D. Quillen

Let N be a countable infinite set, and G the group of all permutations of N. I want to show G is acyclic, i.e. $\tilde{H}_*(G,\mathbb{Z}) = 0$.

First part: The "infinite sum" argument.

Let $\mathcal{N}$ be the groupoid consisting of all countable infinite sets and bijections between them. One has $H_*(G) = H_*(\mathcal{N})$. The disjoint union functor

$$\mathcal{N} \times \mathcal{N} \xrightarrow{\amalg} \mathcal{N}$$

induces a product $\mu : H_*(G) \otimes H_*(G) \longrightarrow H_*(G)$ ~~making $H_*(G)$ into a graded commutative ring~~ [illegible] which is associative and commutative. If e is the obvious generator of $H_0(G) = \mathbb{Z}$, then multiplication by e: $H_*(G) \to H_*(G)$, $\alpha \mapsto e\cdot\alpha$ is the map induced by the embedding $G \hookrightarrow G$ which results from an embedding $N \hookrightarrow N$ with infinite complement.

Let $\Sigma : \mathcal{N} \longrightarrow \mathcal{N}$, $X \longmapsto N \times X$ denote the infinite sum functor. One has

$$\Sigma \amalg \mathrm{id} \simeq \Sigma,$$

i.e. the diagram

$$\begin{tikzcd} \eta \arrow[r, "\Delta"] \arrow[rrr, bend right, "\Sigma"'] & \eta\times\eta \arrow[r, "\Sigma\times id"] & \eta\times\eta \arrow[r, "\mu"] & \eta \end{tikzcd}$$

commutes up to isomorphism. Let $\alpha \in \tilde{H}_*(G)$ and

$$(\Delta_*)(\alpha) = \alpha\otimes e + \sum_i \alpha_i'\otimes\alpha_i'' + e\otimes\alpha$$

with $\deg(\alpha_i'), \deg(\alpha_i'') < \deg(\alpha)$. (I have to pass to mod p or rational homology to have such a formula.) Then since

$$\Sigma_* = \mu(\Sigma_*\otimes id)\Delta_*$$

one has

$$\Sigma_*\alpha = \Sigma_*\alpha\cdot e + \sum_i \Sigma_*\alpha_i'\cdot\alpha_i'' + \Sigma_* e\cdot\alpha$$

Assuming that $e\cdot\beta = 0$ for $0<\deg(\beta)<\deg(\alpha)$, one gets

$$e\cdot\Sigma_*\alpha = \Sigma_*\alpha\cdot e^2 + 0 + e\cdot\Sigma_* e\cdot\alpha .$$

As $e^2 = e$, $\Sigma_* e = e$, one gets $e\cdot\alpha = 0$. Thus:

<u>Lemma 1:</u> <u>Multiplication by e on</u> $\tilde{H}_*(G)$ is zero.

Second part: The "building" argument:

Let X be the following poset. An element of X is a subset x of N such that both x and $N-x$ are infinite. One has $x \leq y$ in X iff either $x=y$ or $x \subset y$ and $y-x$ is infinite.

<u>Lemma 2</u>: X is contractible.

<u>Proof</u>: Let F be a finite subset of X. I am going to produce an element z' of X such that a) $x \mapsto x \cup z'$ is a morphism of posets from F to X, b) $x \leq x \cup z \geq z'$ for all $x \in F$. It follows that the inclusion functor $F \hookrightarrow X$ is homotopic to the constant functor with value z'. Thus F contracts to a point in X, so X is contractible (every element of $\pi_*(X)$ comes from some finite F).

Choose a maximal subset $x_1, \ldots, x_\ell$ of F such that $y = x_1 \cap \cdots \cap x_\ell$ is infinite. For any x in F not equal to one of the x_i, we have $x \cap y$ is finite, so after removing each of these finite subsets from y we obtain an element z of X such that for any x in F, either $z \subset x$ or $z \cap x = \phi$.

Divide z into two infinite pieces $z = z' \sqcup z''$. Let's check z' satisfies a) and b). Let $x \in F$. If

$z \subset x$, then $x \cup z' = x \in F$ and $x \leq x \cup z'$. Also $z' \leq x \cup z'$ because $z'' \subset (x \cup z') - z'$. If on the other hand $z \cap x = \phi$, then $x \cup z' \in F$ because its complement contains z''. Also $x \leq x \cup z' \geq z'$ in this case. This proves b) and part of a).

It remains to show that if $x_1 < x_2$ in F, then $x_1 \cup z' \leq x_2 \cup z'$ in F. This is clear if either $z \subset x_1$ or if $z \cap x_2 = \phi$. Suppose then that we take the remaining case $x_1 \cap z = \phi$, $z \subset x_2$. Then $x_1 \cup z' \leq x_2 = x_2 \cup z'$ because of z''. Q.E.D.

Now we consider the complex of chains on X:

$$\to \bigoplus_{x_0 < \cdots < x_p} \mathbb{Z} \to \cdots\cdots \longrightarrow \bigoplus_{x_0 < x_1} \mathbb{Z} \longrightarrow \bigoplus_{x_0} \mathbb{Z} \longrightarrow \mathbb{Z} \to 0$$

which is acyclic because X is contractible. The group G acts on X, hence the above sequence is a complex of G-modules. It gives rise to a spectral sequence:

$$E^1_{pq} = H_q\Big(G, \bigoplus_{x_0 < \cdots < x_{p-1}} \mathbb{Z}\Big) \Longrightarrow 0$$

Note that if $x_0 < \cdots < x_p$ is a p-simplex in X, then one has a decomposition $N = N_0 \amalg \cdots \amalg N_{p+1}$ with $x_j = N_0 \amalg \cdots \amalg N_j$ and where each N_j is infinite. It follows that G acts transitively on the set of p-simplices and that the stabilizer of a typical p-simplex is isomorphic to $G \times \cdots \times G$ ($p+2$ times). Thus the group of $(p-1)$ chains is an induced module:

$$\bigoplus_{x_0 < \cdots < x_{p-1}} \mathbb{Z} \simeq \mathbb{Z}[G] \otimes_{\mathbb{Z}[G^{p+1}]} \mathbb{Z}$$

and so by Shapiro's lemma:

$$H_*\Big(G, \bigoplus_{x_0 < \cdots < x_{p-1}} \mathbb{Z}\Big) \simeq H_*(G^{p+1}, \mathbb{Z}).$$

It appears that I should be working with coefficient~~s~~ ~~ring~~ a field F, whence by Kunneth:

$$(*) \qquad H_*\Big(G, \bigoplus_{x_0 < \cdots < x_{p-1}} F\Big) \simeq H_*(G^{p+1}, F) \simeq H_*(G, F)^{\otimes (p+1)}$$

The differential d_1 in the spectral sequence is the alternating sum of the "face" maps

$$d_i : H_*\Big(G, \bigoplus_{x_0 < \cdots < x_{p-1}} F\Big) \longrightarrow H_*\Big(G, \bigoplus_{x_0 < \cdots < x_{p-2}} F\Big)$$

which one obtains by deleting x_i from the simplex

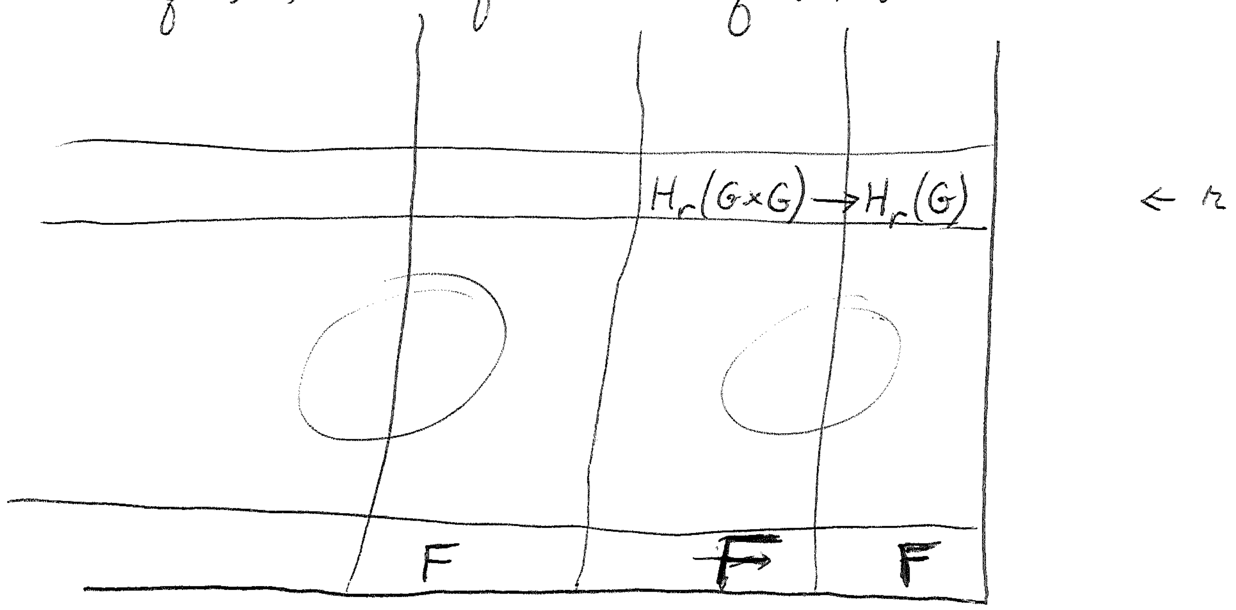
$x_0 < \cdots < x_p$. Under the Shapiro isomorphism d_i will correspond to the map $G^{p+1} \longrightarrow G^p$ sending $(g_1, \ldots, g_i, g_{i+1}, \ldots, g_p) \longmapsto (g_1, \ldots, g_i \oplus g_{i+1}, \ldots, g_p)$, where $\oplus : G \times G \longrightarrow G$ is the map obtained from any isom. $N \amalg N \simeq N$.

One sees that $E^1_{0p} = H_0(G^{p+1}, F) \cong F$ with $d_i = id$, hence $d = d_0 - d_1 + \cdots \pm d_p =$ id if p is odd and 0 if p is even. So E^1_{0*} is

$$\cdots \longrightarrow F \xrightarrow{1} F \xrightarrow{0} F \xrightarrow{1} F$$

and $E^2_{0*} = 0$.

I want to show that $\tilde{H}_*(G, F) = 0$. Suppose that $\tilde{H}_q(G, F) = 0$ for all $q < r$.



By the spectral sequence the map $H_r(G \times G) \xrightarrow{d_1} H_r(G)$ must be onto. But $H_r(G \times G) = H_r(G) \oplus H_r(G)$, so one sees that the map

$$\begin{array}{ccc} H_r(G) \oplus H_r(G) & \longrightarrow & H_r(G) \\ \alpha \qquad \beta & \longmapsto & \alpha \cdot e + e \cdot \beta \end{array}$$

is onto. But by Lemma 1, this map is 0, so $H_r(G) = 0$.

So [redacted] I have proved that $\tilde{H}_*(G, F) = 0$ for all fields F. This implies $\tilde{H}_*(G, \mathbb{Z}) = 0$.

Problem: Let K be a group. Then G acts on $K^N = \{\text{functions}: N \to K\}$, so we can form the group $G \ltimes K^N$ = semi-direct product. (This is a kind of infinite wreath product $G \wr K$). Is the group $G \ltimes K^N$ acyclic? Note: Kan + Thurston have shown this when instead of K^N one takes maps $\mathbb{Q} \to N$ with compact support, and instead of G one takes maps $\mathbb{Q} \to \mathbb{Q}$ with compact support.

of the group in the sense of homological algebra

~~Homology of $B\mathcal{C}$. It is well-known that the homology of the classifying space of a discrete group coincides with the homology. We now describe the generalization of this fact to an arbitrary small category, which is perhaps less well-known.~~

~~$\mathcal{S} \times \mathcal{T} \longrightarrow \mathcal{U}$~~

$\mathcal{S} \xrightarrow{\lambda} \bar{\mathcal{S}} = \underline{\mathrm{Hom}}^{\mathcal{S}^*}(\mathcal{S}, \mathcal{S})$ is an equiv. of monoidal ~~categories~~

$$\begin{tikzcd}
\mathcal{S} \arrow[r, "F"] \arrow[d, "\wr"] & \mathcal{U} \arrow[d] \\
\bar{\mathcal{S}} \arrow[r, dashed] & \bar{\mathcal{U}} \\
\mathcal{S}' \arrow[u, "\wr"] \arrow[ur] &
\end{tikzcd}$$

$\mathcal{S}' : \quad (G, U, FG(0) \simeq U)$

$\mathcal{S} \to \mathcal{S}' \qquad S \longmapsto (\lambda_S, F(S), F(\lambda_S(0)) \simeq F(S)).$

organize as follows.

~~Statement: Given a monoidal cat $\mathcal{S}$ there exists a strict mon. categ. $\tilde{\mathcal{S}}$ and an equiv of monoidal~~

$\tilde{\mathcal{S}}$ characterized up to isom by fact that it is strict & that $\exists\, \xi : \mathcal{S} \to \tilde{\mathcal{S}}$ mon. equiv. $\ni$ $Ob(\tilde{\mathcal{S}}) \cong$ free monoid gen. by $Ob(\mathcal{S})$.

$\mathcal{S} \mapsto \tilde{\mathcal{S}}$ leftadjoint. Any mon func $F : \mathcal{S} \to \mathcal{U}$ with $\mathcal{U}$ strict factors uniquely: $F = F'\xi$, where $F' : \tilde{\mathcal{S}} \to \mathcal{U}$ is a strict mon. functor

---

~~Proof by~~ Constructing a mon. equiv $\lambda : \mathcal{S} \to \bar{\mathcal{S}}$ with $\bar{\mathcal{S}}$ strict.

Construction of
$$\begin{tikzcd} \mathcal{D} \arrow[rr, "F"] \arrow[dr, shift left] & & \mathcal{U} \\ & \mathcal{D}' \arrow[ul, shift left] \arrow[ur] & \end{tikzcd}$$

Proof: Suppose given mon. equiv $\lambda : \mathcal{S} \to \bar{\mathcal{S}}$ with $\bar{\mathcal{S}}$ strict. If S is a set let $W(S)$ be the cat ~~whose objects are~~. Note that $W(S_1 \times S_2) = W(S_1) \times W(S_2)$ of sort that a monoid goes to a monoid cat. Then put

$$\tilde{\mathcal{S}} = W(F.M.(Ob(\mathcal{S}))) \times_{W(Ob\, \bar{\mathcal{S}})} \bar{\mathcal{S}}.$$

In other words $Ob(\tilde{\mathcal{S}})$

Note that if $FM(Ob(\mathcal{S})) \longrightarrow Ob(\bar{\mathcal{S}})$ is an isom $\Longleftrightarrow$ $\tilde{\mathcal{S}} \to \bar{\mathcal{S}}$

Lemma 2: Let $\mathcal{T}$, $\mathcal{U}$ be strict monoidal categories and $F: \mathcal{T} \to \mathcal{U}$ a mon. functor.

Construction: Let $\mathcal{T}$, $\mathcal{U}$ be strict monoidal categories and let $F: \mathcal{T} \to \mathcal{U}$ be a mon. functor. Let $\mathcal{T}'$ be the category whose objects are triples (T, U, α) with $T \in \mathcal{T}$, $U \in \mathcal{U}$, and $\alpha : F(T) \xrightarrow{\sim} U$, and where the morphisms are the obvious ones. Define a product functor in $\mathcal{T}'$ by

$$(T_1, U_1, \alpha_1)(T_2, U_2, \alpha_2) = (T_1 T_2,\ U_1 U_2,\ \alpha_1 * \alpha_2)$$

where $\alpha_1 * \alpha_2$ is the isom.

$$F(T_1 T_2) \xrightarrow{\sim} F(T_1) F(T_2) \xrightarrow[\sim]{\alpha_1 \cdot \alpha_2} U_1 U_2 .$$

Utilizing the axioms for a monoidal functors, one verifies easily that this product functor makes $\mathcal{T}'$ into a strict monoidal category. Moreover we have

$$\begin{tikzcd} \mathcal{T} \arrow[rr, "F"] & & \mathcal{U} \\ & \mathcal{T}' \arrow[ul, "p_1"] \arrow[ur, "p_2"'] & \end{tikzcd}$$

Lemma 0: A monoidal functor $F: \mathcal{S} \to \mathcal{T}$ is an equivalence of monoidal categories iff it is an equivalence of the underlying categories.

Lemma 1: $\lambda: \mathcal{S} \longrightarrow \bar{\mathcal{S}} = \underline{Hom}^{\mathcal{S}^*}(\mathcal{S}, \mathcal{S})$ is an equivalence of monoidal cats.

Lemma 2: Let $F: \mathcal{S} \to \mathcal{U}$ be a mon. functor with $\mathcal{U}$ a strict monoidal category. Then F may be factored

$$\mathcal{S} \xrightarrow{\mu} \mathcal{S}' \xrightarrow{F'} \mathcal{U}$$

where $\mathcal{S}'$ is a strict mon. cat, F' is a strict mon. functor, and μ is an eq. of mon. cats.

Proof: Let $\lambda': \bar{\mathcal{S}} \to \mathcal{S}$ be a mon. functor quasi-inverse to λ above, so that we have compatible isos. of mon. functors: $\lambda'\lambda \simeq id$, $\lambda\lambda' \simeq id$. Let $\mathcal{S}'$ be the cat whose objects are triples (T, U, α), where $T \in Ob(\bar{\mathcal{S}})$, $U \in Ob(\mathcal{U})$, $\alpha: F\lambda'(T) \simeq U$; morphisms in $\mathcal{S}'$ are the obvious ones. Define a product in $\mathcal{T}$ by the formula

$$(T_1, U_1, \alpha_1)(T_2, U_2, \alpha_2) = (T_1T_2, U_1U_2, \alpha_1 * \alpha_2)$$

where $\alpha_1 * \alpha_2$ denotes the isom

$$F\lambda'(T_1T_2) \simeq F\lambda'(T_1) \cdot F\lambda'(T_2) \xrightarrow[\sim]{\alpha_1 \cdot \alpha_2} U_1U_2$$

the former isom being the product isom for the monoidal functor $F\lambda'$. Utilizing the properle

Let $\mathcal{S} \to \mathcal{E}_0$ be a cofinal p.pres. functor

<u>Cor</u>: $\mathcal{S}^{-1}\mathcal{E}_0 \to \mathcal{S}^{-1}\mathcal{E}_M \to \mathcal{S}^{-1}\mathcal{E}_0$ h.eqs.

<u>Proof</u>: cofinality $\Rightarrow$ $\mathcal{S}^{-1}\mathcal{E}_M$ is an H-space with h-inverse and

$$H_*(\mathcal{S}^{-1}\mathcal{E}_M) = (\pi_0\mathcal{S})^{-1}H_*(\mathcal{E}_M) = (\pi_0\mathcal{E}_0)^{-1}H_*(\mathcal{E}_M)$$

so enough to show (by Whitehead) iso on homology.

<u>Proof of Thm.</u> Have canonical isom

$$\mathrm{E}\perp\mathrm{E} \simeq \mathrm{E}\perp ik\mathrm{E}$$

Enough to use coefficients in the field $\mathbb{F}_p$, $\mathbb{Q}$ whence one has a Künneth isom.

$$\cancel{H_*(\mathcal{C}_1)\otimes H_*(\mathcal{C}_2) \xrightarrow{\sim} H_*(\mathcal{C}_1\times\mathcal{C}_2).}$$

~~It is well known if $\mathcal{S}$ is a cat. with~~

$$H_*(\mathcal{X}_1)\otimes H_*(\mathcal{X}_2) \simeq H_*(\mathcal{X}_1\times\mathcal{X}_2).$$

This makes $H_*(\mathcal{X})$ into a cogebra with coproduct and counit induced by the diagonal functor

$$\mathcal{X} \to \mathcal{X}\times\mathcal{X}$$

and $\mathcal{X} \to pt$. Also if $\mathcal{S}$ is a cat with product $H_*(\mathcal{S})$ is an algebra with product + unit induced by $\mathcal{S}\times\mathcal{S} \to \mathcal{S}$ and $pt \to \mathcal{S}$. Recall that we then have a product on the k-module

$$\mathrm{Hom}^{(0)}(H_*(\mathcal{X}), H_*(\mathcal{X}))$$

~~1~~ Example: ~~Given~~ [illegible] $\bar{S} \in \mathcal{S}$, have

$$F(S) = \bar{S}S$$

and
$$(\bar{S}X)S \simeq \bar{S}(XS)$$

In this way we get a monoidal functor

$$\mathcal{S} \longrightarrow \underline{\mathrm{Hom}}^{\mathcal{S}^\circ}(\mathcal{S}, \mathcal{S})$$

Assertion: This is an equivalence of monoidal ~~functors~~ categories.

(need lemma: a monoidal functor is an equiv. of mon. cats $\Leftrightarrow$ it is an equivalence of cats.)

~~the~~ F, φ $\qquad \varphi : F(S_1)S_2 \simeq F(S_1 S_2)$.

Then

$$F(0)X \xrightarrow[\sim]{\varphi} F(0X) \simeq F(X)$$

is an isomorphism of functors of S.

~~[illegible]~~ To show it is an isom in $\underline{\mathrm{Hom}}^{\mathcal{S}^\circ}(\mathcal{S}, \mathcal{S})$.

$$\begin{tikzcd}
(F(0)X)S \arrow[r, dash] \arrow[d, dash] & F(0X)S \arrow[r, dash] & F(X)S \arrow[d, dash] \\
& F((0X)S) \arrow[d, phantom, "\wr"] & \\
F(0)(XS) \arrow[r, dash] & F(0(XS)) \arrow[r, dash] & F(XS)
\end{tikzcd}$$

and the associativity isomorphism gives an isom

$$\boxed{h(S_1S_2) \simeq h(S_1)\cdot h(S_2)}$$

Also

$$h(0) = \begin{cases} X \longmapsto 0X \\ (0X)T \simeq 0(XT) \end{cases}$$

and unit ax. gives $\boxed{h0 \simeq id.}$

Now you must verify that these isos. are compatible with the unity data & assoc data of $\underline{\mathrm{Hom}}^{\mathcal{S}^0}(\mathcal{S},\mathcal{S})$!!!!

So you want

$$\begin{tikzcd}
h((S_1S_2)S_3) \arrow[r, "\simeq"] \arrow[d, "\wr"] & h(S_1S_2)h(S_3) \arrow[dd, "\wr"] \\
h(S_1(S_2S_3)) \arrow[d, "\wr"] & \\
h(S_1)h(S_2S_3) \arrow[r, "\sim"] & h(S_1)h(S_2)h(S_3)
\end{tikzcd}$$

To commute in $\underline{\mathrm{Hom}}^{\mathcal{S}^0}(\mathcal{S},\mathcal{S})$. But two nat. transf. are equal iff equal on objects, so this results from the usual pentagon.

Similarly for

$$\begin{tikzcd}
h(S0) \arrow[r, "\simeq"] \arrow[dr, "\wr"'] & h(S)h(0) \arrow[d, "\wr"] \\
& hS
\end{tikzcd} \qquad 0S$$

$$F(0)X \simeq F(0X) \simeq F(X)$$

$$\begin{tikzcd}
(F(0)X)S \arrow[r, "\varphi \simeq"] \arrow[dd, equal, "\text{assoc}"'] \arrow[ddr, phantom, "\text{satisfied by } \varphi"] & F(0X)S \arrow[r, "\simeq"] \arrow[d, "\varphi S"] \arrow[dr, phantom, "\text{nat of } \varphi"] & F(X)S \arrow[dd, "\varphi"] \\
 & F((0X)S) \arrow[d, equal] \arrow[dr, "\text{unity}"] & \\
F(0)(XS) \arrow[r, "\varphi \simeq"] & F(0(XS)) \arrow[r, "\simeq"] & F(XS)
\end{tikzcd}$$

OKAY.

You have a monoidal functor

$$\mathcal{S} \longrightarrow \underline{\mathrm{Hom}}^{\mathcal{S}^\circ}(\mathcal{S}, \mathcal{S})$$

To prove this it is necessary ~~..~~

$$S \longmapsto \begin{pmatrix} X \longmapsto SX \\ (SX)T \simeq S(XT) \end{pmatrix}$$

fact this is well defined on objects uses pentagon + unity.

Now compose

$$h(S_1 S_2) = \left\{ \begin{matrix} X \longmapsto (S_1 S_2)X \\ ((S_1 S_2)X)T \simeq (S_1 S_2)(XT) \end{matrix} \right\}$$

$$hS_1 \cdot hS_2 = \left\{ \begin{matrix} X \longmapsto S_1(S_2 X) \\ [S_1(S_2 X)]T \simeq S_1[(S_2 X)T] \simeq S_1(S_2(XT)) \end{matrix} \right\}$$

So what I know is that I have

$$\begin{tikzcd}
\mathcal{S} \arrow[r, "\lambda"] \arrow[d, "\text{evident m}"', "\sim"] & \mathrm{Hom}^{\mathcal{S}^*}(\mathcal{S},\mathcal{S}) \\
\tilde{\mathcal{S}} \arrow[ur, "\text{strict}"'] &
\end{tikzcd}$$

mon. equiv.

strict

strict equiv. a morphisms

and then one has an evident monoidal functor $\mathcal{S} \to \tilde{\mathcal{S}}$ which is a monoidal equivalence

Proof: $\tilde{\mathcal{S}}$ is an adjoint.

$$\begin{tikzcd}
\mathcal{S} \arrow[r, "f"] \arrow[d, "\gamma"'] & \mathcal{U} \\
\tilde{\mathcal{S}} \arrow[ur, dashed, shift left, "f_1"] \arrow[ur, dashed, shift right, "f_2"'] &
\end{tikzcd}$$

$\mathcal{U}$ strict

mon. iso

$f_1 \xrightarrow{\sim} f_2$

$f(X) = f_2(X)$

f_1, f_2 strict such that $f_1\gamma = f_2\gamma = f$.

Claim $f_1 = f_2$. Proof: $Ob(f_i) : Ob(\tilde{\mathcal{S}}) \to Ob(\mathcal{U})$ two monoid homos. agree on $Ob(\mathcal{S})$ so are equal.

~~Take an arrow between two objects of $\tilde{\mathcal{S}}$.~~ ?

γ is an equivalence so $\exists!\ f_1 \simeq f_2$ $\quad f_1(X) \simeq f_2(X)$

$$f_1\gamma \simeq f_2\gamma, \qquad f_1\gamma = f = f_2\gamma$$

$$\begin{tikzcd}
\mathcal{S} \arrow[r, "f_0"] \arrow[d, "\gamma"'] & \mathcal{U} \\
\tilde{\mathcal{S}} &
\end{tikzcd}$$

$$\begin{tikzcd}
f_1(X) \arrow[r, "\sim"] \arrow[d, "f_1(u)"'] & f_2(X) \arrow[d, "f_2(u)"] \\
f_1(S) \arrow[r, equal] & f_2(S)
\end{tikzcd}$$

! natural transf.

strict

Let $\delta : \tilde{\mathcal{S}} \to \mathcal{S}$ be quasi-inverse.

$$\mathcal{S}' = \{(T, u, f_0\delta T \simeq u)\}$$

and it maps to both $\tilde{\mathcal{S}}$ and $\mathcal{U}$ strictly

I propose to outline the proof in the following section.

I feel as if the proof should proceed from 2 lemmas.

1) $\lambda : \mathcal{S} \longrightarrow \bar{\mathcal{S}} = \underline{\mathrm{End}}^{\mathcal{S}^*}(\mathcal{S})$ equivalence

2) If $\mathcal{T} \longrightarrow \mathcal{U}$ is a monoidal f. between strict

$$\begin{tikzcd} \mathcal{T} \arrow[r] & \mathcal{U} \\ \mathcal{T}' \arrow[u, "st"', "equiv"] \arrow[ur, "st."'] & \end{tikzcd}$$

& basic construction

$$\begin{tikzcd} \mathcal{S} \arrow[r, "\lambda"] \arrow[d, "\xi"'] & \bar{\mathcal{S}} \\ \tilde{\mathcal{S}} \arrow[ur, "f", "st + f. faithful"'] & \end{tikzcd}$$

$\xi(S) = S$

$\xi(S_1 S_2) \simeq \xi(S_1) \cdot \xi(S_2)$

$$\begin{tikzcd} \mathcal{S} \arrow[r, "F"] \arrow[d, "\xi"'] & \mathcal{T} \text{ st.} \\ \tilde{\mathcal{S}} \arrow[ur, shift left, "F_1'"] \arrow[ur, shift right, "F_2'"'] & \end{tikzcd}$$

$F_1' \cdot \xi = F_2' \cdot \xi$ as mon. f.

$\Rightarrow \exists !\ \theta : F_1' \Rightarrow F_2'$ iso of mon. f.

& $\theta \cdot \xi \quad F_1'\xi \simeq F_2'\xi$ is id

Consider $\{X \in \mathrm{Ob}(\tilde{\mathcal{S}}) \mid \theta_X : F_1'(X) \simeq F_2'(X)$ is an id. map$\}$.

Because θ is an isom. of mon. f. $\Rightarrow$ this set is a submonoid.

$$\lambda : \begin{tikzcd} \mathcal{S} \arrow[r] \arrow[d] & \bar{\mathcal{S}} \\ \tilde{\mathcal{S}} \arrow[ur] & \end{tikzcd}$$ equivalence with $\bar{\mathcal{S}}$ strict

~~next: let $\tilde{\mathcal{S}} \to \bar{\mathcal{S}}$ be strict~~

$$\begin{tikzcd} \mathcal{S} \arrow[r, "f"] \arrow[d] \arrow[dd] & \mathcal{U}_{\text{strict}} \\ \tilde{\mathcal{S}} & \\ \mathcal{S}' \arrow[u] \arrow[uur] & \end{tikzcd}$$

mon.

make the definitions, then mention standard method.

Proposition: Inclusion

$$\begin{Bmatrix} \text{monoids} \\ \text{in Cat} \end{Bmatrix} \subset \begin{Bmatrix} \text{monoidal} \\ \text{cat + mon. f.} \end{Bmatrix}$$

has a left adjoint $\mathcal{S} \longmapsto \tilde{\mathcal{S}}$

Given a monoidal category $\mathcal{S}$ there exists a monoidal f. $\mathcal{S} \to \tilde{\mathcal{S}}$ which is a universal mon. functor to

Given a monoidal category $\mathcal{S}$ there exists a strict monoidal cat $\tilde{\mathcal{S}}$ and a monoidal functor $\mathcal{S} \to \tilde{\mathcal{S}}$ which is a universal monoidal functor from $\mathcal{S}$ to a strict monoidal cat. in the following sense: Given a mon. f. $\mathcal{S} \to \mathcal{T}$ with $\mathcal{T}$ strict $\exists!$ str. mon. f. $\tilde{\mathcal{S}} \to \mathcal{T}$ such that ∇ comm.

Lemma 2: Let $F : \mathcal{S} \to \mathcal{U}$ be a mon. functor with $\mathcal{U}$ strict. Then F may be factored

$$\mathcal{S} \xrightarrow{\mu} \mathcal{T} \xrightarrow{F'} \mathcal{U}$$

where μ is an equiv. of mon. cats., $\mathcal{T}$ is a strict mon. cat, and F' is a strict mon. functor.

Proof: Let $\lambda' : \bar{\mathcal{T}} \to \mathcal{S}$ be a quasi-inverse to $\lambda : \mathcal{S} \to \bar{\mathcal{T}}$ above equipped with its canon. mon. structure. Then $F\lambda' : \bar{\mathcal{T}} \to \mathcal{U}$ is a mon. functor. Let $\mathcal{T}$ be the cat whose objects are triples (T, U, α), $T \in \operatorname{Ob} \bar{\mathcal{T}}$, $U \in \operatorname{Ob} \mathcal{U}$, $\alpha : F\lambda' T \simeq U$, with the evident morphisms. Define a product on $\mathcal{T}$ by

$$(T, U, \alpha)(T', U', \alpha') = (TT'$$

~~Thus it seems that one has determined~~ !!!!!!

But we have a monoidal functor

$$\mathcal{S} \longrightarrow \mathcal{S}'$$

$$S \longmapsto (\gamma S,\ f_0 S,\ f_0\delta(\gamma S) \simeq f_0 S)$$

which is an equivalence of mon. cats. So we get as above a ~~...~~ strict

$$\tilde{\mathcal{S}} \longrightarrow \mathcal{S}'$$

---

uniqueness Lemma: Let $\mathcal{S} \xrightarrow{\gamma} \mathcal{S}'$ be an equiv. of monoidal cats with $\mathcal{S}'$ strict, let $f_1, f_2 : \mathcal{S}' \longrightarrow \mathcal{U}$ be strict mon. functors with $\mathcal{U}$ strict and suppose $f_1\gamma = f_2\gamma$. Then $f_1 = f_2$, provided $Ob(\mathcal{S})$ generates the mon. $Ob(\mathcal{S}')$.

Proof: Because γ is an equivalence, $\exists!$ natural isom $\theta : f_1 \simeq f_2$ of monoidal functors $\ni$

$$\theta \cdot \gamma : f_1\gamma \simeq f_2\gamma$$

is the identity. ~~...~~ Consider $\{X \in Ob(\mathcal{S}') \mid \theta_X = id\}$. Since

$$\begin{array}{ccc} \theta_{X_1X_2} : f_1(X_1X_2) & \simeq & f_2(X_1X_2) \\ \| & & \| \\ f_1(X_1)f_1(X_2) & \underset{\theta_{X_1}\theta_{X_2}}{\simeq} & f_1(X_1)f_2(X) \end{array}$$

follows this set is closed under product, so done.