June 6, 1971.

Let G = group of diffeos. compact support of $\mathbb{R}$. We show (G,G) is perfect.

Let $u \in G$ have support in a closed interval I, let $v \in G$ be such that $vI > I$. Then vuv^{-1} has support in vI and is the transport of u to this interval via the iso. $v : I \longrightarrow vI$. Then vuv^{-1} and u commute. The commutator $vuv^{-1}u^{-1}$ is the analogue of an elementary matrix. Call a $g \in G$ elementary if it has support contained in a union $I \cup J$ of disjoint intervals and if $\exists$ a diffeo. $h : I \longrightarrow J$ such that $h(g|I)h^{-1} \overset{!}{=} g^{-1}|J$. It is clear that the elementary transformations form a normal subset of G and that they generate a normal subgroup E contained in the commutator subgroup (G,G).

Given $g_1, g_2 \in G$ let K be the smallest closed interval containing both of their supports, let $uK > K$ and $vK < K$. Then

$$\big(\underbrace{g_1 u g_1^{-1} u^{-1}}_{\substack{\text{support}\\ K \cup uK}},\ \underbrace{g_2 v g_2^{-1} v^{-1}}_{\substack{\text{support}\\ K \cup vK}}\big) = (g_1, g_2)$$

showing that $(G,G) \subset (E,E)$, hence $E = (G,G)$ and E is perfect.

Next note that given $g \in G$ and a interval I one can move I by an e so that eI is disjoint from the support of g, whence it follows that G_{ab} acts trivially on $H_*((G,G))$.

June 12, 1971. Kummer theory.

Let k be an algebraically closed field of characteristic p. I conjecture then that there is a fibration

$$BGL(\mathbb{F}_q)^+ \longrightarrow BGL(k)^+ \xrightarrow{\ \Psi^q - 1\ } BGL(k)^+$$

and I believe I have proved this when $k = \overline{\mathbb{F}}_p$.

Remark 1: It's a stable phenomenon as

$$BGL_n(\mathbb{F}_q)^+ \longrightarrow BGL_n(k)^+ \longrightarrow BGL_n(k)^+$$

is wrong at the prime p when $k = \overline{\mathbb{F}}_p$.

Remark 2: By Lang's theorem

$$G/G^\sigma \simeq G$$

where σ = frobenius, $G = GL_n(k)$. This has the consequence that the category of k-vector spaces (of finite dimension) endowed with a frobenius operator $F: V \xrightarrow{\sim} V$, $F(av) = a^\sigma F(v)$, is equivalent to the category of $\mathbb{F}_q$-vector spaces. Thus we have the situation where our category with operation $\mathcal{A}$ has an auto. σ and we want to compute the homotopy equalizer of $id, \sigma : \mathcal{A} \rightrightarrows \mathcal{A}$ as the categorical equalizer.

Precisely let $M = \coprod_{n \geq 0} BGL_n(k)$. We want to show that the group completion of $M^\sigma = \coprod_{n \geq 0} BGL_n(\mathbb{F}_q)$ is the homotopy equalizer of $id, \sigma : \overline{M} \rightrightarrows \overline{M}$. (Instead of group completion, which should be OKAY by Anderson's result that M and M^σ are free simplicial monoids, we perhaps might want to use your ~~substitute~~ $B(M \times M, M)$.)

Remark 3: In your stuff on groups G^σ you use the square

$$\begin{array}{ccc} G^\sigma & \longrightarrow & G \\ \downarrow & & \downarrow \Delta \\ G & \xrightarrow{\Gamma = (id, \sigma)} & G \times G \end{array}$$

and the fact that the induced map of homogeneous spaces

$$G/G^\sigma \longrightarrow G \times G / \Delta G$$

an isomorphism. Therefore what you want to solve now is: Problem: Is

$$\Gamma : \quad B(M, M^\sigma) \longrightarrow B(M \times M, M)$$

is a weak equivalence?

Why this does what we want it to. Consider M with left M-action and right M^σ-action and denote the resulting classifying space by $B(M \backslash M / M^\sigma)$. Then we have a weak homotopy equivalence

$$B(M\backslash M/M^{\sigma}) \longrightarrow BM^{\sigma}$$

and maps

$$\begin{array}{ccccc} B(M, M^{\sigma}) & \longrightarrow & B(M\backslash M/M^{\sigma}) & \longrightarrow & BM. \\ \uparrow \text{weq} & & & & \\ B(M\times M, M) & & & & \end{array}$$

Now I think one has $B(M\times M, M) \xrightarrow{\text{weq}} \bar{M}$ (homology OKAY in any case), and as M acts invertibly ~~the~~ on $\bar{M}$, ~~this~~ the above should be a fibration. Thus we should have a fibration

$$\bar{M} \longrightarrow BM^{\sigma} \longrightarrow BM$$

proving (assuming the maps work out) what we want.

June 13, 1971 see July 7, 1972

k = an alg. closed field char p, $\mathbb{F}_q$ = subfield with q elements. $\sigma(x) = x^q$

Let V be a vector space (f.d. always) over k endowed with a σ-linear auto $F: V \longrightarrow V$, $F(xv) = x^\sigma F(v)$. Then

$$k \otimes_{\mathbb{F}_q} V^F \xrightarrow{\sim} V$$

Proof: Let v_j be a basis for V and

$$Fv_j = \sum_k a_{kj} v_k \qquad a_{kj} \in k$$

By Lang's theorem

$$A = B(B^\sigma)^{-1}$$

for some matrix $B = (b_{ij})$. Then

$$F\left(\sum_j v_j b_{ji}\right) = \sum_j Fv_j \cdot b_{ji}^q$$

$$= \sum_{jk} v_k a_{kj} b_{ji}^q$$

$$= \sum v_k b_{ki}$$

showing that $\sum_j v_j b_{ji}$ is a basis for V lying in V^F.

Here's how to prove Lang's theorem by reversing the preceding argument:

Given the matrix $A = (a_{ij})$ define F on k^n by

$$F e_i = \sum_j a_{ji} e_j$$

If we can then find a basis w_i for k^n fixed under F then setting

$$e_i = \sum w_j b_{ji}$$

we have

$$\sum a_{ji} e_j = F(e_i) = \sum w_j b_{ji}^q$$

$$\| $$

$$\sum_{k,j} w_k b_{kj} a_{ji}$$

hence $BA = B^{(q)}$ giving Lang's theorem.

So we have to show that V is spanned by V^F.

We start by showing that $V^F \neq 0$ when $V \neq 0$. Choosing $v \in V$, $v \neq 0$ and let n be least $\ni$ $\{v, Fv, \ldots, F^{n-1}v\}$ are linearly indep. whence a relation

$$F^n v + a_{n-1} F^{n-1} v + \cdots + a_0 v = 0 \qquad a_i \in k.$$

Now observe the identity

$$F^n = (F-x)\left(F^{n-1} + x^{\tau} F^{n-2} + \cdots + x^{\tau + \cdots + \tau^{n-1}}\right) + x^{1+\cdots+\tau^{n-1}}$$

where $\tau(x) = x^{1/q}$ so that $F(x^{\tau} w) = x(Fw)$. This identity tells us there is a division algorithm

$$F^n + a_{n-1}F^{n-1} + \cdots + a_0 = (F-x)\, g(F) + r$$

$$r = x^{1+\cdots+\tau^{n-1}} + a_{n-1}x^{1+\cdots+\tau^{n-2}} + \cdots + a_0$$

[illegible] Then

$$r^{q^{n-1}} = x^{q^{n-1}+\cdots+1} + a_{n-1}^{q^{n-1}} x^{q^{n-1}+\cdots+q} + \cdots + a_0^{q^{n-1}}$$

is a monic polynomial in x, hence x can be chosen so that $r=0$. Then

$$0 = (F-x)\, g(F) v$$

and as $g(F)v \neq 0$, we see v can be chosen so that $Fv = xv$ for some $x \in k$. But then choose $\lambda \in k$ so that

$$F(\lambda v) = \lambda^q x v = \lambda v$$

i.e. [illegible] $\lambda^{q-1} = x^{-1}$ we see [illegible] v can be chosen so that $F(v) = v$. Thus $V^F \neq 0$.

Now let $W \subset V$ be the subspace spanned by V^F, [illegible] and assume $W < V$. Then F

induces an $\bar{F}$ on V/W and so we get a $v \in V$, $v \notin W$ such that $Fv = v + w_0$. Search for $w \ni$

$$F(v+w) = v+w$$

i.e.

$$Fw - w + w_0 = 0$$

~~Because~~ Because W has a basis stable under F, this amounts to solving equations of the form

$$x^q - x + a = 0$$

which is always possible. Conclude V^F spans V, proving the Lang theorem for GL_n.

Suppose [redacted] Θ is a stable isomorphism of V and σ^*V. This means Θ is represented by an iso. $\Theta : W \oplus V \longrightarrow W \oplus \sigma^*V$. ~~By adding~~ You probably want $W = k^n$. Then one has a canonical isomorphism $k^n \longrightarrow \sigma^*(k^n)$, so one gets

$$W \oplus V \longrightarrow W \oplus \sigma^* V \longrightarrow \sigma^* W \oplus \sigma^* V = \sigma^*(W \oplus V)$$

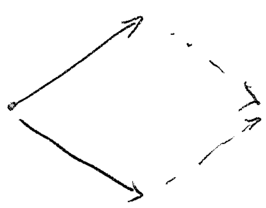
which as we have seen means an isomorphism

$$W \oplus V \xleftarrow{\sim} k \otimes_{F_q} (V_0)$$

where V_0 is an F_q-vector space.

Suppose we now take the category which we know has the homotopy type of $BGL(k)^+$. Its objects are finite sets and a morphism $S \to S'$ is a pair consisting of an injection $u: S \hookrightarrow S'$ and an auto θ of kS'. Suppose I have a chain of maps in this category C joining V and $\sigma^* V$. Meaning is imprecise.

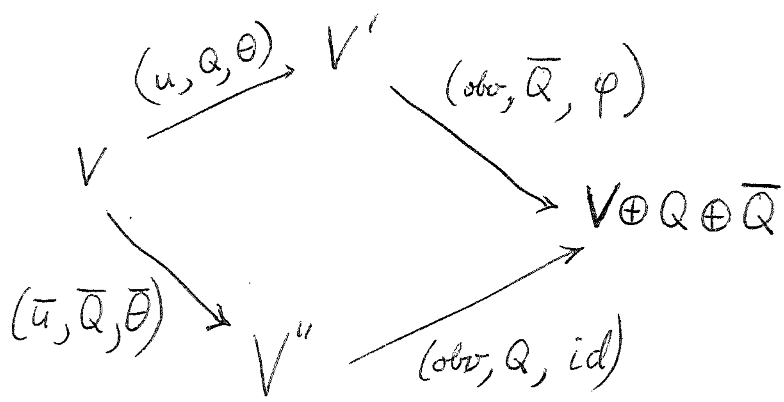
Maybe a better thing is to take the category whose objects are k-vector spaces V in which a morphism from V to V' consists of an injection $u: V \to V'$ plus a complement $Q \subset V'$ for uV, plus an automorphism θ of V'. I claim this category is directed:



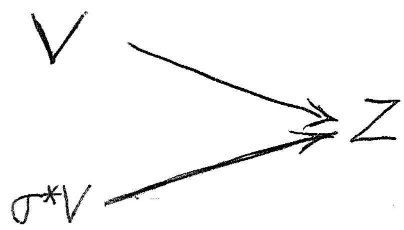
namely given

$$V \longrightarrow V' \qquad (u, Q, \theta)$$
$$V \longrightarrow V'' \qquad (\bar{u}, \bar{Q}, \bar{\theta})$$

consider the composite $V \oplus Q \oplus \bar{Q}$ and the two ways of putting V' and V'' in:



can can choose φ so that the square commutes. So given a chain of arrows joining V and $\sigma^* V$ we can replace it by



in other words we get an isomorphism of $V \oplus Q$ and $\sigma^* V \oplus Q'$. No good.

July 2, 1971: Attempts to construct an Artin-Schreier exact sequence for an algebraically closed field of characteristic p.

k alg. cl. field char p, $\sigma : k \to k$ the Frobenius $\sigma(x) = x^q$, F_q = subfield of k with q elements, M simplicial monoid = nerve of a monoid category equivalent to f.g. k-modules, e.g.

$$M = \coprod_n BGL_n(k),$$

and M' simplicial submonoid obtained from F_q-modules, e.g.

$$M' = \coprod_n BGL_n(F_q).$$

(In this example $M' = M^\sigma$).

By analysis of June 12 I can identify the fibre of the map $BM' \longrightarrow BM$ with $B(M, M')$, where M' acts say to the right of M. ~~[illegible]~~ I feel $B(M,M')$ is homotopy equivalent to the category whose objects are f.g. k-modules E and in which a morphism from E to E' consists of a complemented injection

$$E \underset{i}{\overset{p}{\rightleftarrows}} E' \qquad pi = id_E$$

together with an F_q-reduction of $\mathrm{Ker}(p) = P$. By Lang's theorem such a reduction amounts to an isomorphism $\sigma^* P \simeq P$.

The thing to prove is that this fibre $B(M,M')$ is homotopy equivalent to the connected component

$\bar{M}_e$, where we recall the group completion $\bar{M}$ has the same homotopy type as ΩBM. The way I hoped to do this was by using the map $\Gamma: M \longrightarrow (M \times M)_e$ sending W to $(W, \sigma^* W)$, which induces a map

(*) $$B(M, M') \longrightarrow B((M \times M)_e, M).$$

($(M\times M)_e$ means the submonoid consisting of the components of objects (V', V'') with $\dim V' = \dim V''$.) I would like to prove this map is a homotopy equivalence by some cofinality argument. As $B((M \times M)_e, M) \sim \bar{M}_e$, I would have the desired exact sequence of homotopy groups.

~~Replace the right side of (*)~~

One ought to be able to replace $B((M\times M)_e, M)$ by the category whose objects are pairs (V', V'') of f.g. k-modules, in which a morphism $(V', V'') \to (W', W'')$ consists of a complemented injection

$$V \underset{i}{\overset{p}{\rightleftarrows}} W \qquad (P = \mathrm{Ker}(p) = (\mathrm{Ker}\, p', \mathrm{Ker}\, p'') = (P', P''))$$

together with a diagonal-reduction of P, i.e. an isom $P' \simeq P''$. I found then with (*) replaced by the functor between the appropriate such categories, that (*) was not relatively filtering

Here is a typical obstruction encountered:

Suppose we are given ~~three~~ f.g. k-mod. V, W and a map

$$(W, \sigma^*W) \longrightarrow (V, \sigma^*V),$$

that is, a pair of isomorphisms

$$W + P \xrightarrow{\sim} V$$

$$\sigma^*W + P \xrightarrow{\sim} \sigma^*V$$

where P is a k-module. Then we obtain an isomorphism

$$\sigma^*W + \sigma^*P \xrightarrow{\sim} \sigma^*V \xleftarrow{\sim} \sigma^*W + P,$$

that is, a stable isomorphism of P and σ^*P. If this isom. came from an actual isomorphism of P and σ^*P, then we have a map $W \longrightarrow V$ in the category of type $B(M, M')$ which induces the given map $\Gamma(W) \longrightarrow \Gamma(V)$.

So what appears to be the crux of my ignorance is an understanding of why the category of f.g. k-modules and stable F_q-reductions should be equivalent* to the category of f.g. $\bar{F}_q$-modules.

* after group completion.

So given V with a stable F_δ-reduction

$$V + P \simeq \sigma^* V + P$$

we wish to make it come from an isom. $V \simeq \sigma^* V$. These leads to the cancellation problem $(A + P \simeq B + P \Longrightarrow A \simeq B\ ?)$ which I have some ideas about because of the work on the <u>stability</u> <u>theorem</u>.

July 21, 1970. <u>Thm. 2:</u>

Let $\mathcal{C}$ be a small additive category (assume $0, \times$ exists) I assume $0 \in \mathcal{C}$ is chosen. For each $n \geq 0$ we define categories $\mathcal{C}_n$, $\mathcal{C}_n'$ as follows:

An object of $\mathcal{C}_n$ is a collection $\{X_{ij}, \varphi_{ijk}\}$ where $X_{ij} \in Ob\, \mathcal{C}_n$ $0 \leq i \leq j \leq n$ and

$$\varphi_{ijk} : X_{ij} \oplus X_{jk} \xrightarrow{\sim} X_{ik} \qquad 0 \leq i \leq j \leq k \leq n$$

such that following conditions hold:

(i) $X_{ii} = 0$ (the given 0 object) and φ_{iij} and φ_{ijj} are the canonical maps

(ii) transitivity: Given $i \leq j \leq k \leq l$ want

$$(X_{ij} \oplus X_{jk}) \oplus X_{kl} \longrightarrow$$

$$\| \qquad\qquad\qquad \downarrow$$

$$X_{ij} \oplus (X_{jk} \oplus X_{kl})$$

$$\downarrow$$

$$\longrightarrow$$

to commute.

It is clear what one means by an isomorphism of two such collections and we take such isos as morphisms in $\mathcal{C}_n$.

$n \mapsto \mathcal{C}_n$ is a simplicial object in $\mathcal{C}at$.

An object of C'_n is a collection X_{ij} $0 \le i \le j \le n$ and $\varphi_{ij,i'j'} : X_{ij} \to X_{i'j'}$ defined for $i \le i'$ and $j \le j'$ $\ni$

(i) functorial $\varphi_{ij,ij} = id$, $\varphi_{ij} \circ \varphi \quad \varphi$

(ii) $X_{ii} = 0$ (the given 0)

(iii) for $i\,j\,k$ $0 \le i \le j \le k \le n$ have ~~exact sequence~~

$$0 \longrightarrow X_{ij} \xrightarrow{\varphi_{ij,ik}} X_{ik} \xrightarrow{\varphi_{ik,jk}} X_{jk} \longrightarrow 0$$

is split exact, i.e. $\exists$ map $X_{jk} \longrightarrow X_{ik}$ inducing isomorphism of

$$X_{ij} \oplus X_{jk} \longrightarrow X_{ik}.$$

Morphisms in C'_n are isomorphisms

Again given a monotone map $s : [m] \longrightarrow [n]$ we ~~pull back an object~~ get a functor $C'_n \to C'_m$ so $n \mapsto C'_n$ is a simplicial object in Cat. There is an evident morphism of simplicial categories $C_\cdot \longrightarrow C'_\cdot$

Now form simplicial spaces $BC_\cdot \longrightarrow BC'_\cdot$

Theorem: $|BC_\cdot| \longrightarrow |BC'_\cdot|$ is a homotopy equivalence.

~~Proof. The proof of ... Segal ...~~

Example: Take C to be full subcategory of A-modules consisting of A^n $n \ge 0$, let $G_n = GL_n(A) = Aut(A^n)$ and let $G_{n_1,\ldots,n_\nu}$ = the group of autos. of $A^{n_1+\cdots+n_\nu}$ which fixes the submodules $A^{n_i+\cdots+n_\nu}$ $i = 1, \ldots, \nu$. Then the theorem says that the map of simplicial spaces

$$\begin{array}{ccccc} \equiv \coprod_{n_1,n_2} B(G_{n_1}\times G_{n_2}) & \equiv & \coprod_{n} BG_n & = & e \\ \downarrow & & \downarrow & & \downarrow \\ \equiv \coprod_{n_1,n_2} BG_{n_1,n_2} & \equiv & \coprod_{n} BG_n & = & e \end{array}$$

is a homotopy equivalence.

Corollary: ~~[illegible]~~ Let

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

be an exact sequence of representations of G over A. Then the two maps $BG \rightrightarrows K_0A \times BGL(A)^+$ associated to $E'\oplus E''$ and E are homotopic.

Proof: We can assume E', E'' are free A-modules of ranks n_1, n_2; it's enough to show that the two maps

$$BG_{n_1,n_2} \rightrightarrows \Omega\big(B \textstyle\coprod_n BG_n\big)$$

associated to the two maps $G_{n_1,n_2} \rightrightarrows G_{n_1+n_2}$ are homotopic. But this is clear in the second simplicial space as there is a map $BG_{n_1,n_2} \times \Delta(2) \longrightarrow |BC'|$.

<u>*Proof of theorem 2:*</u> By Whitehead theorem enough to show $|BC.| \to |BC'_.|$ induces isos. on π_1's and for homology with coefficients in local coefficient systems L which are k-modules with k a field. Assertion about π_1's is clear because

$$\pi_0 C_\nu \xrightarrow{\sim} \pi_0 C'_\nu \xrightarrow{\sim} (\pi_0 C)^\nu$$

hence $\pi_1 |BC.| = \pi_1 |BC'_.| = K_0 C$ the group associated to the monoid $\pi_0 C$. We consider map of spectral sequences

$$\begin{array}{ccc} E^1_{rs} = H_s(BC_r, L) & \Longrightarrow & H_{r+s}(|BC.|, L) \\ \downarrow & & \downarrow \\ {}'E^1_{rs} = H_s(BC'_r, L) & \Longrightarrow & H_{r+s}(|BC'_.|, L) \end{array}$$

and we are going to prove that the induced map $E^2 \to {}'E^2$ is an isom.

Recall that L is just a $k[K_0]$-module. Also C_ν and C'_ν are the groupoids associated to additive categories hence $H_*(BC_.)$ and $H_*(C'_.)$ are simplicial (anti-)commutative rings whose homomorphisms into a graded K-alg. $S.$ are the exponential characteristic classes. $E^1_{*r*} = H_*(BC_r, L)$ is a simplicial module over $H_*(BC_.)$.

Lemma: Let $S.$ be a simplicial subset of the center of a simplicial ring $A.$ and let $M.$ be a (left) simplicial $A.$-module. Denote by $(\pi_0 S.)^{-1} H_i(M.)$ the localization of the $H_0(A.)$-module $H_i(M)$ with respect to the subset of $H_0(A.)$ which is the image of the map $S_0 \to H_0(A.)$. Then the canonical map $M. \to S.^{-1}M.$ induces an isom

$$(\pi_0 S.)^{-1} H_i(M.) \xrightarrow{\sim} H_i(S.^{-1}M.).$$

Proof: We may suppose $S.$ is a multiplicative system in $A.$; localizing (M. and A.) with respect to the "constant" simplicial subsets of $A.$ consisting of degeneracies of S_0, we reduce to proving that $H_*(M.) \xrightarrow{\sim} H_*(S.^{-1}M)$ if S_0 is a group. However if $G.$ is the simplicial abelian group generated by the monoid $S.$ then $\mathbb{Z}[S.] \to \mathbb{Z}[G.]$ is a flat map of simplicial $\mathbb{Z}[S.]$-modules and

$$S.^{-1}M. = \mathbb{Z}[G.] \otimes_{\mathbb{Z}[S.]} M.$$

so by the Kunneth spectral sequence to prove $M. \to S.^{-1}M.$ is an isomorphism it will suffice to show that $\mathbb{Z}[S.] \to \mathbb{Z}[G.]$ induces isomorphisms on homology, i.e. we are reduced to proving that $S. \to G.$ is a homotopy equivalence when $S.$ is a group. Using Segal's theorem that $S. = \Omega|BS.|$ and $G. = \Omega|BG.|$ we have to prove only that $BS_\nu \to BG_\nu$ is a homotopy equivalence for each ν, or only that

$$\mathrm{Tor}_*^{H_*(BS_\nu)}(k, L) \xrightarrow{\sim} \mathrm{Tor}_*^{H_*(BG_\nu)}(k, L)$$

for any $k[G_\nu]$-module L and field k, but this is immediate from the fact that ~~Tor~~ commutes with localization (ref.)

According to the lemma since the multiplicative system $\pi_0 C_\cdot \subset H_*(BC_\cdot)$ is reduced to identity in **dimension 1**, we may localize the simplicial module $H_*(BC_\cdot; L) = H_*(BC_\cdot) \otimes_{k[\pi_0 C_\cdot]} L$ with respect to $\pi_0 C_\cdot$ without changing its homology. But

$$H_*(BC_r)[\pi_0 C_r^{-1}] \cong V_r \otimes_k k[K_0^r]$$

where $V_r = \varinjlim_{\alpha \in \pi_0 C_r} H_*(B\mathrm{Aut}(X_\alpha))$ is the algebra whose homomorphisms to $S_\cdot$ are the same as exponential classes for representations of groups on objects of the additive category C_r which vanish for trivial representations. Thus we see that

$$H_*(BC_\cdot, L)[\pi_0 C_\cdot^{-1}] \simeq V_\cdot \otimes_k (k[K_0^\cdot] \otimes_k L)$$

where $k[K_0^\cdot] \otimes L$ is the chain complex of the group K_0 acting on L, so

$$H_r(k[K_0^\cdot] \otimes_k L) = H_r(BK_0, L)$$

Similarly as $\pi_0 \mathcal{C}_r \cong \pi_0 \mathcal{C}'_r \cong (\pi_0 \mathcal{C})^r$ we have that

$$H_*(BC'_\cdot, L) \simeq V'_\cdot \otimes_k (k[K_0^\cdot] \otimes_k L)$$

so to prove the theorem it suffices to show that $V_r \xrightarrow{\sim} V'_r$, i.e. that exponential classes for objects of $\mathcal{C}'_r$ are the same as those for $\mathcal{C}_r$.

I note that

$$V'_r = V_r \otimes T_r$$

where T_r is a Hopf algebra whose homomorphisms to S. are the same as exp. characteristic classes of representations on objects of $\mathcal{C}'_r$ which vanish if the representation splits, i.e. preserves a direct sum decomposition. To show $T_r = k$ it suffices to show $Q(T_r) = 0$, so let $u : QT_r \to k$ be a k-linear map, i.e. u is an additive natural transf. to $H^*(G)$ for representations ρ of groups G on direct sums $C_1 \oplus \cdots \oplus C_r$ which preserves the subobjects $C_j \oplus \cdots \oplus C_r$ for each $j = 1, \ldots, r$ and ~~moreover~~ $u = 0$ if the representation preserves C_j. I suppose u is of dimension $i > 0$ and propose to show that u is zero.

By induction on r I can suppose $u(\rho) = 0$ if ~~[illegible]~~ $\rho(C_1) = C_1$, ~~since~~ then ρ is the direct sum of the representation $C_1 \oplus 0 \oplus \cdots \oplus 0$ and the representation $0 \oplus C_2 \oplus \cdots \oplus C_r$. ~~Consider the split extension of groups~~

~~$$1 \longrightarrow \mathrm{Hom}(C_2 \oplus \cdots \oplus C_r, C_1) \longrightarrow \mathrm{Aut}'(C_1 \oplus \cdots \oplus C_r) \longrightarrow \prod_i \mathrm{Aut}(C_i) \longrightarrow 1.$$~~

~~and suppose chosen a finite free extension Λ of $\mathbb{Z}$ as in the key lemma, [illegible] such that $i < \frac{l-1}{2}$ or d depending on the characteristic and that $l-1$ or d is prime to the characteristic of k. Suppose first that the representation C_1 of G admits a Λ-module structure. [illegible] We consider~~

~~Choose an integer d prime to the characteristic of~~

suppose chosen a finite free extension Λ of $\mathbb{Z}$ as in key lemma with $i < \frac{l-1}{2}$ (or d) and with $l-1$ (or d) prime to the characteristic of k. Given ρ then $u(\rho^{\oplus e})$ $(e = [\Lambda:\mathbb{Z}])$ $= e\,u(\rho)$ so it suffices to prove $u(\rho^{\oplus e}) = 0$ and we therefore assume that $C_1, \ldots, C_n$ have Λ-action such that ρ commutes with the Λ-structure on $C_1 \oplus \cdots \oplus C_r$. Consider the extension

$$1 \longrightarrow \underset{\underset{M}{\shortparallel}}{\mathrm{Hom}_\Lambda(C_2 \oplus \cdots \oplus C_r, C_1)} \longrightarrow \underset{\underset{G'}{\shortparallel}}{\mathrm{Aut}'_\Lambda(C_1 \oplus \cdots \oplus C_r)} \longrightarrow \underset{\underset{G}{\shortparallel}}{\prod_{i=1}^{r} \mathrm{Aut}_\Lambda(C_i)} \to 1$$

where Aut'_Λ denotes the group of Λ-linear autos of $C_1 \oplus \cdots \oplus C_r$ which preserves the filtrations. Define a hom.

$$\Lambda^\times \longrightarrow G'$$

by letting $\Lambda^\times$ act by multiplication on C_1 and as the identity on $C_2 \oplus \cdots \oplus C_n$. Let $\Lambda^\times$ act by conjugation on the extension & observe that it acts trivially on G and as multiplication on the Λ-module M. Now look at the Hochschild - Serre spectral sequence

$$E^2_{rs} = H_r(G', H_s M) \Longrightarrow H_{r+s}(G, M)$$

as representations of $\Lambda^\times$ over k, k assumed to be alg. closed. Then $\Lambda^\times$ acts only on $H_s M$ and the action is semi-simple so we may split up the spec. seq. into trivial and non-eigenspaces obtaining

$$'E^2_{rs} = H_r(G', (H_s M)^{\Lambda^\times}) \Longrightarrow H_{r+s}(G, M)$$

By the lemma there are no invariants for $0 < s \leq i$, hence

$$H_i(G') \xrightarrow{\sim} H_i(G)$$

But if ρ' is the standard rep of $G' = \mathrm{Aut}'_\Lambda(C_1 \oplus \cdots \oplus C_r)$ we know that $u(\rho')(G) = 0$ by assumption, hence $u(\rho') = 0$.

June 14, 1971. Summary of work on K-theory.

K-theory is a kind of localization; one starts with a category with operation such as projective modules and makes the operation invertible to get the K-theory. From the homotopy point of view one is going from M to ΩBM, and there are the following things to be understood: M, i.e. what is a category with operation, BM, ie. what is its classifying space, and finally what is the loop space.

Nature of M: examples: topological categories, categories with internal operation, 2-categories. The one idea I have is that the way to express the fact one has a category with internal operation is to say one can form a pseudo-simplicial category, i.e. a cofibred category over Δ° with fibre C^n. One then gets a category which is the classifying space for C. (The idea comes from Segal, who shows that instead of trying to describe A_∞-associativities, just say that there is a special simplicial space.)

Nature of BM. As a homotopy type it is the realization of the simplicial gadget associated to M. I wanted to have a more concrete understanding of BM through the torsors for M. Torsors for a groupoid in a topos. (torsor for a category) If C is a category one has an understanding of what one means by a C-torsor, namely, a sheaf P over X endowed with right C-action, such that the stalks are all pro-objects. One can ask when every in the homotopy category from BC' to BC is represented by a C-torsor over BC; i.e. a map C'→ Pro(C), but this seems to be false; for simplicial complexes one needs to be able to subdivide. Do not therefore have a good understanding of the homotopy classification of C-torsors. One wants a generalization for 2-categories - it would seem that a torsor would be a kind of stack; this is what happens in the Mather situation. Do not ~~have a good~~ understand ~~ing of~~ torsors with respect to a 2-category.

Loop space. Here my understanding is nil. The best approximation so far is to find a special simplicial gadget whose realization has the correct homotopy type; then one has the loop space. But still need a proof of this and Segal's $|\Omega.Y| \xrightarrow{\sim} Y$.

ΩBM: In the case of a category with operation, say modules, I have essentially two different possible concrete ~~approa~~ approaches. First is the one dating from April 30 - May 5,1971. One considers the category of modules and complemented injections, and over this one constructs the cofibred category associated to the functor Aut. Then one shows this has the right homotopy type, using an acyclicity argument of ~~fal~~ flasqueness. At the moment it remains to find ~~and~~ a good torsor interpretation of this construction. Secondly, there is the B(MxM,M) approach. If this works ~~andxif~~ (it has not been carefully checked) and if we get some kind of understanding of torsors for a bicategory (which, after all, is what we are looking at when we talk about the category of modules acting diagonally by direct sum on the category of pairs of modules), then we will have ~~anxinterpe~~ an interpretation if K-classes as torsors. Another thing to keep in mind when working with the B(MxM,M) approach is the possibility of using abstract homotopy theory.

perhaps you have an idea about the J-homomorphism.
We feel certain that

$$\begin{array}{ccccc}
BGL_n(\bar{\mathbb{F}}_p) & \xrightarrow{\partial} & BBGL(\bar{\mathbb{F}}_p) & \longrightarrow & B(B\Sigma^{\times}_{p^\infty}) \\
\downarrow & & & & \swarrow \\
BU & \xrightarrow{J} & BG & \longrightarrow & BG[\tfrac{1}{p}]
\end{array}$$

commutes. Therefore we should try to understand the sphere bundle of an

---

why over $\boxed{B(B\Sigma^{\times}_{p^\infty})}$ should there be a stable ~~and~~ p^{-1}-spherical fibre space.

---

p prime no.
~~so~~ look at sets of order a power of p under product. Given a fibre space of that sort - covering of degree p^a, then over its suspension I want a spherical fibre space.

Covering

$$\begin{array}{c} Y \\ \downarrow \\ X \end{array}$$

June 24, 1971.

Fix a ring R and let C denote the category whose objects are f.g. proj. R-modules, in which a morphism $u: E \longrightarrow E'$ is a pair (i, Q), where $i: E \longrightarrow E'$ is a direct injection and Q is a complement to the image of i. The nerve of C has for its n-simplices diagrams in $\mathcal{P}_R$

$$E_0 \underset{i_0}{\overset{p_0}{\rightleftarrows}} E_1 \cdots \rightleftarrows E_n$$

such that $p_j i_j = \text{id}$.

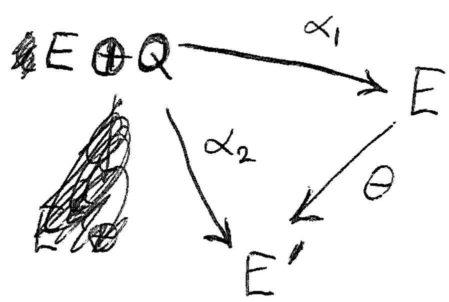
On the other hand we can consider the simplicial category $\mathcal{B}$ such that $\mathcal{B}_n$ is the category of such diagrams and their isomorphisms. It is clear that $\mathcal{B}_n$ is equivalent to $(\mathcal{P}_R)^{n+1}$, that in fact $\mathcal{B}$ is the simplicial category obtained by letting $\mathcal{P}_R$ act on itself. (Topologically $\mathcal{B} = B(M, M)$ $M = B\mathcal{P}_R$). Clearly $\mathcal{B}$ is contractible.

So is C because it has 0 for initial object.

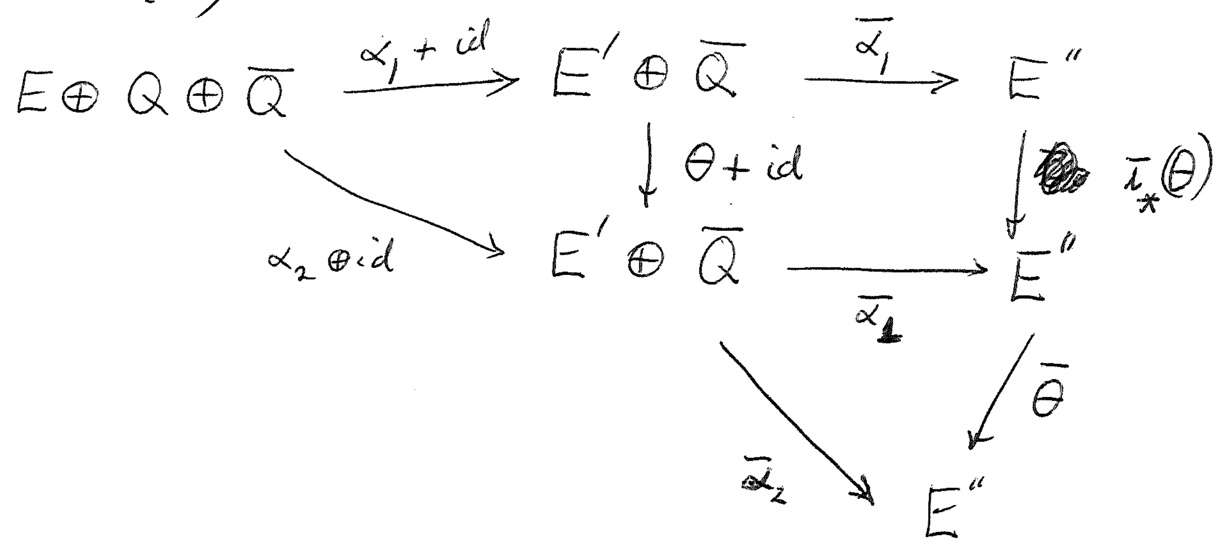
But I am really interested in the category whose objects are pairs (E_1, E_2) in which a morphism $(E_1, E_2) \longrightarrow (E_1', E_2')$ is constituted by (Q, α_1, α_2) where

$$\alpha_1 : E_1 \oplus Q \xrightarrow{\sim} E_1' \qquad \alpha_2 : E_2 \oplus Q \xrightarrow{\sim} E_2'$$

These morphism form a groupoid with no automorphisms, hence we can normalize by requiring $Q \subset E_1'$, that is, α_1 should be given by a direct injection $E_1 — E_1'$ and a complement for its images. Note that at least over a field the component of this category where $\mathrm{rank}(E_1) = \mathrm{rank}(E_2)$ contains the equivalent full subcategory of pairs with $E_2 = E_1$. Thus this subcategory has objects E and for its morphisms from E to E' triples (i, Q, θ) where $i : E \hookrightarrow E'$, $Q \subset E'$ $\ni$ $iE \oplus Q \simeq E'$, and θ is the autom of E' such that



commutes. Now given $\bar{\imath} : E' \hookrightarrow E''$, $\bar{Q} \subset E''$, and $\bar{\theta} \in \mathrm{Aut}(E'')$



the composite is the triple $(\bar{\imath} i, \bar{\imath} Q + \bar{Q}, \bar{\theta}\, \bar{\imath}_*(\theta))$.

This shows the subcategory is the cofibred category over $\mathcal{C}$ associated to the functor $E \mapsto \mathrm{Aut}(E)$ to groups.

What I want to show is that this subcategory is of the same homotopy type as the simplicial category consisting of

$$(E', E'', V_1, \ldots, V_n) \qquad \text{rank } E' = \text{rank } E''$$

(topologically ~~$B(M \times M, M)$~~ $B((M \times M)_0, M)$ where $M = B\mathcal{P}_R$ and $(M\times M)_0$ is the union of the $B\,\mathrm{Aut}(P) \times B\,\mathrm{Aut}(P) \subset M \times M$.

June 27, 1971:

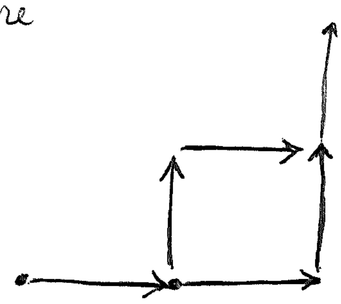
Let $\mathcal{B}$ be a bicategory and let X be the associated bisimplicial set. Following Artin – Mazur, we consider the simplicial set whose n-simplices are ~~the~~ elements

$$(x_{pq})_{p+q=n} \in \prod_{p+q=n} X_{pq}$$

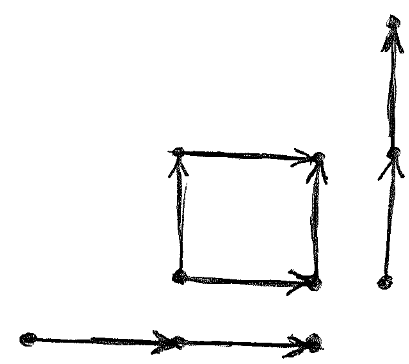
such that

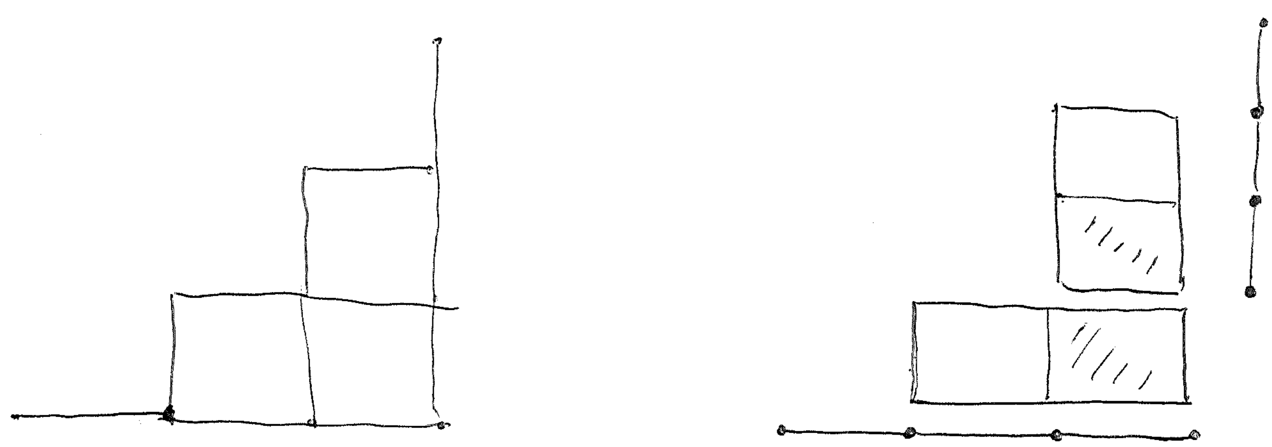
$$d_0^h x_{pq} = d_q^v x_{pq}.$$

Picture



made up of:





We call this simplicial set $AM(X)$ and we want to know when this might be the nerve of a 2-category.

<u>Example</u>: Let $\mathcal{B}$ be $(M \times M, M)$. Thus the vertical object cat. is pairs (E', E'') of vector spaces, say $(M = \text{Nerv}(\mathcal{P}_R))$. A one-simplex of $AM(X)$ consists of $(E_0', E_0'', F, E', E'', \alpha', \alpha'')$ such that

$$\alpha' : E_0' \oplus F \xrightarrow{\sim} E'$$
$$\alpha'' : E_0'' \oplus F \xrightarrow{\sim} E''$$

This is ~~the composite of the~~ made up of $(E_0', E_0'', F) \in X_{10}$ and $(\alpha', \alpha'') \in X_{01}$,

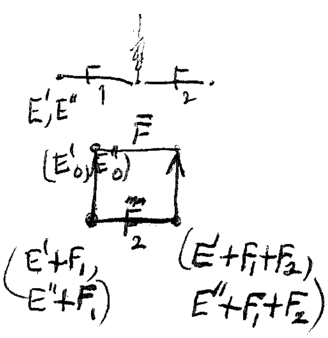
$$d_0(E_0', E_0'', F) = (E_0' + F, E_0'' + F).$$

Denote this one simplex by $(E_0', E_0'', F, E', E'', \alpha', \alpha'')$. Its source is E_0', E_0'' and target is E', E''.

A two simplex of $AM(X)$ consists of three things.

(i) (E', E'', F_1, F_2)

(ii) $(\alpha', \alpha'', \bar{F}, \beta)$

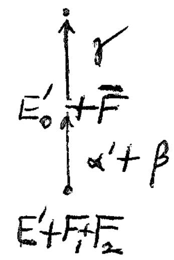


$$\alpha' : E' + F_1 \simeq E_0'$$
$$\alpha'' : E'' + F_1 \simeq E_0''$$

~~$\beta' : E' + F_1 + F_2 \xrightarrow{\sim} E_0' + \bar{F}$~~
~~$\beta'' : E'' + F_1 + F_2 \xrightarrow{\sim} E_0'' + \bar{F}$~~

$$\beta : F_2 \simeq \bar{F}$$

(iii)

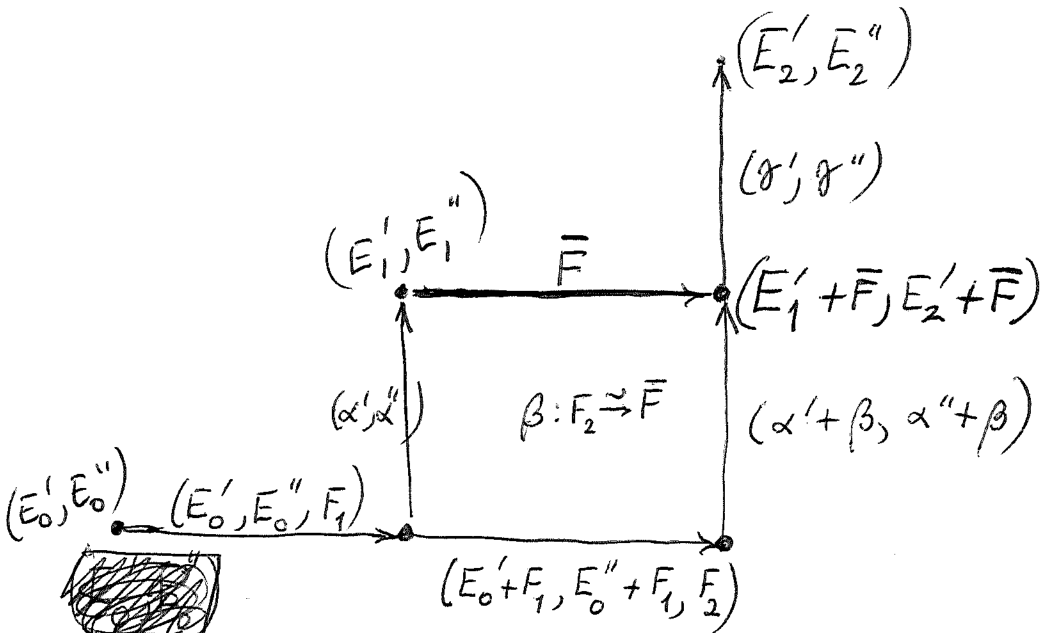


$$(\gamma', \gamma'')$$

$$\gamma' : E_0' + \bar{F} \xrightarrow{\sim} E_1'$$

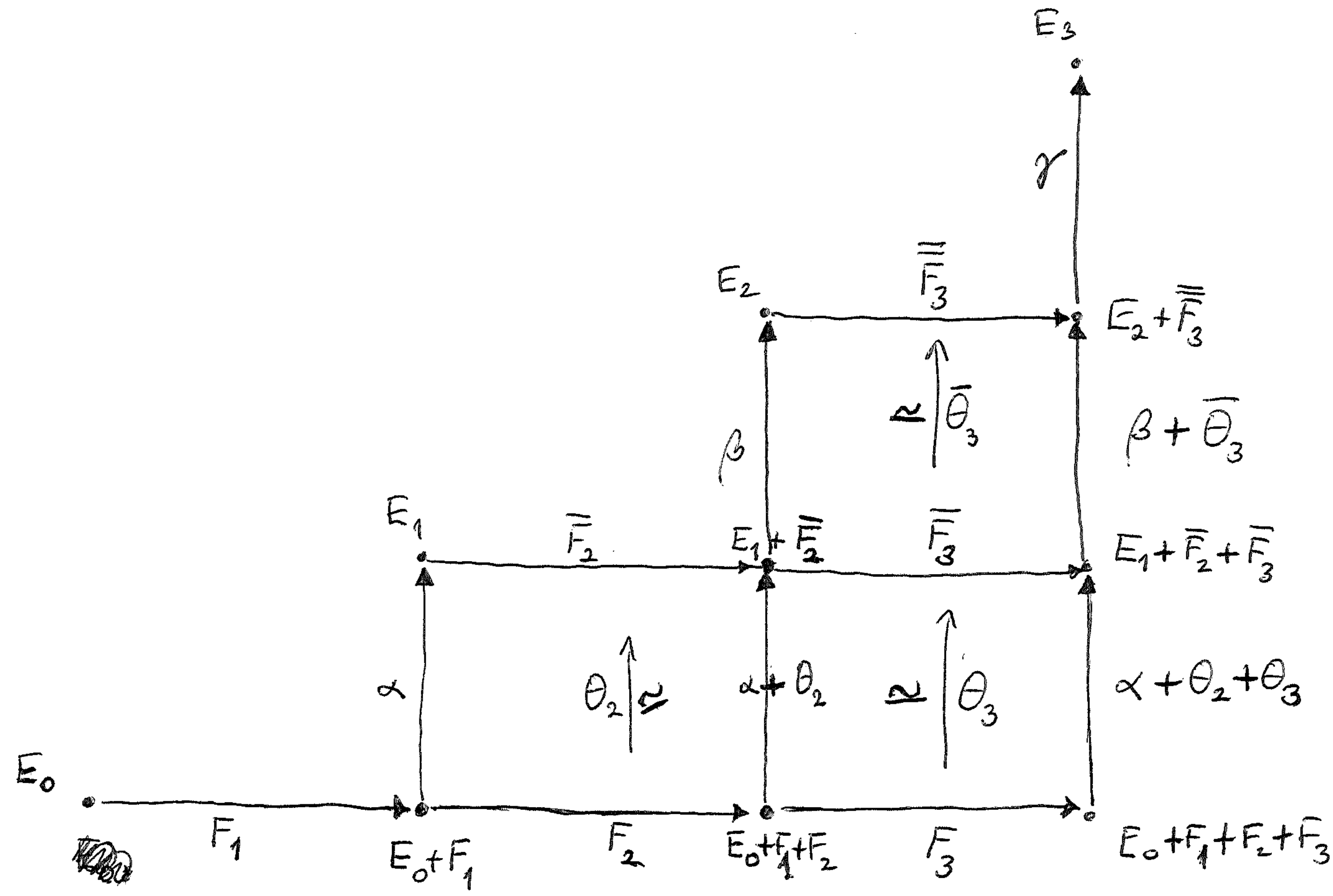
$$\gamma'' : E_0'' + \bar{F} \xrightarrow{\sim} E_1''$$

Perhaps it would be better to write this



Now ~~that~~ a three simplex looks like the scheme on the following page.

~~[illegible]~~



where we have abbreviated (E_i', E_i'') to E_i, and (α', α'') [redacted] to α, etc.

It is clear that an n-simplex is simply a bunch of segments and squares fitting together, hence $AM(X)$ is equal to its [redacted] 2-coskeleton.

It seems reasonable to conjecture that $AM(X)$ is homotopy equivalent to [illegible] its simplicial subset which is equal to its 2-coskeleton and which has the same 0 & 1 simplices and only those 2-simplices for which $F = \bar{F}$ and the θ-ism: $\theta : F \to \bar{F}$ is the identity. If this conjecture were true, then by Artin-Mazur one has that $X \sim AM(X)$, while by the conjecture $AM(X) \sim$ sub-s.set = nerve of the category with morphisms $(E', E'') + F \xrightarrow{\sim} (E'_0, E''_0)$. Hence we would have a good proof that this category has the homotopy type of $BGL(R)^+$.

June 28, 1971

From preceding day we can be more certain that $B(M\times M, M)$ is homotopy equivalent to the category of pairs (E', E'') in which a morphism from (E', E'') to (E'_0, E''_0) is a pair of complemented injections $E'+Q'\simeq E'_0$, $E''+Q''\simeq E''_0$ together with an isom. of $Q'\simeq Q''$. Also we can identify $B(M,M')$ up to homotopy with the category whose objects are k-spaces E in which a morphism from E to E_0 is a complemented direct injection $E+Q\simeq E_0$ together with a reduction $Q\simeq\sigma^*Q$ of Q to $\mathbb{F}_q$.

Unfortunately the functor $B(M,M')\longrightarrow B(M\times M,M)$ in these terms did not appear to be relatively filtering.

Acyclicity of the map $f : \underline{\underline{J}} \longrightarrow \Gamma$:

It suffices to show that if L is an injective Γ-sheaf, then $R^q f_*(f^*L) = 0$ for $q > 0$ and $= L$ for $q = 0$. We consider the standard resolution of the final object of $J^\sim$:

(*) $$Ar_2 J \;\overrightarrow{\overrightarrow{\longrightarrow}}\; Ar_1 J \;\overrightarrow{\longrightarrow}\; Ob\, J.$$

(It is the nerve of $(Ar\, J, J)$.) Since for any sheaf X over $Ob\, J$ we have an equivalence

$$J^\sim/X \times_{Ob\, J} Ar\, J \longrightarrow (Ob\, J)^\sim/X \quad (= X^\sim)$$

we obtain on taking $X = Ar_p J$ endowed with the last vertex operator $k_p : Ar_p J \longrightarrow Ob\, J$, an isom.

$$H^q(J^\sim/Ar_{p+1}J \; ; \; j_{Ar_{p+1}J}{}^*(f^*L)) = H^q(Ar_p J \; ; \; k_p{}^*(f^*L)).$$

L injective in Γ implies L injective in $(Ob\, \Gamma)^\sim$, hence f^*L is injective in $(Ob J)^\sim$, hence as k_p is etale, one knows that $k_p{}^*f^*L$ is injective in $(Ar_p J)^\sim$. This means that the above cohomology groups vanish for q 0, hence that the resolution (*) can be used to calculate the cohomology of f^*L:

$$H^q(J^\sim ; f^*L) = H^q(n \longmapsto Hom_{J^\sim}(Ar_{n+1}J;\ f^*L))$$

$$= H^q(n \longmapsto Hom_{\Gamma^\sim}(Ar_n J \times_{Ob\,\Gamma} Ar\,\Gamma ; L)).$$

$$= H^q(Hom_{(\Gamma^\sim)_{ab}}(Z[Ar_* J \times_{Ob\,\Gamma} Ar\,\Gamma];\ L)).$$

As L is an injective Γ-sheaf, we will have

$$H^q(J^\sim ; f^*L) = \begin{cases} H^0(\Gamma \; ; L) & q = 0 \\ 0 & q > 0 \end{cases}$$

provided that the simplicial object of $\Gamma^\sim$

$$Ar_* J \times_{Ob\,\Gamma} Ar\,\Gamma$$

is acyclic. However the stalk of this simplicial object over the point y of $Ob\,\Gamma$ is just the nerve of the category of arrows $y \longrightarrow f(x)$ with x on object of J. This category is equivalent to the fibre category of J over y, whose objects are the x with $f(x) = y$ and morphisms induce the identity of y.

associated to an arithmetic group. In virtue of results of Serre [23], this means that our theorems on the cohomology ring apply to S-arithmetic groups. Some consequences are discussed in section 14.

The numbering of sections and references is continued from part I.

Fix y R
We consider the category of pairs consisting ~~of~~ an object $(x,a_o,..,a_n)$ of J and an arrow $u : y \longrightarrow x$. Morphisms must preserve the u maps so this category is discrete. Consider the full subcategory with x = y and u the identity germ of y. Want to show that this subcategory is equivalent. So given u : y --- x and $(x,a_o,..,a_n)$ it is only necessary to know that u can be extended to an isomorphism of $(y,a_o,..,a_n)$ with $(x,a_o,..,a_n)$.

The next point is the acyclicity of the nerve of the fibre category J. Identification of the fibre category: Start with I_1 consisting of (0,a). It is a monoid category and acts on the category of (x,a) (use x instead of y). One then can form the simplicial category associated to the bicategory (J_{0x},I_1). The nerve of this total category is apt to be a mess. So instead perhaps what one wants to do is to first identify the homology of the nerve of a category with the derived functors of the inductive limit, and then go through the nonsense of the cofibred category homology.

We have the following situation. We are given a simplicial category, we form the total category and we want to know about the homology of the nerve of the total cat. First thing to do is to identify the homology of the categoryand of its nerve.(reference to G-Z.) Let C be a category and consider covariant functors to sets. Have left-derived functors of the inductive limit.

So I want to prove that the homology of a category and of its nerve are the same. One observes that any covariant functor of the form

$$\bigoplus \mathrm{Hom}(\ ,\ )$$

Haefliger structures

Questions:

Instead of considering the stack of Γ-torsors, consider the ~~site~~ category whose objects are manifolds with a codim. 1 foliation, whose arrows ~~are maps ... to the folia. preserving~~ $f: X \to Y$ are C^∞ maps ∋ f is transv. to foliation on Y and ∋ foliation on X induced by f. Then does this site have the same cohomology?

Examples of a sheaf on $\mathbb{R}$ with Γ-action: C^∞ functions, jets, vector fields, tensor fields, etc. In particular, if we want to compute $\mathbb{R}$-cohomology of $B\Gamma$ we might want to use de Rham complex on $\mathbb{R}$.

$$0 \longrightarrow \mathbb{R} \longrightarrow \Omega^0 \longrightarrow \Omega^1 \longrightarrow 0$$

~~This gives~~ By nonsense given any Γ-structure P over X we get a complex of sheaves on X:

$$0 \longrightarrow \mathbb{R}_X \longrightarrow P \times^\Gamma \Omega^0 \longrightarrow P \times^\Gamma \Omega^1 \longrightarrow 0$$

If P comes from a codim. 1-foliation on X, then this should be the complex of functions and forms constant along the leaves.

Haefliger structures again.

~~Hab~~ cocycles.

E

$U \subset X$ — Bott's idea was to assume that E over U becomes foliated? ~~NO~~

$$0 \longrightarrow E \longrightarrow T_X \longrightarrow Q \longrightarrow 0$$

Pont $Q \subset H^*(X)$ vanishes above a certain degree.

idea is that Q develops a partial connection

~~[illegible]~~

$$(\Gamma E, \Gamma E) \subset \Gamma E$$

hence $s \in \Gamma E$, $t \in \Gamma(Q)$

$$\pi u = t$$

$$\pi[s, u] \in \Gamma(Q)$$

well-defined

$$\Gamma E \times \Gamma Q \longrightarrow \Gamma Q$$

$$s \quad t \quad \nabla_s(t)$$

now the idea is to make this independent of tangent bundle nonsense. Thus ~~[illegible]~~ the point is one has Q which has a Haefliger structure. ~~[illegible]~~

The point is that one has a Haefliger bundle Q and Bott proves that Pont Q vanishes in dim $> 2q$. Thus he is ~~doing something about~~ showing the map

$$H^i(BO_q) \longrightarrow H^i(B\Gamma_q)$$

is zero for $i > 2q$. ~~[illegible] is completely [illegible] the point [illegible]~~ There is a nice trick to get straight here — thus he proves a theorem about ~~a vector bundle~~ a quotient of the tangent bundle. ~~Next he establishes a criterion~~ But this implies a result about a Haefliger structure, i.e. given $f: Q \longrightarrow X$ submersion with foliation transversal to fibres $\exists$ on Q exact sequence

~~$0 \longrightarrow E \longrightarrow T_Q \longrightarrow f^*T_X \longrightarrow 0$~~

~~with $E = T_f$ so one knows that~~ ?

~~Now start by trying to induce~~

~~Confusing Yes.~~

~~$T_Q = f^*T_X \oplus f^*Q$~~

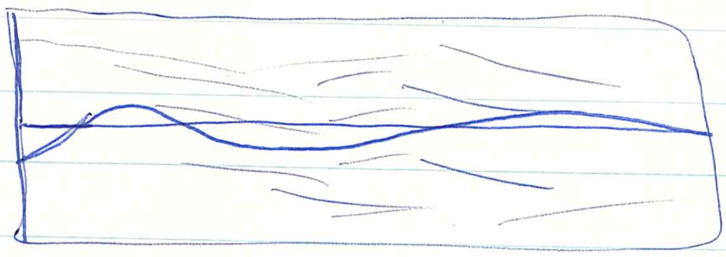
$$0 \longrightarrow E \longrightarrow T_Q \longrightarrow f^*Q \longrightarrow 0$$

Philips-Gromov theorem: transversality thm.

$$\text{Hom}_{\text{surj}}(T(X), T(Y)) \longleftarrow \text{Hom}_{\text{nice}}(X, Y).$$

look this way. you have $Q \downarrow X$ foliation transversal to fibres

you want a map $X \longrightarrow Q$ section which is transversal to foliation i.e. such that $T(X) \longrightarrow Q$ onto.



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transversal to fibres.

$$X \xrightarrow{s} Q \qquad f : Q \longrightarrow X.$$

$$T(X) \longrightarrow T(Q) \twoheadrightarrow f^*Q$$

so you want to see that $T(X) \twoheadrightarrow f^*Q$ is possible

point was that given Q foliated then space of maps $X \longrightarrow Q$ transversal to foliation same homotopy type as the surjective bundle maps $T(X) \longrightarrow T(Q)/E$

where ~~The~~ is tangent space to the foliation. Thus Bott thm $\Longrightarrow$ Pont(f^*Q) vanishes, but f heq $\Longrightarrow$ Pont(Q)

---

refinement of the idea. Assume X, U and a bundle on X which picks up a Haefliger st. on U.

$$\begin{array}{ccc} U & \longrightarrow & B\Gamma_q \\ \downarrow & & \downarrow \\ X & \longrightarrow & BO_q \\ \downarrow & & \\ (X, U) & & \end{array}$$

then he defines elements in $H^{4i}(BO_q, B\Gamma_q)$ for $4i > 2q$ because the

$$H^{4i-1}(B\Gamma_q) \longrightarrow H^{4i}(BO_q, B\Gamma_q) \longrightarrow H^{4i}(BO_q) \xrightarrow{\ 0\ } H^{4i}(B\Gamma_q)$$

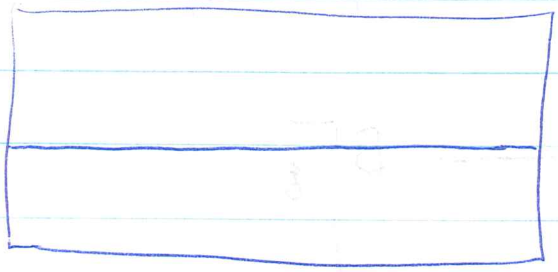
I conjecture that he should be able to construct elements in $H^{4i-1}(B\Gamma_q)$ with some work.

theorem of Milnor $\quad B\Gamma_q \longrightarrow BO_q \quad \sim q$-connected

idea was to take $S^i \longrightarrow B\Gamma_q$

and look at as a Haef. structure on $S^i \times \mathbb{R}^q$

foliation on $S^i \times \mathbb{R}^q$



~~form disk in $R^i - 0$~~

$(R^i - 0)$ foliation here of codim q

trivial normal bundle

$GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \times \mathbb{R}^n \rightrightarrows GL_n(\mathbb{R}) \tilde{\times} \mathbb{R}^n \Longrightarrow \mathbb{R}^n$!!!!!

$H^*_{GL_n(\mathbb{R})}(\mathbb{R}^n$

pseudogroup affine transformations here! $\quad F_1, F_2, \dots, F_m$.

objects $\mathbb{R}^n$

pseudogroup of affine transformations.
affine transf.

Toward an understanding of Ω:

$$\underline{\Omega X} \longrightarrow PX \longrightarrow \underline{X}$$

ΩX some sort of space or topos with assoc. inv. composition hence must be understand in terms of a s. object

$$B(\Omega X): \quad \underline{(\Omega X)^2} \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \underline{\Omega X} \rightrightarrows pt.$$

Why is the realization of

$$\Omega_2 X \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \Omega X = pt$$

homotopy equivalent to X when $\pi_0 X = pt$.

Given a (reduced) simplicial space $X_\bullet$ there is a basic adjunction arrow

$$X_\bullet \longrightarrow \Omega|X_\bullet|$$

which we want to show is a heq. when $X_\bullet$ is special + $\pi_0 X_1$ is a group. The method consists of establishing spectral sequences with the same abutment then using comparison thm.

Given an open ~~set~~ $U \subset \mathbb{R}_{<0}$

$\mathcal{F}(U) =$ smooth functions of on

$\{(x,y) \mid x \in U \quad x \leq y \leq 0\}$

$\mathcal{G}(U) =$ smooth functions on $[x, 0]$

map

$$\mathcal{G}(U) \longrightarrow \mathcal{F}(U)$$

$$g \longmapsto \big((x,y) \longmapsto g(x) \big)$$

NO further!!!!!

---

formal groupoid approach.
idea here is to take Γ and formally complete along origin.

~~[illegible]~~ $\mathcal{O}$ functions on line
i.e. if you take

$$\mathrm{Ar}\,\Gamma \longleftarrow \mathrm{Ob}\,\Gamma .$$

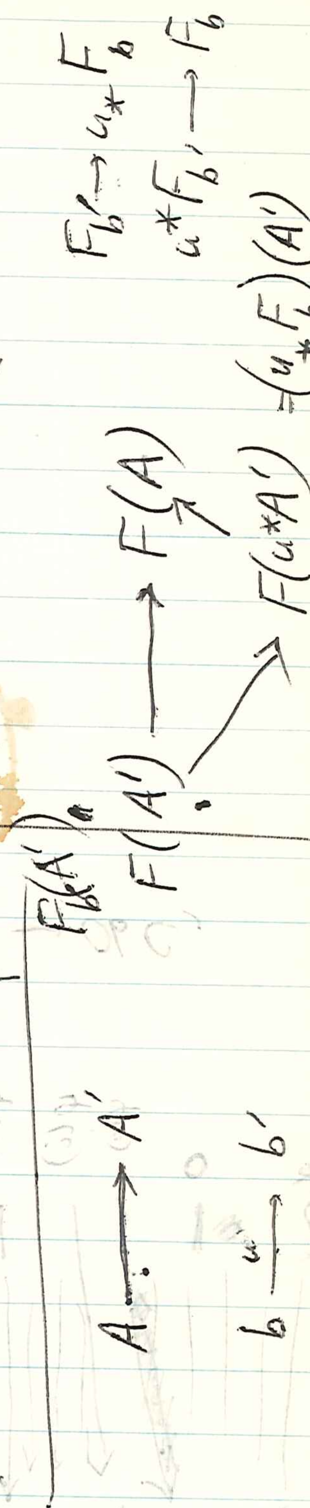
must consider (smooth) functions on $\mathrm{Ar}\,\Gamma$ relative to source say

example: $G \times X \quad X$

then want $\mathcal{O} \longrightarrow \mathcal{O} \otimes \mathfrak{g}^* \longrightarrow \mathcal{O} \otimes \Lambda^2 \mathfrak{g}^*$

$$C^*(\mathfrak{g}, \mathcal{O})$$

In general should be the same, namely $\mathfrak{g}$ sheaf of (germs of) vector fields on X. Then take $C^*(\mathfrak{g}, \mathcal{O})$

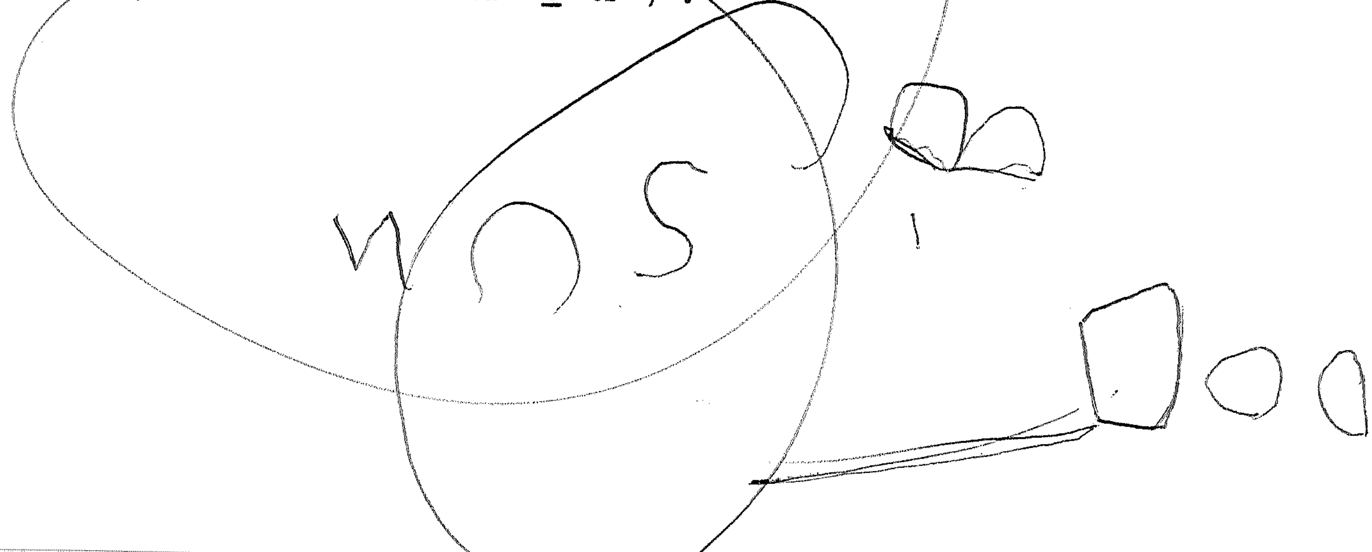


MXXKKXXX Question: Can you say anything about the Lie algebra of vector fields on R^1 with compact support? Form something like a simplicial Lie algebra. Now a Lie algebra Maybe should be thought of as a kind of formal groupoid, or rather as having a formal groupoid for its classifying space. Better: Lie algebra = formal group. The analogy to follow is to recast everyhing for formal groupoids, and then try to carry out the Mather theorem. Thus

The analogy: Think of a Heafliger structure as having a formal completion along the distingushie section. NO.

The analogy. Let G be an algebraic group, and G* its completion at the identity. G is an fpqc sheaf, and possibly also is G*; in fact G* is axlimitxof an inductive limit of representable sheaves, although not sheaves of groups. There is a canonical map G*---G. Now take the groupoid Γ; it should have a formal completion, and a map Γ^* Γ. THIS thereisatbig. The problem is to define the cohomology of this formal completion and to understand its torsors, etc. The hope somehow is that there might be an analogue of Mather's theorem involoving the vector fields on R with compact support. The idea I have is that there isxaxex it should be possible to kill the fundamental group of the Lie algebra, then I can form some king of group which should have a classifying space with which should be the completion of Γ. In any case we can speak of the homology of the Lie algebra of vector fields wiht compact support, and once we have a formal classifying space we can ask whether or not there it has the same homology.as this Lie algebra. Preliminary questions- does the homology of the Lie algebra have a Pontryagin product structure? Does the group of diffeomorphisms of the line act trivially on ~~the homology of the Lie algebra~~.??? Thexschemexgeneralxschemexisxsie

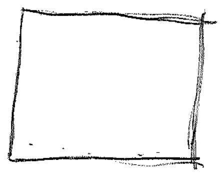
Cohomology of the structural sheaf $\underline{O}$ of Γ.



Mather theorem:

~~Γ~~ Γ, $\mathbb{R}$, Θ clean things
topological ~~categories~~ groupoids
or stacks.

~~In 2 dimensions~~: $\Omega^2(B\Gamma)$ parameterizes (foliated) $I \times I$ bundles with flat foliation along the edges.



so $\exists$ canonical map $BG_c(I \times I) \longrightarrow \Omega^2(B\Gamma_2)$ which we, of course, conjecture is an isomorphism, ~~[illegible]~~ ~~One might ask what is lost~~ no $\Omega^2(F\Gamma_2)$ where

$$F\Gamma_2 \longrightarrow B\Gamma_2 \longrightarrow BO_2$$

is the fibre. Now one might ask what we lose in going from $F\Gamma_2$ to $\Omega^2 (F\Gamma_2)$. Answer: only the π_0 and π_1.

$$F\Gamma_1 \times F\Gamma_1 \longrightarrow F\Gamma_2$$

$$\Sigma G_1 \qquad \Sigma G_1$$

---

so given an $F\Gamma_2$-structure,

Γ_2 st. $x \in \mathbb{R}^2$
$y \in \mathbb{R}^2$ germs of diffeos.
$g: X \longrightarrow \mathbb{R}^2$ $\Theta: x \longrightarrow y$
$f: X \longrightarrow \mathbb{R}^2$ $\ni d\Theta = id$ at x.
want h_x germ of diffeo

If I want to understand Γ_2-structures I need first to understand the product map.

May should first try to ~~[illegible]~~ understand Ω^2

$\Omega X \longrightarrow \Lambda X \longrightarrow X$. $\quad \Lambda X$ = paths ~~[illegible]~~ $x \longrightarrow x_0$ where x_0 is basepoint, and ΩX paths $x_0 \longrightarrow x_0$.

~~hence need to know Haefliger structures along the~~

maybe the idea is that for Γ_n suitable to use non-degenerate manifolds of dim. n. don't use arcs but instead use 2-manifolds, ~~better~~ germs of.

~~So the next thing is to count up the situation in so far~~

One thing that perhaps will be important to ~~[illegible]~~ us is the category whose objects are points (x_1, x_2) with $x_1 < 0$ and whose ~~[illegible]~~ morphisms are the ~~[illegible]~~ germs of diffeos. ~~[illegible]~~ of the segment (x_1, x_2) —— $(0, x_2)$

(y_1, y_2) —— $(0, y_2)$

which preserve the x coordinate at the ends.

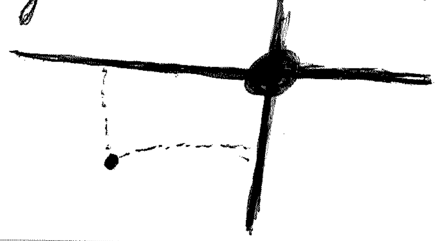
~~A foliation on a [illegible]~~

<u>Situation</u>: ~~Let [illegible]~~ $\begin{array}{c} E \\ \downarrow \\ X \end{array}$ diff. microbundle with foliation trans. to fibres.

then E trivialized as microbundle

$X \times \mathbb{R}^2$ foliations transversal to fibres

then want to join to trivial structure



convex coordinates

P

July 9, 1971: Friedlander's thesis

Recall the Verdier theorem. Let X be a topos, F an abelian object of X, and let U be a hypercovering of the final object e. There is a spectral sequence

$$E_2^{pq} = H^p(\nu \longmapsto H^q(U_\nu; F)) \Longrightarrow H^{p+q}(X; F).$$

Consider the category ($HR(X)$) of hypercoverings with homotopy classes of maps of simplicial objects as morphisms.

Assigning to U the above spectral sequence defines a spectral sequence functor on $HR(X)^\circ$. One shows that $HR(X)^\circ$ is filtering and that

$$\varinjlim_U E_2^{p+}(U) = 0,$$

whence the edge homomorphism becomes an isom. in the limit

$$\varinjlim_U H^p(\nu \longmapsto \Gamma(U_\nu; F)) \xrightarrow{\sim} H^p(X; F).$$

To simplify the following we work with spaces instead of topoi.

Let $f: X \longrightarrow Y$ be a map and F an abelian sheaf on X. We then have the Leray spectral sequence

$$E_2^{pq} = H^p(Y; R^q f_*(F)) \Longrightarrow H^{p+q}(X; F)$$

We ask whether it is possible to compute

$R^q f_*(F)$ simplicially. Thus given a hypercovering $\mathcal{U} = \{U_\nu\}$ of X let $a_{U_\nu}: U_\nu \to X$ be the augmentation map. There is a spectral sequence

$$E_2^{pq} = H^p(\nu \longmapsto R^q(fa_{U_\nu})_* a_{U_\nu}^* F) \Longrightarrow R^{p+q} f_*(F)$$

obtained as follows. Let $I^\cdot$ be an injective resolution of F and consider the double complex

$$f_* \underline{Hom}(\mathbb{Z}_{U_p}, I^q) = (fa_p)_* a_p^*(I^q) \qquad a_p = a_{U_p}$$

of abelian sheaves on Y. Compute homology wrt p keeping q fixed: Then $p \mapsto \underline{Hom}(\mathbb{Z}_{U_p}, I^q) = a_{p*} a_p^* I^q$ is an injective complex as a_p^* preserves injectives; moreover it resolves I^q, hence is homotopy equivalent to I^q. $p \mapsto f_* \underline{Hom}(\mathbb{Z}_{U_p}, I^q)$ is therefore an injective resolution of $f_* I^q$, so spectral sequence degenerates, showing that the total complex has homology $R^q f_*(F)$. The other spectral sequence gives the spec. sequence above. (Alternative approach is to use that for any $V \subset Y$, $\mathcal{U} \times_Y V = \mathcal{U} \times_X f^*V$ is a hypercovering of f^*V, hence there is a spectral sequence of presheaves on Y

$$E_2 = H^p(\nu \longmapsto H^q(U_\nu \times_X f^*V; F)) \Longrightarrow H^{p+q}(f^*V; F)$$

and then take the spectral sequence of associated sheaves.)

Now we want to know if the limit of E_2^{p+} as $\mathcal{U}$ runs HR(X) is zero. This would be proved by showing that given a surjective map $U_\nu \to X$ and $V \hookrightarrow Y$ and $x \in H^q(U_\nu \times_X f^*V; F)$, we can kill χ by refining U_ν and V. However keeping V fixed we can ask if it possible to kill χ by refining U_ν. Thus if $j: f^*V \to X$ is the inclusion the sheaf on X associated to the presheaf

$$U \longmapsto H^q(U \times_X f^*V; F)$$

is

$$R^q j_* (j^*F).$$

~~[illegible]~~ (Question: ~~Is~~ it true that for any presheaf G on X we have

$$\varinjlim_{\mathcal{U}} \check{H}^n(\mathcal{U}, G) = H^n(X, \tilde{G}) ?$$

Yes, by Verdier's appendix

If this were so then the limit [redacted] over $\mathcal{U}$ of the spectral sequence at the bottom of page 2 ought to be the Leray spectral sequence for $j: f^*V \longrightarrow X$ and j^*F.)

So the moral is this: Hypercoverings [redacted] of X are not sufficiently fine to compute ~~[illegible]~~ $H^q(Z; F)$, $Z \to X$ etale, and probably not sufficiently fine to compute $R^q f_*(F)$.

What you know about stability

A = field k. If V is a vector space over k, let $U(V)$ be the simplicial complex of independent vectors in V. Assuming k infinite it follows that $U(V)$ is a bouquet of $(n-1)$-spheres if $\dim(V)=n$. In effect any finite subcomplex of $\dim < n-1$ is contained in a cone. Now write down the complex of chains

$$0 \to C_{n-1}(U(V)) \longrightarrow C_1(U(V)) \longrightarrow C_0(U(V)) \to \mathbb{Z} \to 0$$

where $C_i(U(V)) \cong \mathbb{Z}[\mathrm{Aut}(V)] \otimes \mathbb{Z}[\text{stab of } i\text{-simplex}]\ \mathbb{Z}^{\text{sign}}$

$V = k^n$ stand. basis $e_1, \dots, e_n$. Then

$$C_i(U(V)) = \mathbb{Z}\left[GL_n(k) / \text{stab. of } e_1 \dots e_{i+1}\right]$$

$$\text{stab. of } e_1 \dots e_{i+1} = \left(\begin{array}{c|c} \Sigma_{i+1} & 0 \\ \hline * & * \end{array}\right), \quad * \uparrow GL_{n-i-1}$$

so we get a spect. seq.

$$E^1_{pq} = H_q\left(\left(\begin{array}{c|c} \Sigma_p & 0 \\ \hline * & GL_{n-p} \end{array}\right), \mathbb{Z}^{\text{sign}}\right) \Longrightarrow \text{begins in dim } n.$$

now replace $\mathbb{Z}$ by $\mathbb{Q}$, ~~suppose~~ The $*$ goes by unstable splitting thm, and rest goes because Σ_p works on sign. So get that C_p are coh. trivial for $p \geq 2$.

Introduce operators T from D^- to D^-.

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so there is some hope left that the situation will improve. for each n we have $Sd^n(0<1)$ and

---

So let me start with $Sd^n \leftrightarrow Sd^{n+1} \leftrightarrow Sd^{n+2} \to \cdots$

$$\mathcal{M}_n(\mathcal{C}) = \underline{Hom}(Sd^n(0<1)^\circ, \mathcal{C}) \qquad n \geq 1.$$

and the result is that we can try to take the limit over n of this. But there isn't much to do at first. Thus one gets a sheaf.

2 possibilities for taking limits.

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get nothing but the standard loop space

Question: When might $\Phi(F)$ be locally constant for all F?

Or when might $\mathcal{M}_\Delta \to \mathcal{C}\times\mathcal{C}$ be cofibred with heq fibres. ~~Thus to show~~ In this case

$\Delta/(X,Y)$ would have the h-type of the paths joining X to Y.

$$\Phi(\mathbb{Z}[Hom(A,?)])(X) = \varinjlim_{X \leftarrow Z \to Y \leftarrow A} \mathbb{Z}$$

$$\mathbb{Z}[Hom(A,Y)] = \varinjlim_{A \to Y} \mathbb{Z}$$

for Φ to make every F loc. const means that if A is fixed, then for any $X \to X'$ we have that

$$(\text{paths } X \leftarrow Z \to Y \leftarrow A) \longrightarrow (\text{paths } X' \leftarrow Z \to Y \leftarrow A)$$

is a homotopy equivalence.

$M_\Delta \to \mathcal{C} \times \mathcal{C}$ fibration means that $\forall\ X \to X'$, Y fixed

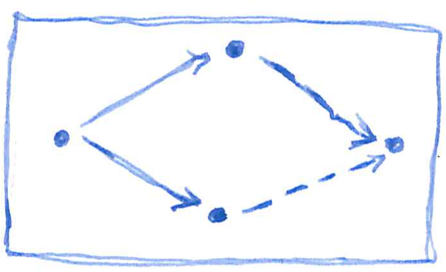
$$(\text{paths } X \leftarrow Z \to Y) \longrightarrow (\text{paths } X' \leftarrow Z \to Y)$$

is a heq.

~~Ok~~ Probably second implies the first.

Suppose now that ~~$X \leftarrow Z \to Y$~~

we try category of pairs (V^+, V^-) and see if this condition holds in this case. The point is that I know we can complete



$$\begin{array}{ccc} & X\oplus A & \\ X & & X\oplus A\oplus B \\ & X\oplus B & \end{array}$$

in a functorial way, which means that I have a functorial way of collapsing multiple paths into simpler ones only partially complete, the goal being to prove that $BGL(A)^+$ is the connected component of $\Omega B \coprod_n BGL_n A$ by producing a map + showing it gives $\cong$ on homology.

§5 $BGL(A)^+$ as an infinite loop space.

In [ ] Graeme Segal has simplified and generalized the Boardman–Vogt theory of "homotopy-everything" H-spaces. In particular he has shown that to every small category $\mathcal{A}$ with a coherently commutative associative and unitary internal operation there is canonically associated an Ω-spectrum. In this section we show that when $\mathcal{A}$ is the category of finitely generated projective A-modules with the direct sum operation then the zero-th space of Segal's spectrum is homotopy equivalent to $K_0A \times BGL(A)^+$. On one hand this shows $BGL(A)^+$ is an infinite loop space and on the other it puts the algebraic K-theory of this paper into a broader context.

We begin by reviewing Segal's definition. Denote the given operation of $\mathcal{A}$ by $\oplus$ and the given neutral object by 0. For each $n \geq 0$ define a category $\mathcal{A}_n$ as follows. An object of $\mathcal{A}_n$ is a function which assigns objects P_σ of $\mathcal{A}$ to each subset σ of $\{1, \dots, n\}$ and isomorphisms $P_\sigma \oplus P_{\sigma'} \xrightarrow{\sim} P_{\sigma \cup \sigma'}$ to each pair σ, σ' of disjoint subsets such that $P_\emptyset = 0$ and such that these isomorphisms are compatible with given associativity, commutativity and unit isomorphisms of $\mathcal{A}$. Morphisms in $\mathcal{A}_n$ are isomorphisms of such functions. An object of $\mathcal{A}_n$ may be thought of as a collection $\{P_\sigma\}$, $\sigma \subset \{1,..,n\}$, of objects of $\mathcal{A}$ together with the data expressing P_σ as the direct sum of $P_{\{i\}}$ for i running over σ, hence the functor

$$\mathcal{A}_n \longrightarrow (\mathcal{A}_1)^n$$

which assigns to $\{P_\sigma\}$ the family $P_{\{1\}}, \ldots, P_{\{n\}}$ is an equivalence of categories.

~~Let Γ be the category whose objects are the integers $n \geq 0$ with $u \in \mathrm{Hom}_\Gamma(m,n)$ defined to be a partition $\{1,..,n\} = \sigma_1 \amalg \ldots \amalg \sigma_m$, or equivalently a map from subsets of $\{1,..,m\}$ to subsets of $\{1,..,m\}$ preserving empty sets and disjoint unions. To u we associate the functor $u^*: \mathcal{A}_n \to \mathcal{A}_m$ which sends $\{P_\sigma\}$ to $\{P_{u\tau}\}$ where if $\tau \subset \{1,..,m\}$ then~~

Let Γ be the category whose objects are the integers $n \geq 0$ with $u \in \mathrm{Hom}_\Gamma(m,n)$ defined to be a map $\tau \mapsto u\tau$ from subsets of $\{1,..,m\}$ to subsets of $\{1,..,n\}$ preserving empty subsets and disjoint unions. To u we associate the functor $u^*: \mathcal{A}_n \to \mathcal{A}_m$ sending $\{P_\sigma\}$

to $\{P_{u\tau}\}$. It is clear that $n \mapsto \mathcal{A}_n$ is a functor from Γ° to Cat, the category of small categories. Let $B\mathcal{C}$ denote the geometric realization of the standard simplicial set associated to the category $\mathcal{C}$. Then $n \mapsto B\mathcal{A}_n$ is a functor from Γ° to spaces i.e. it is a Γ-space in Segal's terminology, in fact it is a special Γ-space, i.e. $B\mathcal{A}_0 = pt$ and $B\mathcal{A}_n$ is homotopy equivalent to $(B\mathcal{A}_1)^n$.