism $(1-q^r)\ell^{-1}\cdot\chi : C \longrightarrow \mathbb{Z}/\ell$. Then

$$c(\alpha) = 1 + (-1)^{r-1}u^{r} + (-1)^{r-1}u^{r-1}v\cdot\varepsilon$$

Remarks: This theorem combined with the Brauer map gives an alternative construction of the arithmetic Chern classes of a representation of a finite group. The only things to check and clean up are the effect of the choice of ϕ which here should amount only to a choice of a generator of μ_ℓ.

have to decide which are the thms. and the corollaries.

Theorem 2: 1) (Cohomology of $E\Psi^q$) There is an additive isomorphism

$$S[c'_r, \ldots\ldots] \otimes \Lambda[c''_r, \ldots\ldots] \cong H^*(E\Psi^q)$$

which is a ring isomorphism except when $\ell=2$ and 4 doesn't divide q-1.

2) The natural transformation $kF_6(X) \longrightarrow E\Psi^6(X)$ induces an isomorphism on transf. to cohomology with coefficients in a field.

3) $H_*(E\Psi^6) \cong S[\xi_1, \xi_2, \ldots] \otimes \Lambda[\eta_1, \ldots]$ as in the first section.

<u>Proof of theorem 2:</u> We use the Eilenberg-Moore spectral sequence associated to the fibre square

$$\begin{tikzcd} E\Psi^\theta \arrow[r] \arrow[d, "i"] & BU^I \arrow[d, "(e_0, e_1)"] \\ BU \arrow[r, "\Gamma"] & BU \times BU \end{tikzcd}$$

which after identifying $H^*(BU^I) = H^*(BU)$ and $(e_0, e_1)^*$ with Δ^* is

$$E_2^{p\theta} = \mathrm{Tor}_{-p}^{H^*(BU\times BU)}\left(H^*(BU)_\Gamma, H^*(BU)_\Delta\right)^\theta \Rightarrow H^*(E\Psi^\theta).$$

Now write

$$H^*(BU) = S(V)$$

where V is vector space with basis $c_1, c_2, \cdots$. Then in the notation of our technical section the E_2 term is

$$\mathrm{Tor}_*^R(R/I, R/J) \simeq S(V_\sigma) \otimes \Lambda^*(V^\sigma)$$

where $\sigma = (\Psi^\theta)^*$. Thus the E_2 term is

$$E_2 \cong S[\bar{c}_r, \bar{c}_{2r}, \cdots] \otimes \Lambda[\bar{\bar{c}}_r, \bar{\bar{c}}_{2r}, \cdots]$$

where $\bar{c}_{jr} \in E_2^{0,2jr}$ and $\bar{\bar{c}}_{jr} \in E_2^{-1,2jr}$. ~~The spectral sequence collapses because E_2 is generated by E_2^{0*} and $E_2^{-1,*}$ (compare theorem . of [J]). So we know that the Poincaré series of $H^*(E\Psi^\theta)$ is equal~~

~~to that of a symmetric algebra with generators~~ This shows that the Poincaré series of $H^*(E\Phi^8)$ is dominated by that of a symmetric algebra with one generators of degree $2rj$ for each $j \geq 1$ tensored with an exterior algebra with one generator of degree $2rj-1$ for each $j \geq 1$. (In fact equal since the spectral sequence degenerates, ~~compare~~ thm. of [ ]).

The good way of ~~doing this~~ doing this is to put down the maps

$$S[c_r', \ldots] \otimes \Lambda[c_r'', \ldots] \longrightarrow H^*(E\Phi^8) \longrightarrow H^*(BGL_\infty(\mathbb{F}_8))$$

Then the composition is an isomorphism by previous work and the spectral sequence furnishes the bound required to prove all three maps are isos.

Assertion 2: Comes from fact that we showed that any transformation $k\mathbb{F}_8 \longrightarrow H^i$ is a polynomial in the $\bar{c}_i$, ~~On the other hand the ... prime to the characteristic~~ for $l \neq p$. For $l = p$ and $l = 0$, all natural transformations are trivial.

Cateory theory aspect: Start with the 2-category $\underline{T}$ with objects = spaces, and (1-)morphims = continuous maps, and 2-morphisms = homotopy classes of homotopies of maps. Form a new category whose objects are the 1-morphisms of $\underline{T}$ and whose morphisms are the 2-commutative squares, i.e. such a thing is a collection of four maps

$$\begin{tikzcd} A \arrow[r, "g'"] \arrow[d, "f'"'] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

together with a 2-arrow $h: gf' \Rightarrow fg'$. Composition is evident. Now notice that this new category $Arr_2(\underline{T})$ is itself a 2-category, where a 2-morphism from the 1- morphism represented by the above square, which we shall denote $\underline{g} = (g, g', h): f' \quad f$, to another 1-morphism $\underline{g}_1 = (g_1, g_1', h_1)$ is defined to be a pair of 2-morphisms $k: g \Rightarrow g_1$ and $k': g' \Rightarrow g_1'$ such that the square of 2-morphisms

$$\begin{tikzcd} gf' \arrow[r, Rightarrow, "h"] \arrow[d, Rightarrow, "(k)f'"'] & fg' \arrow[d, Rightarrow, "f(k')"] \\ g_1f' \arrow[r, Rightarrow, "h_1"'] & fg_1' \end{tikzcd}$$

is commutative.

The point of this terminology is that to a morphism $f: X \rightarrow Y$ in $\underline{T}$ one has associated a long exact sequence in cohomology

$$H^{i-1}(Y) \rightarrow H^i(\text{Cone } f, pt) \longrightarrow H^i(X) \longrightarrow H^i(Y)$$

that the long exact sequence is a functor on the category $Arr_2(\underline{T})$ and that two morphisms in $Arr_2(\underline{T})$ which are 2-isomorphic give the same morphism of long exact cohomology sequences.

~~Now what you have to consider is a 2-commutative square in~~ ~~$Arr_2(\underline{T})$ whose four vertices are the arrows i^2, Δ^2, i, Δ. You must show the the~~

Standard notation: $i : E\Phi^8 \longrightarrow BU$ and the basic homotopy $h : i \Longrightarrow \Phi^8 \circ i$. Then we have an arrow $\alpha : i \longrightarrow \Delta$ in $\mathrm{Arr}_2(\underline{T})$ furnished by (Γ, i, H) where $H = (\text{trivial homot.}, h^{-1})$

$$\begin{tikzcd} E\Phi^8 \arrow[r, "i"] \arrow[d, "i"'] & BU \arrow[d, "\Delta"] \arrow[dl, phantom, "\overset{H}{\Rightarrow}"] \\ BU \arrow[r, "\Gamma"] & BU \times BU . \end{tikzcd}$$

Next consider the arrow $\alpha^2 : i^2 \longrightarrow \Delta^2$ which one obtains by squaring all the above data. Now form a 2-commutative square in $\mathrm{Arr}_2(\underline{T})$

$$(*) \qquad \begin{tikzcd} i^2 \arrow[r, "\mu"] \arrow[d, "\alpha^2"'] & i \arrow[d, "\alpha"] \\ \Delta^2 \arrow[r, "\mu"] & \Delta \end{tikzcd}$$

Here μ denotes on ~~the hand~~ the top the commutative square

$$\begin{tikzcd} (E\Phi^8)^2 \arrow[r, "\mu_{E\Phi^8}"] \arrow[d, "i^2"'] & E\Phi^8 \arrow[d, "i"] \\ (BU)^2 \arrow[r, "\mu_{BU}"] & BU \end{tikzcd}$$

with trivial homotopy and on the bottom the square

$$\begin{tikzcd} BU^2 \arrow[r, "\mu_{BU}"] \arrow[d, "\Delta"'] & BU \arrow[d, "\Delta"] \\ (BU \times BU)^2 \arrow[r, "(\mu_{BU} \times \mu_{BU}) \circ (id \times T \times id)"'] & BU \times BU . \end{tikzcd}$$

Finally the homotopy needed to make (*) commute is obtained as follows: The composite

$$\begin{tikzcd}
(E\Phi^{g})^2 \arrow[r, "i^2"] \arrow[d, "i^2"'] & BU^2 \arrow[r, "\mu_{BU}"] \arrow[d, "\Delta^2"] \arrow[dl, phantom, "H^2"] & BU \arrow[d, "\Delta"] \\
(BU)^2 \arrow[r, "\Gamma^2"] & (BU\times BU)^2 \arrow[r, "\mu_{BU\times BU}"] & BU\times BU
\end{tikzcd}$$

and the composite

$$\begin{tikzcd}
(E\Phi^{g})^2 \arrow[r, "\mu_{E\Phi^{g}}"] \arrow[d, "i^2"'] \arrow[dr, phantom, "\text{comm.}"] & E\Phi^{g} \arrow[r, "i"] \arrow[d, "i"] & BU \arrow[d, "\Delta"] \\
(BU)^2 \arrow[r, "\mu_{BU}"] & BU \arrow[r, "\Gamma"] & BU\times BU
\end{tikzcd}$$

are 2-isomorphic: on the top use trivial homotopy on the bottom use the homotopy (id, can) where.

$$\text{can}: \Phi^{g}(x\oplus y) \simeq \Phi^{g}x \oplus \Phi^{g}y$$

Now you must check that the square of homotopies commutes

$$\begin{tikzcd}
(\Gamma\,\mu_{BU})\circ(i^2) \arrow[r, Rightarrow, "H"] \arrow[d, Rightarrow, "\text{can}\ i^2"'] & (\Delta)\circ(i\,\mu_{E\Phi^{g}}) \arrow[d, Rightarrow, "\text{iden}"] \\
(\mu_{BU\times BU}\circ\Gamma^2)(i^2) \arrow[r, Rightarrow, "\mu_{BU\times BU}H^2"] & (\Delta)\circ(\mu_{BU}\, i^2)
\end{tikzcd}$$

but what this does is to take a point $((x,\lambda),(x',\lambda')) \in (E\Psi^\theta)^2$

$$\begin{tikzcd}
(x\oplus x', \Psi^\theta(x\oplus x')) \arrow[r, Rightarrow, "{id,\ \lambda''}"] \arrow[d, Rightarrow, "can."', "id"] & (x\oplus x', x\oplus x') \arrow[d, Rightarrow, "id"] \\
(x\oplus x', \Psi^\theta x \oplus \Psi^\theta x') \arrow[r, Rightarrow, "{id,\ \lambda\oplus\lambda'}"'] & (x\oplus x', x\oplus x')
\end{tikzcd}$$

but this commutes precisely by the definition of λ''.

So what I have just computed is that I do get a ~~map~~ commutative square of long exact cohomology sequences associated with the square $(*)$, which I shall write

$$\begin{tikzcd}
(E\Psi^\theta)^2 \arrow[r, "i^2"] \arrow[d, "i^2"] & (BU)^2 \arrow[d, "\Delta^2"] \\
(BU)^2 \arrow[r, "\Gamma^2"'] & (BU\times BU)^2
\end{tikzcd}
\xrightarrow{\ \mu\ }
\begin{tikzcd}
E\Psi^\theta \arrow[r, "i"] \arrow[d, "i"] & BU \arrow[d, "\Delta"] \\
BU \arrow[r, "\Gamma"'] & BU\times BU
\end{tikzcd}$$

Rewrite the first square to

$$\begin{tikzcd}
(E\Psi^\theta)^2 \arrow[r, "i^2"] \arrow[d, "i^2"] & (BU)^2 \arrow[d, "\Delta_{(BU)^2}"] \\
(BU)^2 \arrow[r, "\Gamma"] & (BU)^2\times(BU)^2
\end{tikzcd}$$

and if $\alpha \in H^*(BU)$ denote by $\alpha^{(i)} = pr_i^*(\alpha)$

where $pr_i : (BU)^2 \to BU$ are the two projections. Then by naturality of Φ operation I know that

$$c_i''(u^{(1)} \oplus u^{(2)}) = \Phi(\mu^*(c_i \otimes 1 - 1 \otimes c_i))$$

$$= \sum_{a+b=i} \Phi\left(c_a^{(1)} c_b^{(2)} \otimes 1 - 1 \otimes c_a^{(1)} c_b^{(2)}\right).$$

$$\cancel{\sum_{a+b=i} \Phi\left\{(c_a^{(1)} \otimes 1 - 1 \otimes c_a^{(1)})(c_b^{(2)} \otimes 1) + (1 \otimes c_a^{(1)})(c_b^{(2)} \otimes 1 - 1 \otimes c_b^{(2)})\right\}}$$

To evalute a term in this sum we use the derivation property which tells us that in a situation

$$\begin{tikzcd}
S(V) \arrow[r] & S(V_\sigma) & \longrightarrow & H^*(BU^2) \\
S(V) \otimes S(V) \arrow[u, "\Delta^*"] \arrow[r, "\mu^*"] & S(V) \arrow[u] & & H^*(BU^2 \times BU^2) \arrow[u, "\Delta^*"] \arrow[r, "\mu^*"] & H^*(BU^2)
\end{tikzcd}$$

one has the formula for $f \in S(V)^\sigma$

$$\begin{tikzcd}
S(V)^\sigma \arrow[r] \arrow[d, "(\pi \otimes id)d"'] & I \cap J \arrow[d] \\
S(V_\sigma) \otimes V^\sigma \arrow[r] & I \cap J / IJ
\end{tikzcd}$$

$$f \longmapsto f \otimes 1 - 1 \otimes f$$

$$a \otimes v \longmapsto (a \otimes 1)(v \otimes 1 - 1 \otimes v)$$

To evaluate a typical term in the sum we use the derivation property proved in the technical appendix, which tells us that if ~~[illegible]~~ V is a vector space with endomorphism σ^* and if $\rho : V \longrightarrow H^*((BU)^2)$ ~~[illegible]~~ is a $\mathbb{Z}/\ell$ linear map compatible with σ^* on V and ~~$(\Phi\delta)^*$ on~~ $((\Phi\delta)^2)^*$ on ~~[illegible]~~ $H^*((BU)^2)$, then for $f \in S(V)^\sigma$

$$\Phi\left(\rho(f)\otimes 1 - 1\otimes \rho f\right) = \sum_{i=1}^{m} i^*\left(\rho\left(\frac{\partial f}{\partial v_i}\right)\right)\cdot \Phi(v_i)$$

where $v_1, \ldots, v_m, \ldots, v_n$ is a basis for V such that $v_1, \ldots, v_m$ is a basis for V^σ and

$$df = \sum_{i=1}^{m} \frac{\partial f}{\partial v_i} \otimes dv$$

Here we take V to be the subspace of $H^*((BU)^2)$ generated by $c_a^{(1)}$ and $c_b^{(2)}$, and we take $\sigma^* =$ effect of $((\Phi\delta)^2)^*$. Here $a \not\equiv 0 \ (r)$, then $V^\sigma = 0$ so ~~[illegible] the corresponding term is~~

$$\Phi\left(c_a^{(1)} c_b^{(2)} \otimes 1 - 1 \otimes c_a^{(1)} c_b^{(2)}\right) = 0$$

On the other hand if $a, b \equiv 0 \quad (r)$, then $V^\sigma = V$, so the formula gives

$$\Phi\left(c_a^{(1)} c_b^{(2)} \otimes 1 - 1 \otimes c_a^{(1)} c_b^{(2)}\right) = c_a^{\prime(1)} c_b^{\prime\prime(2)} + c_b^{\prime(2)} c_a^{\prime\prime(1)}.$$

Thus

$$\mu^* c_i'' = \sum_{\substack{a+b=i \\ a,b\equiv 0 \ (r)}} c_a^{\prime(1)} \otimes c_b^{\prime\prime(2)} + c_a^{\prime\prime(1)} \otimes c_b^{\prime(2)}$$

which proves the product formula.

Proof of normalization assertion: We can suppose that $\chi : C \longrightarrow S^1$ is the embedding of the (q^r-1)-th roots of 1. Let $T = (S^1)^r$ and let σ be the endomorphism of T

$$\sigma(z_1,\ldots,z_r) = (z_r^q, z_1^q, \ldots, z_{r-1}^q).$$

Let $j : C \longrightarrow T$ be given by

$$j(c) = \left(\chi(c)^q, \ldots, \chi(c)^{q^r}\right)$$

whence j is an isomorphism of C with the subgp. of fixpoints T^σ. I claim there is a map of squares

$$A:\quad \begin{tikzcd} BC \arrow[r, "j"] \arrow[d, "j"'] & BT \arrow[d, "\Delta_{BT}"] \arrow[r, phantom, "\longrightarrow"] & B\Phi^q \arrow[r, "i"] \arrow[d, "i"'] & BU \arrow[d, "\Delta_{BU}"] \\ BT \arrow[r, "{(id,\sigma)}"] & BT\times BT & BU \arrow[r, "\Gamma"] & BU\times BU \end{tikzcd}$$

~~where the maps BT → BU is the standard repr.~~

~~[illegible]~~ defined by the ~~[illegible]~~ maps

$BC \xrightarrow{\alpha} E\Phi^{\delta}$ virtual bdle assoc. to representation $E = \bigoplus_{a=1}^{5} \chi^{\delta^a}$ plus the (unique ~~[illegible]~~ by Atiyah) isomorphism $E \cong \Phi^{\delta}(E)$.

$BT \xrightarrow{s} BU$ virtual bdle assoc. to standard repn. of T on $\mathbb{C}^r$

$BT \times BT \xrightarrow{s \times s} BU \times BU$.

together with the $\overbrace{\text{homotopies}}^{\text{trivial}}$ in the squares

$$\begin{tikzcd}
BC \arrow[r, "j"] \arrow[d, "j"'] & BT \arrow[r, "s"] \arrow[d, "\Delta_{BT}"] & BU \arrow[d, "\Delta_{BU}"] \\
BT \arrow[r, "\Gamma_{BT}"] & BT \times BT \arrow[r, "s \times s"] & BU \times BU
\end{tikzcd}$$

and the homotopies

$$\begin{tikzcd}
BC \arrow[r, "\alpha"] \arrow[d, "j"'] \arrow[dr, phantom, "\text{trivial homotopy}"] & E\Phi^{\delta} \arrow[r, "i"] \arrow[d, "i"] \arrow[dr, phantom, "\text{st. homotopy}"] & BU \arrow[d, "\Delta_{BU}"] \\
BT \arrow[r, "s"] & BU \arrow[r, "\Gamma_{BU}"] & BU \times BU
\end{tikzcd}$$

plus the homotopy $i\alpha \Longrightarrow sj$ (trivial one)

+ the homotopy $\Gamma_{BU} \circ s \Longrightarrow s \times s \circ \Gamma_{BT}$ (unique one here),

and they fit together OKAY since $[S(BC), BU \times BU] = 0$.

The existence of the map A shows that $c''_{jr}(\alpha)$ may be computed as $\Phi(c_{jr}(S)\otimes 1 - 1\otimes c_{jr}(S))$. But

$$c_{jr}(S) = \begin{cases} 0 & j>1 \\ \alpha_1\cdots\alpha_r & j=1 \end{cases}$$

where $\alpha \in H^2(BS^1)$ is the universal first Chern class and $\alpha_i = pr_i^*(\alpha)$, $pr_i : T \to S^1$ denoting the ith projection. To compute this Φ we use the derivation formula and take V ~~to be the subspace~~ $= H^2(BT)$ which has the basis $\alpha_1, \ldots, \alpha_r$ and

$$\sigma^*(\alpha_i) = \begin{cases} q\,\alpha_{i-1} & 1<i\leq r \\ q\,\alpha_r & i=1 \end{cases}$$

Then V^σ is ~~...~~ of dimension 1 spanned by

$$\beta = \sum_1^r q^{r-i}\alpha_i$$

and V_σ is one-dimensional with generator γ satisfying

$$\pi(\alpha_i) = q^i\gamma$$

$$j^*(\gamma) = u \ (= c_1(X)).$$

Now β is the first Chern class of the character

$$X(z_1,\ldots,z_r) = \prod_1^r z_i^{q^{r-i}}$$

and $$\left(\chi \cdot \sigma^* \chi^{-1}\right)(z_1, \ldots, z_r) = z_r^{1-g^r}$$
$$= \chi_1(z_1, \ldots, z_r)^{\ell}$$

where
$$\chi_1(z_1, \ldots, z_r) = z_r^{(1-g^r)/\ell}$$

Thus by (lemma ?) $\Phi(\beta \otimes 1 - 1 \otimes \beta) \in H^1(BC)$ is the class of the homomorphism
$$\chi_1 j = \chi^{(1-g^r)/\ell} : C \longrightarrow \mathbb{Z}/\ell$$
which is v by definition. ~~By the derivation~~ ~~for~~ Now if $\pi : S(V) \to S(V_\sigma)$ is the natural map and $d : S(V) \to S(V) \otimes V$ is the exterior derivative, then
$$(\pi \otimes id)\, d(\alpha_1 \ldots \alpha_r) = \sum_1^r \pi(\alpha_1 \ldots \hat{\alpha}_i \ldots \alpha_r) \otimes \alpha_i$$
$$= \left(\prod_1^r g^i\right) \gamma^{r-1} \otimes \sum_1^r g^{r-i} \alpha_i$$
$$= (-1)^{r-1} \gamma^{r-1} \otimes \beta$$
hence by the derivation property and the above computation
$$c_r''(\alpha) = \Phi(\alpha_1 \ldots \alpha_r \otimes 1 - 1 \otimes \alpha_1 \ldots \alpha_r) = (-1)^{r-1} u^{r-1} v.$$

But
$$c_r'(\alpha) = \alpha^* \iota^* c_r = c_r(E) = \prod_1^r g^i u = (-1)^{r-1} u^{r}$$

completing the proof.

The Brauer map:

Let $\overline{\mathbb{F}}_q$ be an algebraic closure of $\mathbb{F}_q$ and let $\phi: \overline{\mathbb{F}}_q^{*} \longrightarrow S^1$ be a homomorphism. Following Green one can use ϕ to lift representations of a finite group G to virtual complex representations

$$\phi: \ k\overline{\mathbb{F}}_q(G) \longrightarrow k\mathbb{C}(G).$$

This is the unique transformation ~~[illegible]~~ of functors on the category finite groups extending the obvious map for one-dimensional representations. ϕ is a homomorphism of λ_N-rings; note that the image of ϕ is stable under Ψ^q for N sufficiently large (multiplicatively). By computing on cyclic groups one knows that Ψ^q on $k\overline{\mathbb{F}}_q(G)$ is induced by the Frobenius automorphism, hence there is a sequence

$$0 \longrightarrow k\mathbb{F}_q(G) \xrightarrow{\ \phi\ } k\mathbb{C}(G) \xrightarrow{\Psi^q - 1} k\mathbb{C}(G)$$

<u>Proposition 1:</u> This sequence is exact.

<u>Proof:</u> According to the Brauer theory if $|G| = p^a h$ with $(h,p) = 1$ and if N is so large that p^a / q^N and $h / q^N - 1$, then Ψ^{q^N} is a projection operator on $k\mathbb{C}(G)$ whose image is $\phi[k\overline{\mathbb{F}}_q(G)]$; Moreover ϕ is injective hence

$$0 \longrightarrow k\mathbb{F}_q(G) \longrightarrow k\mathbb{C}(G) \xrightarrow{\Psi^{q^N} - 1} k\mathbb{C}(G)$$

is exact and we have to prove that $k\mathbb{F}_q(G)$ is the Φ^q-invariant subring of $k\overline{\mathbb{F}}_q(G)$. Now the former is a free abelian group with isomorphism classes of irred. reps. over $\mathbb{F}_q$ as generators, and the Φ^q-invariant subgroup of $k\overline{\mathbb{F}}_q(G)$ is a free abelian group with generators corresponding to the orbits of Φ^q on the ~~[illegible]~~ irreducible representations over $\overline{\mathbb{F}}_q$. It suffices therefore to show that if V is an irred. rep. over $\overline{\mathbb{F}}_q$, ~~[illegible]~~ and d is the least ~~[illegible]~~ positive integer such that $\Phi^{q^d}[V] \cong [V]$, then $W = V + \Phi^q V + \cdots + \Phi^{q^{d-1}} V$ is defined over $\mathbb{F}_q$. But this is clear from the following.

<u>Lemma:</u> Let W be ~~a~~ a semi-simple representation of G over $\overline{\mathbb{F}}_q$ such that $\Phi^q[W] = [W]$, then W is defined over $\mathbb{F}_q$.

$\Phi^q[W] = \sigma^* W$ where σ is the Frobenius, and since equality for two semi-simple representations in the Grothendieck group implies ~~[illegible]~~ isomorphism, it follows that $\exists$ an isomorphism $\theta : \sigma^* W \cong W$. Choosing a basis and letting $U(g)$ be the matrix associated to g we have an equation

$$\sigma(U(g)) = A\,U(g)A^{-1} \qquad g \in G$$

for some θ. By Lang $A = (\sigma B)^{-1} B$ hence

$$\sigma(B\,U(g)B^{-1}) = B\,U(g)\,B^{-1}$$

and so $B\,U(g)B^{-1}$ is a matrix with entries over $\mathbb{F}_q$.