January 1, 1971:

Complements to the preceding:

1) One might hope to use it to get a hold on the cohomology of p-groups. Thus if G is a p-group and Z is a cyclic order p central subgroup one chooses a representation V of G whose restriction to Z has no trivial component. If S is the sphere, then $e(V)$ is a non-zero divisor in H^*_G and

$$H^*_G/(e(V)) = H^*_G(S) = H^*_{G/Z}(S/Z)$$

Now if we choose a cyclic order p central subgroup Z' of G/Z and denotes by $B_1, \ldots, B_k$ those cyclic $[p]$-subgroups mapping isomorphically onto Z', we have exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_i H^*_G(S^{B_i}) \arrow[r, "i_*"] \arrow[dr, hook] & H^*_G(S) \arrow[r, "j^*"] \arrow[d] & H^*_G(S - \cup S^{B_i}) \arrow[r] & 0 \\
& & \bigoplus_i H^*_G(S^{B_i}) & &
\end{tikzcd}$$

~~But the problem vanishes~~ (this is what corresponds to $(S/Z)^{Z'}$ in S/Z.) But the problem is to lift this exact sequence back up to an exact sequence for $H^*(G)$.

2) Example of a p-group G such that $J(G)$ as defined by Thompson or Gorenstein differ. Take $G = \mathbb{Z}/p\mathbb{Z} \tilde{\times} \mathbb{Z}/p^2\mathbb{Z}$, $x^p = y^{p^2} = 1$, $xyx^{-1} = y^{1+p}$. Then every subgroup is abelian and the maximal ones are of order p^2, so Gorenstein's $J(G) =$ subgroup generated by A of maximal order is all of G. But $\langle x, y^p \rangle$ is the unique abelian subgroup with two generators, so it is the Thompson $J(G)$.

Computation of the cohomology of this group. Use Hochschild-Serre for extension

$$1 \to \langle y \rangle \longrightarrow G \longrightarrow \langle x \rangle \longrightarrow 1$$

which splits

u^p				
vu^{p-1}	b			
$\cancel{u}$	$\times$	$\times$	$\times$	$\times$
vu	bvu	$\cancel{avu}$	$\times$	$\times$
$\times$	$\cancel{bu}$	$\cancel{au}$	$\times$	$\times$
v	bv	$\cancel{av}$	$\cancel{ba}v$	$\cancel{a^2}v$
1	b	a	ba	a^2

i) u^p is an infinite cycle because if one induces a character of $\langle y \rangle$ non-trivial on y^p one get a repn. V whose Euler class restricts to u^p.

ii) No differentials come into bottom row as extension splits, hence $d_2 v = 0$.

iii) $d_2 u = 0$ impossible, because then $d_3 u = 0$ by ii)

And so u is an infinite cycle. This means that There is a central extension

$$1 \to Z \to \tilde{G} \xrightarrow{\pi} G \to 1$$

where Z cyclic order p such that $\pi^{-1}\langle y^p\rangle$ is cyclic order p^2. If $\pi\tilde{y} = y$, then $\tilde{y}^{p^3} = 1$; if $\pi\tilde{x} = x$ then

$$\tilde{x}^p = \tilde{y}^{p^2\alpha}$$

$$\tilde{x}\tilde{y}\tilde{x}^{-1} = \tilde{y}^{1+p+\beta p^2}$$

But

$$1 = \tilde{x}^p\tilde{y}\tilde{x}^{-p} = \tilde{y}^{(1+p+\beta p^2)^p}$$

and

$$(1+p+\beta p^2)^p \equiv 1 + p(p+\beta p^2) + p\left(\tfrac{p-1}{2}\right)(p+\beta p^2)^2$$

$$\equiv 1+p^2 \qquad \text{mod } p^3$$

so we reach a contradiction.

iv) thus $d_2 u = \lambda a v$, $\lambda \neq 0$, and so row $t=2$ kills the $t=1$ row except for v and bv.

v) $d_2 u^i = i u^{i-1}\lambda a v = (i\lambda)\, a v u^{i-1}$ $\qquad i\lambda \neq 0$ if $1 \le i \le p-1$ so the $t = 2i$ row kills all of the $t = 2i-1$ row except for vu^{i-1} and bvu^{i-1}, for $1 \le i < p$.

vi) All other differentials have nowhere to go so are zero.

~~Thus we obtain the following basis for E_∞~~

~~vi) Another proof of iii): If $d_2 u = 0$, then u is an infinite cycle and the spectral sequence degenerates~~

~~The spectral sequence for $0 < t \leq 2p-1$ is that of the G-space SV, V being as in 1. By the P.A. finitely theorem~~

$$\cancel{H_G(SV) = H_{\langle x\rangle}(}$$

vii) A way of checking this computation is as follows. The spectral sequence we are looking at is that of

$$H^*_G(SV) = H^*_{\langle x\rangle}(SV/\langle y\rangle)$$

where V is the representation induced from $\chi : \langle y\rangle \longrightarrow \mathbb{C}^*$ which is non-trivial on $\langle y^p\rangle$. (This procedure gives an irreducible representation as the degree is p and one gets at least $p-1$ different ones; these are all the non-abelian repns as

$$\underbrace{(p-1)}_{\text{no}}\ \underbrace{p^2}_{\text{deg}} + \underbrace{p^2}_{\text{no}}\cdot\underbrace{1}_{\text{deg}} = p^3 = \text{order.})$$

for repns of $G/\langle y^p\rangle$.

Now $SV/\langle y\rangle$ has $\langle x\rangle$-fixpoints hence contains two copies of $H^*_{\langle x\rangle}$ correspond to the images of f^* and i_*

$$SV\langle y\rangle \underset{i}{\overset{f}{\rightleftarrows}} pt$$

These are the elements corresponding to the two infinite rows. Now the fixpoint set $(SV/\langle y\rangle)^{\langle x\rangle}$ is covered by the fixpoint sets of the complementary subgroups $\langle x y^{ap}\rangle$ $a = 0, 1, \ldots, p-1$. Some notation

$\pi : SV \longrightarrow SV/\langle y\rangle$. Then

$$\pi^{-1}\left\{(SV/\langle y\rangle)^{\langle x\rangle}\right\} = \coprod_{a=0}^{p-1} SV^{\langle xy^a\rangle}$$

These subgroups $\langle xy^a\rangle$ are conjugate under y, so

$$H^*_{G/\langle y\rangle}\left((SV/\langle y\rangle)^{\langle x\rangle}\right) = H^*_G\left(\coprod_{a=0}^{p-1} SV^{\langle xy^a\rangle}\right)$$

$$= H^*_{\langle x, y^p\rangle}\left(SV^{\langle x\rangle}\right)$$

Now V is the regular representation of $\langle x\rangle$ so $V^{\langle x\rangle}$ is one-dimensional.

$$H^*_{\langle x, y^p\rangle}\left(SV^{\langle x\rangle}\right) = H^*_{\langle x\rangle} \otimes H^*(\underbrace{SV^{\langle x\rangle}/\langle y^p\rangle}_{\parallel \atop S^1})$$

This also shows $d_2 u = 0$ is impossible because one would otherwise get the wrong rank for $H^*_{\langle x\rangle}(SV/\langle y\rangle)$.

So at this point I am satisfied that my computations are correct and want to see if the cohomology is detected by primary subgroups.

Looking at the formula for E^∞ we see that $H^{2p-1}(G)$ is of rank 2 with

$$0 \underset{1}{\subset} F_{2p-1} H^{2p-1} = F_1 H^{2p-1} \underset{1}{\subset} H^{2p-1}$$

and where $F_{2p-1} H^{2p-1}$ is generated by ba^{p-1}.
Precisely we have an isomorphism

$$H^{2p-1}(G) \xrightarrow{\sim} H^{2p-1}(\langle x \rangle) \oplus H^{2p-1}(\langle y \rangle)$$

Let $\gamma \in H^{2p-1}(G)$ be the element such that $\gamma | \langle x \rangle = 0$ and $\gamma | \langle y \rangle = vu^{p-1}$. Then

i). multiplication by γ

$$\begin{array}{ccc} H^*(\langle x \rangle) & \longrightarrow & H^*(G) \\ \alpha & \longmapsto & (\pi^* \alpha) \cdot \gamma \end{array}$$

is injective. In other words the $b^{\varepsilon} a^{i} \gamma$ are all non-zero in $H^*(G)$.

ii). What is $\gamma | \langle x, y^p \rangle$?

Let us consider the analogue of the representation V over a finite field $\mathbb{F}_q$ having $v_p(q-1) = p^2$, i.e. choose $q \equiv 1 \ (p^2)$ using Dirichlet's theorem. Then one takes a faithful character $\chi : \langle y \rangle \xrightarrow{\sim} \mu_{p^2} \subset \mathbb{F}_q$ and makes V the induced representation. Now I know that

$$c_1(\chi) = u + v\varepsilon$$

u generates $H^2(\langle y \rangle)$
v ——— $H^1(\langle y \rangle)$

and

$$V | \langle y \rangle = \bigoplus_{a=0}^{p-1} \chi^{(1+p)^a} .$$

But $(1+p)^a \equiv 1 + ap \quad (p^2)$

so
$$V|\langle y\rangle = \bigoplus_{a=0}^{p-1} \chi^{\otimes(1+ap)}$$
$$c_1(\chi^{\otimes j}) = j\, c_1(\chi)$$

so
$$\begin{aligned} c(V)|\chi &= \prod_{a=0}^{p-1} 1+(1+ap)c_1(\chi) \\ &= (1+c_1(\chi))^p \\ &= 1+(u+v\varepsilon)^p \\ &= 1+u^p . \end{aligned}$$

and so I don't get the desired element of degree $2p-1$.

January 2, 1971:

The group $G = \mathbb{Z}/p\mathbb{Z} \tilde{\times} \mathbb{Z}/p^2\mathbb{Z}$ is an example of a p-group such that $H^*(G)$ has embedded components. Indeed the subgroup $A = \langle x, y^p \rangle$ is the only maximal [p]-subgroup, so represents the minimal prime. On the other hand A is its own centralizer so the kernel of $H^*(G) \longrightarrow H^*(A)$ is the intersection of the other primary ideals. But $H^1(G) \longrightarrow H^1(A)$ is not injective, so one must have other associated primes than A.

I want now to compute the restriction homomorphism from G to A. So for degrees $< 2p$ we can look instead at the map

$$\begin{array}{ccc} H^*_G(SV) & \longrightarrow & H^*_A(SV) \\ \| & & \| \\ H^*_{\langle x \rangle}(SV/\langle y \rangle) & \longrightarrow & H^*_{\langle x \rangle}(SV/\langle y^p \rangle) \end{array}$$

For general reasons this should be injective after inverting the generator $a \in H^2_{\langle x \rangle}$. Precisely we have a commutative square

$$\begin{tikzcd}
H^*_G(SV) \arrow[r] \arrow[d] & H^*_A(SV) \arrow[d] \arrow[r] & H^*_A\left(\coprod_{a=0}^{p-1} SV^{\langle xy^{pa}\rangle}\right) \arrow[dl] \\
H^*_G\left(\coprod_{a=0}^{p-1} SV^{\langle xy^{pa}\rangle}\right) \arrow[r, "\sim"] & H^*_A(SV^{\langle x\rangle}) &
\end{tikzcd}$$

isomorphism after inverting a.

where the first vertical maps become isomorphisms after inverting a. So the next thing to do is compute $H^*_A(SV)$.

So V as a rep. of $A = \langle x, y^p\rangle$ is the regular repn of $\langle x\rangle$ tensor product with the character χ on y^p whose first Chern class is the generator u of $H^2(\langle y\rangle) \xrightarrow{\sim} H^2(\langle y^p\rangle)$. So

$$e(V)\,|\,A = \prod_{i=0}^{p-1} (ia+u)$$
$$= u^p - a^{p-1}u.$$

Consequently

$$H^*_A(SV) = H^*_A \,/\, (e(V)) = \mathbb{Z}/p\mathbb{Z}\,[b,a,w,u]/(u^p - a^{p-1}u)$$
$$= \Lambda[b,w] \otimes S[a,u]/(u^p - a^{p-1}u)$$

<u>January 3, 1971</u>:

Let $G = \langle x, y \rangle$, $x^p = y^{p^2} = 1$, $xyx^{-1}y^{-1} = y^p$. Then $H^1(G)$ has for basis the elements b, v defined by

$$b(x) = 1 \qquad v(x) = 0$$
$$b(y) = 0 \qquad v(y) = 1.$$

Then $bv \in H^2(G)$ is non-zero by the spectral sequence (see p. 14) yet it vanishes on any cyclic subgroup C as the multiplication $H^1(C) \otimes H^1(C) \to H^2(C)$ is identically zero. In addition it vanishes on the subgroup $\langle x, y^p \rangle$ because v does. <u>Consequently G is an example of a p-group whose mod p cohomology is not detected by primary subgroups.</u>

~~Similarly we know that $H^{2i-1}(G) \to H^{2i-1}(\langle y \rangle)$ is non-zero so if $\gamma_{2i-1} \in H^{2i-1}(G)$ has non-zero restriction to $\langle y \rangle$, then we know from the spectral sequence that $b\gamma_{2i-1}$ is non-zero~~

One might hope to be able to prove something by the following inductive method. Suppose G is a p-group and

$$0 \subset Z_1 \subset Z_2 \subset \cdots \subset Z_n = G$$

is a chief series, i.e. $Z_i \lhd G$ and Z_i/Z_{i-1} cyclic of order p. Then

$$H^*_G(X) \cong \underbrace{H^*_G(X^{Z_1})}_{\text{needs a dimension shift depending on codimension of } X^{Z_1} \text{ in } X.} \oplus H^*_G(X - X^{Z_1})$$

The second factor reduces to a smaller group

$$H^*_G(X - X^{Z_1}) = H^*_{G/Z_1}((X - X^{Z_1})/Z_1)$$

while for the first we have that $e(V)$ is a non-zero divisor and

$$H^*_G(X^{Z_1}) / e(V) H^*_G(X^{Z_1}) = H^*_G(X^{Z_1} \times SV)$$

$$\cong H^*_{G/Z_1}(X^{Z_1} \times (SV/Z_1))$$

where V is any representation of G whose restriction to Z_1 has no trivial components.

For example we can prove the dimension theorem this way because

$$\dim\{H^*_G(X)\} = \max\{\dim\{H^*_{G/Z_1}(X^{Z_1} \times (SV/Z_1))\} + 1, \dim(H^*_{G/Z_1}(X - X^{Z_1}/Z_1)\}$$

and hence we get to a smaller group G/Z_1.

The problem with using this method stems from ignorance of $X - X^{Z_1}$ even when X is nice. Thus in the example before we had to contend with

$$SV - \coprod_{i=0}^{p-1} SV^{\langle xy^{pi}\rangle}$$

$$= \bigcup_{i=0}^{p-1} \left[SV^{\langle xy^{pi}\rangle} \times S \bigoplus_{j\neq i} V^{\langle xy^{pj}\rangle} \right]$$

(here $SV = V - 0$). Perhaps this kind of space is not too unreasonable, e.g. for $p=2$ it is just the product

$$SW \times SW$$

where x flips the two factors and y acts by the characters ζ and ζ^3.

February 12, 1971 K-groups for a curve over a finite field.

Let X be a complete non-singular curve ~~[illegible]~~ over a finite field k. Assume that $k = \Gamma(X, \mathcal{O}_X)$. In the classical picture the function field F of X is a finitely generated extension of k of tr.d. 1, and k is the set of elements algebraic over the prime field $\mathbb{F}_p$. The closed points of X correspond to valuations v on F, $\mathcal{O}_v$ is the valuation ring. Writing F as a finite extension of $k[z]$ corresponds to viewing X as a ramified covering:

$$X \longrightarrow \mathbb{P}^1_k \qquad \text{finite flat}$$

given by the rational function z.

I want to determine all characteristic classes for representations of groups acting on vector bundles over X, with coeffs. mod l. Given E over X

$$\mathrm{End}(E) = \Gamma(X, \underline{\mathrm{Hom}}(E,E))$$

is finite dimensional over k, hence $\mathrm{End}(E)$ and $\mathrm{Aut}(E)$ are finite. Thus we can ~~[illegible] vector bundles~~ restrict to considering only reps. of finite groups. (Quite generally this argument shows that when X is proper over a field k, we can restrict attention to affine algebraic groups over k, for which one may eventually have a good hold on the cohomology of its underlying discrete group.)

Denote by $R_X(G)$ the Grothendieck group of the G bundles on X, and by $R_X(G)_c$ the Groth. gp. of coherent G-sheaves. Then

$$R_X(G) \xrightarrow{\sim} R_X(G)_c$$

Indeed if F is coherent and L is ample on X, then

$$\Gamma(X, F \otimes L^{\otimes N}) \otimes_k L^{\otimes -N} \longrightarrow F$$

is surjective for N sufficiently large. If G acts on F, $\Gamma(X, F\otimes L^{\otimes N})$ is a representation of G and the above map is equivariant. This shows F is the quotient of a G-bundle, hence we can resolve F by bundles. Thus we get an exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow F \longrightarrow 0$$

where E', E are G-bundles, and the rest ~~of~~ Groth's argument should go through. (This should work when X non-singular and either G finite or X proper.)

Next: there should be an exact sequence for G finite

$$\boxed{\bigoplus_{v \in X} R_{k(v)}(G) \xrightarrow{i_*} R_X(G) \xrightarrow{j^*} R_F(G) \longrightarrow 0}$$

defined in usual way: Given a repn. over $k(v)$ view it as a coherent G-sheaf supported on $\{v\} \subset X$; this together with the above gives ι_*. $j^*E =$ stalk at generic point. (Will assume above OK; surjectivity of j^* clear: given a repn. V of G over fn. field F, choose a coherent sheaf $\mathcal{M}$ (ugly notation) over X with $j^*\mathcal{M} = V$, i.e. $\mathcal{M} = \{M_V\} \ni M_V \subset V$ is an $\mathcal{O}_V$ module; then as G finite, we can make $\mathcal{M}$ stable under G by averaging.)

Lemma: $R_k(G) \xrightarrow{\sim} R_F(G)$

Proof: (Need to recall following fact from Galois theory: Given fields

$$\begin{array}{ccc} F & \text{---} & F.k_1 \\ | & & | \\ k & \longrightarrow & k_1 \end{array}$$

with k_1/k Galoisian and $F \cap k_1 = k$, then $F.k_1/F$ is Galoisian and

$$\mathrm{Gal}(F.k_1/F) \xrightarrow{\sim} \mathrm{Gal}(k_1/k).$$

If F/k sep. alg. follows as $\mathrm{Gal}(\bar{k}/k)$ gen. by $\mathrm{Gal}(\bar{k}/k_1)$ and $\mathrm{Gal}(\bar{k}/F)$. OKAY if F/k purely trans.; OKAY if F/k purely insep.; $\therefore$ done)

Let E be an irred. representation of G over k, i.e. $\mathrm{End}_{k[G]}(E)$ skew-field, hence (k finite) a finite extension k_1 of k. As k_1/k Galoisian and $k_1 \cap F = k$, above fact recalled shows that

$$\mathrm{End}_{F[G]}(F \otimes_k E) = F \otimes_k \mathrm{End}_{k[G]}(E)$$

is a field, hence $F \otimes_k E$ irreducible. So have

$$k[G]/\mathrm{rad} = \prod_{i=1}^{r} M_{n_i}(k_i)$$

$$F \otimes (k[G]/\mathrm{rad}) = \prod_{i=1}^{r} M_{n_i}(k_i \otimes_k F)$$

and $k_i \otimes_k F$ is a field. Thus
$\mathrm{rad}\{F[G]\} = F \otimes_k \mathrm{rad}\{k[G]\}$ and so $E_i \longmapsto F \otimes_k E_i$ bijective map of iso classes of irred. $k[G]$-modules & irred $F[G]$-modules. As $R_k(G)$ & $R_F(G)$ are free abelian groups on these iso. classes resp., the lemma follows.

The lemma implies that j^* has a section

$$\begin{tikzcd} & & R_k(G) \arrow[dll, "s"'] \arrow[d, "\simeq"] & \\ R_X(G) \arrow[rr, "j^*"] & & R_F(G) \arrow[r] & 0 \end{tikzcd}$$

where $s(V) = V \otimes_k \mathcal{O}_X$.

<u>Proposition</u>: Let $\theta : R_X(?) \longrightarrow H^i(?, S.)$ be an exponential char. class. Assume $l \mid q-1$. If $\theta = 1$ on the following representations, then $\theta \equiv 1$:

- $k(v)^{\times}$ acting by multiplication on $k(v)$, $\quad v \in X$
- $k^{\times}$ acting by multiplication on $\mathcal{O}_X$

Proof. I have to recall my theorem that an exponential class $\theta : R_{k_0}(?) \longrightarrow H^0(?, S.)$ with k_0 a finite field $\ni$ $\mu_\ell \subset k_0^\times$ is ~~[illegible]~~ determined by the element of $H^0(k_0^\times, S.)$ which is θ applied to the multiplication repn. of $k_0^\times$ on k_0. Denote this repn. by $m(k_0)$; then ~~[illegible]~~ the map $\theta \mapsto \theta(m(k_0))$ is a bijection of the set of exp. classes θ and elements of $H^0(k_0^\times, S.)$ with augmentation 1. This bijection preserves the group structures also. ~~[illegible]~~

Using this, the proposition follows from the fact that

$$R_k(G) \oplus \bigoplus_{v \in X} R_{k(v)}(G) \xrightarrow{s + i_*} R_X(G)$$

is surjective. Also this surjectivity implies.

<u>Proposition</u>: There are no $\overbrace{\text{mod } p}^{\text{non-trivial}}$ charac. classes for $R_X(?)$

In effect there are none for $R_{k_0}(G)$, k_0 finite. In more concrete terms, suppose θ an exp. class mod p for R_X. To show $\theta \equiv 1$ enough to consider G a p-group. If a p-group G acts on a vector bundle E, then there is a filtration on the generic stalk of E stable under G ~~[illegible]~~ such that G acts trivially on the quotients. This filtration extends uniquely to a flag $0 \subset E_1 \subset \cdots \subset E$ of E (valuative criterion for separation + properness of ~~[illegible]~~ flag bundle) and G acts trivial on the quotient line bundles, so $\theta(E) = \prod \theta(E_i/E_{i-1}) = 1$.

<u>January 15, 1971:</u> Cohomology of $GL_n(\Lambda)$, $[\Lambda : \mathbb{Z}_p] < \infty$

Let Λ be a complete d.v.r finite over $\mathbb{Z}_p$ and such that $\Lambda \supset \mu_p$. Let an elementary abelian p-group A act on a free Λ-module E, in such a way that the action is trivial on $E/(\zeta-1)E$. I claim that then E is the direct sum of its eigen-modules

$$E = \bigoplus_{\chi \in A^\vee} E_\chi \qquad E_\chi = \{e \mid a \cdot e = \chi(a) e\}.$$

It is obviously enough to consider the case where A is cyclic, since the eigenmodules of a cyclic subgroup of A are stable under A. Write

$$E \otimes_\Lambda F = \bigoplus_{i=0}^{p-1} V_i \qquad F = \text{q.f. of } \Lambda$$

where a fixed generator σ of A has eigenvalue ζ^i on V_i. Let j be least such that

$$v_j + \dots + v_{p-1} \in E \implies v_j, v_{j+1}, \dots, v_{p-1} \in E.$$

(then $j \leq p-1$). Suppose $j > 0$ and let

$$e = v_{j-1} + \dots + v_{p-1} \in E.$$

By hypothesis σ acts trivially on $E/(\zeta-1)E$, i.e. $\sigma e - e \in (\zeta - 1)e$, so

$$\frac{\sigma - 1}{\zeta - 1} e = \frac{\zeta^{j-1} - 1}{\zeta - 1} v_{j-1} + \frac{\zeta^j - 1}{\zeta - 1} v_j + \dots + \frac{\zeta^{p-1} - 1}{\zeta - 1} v_{p-1}$$

belongs to E. Thus

$$\frac{\sigma-1}{\zeta-1}e-\frac{\zeta^{j-1}-1}{\zeta-1}e=\frac{\zeta^{j}-\zeta^{j-1}}{\zeta-1}v_j+\frac{\zeta^{j+1}-\zeta^{j-1}}{\zeta-1}v_{j+1}+\cdots+\frac{\zeta^{p-1}-\zeta^{j-1}}{\zeta-1}v_{p-1}$$

$$=\zeta^{j-1}\left\{v_j+\frac{\zeta^2-1}{\zeta-1}v_{j+1}+\cdots+\frac{\zeta^{p-j}-1}{\zeta-1}v_{p-1}\right\}\in E$$

By choice of j we have

$$v_j,\ \frac{\zeta^2-1}{\zeta-1}v_{j+1},\ \ldots,\ \frac{\zeta^{p-j}-1}{\zeta-1}v_{p-1}\ \in E$$

But $\dfrac{\zeta^i-1}{\zeta-1}$ is a unit for $0<i<p$, hence

$v_j, v_{j+1}, \ldots, v_{p-1}\in E$, hence also $v_{j-1}\in E$. This contradicts j least unless $j=0$. Thus

$$E=\bigoplus_{i=0}^{p-1}E_i$$

a claimed.

Remark: The above argument is completely general and reversible.

Theorem: Let Λ be an integral domain containing μ_p, $(p\neq 0)$ and let an elementary abelian p-group A act on a projective Λ-module E. Then

$$E=\bigoplus_{\chi\in A^\vee}E_\chi \iff A \text{ acts trivially on } E/(\zeta-1)E.$$

The hope might be that we can compute the cohomology of $GL_n(\Lambda, (\zeta-1)\Lambda)$ = subgroup of $GL_n(\Lambda)$ of $A \equiv 1 \mod (\zeta-1)\Lambda$, because this group has such a nice $[p]$-subgroup structure. The conjecture, of course, is that this group has no embedded components, whence the cohomology is detected on the centralizer.

$\Lambda = \mathbb{Z}_p[\mu_p]$, $SL_2(\Lambda)$ is special case to consider.
Thus $SL_2(\Lambda, (\zeta-1)\Lambda)$ consists of matrices

$$\begin{pmatrix} 1+\alpha\pi & \beta\pi \\ \gamma\pi & 1+\delta\pi \end{pmatrix} \qquad \pi = \zeta - 1$$

such that

$$(1+\alpha\pi)(1+\delta\pi) - \beta\gamma\pi^2 = 1$$

$$\boxed{\alpha+\delta + (\alpha\delta - \beta\gamma)\pi = 0}$$

Remark 1: $SL_2(\Lambda, (\zeta-1)\Lambda) = \Gamma$ has at least 3 generators as a pro-p-group since

$$\begin{pmatrix} 1+\alpha\pi & \beta\pi \\ \gamma\pi & 1+\delta\pi \end{pmatrix} \longmapsto \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \pmod{\pi}$$

is a homomorphism onto $(\Lambda/\pi\Lambda)^3 = (\mathbb{Z}/p\mathbb{Z})^3$.

Remark 2: The subgroup $\Gamma_0 \subset \Gamma$ of index p for which $\alpha \equiv \delta \equiv 0 \ (\pi)$ has no torsion. In effect we've seen that [redacted] there is only one cyclic subgroup of order p up to conjugacy and that it is generated by $\begin{pmatrix}\zeta & 0\\ 0 & \zeta^{-1}\end{pmatrix}$ which has $\alpha = 1$. Consequently Γ_0 is a torsion-free analytic p-group of dimension $3(p-1)$, hence by Serre-Lazard is a Poincaré pro p-group of dimension $3(p-1)$.

Remark 3: If you have an analytic p-group G with a filtration G_i $i \geq 1$ such that $(G_i, G_j) \subset G_{i+j}$, $G_i^p \subset G_{i+1}$ and pth power

$$G_i/G_{i+1} \xrightarrow{\cong\, p} G_{i+1}/G_{i+2}$$

is an isomorphism, then its cohomology is an exterior algebra on the dual of G_1/G_2. This applies to

$$G_1 = \begin{pmatrix} 1+\alpha p & \beta p \\ \gamma p & 1+\delta p \end{pmatrix} \qquad G_i = \begin{pmatrix} 1+\alpha p^i & \beta p^i \\ \gamma p^i & 1+\delta p^i \end{pmatrix}$$

because

$$(1+p^i\Delta)^p = 1 + p^{i+1}\Delta + \underbrace{\cdots + p^{pi}\Delta^p}_{\text{higher}}$$

Thus $H^*(SL_2(\Lambda, p\Lambda))$ is an exterior algebra on $3(p-1)$ generators.

<u>Remark 4</u>: The cohomology of $\Gamma =$ [redacted] $\{1+\pi\Delta\}$ is not detected by the centralizer of [redacted] $\begin{pmatrix}\xi & 0\\ 0 & \xi^{-1}\end{pmatrix}$ since that centralizer is $\begin{pmatrix}* & 0\\ 0 & *\end{pmatrix}$ and the homomorphisms

$$\begin{pmatrix}1+\alpha\pi & \beta\pi\\ \gamma\pi & 1+\delta\pi\end{pmatrix} \longmapsto \beta,\gamma \mod \pi$$

vanish on this centralizer. Thus either the basic Chern class $c_2 \in H^4(\Gamma)$ is a zero-divisor, or the ~~[illegible]~~ primary component thm. for [redacted] maximal $[p]$-subgroups doesn't extend to pro-p-groups.

January 17, 1971: Spectrum of $SL_2(F)$, $[F:\mathbb{Q}_p]<\infty$.

The building X of $SL_2(F)$ gives amalgamated product representation

$$SL_2(F) = G_0 *_{G_{01}} G_1$$

where G_0 is stabilizer of the lattice $L_0 = \Lambda \oplus \Lambda < F^2$ and G_1 is the stabilizer of $L_1 = \Lambda \oplus \pi\Lambda$ and $G_0 \cap G_1 = G_{01}$. Consequently the Mayer-Vietoris sequence shows that

$$H^*(SL_2(F)) \longrightarrow H^*(G_0) \times_{H^*(G_{01})} H^*(G_1)$$

is an F-isomorphism. Now this formula tells us what the spectrum of $H^*(SL_2F)$ is, hence we can check agreement with our conjecture.

i) every cyclic A in SL_2F conjugate to $\begin{pmatrix} \gamma^i & 0 \\ 0 & \gamma^{-i} \end{pmatrix}$ hence comes from a cyclic subgroup of G_{01}.

ii) every $A < G_0$ conjugate to a subgroup of G_{01}. Clear as G_{01} contains the Sylow group.

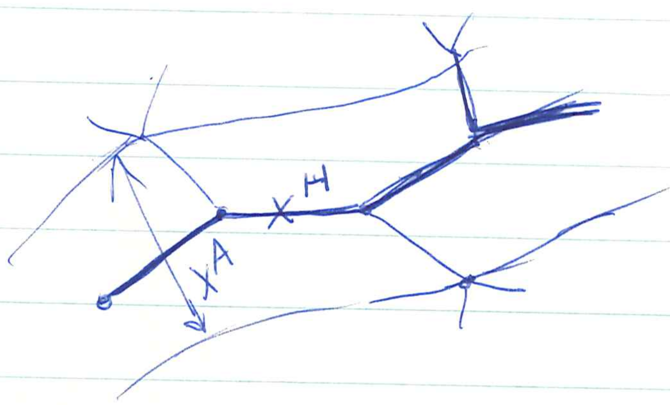
Thus

$$\begin{tikzcd} & H(G_0)(\Omega) \arrow[dr] & \\ H(G_{01})(\Omega) \arrow[ur] \arrow[rr, two heads] \arrow[dr, two heads] & & H(SL_2F)(\Omega) \\ & H(G_1)(\Omega) \arrow[ur] & \end{tikzcd}$$

iii) Given $A \subset G_{01}$ either L_0 or L_1 is good i.e. A acts trivially on $L/\pi L$, whence $L \cong \Lambda \oplus \Lambda$ with A standard diagonal form. (In effect given $A \subset SL_2(F)$ one sees that the lattices fixed by A are of two types: either good or bad and that every neighbor of a good lattice is fixed by A while, every bad lattice is joined directly to a good one

good $A + \pi^i \Lambda$, bad $\Lambda + \pi^i \Lambda + \varepsilon$

This means that X^A is a "halo" around X^H, $H =$ the max. compact subgroup of the centralizer of A



)

By iii) if $A < G_{01}$ then A is conjugate in G_0 or G_1 to standard A. This shows variety of $H(SL_2F)$ has at most one non-trivial stratum. The Weyl group for the standard A in G_0 being same as for SL_2F one sees the conjecture is true.

Computations for the prime 2: The hope is that working with $SL_2^e(F) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, ad-bc = \pm 1 \right\}$ avoids the symplectic pathology at 2. Hence

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}_2) \,\middle|\, \begin{array}{l} \equiv 1 \mod 2 \\ \det \pm 1 \end{array} \right\}$$

perhaps should be computable. More generally one should look at the subgroup $GL_n(\mathbb{Z}_2; 2\mathbb{Z}_2)$ of $GL_n(\mathbb{Z}_2)$ consisting of matrices $\equiv 1 \mod 2$, as this has a nice spectrum. Now $GL_n(\mathbb{Z}_2; 4\mathbb{Z}_2)$ is susceptible to Lazard theory.

$$0 \longrightarrow SL_2(\mathbb{Z}_2; 4\mathbb{Z}_2) \longrightarrow \Gamma \longrightarrow (\mathbb{Z}/2\mathbb{Z})^4 \longrightarrow 0$$

$$\begin{pmatrix} 1+2\alpha & 2\beta \\ 2\gamma & 1+2\delta \end{pmatrix} \longmapsto \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mod 2$$

onto because of $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

By Lazard $H^*(SL_2(\mathbb{Z}_2; 4\mathbb{Z}_2))$ should be an exterior algebra on three generators of dimension 1, dual to

$$\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 21 & -8 \\ 8 & -3 \end{pmatrix}$$

(We can check this by showing that for each n we get matrices congruent to $1 +$

$$2^n \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad 2^n \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad 2^n \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \mod 2^{n+1}$$

But we have the identity

$$\underbrace{\underbrace{\begin{pmatrix} 1 & \pi \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}}\begin{pmatrix} 1 & -\pi \\ 0 & 1 \end{pmatrix}}\begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1+\pi a & \pi \\ a & 1 \end{pmatrix}\begin{pmatrix} 1 & -\pi \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1+\pi a & -\pi^2 a \\ a & 1-\pi a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+\pi a+\pi^2 a^2 & -\pi^2 a \\ \pi a^2 & 1-\pi a \end{pmatrix}$$

take $\pi = 2^i$ $a = 2^j$ we see that

$$\begin{pmatrix} 1+2^{i+j} & 0 \\ 0 & 1-2^{i+j} \end{pmatrix} \mod 2^{i+j+1}$$

so we get the right diagonal terms.

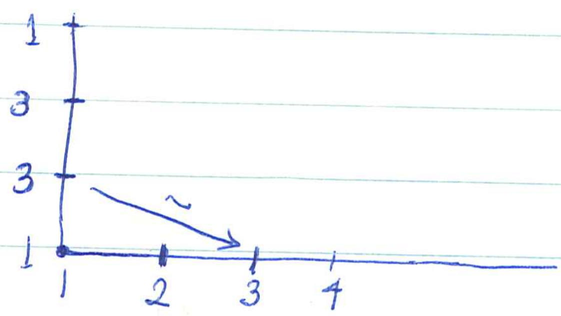
<u>Conclusion</u>: The subgroup $\Gamma_0 = \begin{pmatrix} 1+4\alpha & 2\beta \\ 2\gamma & 1+4\delta \end{pmatrix}$ of Γ has no torsion and has 2 generators, namely

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

Thus by Poincare duality the Betti nos. of Γ_0 are $1, 2, 2, 1$. If one considers the spectral sequence of

$$1 \longrightarrow SL_2(\mathbb{Z}_2; 4\mathbb{Z}_2) \longrightarrow \Gamma_0 \longrightarrow (\mathbb{Z}/2)^2 \longrightarrow 1$$

one has



where the indicated d_2 is an isomorphism as H^1 has 2 generators. This shows that the product

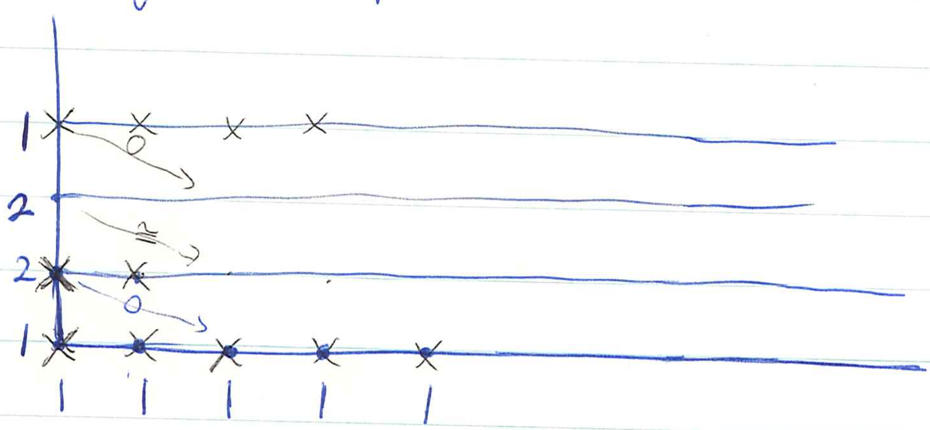
$$H^1(\Gamma_0) \otimes H^1(\Gamma_0) \longrightarrow H^2(\Gamma_0)$$

is zero.

Now let $\Gamma_1 = \begin{pmatrix} 1+4* & 2* \\ 2* & 1+2* \end{pmatrix}$ be the subgroup generated by Γ_0 and $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, i.e., the semi-direct product. Then A acts trivially on $H^1(\Gamma_0)$, since the commutator: $(\{1+2\Delta\}, \{1+2\Delta\}) \subset \{1+4\Delta\}$. As A must trivialize $H^3(\Gamma_0)$ and preserve Poincare duality it follows that A acts trivially on $H^*(\Gamma_0)$, hence the E_2 of the spectral sequence for

$$1 \longrightarrow \Gamma_0 \longrightarrow \Gamma_1 \longrightarrow A \longrightarrow 1$$

is of the form of a tensor product



Now Γ_0 has 3 generators so $d_2(E_2^{01}) = 0$.

Now consider the conjugation action of the matrix

$$\omega = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \qquad \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}^2 = \begin{pmatrix} -1 & +1 \\ -1 & 0 \end{pmatrix}$$

of order 3. [Finished Feb 3: I computed that Γ_1 is stable under this and that, one gets the irred. 2-dim'l rep'n for $H^1(\Gamma_0)$ and $H^2(\Gamma_0)$ while ω acts trivially on $H^3(\Gamma_0)$. This shows that d_2, d_3 vanish on E_2^{03} in above spectral sequence. Only problem left is whether $d_2 E_2^{02} \neq 0$, which is probably the case from past experience: (centralizer of A in Γ_1 is $\begin{pmatrix} a & 0 \\ 0 & \pm a \end{pmatrix}$ $a \equiv 1(4)$ has cohomology of rank 2 after localization.)

COMPUTATION INCONCLUSIVE

January 24, 1971. (David is 7)

Notes on trip to Atlantic City & Institute.

Theorem of Borel: Let Λ be a Dedekind domain in a number field F with r_1 real places and r_2 complex places. Then

$$K_{2i}\Lambda \otimes \mathbb{Q} = 0 \qquad i \geq 1$$

$$\dim K_{4i+1}\Lambda \otimes \mathbb{Q} = r_1 + r_2 \qquad \text{"}$$

$$\dim K_{4i-1}\Lambda \otimes \mathbb{Q} = r_2 \qquad \text{"}$$

(If Λ is the ring of S-units in F, then $K_1\Lambda \otimes \mathbb{Q}$ by Dirichlet has rank $r_1 + r_2 - 1 +$ no. of non-arch. places in S.)

Borel's theorem amounts to a computation of the homology $H_*(SL(\Lambda), \mathbb{Q})$. Here are the essential steps in his argument.

Now Λ is the ring of S-units for some possibly infinite set S of places in F, because an integral closed domain is the intersection of the discrete valuation rings associated to its minimal primes. By passage to the limit can assume S finite.

~~[struck-through line]~~

The case where Λ is the ring of integers: Then one makes $\Gamma = SL_n(\Lambda)$ act on the symmetric space

$$X = G/K = \prod_{\text{real places}} SL_n(\mathbb{R})/SO_n \times \prod_{\text{cx. places}} SL_n(\mathbb{C})/SU_n$$

Consider real cohomology. If $\Gamma_0 \triangleleft \Gamma$ is torsion-free then we have maps

$$H^*_{\Gamma} \xrightarrow{\sim} \left(H^*_{\Gamma_0}\right)^{\Gamma/\Gamma_0} \hookrightarrow H^*_{\Gamma_0} \xrightarrow{\sim} H^*_{\Gamma_0}(G/K) = H^*(\Gamma_0 \backslash G/K)$$

and on the other hand we have a map

$$\begin{array}{ccc} \text{complex of } G\text{-inv forms on } X & \longrightarrow & \text{DR cx of } \Gamma_0\backslash G/K \\ \| & & \\ \left(\Lambda(\mathfrak{g}/\mathfrak{k})^\vee\right)^k = I^* & & \end{array}$$

and one knows that $d=0$ on I^* (this is standard symmetric space version of fact that for reductive $\mathfrak{g}$ $(\Lambda\mathfrak{g}^\vee)^{\mathfrak{g}}$ has zero-differentials). Thus we get a map

$$I^* \longrightarrow H^*(\Gamma_0\backslash G/K)$$

and it's clear that the image lies in Γ/Γ_0-invariants since I^* forms are G-invariant. So we obtain a map

$$(*) \qquad I^* \longrightarrow H^*_\Gamma .$$

Borel proves this map is an isomorphism in a range which increases with n. The method is to ~~[illegible]~~ use the cornered extension $\bar{X}$ to show that cohomology class of $\Gamma_0\backslash X$ can be represented by L^2-forms in the good range; then one can generalize the Matsushima treatment when $\Gamma\backslash X$ is compact. (The surjectivity part comes by an integration by parts

$\cancel{\sum \|\Delta X_i\omega\|^2 \leq (\omega, \Delta\omega) + \text{Const.} \|\omega\|^2}$ ~~approximate~~

~~Use the constant~~

$$(\Delta\omega, \omega) = \sum a_{ij} (X_i\omega, X_j\omega) + b\|\omega\|^2$$

where the constants are such that these things imply a harmonic form on $\Gamma_0\backslash X$ is actually G-invariant.)

I should be able to understand the above map $(*)$ in concrete representation terms. Thus ~~[illegible]~~ it should be possible to construct additive characteristic classes using the real ~~[illegible]~~ and complex places.

February 7, 1971:

Borel's characteristic classes: Let G be a connected real Lie group and K a maximal compact subgroup. Look at the de Rham cx

$$0 \longrightarrow \mathbb{R} \longrightarrow A^0(X) \longrightarrow A^1(X) \longrightarrow A^2(X) \longrightarrow \cdots$$

$G_d = G$ disc. top

as a resolution of $\mathbb{R}$ in category of G_d-modules. Here $X = G/K$ which is contractible; by G-modules we mean $G = G_d$ with discrete topology, (although it makes sense to work with continuous G-modules and then, so it seems, the classes to be defined lie in the continuous EM coh. of G coeffs. in $\mathbb{R}$). This resolution gives rise therefore to a spectral sequence

$$E_1^{pq} = H^q(G_d, A^p(X)) \Longrightarrow H^{p+q}(G_d, \mathbb{R})$$

~~with edge homomorphism~~

$$\cancel{I^p(X) = H^0(G, A^p(X)) \longrightarrow H^p(G_d, \mathbb{R})}$$

~~where $I^p(X)$ = the invariant differential forms on the symmetric~~

Now we want to look at edge homo

$$E_2^{p0} \longrightarrow H^p$$

and use that the complex of invariant differential forms

$$H^0(G, A^\cdot(X)) = I^\cdot(X)$$

on the symmetric space X has zero differentials.
Hence the edge homomorphism gives map

$$\begin{array}{ccc} I^p(X) & \longrightarrow & H^p(G_d, \mathbb{R}) \\ \| & & \\ (\Lambda^p p^*)^k & & \end{array}$$

(As mentioned these classes lie in the Eilenberg-Maclane cont. coh. of G and I think it's known that the ~~[illegible]~~ above map is an isomorphism with this cohomology.)

February 16, [redacted] 1971

X complete non-singular curve over an alg. closed field k. Then for G finite (order prime to p) we have

$$R_k(G) \otimes K(X) \xrightarrow{\sim} R_X(G).$$

Indeed let G act on a vector bundle E. Then one [redacted] decomposes E according to the reps. of G.

$$E = \sum V_i \otimes \operatorname{Hom}_G(V_i, E)$$

as in the compact group case.

This being so, it follows that an exponential class

$$R_X(G) \longrightarrow H^0(G, S.)^\times$$

where $S.$ is a graded anti-commutative algebra over $\mathbb{Z}/\ell\mathbb{Z}$, will be trivial on the divisible part of $\operatorname{Pic}(X)$. Recall that

$$\begin{aligned} K(X) &\cong \mathbb{Z} \oplus \operatorname{Pic} X \\ &\cong \mathbb{Z} \oplus \operatorname{Pic}^\circ X \oplus \mathbb{Z}. \end{aligned}$$

The point is that $R_k(G) \otimes \operatorname{Pic}^\circ X$ is ℓ-divisible, hence there can be no homomorphism of it into an ℓ-complete group.

Now I want to consider the case where k is a finite field of order q. Let $\bar{k}$ = alg. closure of k and $\bar{X} = Sp(\bar{k}) \times_{Sp(k)} X$. I want to prove that

$$\boxed{*} \qquad \boxed{R_X(G) \xrightarrow{\sim} R_{\bar{X}}(G)^{Gal(\bar{k}/k)}}$$

for any finite group G, say of order prime to p. Let $V_1, \dots, V_m$ be the distinct irreducible representations of G over k and let $L_i = End_{k[G]}(V_i)$. Then L_i is a finite extension of k. Now

$$End_{\bar{k}[G]}(\bar{k} \otimes_k V_i) = \bar{k} \otimes_k L_i \xrightarrow{\sim} \bar{k}^{d_i}$$

where $d_i = [L_i : k]$. Thus $\bar{k} \otimes_k V_i$ is a sum of d_i inequivalent representations over $\bar{k}$, in fact an orbit under the Galois group:

$$\bar{k} \otimes_k V_i \simeq \bigoplus_{a=0}^{d_i - 1} \Psi^{\delta^a}(W_i).$$

where W_i is irreducible over $\bar{k}$. Now by Wedderburn

$$k[G] \simeq \prod_{i=1}^{m} M_{n_i}(L_i) \qquad |G| = \sum n_i^2 d_i$$

$$n_i = \dim_{L_i} V_i$$

$$\bar{k}[G] \simeq \prod_{i=1}^{m} M_{n_i}(\bar{k} \otimes_k L_i)$$

so its clear that $\{\Psi^{\delta^a} W_i,\ 0 \le a < d_i,\ 1 \le i \le m\}$ is the complete set of irreducible representations of G over k. Thus

$$R_k(G) \xrightarrow{\sim} R_{\bar{k}}(G)^{Gal(\bar{k}/k)} .$$

Now given a G-vector bundle E over X, we have

$$(*) \qquad E \simeq \bigoplus_{i=1}^{m} \underline{Hom}_{k[G]}(V_i, E) \otimes_{L_i} V_i$$

where $\underline{Hom}_{k[G]}(V_i, E)$ is the G-invariant subbundle (recall $|G|$ prime to p) of $\underline{Hom}_X(O_X \otimes_k V_i, E)$. Now set

$$E_i = \underline{Hom}_{k[G]}(V_i, E).$$

It has L_i as endomorphisms, hence is the restriction of a bundle on $L_i \otimes_k X$. Thus we obtain an iso.

$$\begin{array}{ccc} R_X(G) & \xleftarrow{\sim} & \bigoplus_{i=1}^{m} K(L_i \times_k X) \\ \sum_{i=1}^{m} E_i \otimes_{L_i} V_i & \longleftarrow\!\!\shortmid & (E_i). \end{array}$$

On the other hand we have

$$\begin{array}{ccc} R_{\bar{X}}(G) & \xleftarrow{\sim} & \bigoplus_{i=1}^{m} \bigoplus_{a=0}^{d_i-1} K(\bar{X}) \\ \sum E_{ia} \otimes_{\bar{k}} \Phi^{q^a} W_i & \longleftarrow\!\!\shortmid & (E_{ia}) \end{array} .$$

Since

$$\bar{k} \otimes_k (E_i \otimes_{L_i} V_i) = (E_i \otimes_{L_i} \bar{k}) \otimes_{\bar{k}} (\bar{k} \otimes_k V_i)$$
$$= (E_i \otimes_{L_i} \bar{k}) \otimes_{\bar{k}} \bigoplus_{a=0}^{d_i - 1} \Psi^{\delta^a} W_i$$

all we have to show is that

$$K(L_i \times_k X) \xrightarrow{\sim} K(\bar{X})^{Gal(\bar{k}/L_i)}$$
$$E_i \longmapsto E_i \otimes_{L_i} \bar{k}$$

is an isomorphism.

So we are reduced to showing that

$$K(X) \xrightarrow{\sim} K(\bar{X})^{Gal(\bar{k}/k)}$$

and recall that

$$K(X) \simeq \mathbb{Z} \oplus \operatorname{Pic} X \qquad \text{canonical}$$
$$\operatorname{Pic}(X) \simeq \mathbb{Z} \oplus \operatorname{Pic}^0 X \qquad \text{depends on Jence of a divisor of degree 1}$$

also for $\bar{X}$, hence want to show

$$\operatorname{Pic}(X) \xrightarrow{\sim} \operatorname{Pic}(\bar{X})^{Gal}.$$

But we have

$$0 \longrightarrow \bar{k}^\times \longrightarrow F^\times \xrightarrow{I} \operatorname{Div}(\bar{X}) \longrightarrow \operatorname{Pic}(\bar{X}) \longrightarrow 0$$

and $\left.\begin{array}{rl} H^1(\mathrm{Gal}, \bar{k}^\times) & = 0 \\ \bar{F}^\times & = 0 \end{array}\right]$ Hilbert

$H^2(\mathrm{Gal}, \bar{k}^\times) = 0$ Wedderburn. (← stupid $\mathrm{Gal} \cong \hat{\mathbb{Z}}$)

hence $H^1(\mathrm{Gal}, I) = 0$. so

$$F^\times \longrightarrow \mathrm{Div}(\bar{X})^{\mathrm{Gal}} \longrightarrow \mathrm{Pic}(\bar{X})^{\mathrm{Gal}} \longrightarrow 0$$

exact. Since $\mathrm{Div}(\bar{X})^{\mathrm{Gal}} = \mathrm{Div}(X)$

one sees that $\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Pic}(\bar{X})^{\mathrm{Gal}}$, q.e.d.

(As a check we expect 5 terms

$$\begin{array}{ccccccc} 0 \to H^1(\mathrm{Gal}, H^0(\bar{X}, \mathbb{G}_m)) & \to & H^1(X, \mathbb{G}_m) & \to & H^0(\mathrm{Gal}, H^1(\bar{X}, \mathbb{G}_m)) & \to & H^2(\mathrm{Gal}, H^0(\bar{X}, \mathbb{G}_m)) \\ \| & & & & & & \| \\ H^1(\mathrm{Gal}, \bar{k}^\times) & & & & & & H^2(\mathrm{Gal}, \bar{k}^\times) \\ \| & & & & & & \| \\ 0 & & & & & & 0 \end{array}$$
)

Next problem: Compute natural transf. from

$$R_X(G) = R_k(G) \oplus R_k(G) \oplus \left[R_{\bar{k}}(G) \otimes \mathrm{Pic}^\circ(\bar{X})\right]^{\mathrm{Gal}(\bar{k}/k)}$$

to $H^0(G, S.)$.

January 27, 1971. Stability theorem.

It is now clear that your earlier version of the stability theorem has a mistake due to the fact that a non-degenerate simplex in the singular semi-simplicial set of a simplicial complex may have repeated vertices, e.g. $(x\ y\ x)$.

Special case: Assume that Λ is an alg. over $\mathbb{Z}[\ell^{-1}]$ for some prime no. ℓ and that the coefficients of cohomology is $\mathbb{Q}$. Then we know that the homomorphism

$$GL_i \longrightarrow \left[\begin{array}{c|c} I_{n-i} & 0 \\ \hline \Lambda^{(n-i)i} & GL_i \end{array}\right]$$

induces an isomorphism on cohomology. (Recall: Let the group $GL_i \tilde{\times} \mathrm{Hom}_\Lambda(\Lambda^{n-i}, \Lambda^i)$ be denoted $G \tilde{\times} M$ and consider H-S for extension

$$0 \to M \to G \tilde{\times} M \to G \to 0$$

$$E_2^{rs} = H^r(G, H^s(M)) \Longrightarrow H^{r+s}(G \tilde{\times} M).$$

Consider the automorphism θ of the spectral sequence produced by conjugating by $\ell I_i \in G$. Recall

$$H^i(M) = \mathrm{Hom}_{\mathbb{Z}}(\Lambda^i M, \mathbb{Q})$$

this isomorphism is functorial in the abelian group M. As

$$\theta\begin{bmatrix} I & \\ m & I\end{bmatrix} = \begin{bmatrix} I & \\ & \lambda I\end{bmatrix}\begin{bmatrix} I & \\ m & I\end{bmatrix}\begin{bmatrix} I & \\ & \lambda^{-1} I\end{bmatrix}$$

$$= \begin{bmatrix} I & \\ \lambda m & I\end{bmatrix}$$

we have that θ acts on $H^i(M)$ by multiplying by l^i, i.e. $\theta^*(u) = l^i u$. On the other hand θ acts trivially on G since λI is in the center of G. Thus θ acts on E_2^{rs} by multiplying by l^s, i.e. $\theta^* u = l^s u$ if $u \in E_2^{rs}$. Naturality of the spectral sequence and fact that $l \neq 1 \Longrightarrow$ all differentials are zero. But θ acts trivially on abutment, hence $E_2^{rs} = 0$ $s > 0$ and $E_2^{r0} = H^r(G) \xrightarrow{\sim} H^r(G \tilde{\times} M)$.)

Now we consider the following simplicial complex, denoted X^n. A vertex of X^n is a unimodular vector v in Λ^n, i.e. a direct injection $v : \Lambda \to \Lambda^n$. An i-simplex of X^n is by definition a subset $\{v_0, \dots, v_i\}$ such that

$$\Lambda^{i+1} \longrightarrow \Lambda^n$$

$$(\alpha_j)_{0 \le j \le i} \mapsto \sum_{0 \le j \le i} \alpha_j v_j$$

is a direct injection. We assume for some integer d

(i) $GL_n\Lambda$ acts transitively on the set of direct injections from Λ^j to Λ^n for $1 \le j \le n-d$

(ii) $\tilde{H}_{j-1}(X^n, \mathbb{Q}) = 0 \qquad 1 \le j < n-d.$

For example if Λ is a field, or more generally a local ring, then (i) holds with $d=0$. If Λ is an infinite field, then (ii) holds with $d=0$, since I know that X^n has the homotopy type of a wedge of S^{n-1}'s.

In virtue of (ii), the complex of rational chains

$$(*) \qquad \to C_{n-d-1}(X^n) \cdots \longrightarrow C_1(X^n) \longrightarrow C_0(X^n) \longrightarrow \mathbb{Q} \longrightarrow 0$$

has homology beginning in dimension $n-d-1$.
In virtue of (i) we have with $\Delta_j = \left[\begin{array}{c|c} I_j & 0 \\ \hline * & GL_{n-j} \end{array}\right]$ that

$$\text{set of } (j-1)\text{-simplices} = GL_n\Lambda \Big/ \left[\begin{array}{c|c} \Sigma_j & 0 \\ \hline * & GL_{n-j} \end{array}\right]$$

for $1 \le j \le n-d$, hence

$$C_{j-1}(X^n) = \operatorname{Ind}_{\left[\begin{array}{c|c} \Sigma_j & 0 \\ \hline * & GL_{n-j} \end{array}\right] \to GL_n} (sgn)$$

where sgn denotes the sign representation of $\left[\begin{array}{c|c} \Sigma_j & 0 \\ \hline * & GL_{n-j} \end{array}\right]$ over $\mathbb{Q}$.
Thus

$$H_q(GL_n, C_{j-1}(X^n)) = H_q\left(\left[\begin{array}{c|c} \Sigma_j & 0 \\ \hline * & GL_{n-j} \end{array}\right], sgn\right)$$

$$= H_q\left(\left[\begin{array}{c|c} I & 0 \\ \hline * & GL_{n-j}\end{array}\right], sgn\right)^{\Sigma_j}$$

(over $\mathbb{Q}$ used here)

$$= \left(H_q\left(\left[\begin{array}{c|c} I & 0 \\ \hline * & GL_{n-j}\end{array}\right]\right) \otimes sgn\right)^{\Sigma_j}$$

$$= 0 \quad \text{if} \quad j \geq 2$$

Now use acyclicity of

$$0 \longrightarrow Z_{n-d-1}(X^n) \longrightarrow C_{n-d-1}(X^n) \longrightarrow \cdots \longrightarrow C_1(X^n) \to Z_0(X^n) \to 0$$

and you have

$$H_q^{GL_n,}(Z_0(X^n)) \xrightarrow{\sim} H_{q-n+d+1}(GL_n, Z_{n-d-1}(X^n))$$

i.e.

$$H_q^{GL_n,}(Z_0(X^n)) = 0 \quad \text{if} \quad q < n-d-1$$

so as

$$\cdots \xrightarrow{\partial} H_q^{GL_n,}(Z_0(X^n)) \longrightarrow H_q\left(\left[\begin{array}{c|c} 1 & 0 \\ \hline * & GL_{n-1}\end{array}\right]\right) \longrightarrow H_q(GL_n) \longrightarrow H_{q-1}(GL_n, Z_0(X^n)) \longrightarrow \cdots$$

we see that

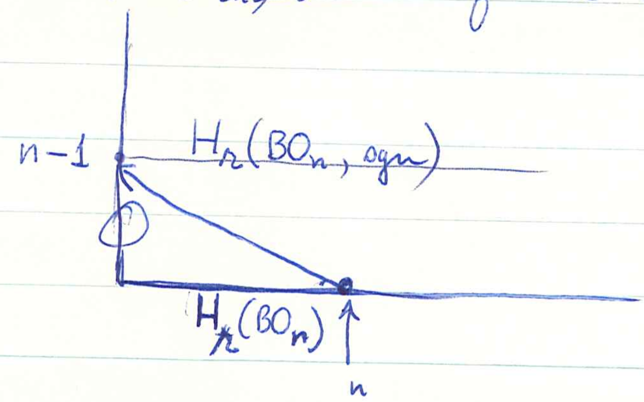
$$\boxed{H_q(GL_{n-1}) \longrightarrow H_q(GL_n) \quad \begin{array}{l}\text{surjective } q = n-d-1 \\ \text{isomorphism } q < n-d-1\end{array}}$$

Remarks concerning the range:

Topologically one obtains the same estimate

$$H_q(BO_{n-1}) \longrightarrow H_q(BO_n) \qquad \begin{array}{l} \text{surj. for } q = n-1 \\ \text{isom. } q < n-1 \end{array}$$

because of the Gysin-sequence



$$H_{q-n+1}(BO_n, sgn) \longrightarrow H_q(BO_{n-1}) \longrightarrow H_q(BO_n) \longrightarrow H_{q-n}(BO_n, sgn) \longrightarrow \cdots$$

Thus this range is what we expect.

Actually it appears that we get a more general result. Let M be a $\mathbb{Q}[GL_n\Lambda]$-module.

$$H_q(GL_n, C_{j-1}(X^n) \otimes_{\mathbb{Q}} M)$$

$$= H_q\left(\left[\begin{array}{c|c} \Sigma_j & 0 \\ \hline * & GL_{n-j} \end{array}\right], sgn \otimes M\right)$$

$$\Rightarrow H_q\left(\left[\begin{array}{c|c} I & 0 \\ \hline * & GL_{n-j} \end{array}\right], sgn \otimes M\right)^{\Sigma_j}$$

Now assume $E_n\Lambda$

hence M is a $K_1(\Lambda)/\{\pm \ell^{n\mathbb{Z}}\}$ module in Bass-stable range

acts trivially on M ~~and thus can be rewritten~~

as well as the permutation matrices and the diagonal matrices with entries ℓ^n, $n \in \mathbb{Z}$. Then the above argument goes through and shows

$$H_q\left(\begin{array}{c|c} I & 0 \\ \hline * & GL_{n-j} \end{array}, \operatorname{sgn} \otimes M\right)^{\Sigma_j} = H_q(GL_{n-j}, M) \otimes (\operatorname{sgn}^{\Sigma_j})$$
$$= 0 \qquad \text{for } j \geq 2$$

hence

$$H_q(GL_{n-1}, M) \xrightarrow{\sim} H_q(GL_n, M) \qquad q < n-d-1$$
$$\text{surj } q < n-d.$$

What one needs now,

(i) argument for $\mathbb{Z}/p\mathbb{Z}$ - cohomology

(ii) argument for π_1

~~*acyclicity*: of standard complex for a finite field. Chern classes using a geometric thing.~~

~~One method of proving acyclicity is to reduce to building. v_0~~

~~other possibility is to make a base extension~~

~~$n-1 \longrightarrow n$~~

[q odd $\quad 4 | q-1.$?]

$$H^*(BO_n(\mathbb{F}_q)) = \mathbb{Z}/2\mathbb{Z}\left[\begin{matrix} w_1', \ldots, w_n' \\ w_2'', \ldots, w_n'', \end{matrix}\right]$$

and w_n' w_n'' disappear on $BO_{n-1}(\mathbb{F}_q)$.

so the map

$$H_{n-1}(BO_{n-1}^{(\mathbb{F}_q)}) \hookrightarrow H_{n-1}(BO_n(\mathbb{F}_q))$$

is injective but not surjective

$$O_2(\mathbb{F}_q) = \mathbb{Z}_2 \tilde{\times} \mathbb{F}_{q^2}^*$$

$$\mathbb{Z}_2[t_1, t_2, e] / (t_1^2 + t_1 t_2) \qquad e \in H^2 \qquad w_2$$

$$t_1, t_2 \in H^1$$

$$O_1(\mathbb{F}_q) = \mathbb{Z}/_2\mathbb{Z} \qquad \pm 1$$

And only single generators.

Thus $\quad H^1(BO_2(\mathbb{F}_q)) \longrightarrow H^1(BO_1(\mathbb{F}_q))$ not surjective

	1	2	3	4	5	6	7	8
$n=0$	1							
$n=1$	σ							
$n=2$		a_1	a_2	a_3	$\cdots$			
$n=3$		σa_1	σa_2	σa_3				
$n=4$			a_1^2	b_3				
5					$\Sigma\text{-}0$			
6				a_1^3				
7								
8							c_7	

so ~~it~~ seems that

$$a_1^2 \in H_2^{\boxed{}}(\Sigma_4)$$

doesnot come from $H_2(\Sigma_3)$

$$a_1^n \in H_n(\Sigma_{2n})$$

doesnot come from Σ_{2n-1}.

p odd first class in degree $\quad c''_{p-1} \quad 2p-3$

$$a_{2p-3} \in H_{2p-3}(\Sigma_p)$$

0

OKAY because $p \leq 2p-3$ for p odd.

$\boxed{\Sigma_{n-1} \subset \Sigma_n}$

want stability
simplicial complex

not true that what you wants holds ~~un~~stably.

$\boxed{\text{first classes.}}$

question: Given finite field k when is

$H^*(GL_n(k))$ $\mathbb{Z}/p\mathbb{Z}$ coeff

first non-trivial?

for the symmetric groups one knows the basic classes

$H \quad \Sigma_{2^n} \qquad\qquad w_{2^n-1}$

$$H^*(\Sigma_{2^n-1}) \longrightarrow H(\Sigma_{2^n})$$

not onto
in dim $2^{n}-1$

$$H_i(BO_{n-1}) \xrightarrow[\substack{\text{surj } i<n \\ \text{inj always}}]{} H_i(BO_n)$$

$$\hookrightarrow H^{i-n+1}(BO_n) \leftarrow H^i(BO_{n-1}) \longleftarrow H^i(BO_n) \xleftarrow{w_n} H^{i-n}(BO_n)$$

inj $i<n$
surj always.

$$
\begin{tikzcd}
P \arrow[r, shift left, "i"'] \arrow[dr, shift right, "i"'] & Q \arrow[l, shift left, "j"'] \arrow[d, shift right, "\theta"'] \\
 & Q \arrow[u, shift right, "\theta^{-1}"'] \arrow[ul, shift right, "j"']
\end{tikzcd}
$$

$$\boxed{\theta i = i}$$

$$j\theta^{-1} = j$$

or $\boxed{j\theta = j}$

~~and so it is now clear that θ can be any automorphism of the exact sequence~~

$$\cancel{0 \to P \xrightarrow{i} Q}$$

Thus θ is essentially an arbitrary auto of $\mathrm{Ker}(j)$.

So go back to preceding page.

~~$j(x,r)$~~

$$
\begin{tikzcd}
P \arrow[r, shift right, "i"'] \arrow[dr, shift right, "i'"'] & P \oplus R \arrow[l, shift right, "j"'] \arrow[d, shift right, "i''"'] \\
 & P \oplus R' \arrow[u, shift right, "j''"'] \arrow[ul, shift right, "j'"']
\end{tikzcd}
$$

$$i''(p,r) = i''(p,0) + i''(0,r)$$

$$= (p,0) + (x,y)$$

$$x = j'(x,y) = jj''(x,y) = jj''i''(0,r) = j(0,r) = 0$$

$$\therefore\ i''(R) \subset R'. \qquad i''(P) = P$$

$$jj''(0,r') = j(0,r') = 0. \qquad \therefore\ j''(R') \subset R$$

$$j''(P) = j''(i''P) = P.$$

to the full subcategory whose objects are diagrams

$$(P,Q) \longrightarrow (P',P') \xleftarrow{id_{P'}} (0,0).$$

Another possibility would be to associate to a pair consisting of a module R and an isom.

$$\alpha : P \oplus R \simeq Q \oplus R$$

the object of $\mathcal{S}\mathcal{T}(P,Q)$ consisting of the diagram

$$\langle R,\alpha\rangle : \quad (P,Q) \xrightarrow[((in_1,pr_1),(in_2,pr_2);\, id_R)]{} (P\oplus R, Q\oplus R) \xleftarrow[((0,0),(0,0);\,\alpha)]{\alpha} (0,0).$$

Clearly, every object of $\mathcal{S}\mathcal{T}(P,Q)$ is isomorphic to such a one.

So what I want now is to calculate the set of maps from $\langle R,\alpha\rangle$ to $\langle R',\alpha'\rangle$.

$$\begin{tikzcd}
 & P\oplus R \arrow[dl, shift right, "p"'] \arrow[d, shift right, "j"'] \\
P \arrow[ur, shift right, "i"'] \arrow[dr, shift right, "i'"'] & P\oplus R' \arrow[u, shift right, "q"'] \arrow[l, shift right, "p'"']
\end{tikzcd}$$

$$q(p\oplus r') = q(p\oplus 0) + q(0\oplus r')$$
$$= p \oplus \theta r'$$

$j(p\oplus 0) = (p\oplus 0).$

$q(p\oplus r') = (p, \ \theta(p+r')$

$q\, j(0\oplus r) = 0$

So suppose (P, Q) is in the identity component of $\mathcal{SS}$. I propose to define the category of <u>stable trivializations</u> of (P,Q), or of <u>stable isomorphisms</u> of P and Q. Its objects should be diagrams

$$(P,Q) \longrightarrow (P',Q') \longleftarrow (0,0)$$

in $\mathcal{SS}$ with morphisms given by comm. diagrams

$$(*) \qquad \begin{tikzcd} (P,Q) \arrow[r] \arrow[dr] & (P'',Q'') & (0,0) \arrow[l] \arrow[dl] \\ & (P',Q') \arrow[u] & \end{tikzcd}$$

As a start I should try to determine the components of this category. Observe it is non-empty iff $cl(P) = cl(Q)$ in $K_0\mathcal{P}$.

NOTATION: $\mathcal{ST}(P,Q)$

Notice that the map $(*)$ is an isomorphism iff

$$(P',Q') \longrightarrow (P'',Q'')$$

is a pair of isom. $P' \xrightarrow{\sim} P''$, $Q' \xrightarrow{\sim} Q''$. Thus I may replace $\mathcal{ST}(P,Q)$ by the equivalent subcategory whose objects are diagrams of the form

$$(P,Q) \longrightarrow (P',P') \longleftarrow (0,0)$$

Assuming the second arrow is given by an autom $\alpha : P' \xrightarrow{\sim} P'$, if I have

$$(P',P') \xrightarrow{(\theta_1,\theta_2)} (P',P')$$

α gets changed to $\theta_2 \alpha \theta_1^{-1}$. Therefore $\mathcal{ST}(P,Q)$ is equivalent will send $(\theta_1, \theta_2) \longmapsto id$ if $\exists\, \alpha \in Aut(P)$ such that

$$\alpha = \theta_2 \alpha \theta_1^{-1}$$

i.e. $$\alpha \theta_1 = \theta_2 \alpha$$

i.e. $$\alpha \theta_1 \alpha^{-1} = \theta_2 .$$ In particular $g_1 g_2$ is conjugate to $g_2 g_1$, so we see that

$$(g_1 g_2, g_2 g_1) \longmapsto id \qquad (x, x) \longmapsto id$$

and therefore $\exists$ factorization

$$\begin{tikzcd}
Aut(P) \times Aut(P) \arrow[rr] \arrow[dr] & & Aut(F(P)) \\
& Aut(P)^{ab} \arrow[ur] &
\end{tikzcd}$$

$$(\theta_1, \theta_2) \longmapsto \bar{\theta}_1 \bar{\theta}_2^{-1}$$

so let us compute the fundamental group, which must be abelian as we have an h-space.

$F(P)$ morp. inv. functor

can suppose ~~the~~ the pair $(0 \leftrightarrows P, id_P)$ gives the identity $F(0) \longrightarrow F(P)$.

Then given P $\alpha \in Aut(P)$

$$(0 \leftrightarrows P, \alpha)_* : F(0) \longrightarrow F(P) = F(0)$$

so we get an element $[P,\alpha] \in Aut(F(0))$.

Observe it depends only on the conj. class?

$$(P \underset{id}{\overset{id}{\rightleftarrows}} P, \beta)\,(0 \leftrightarrows P, \alpha)_*$$

$$(P \underset{id}{\overset{id}{\rightleftarrows}} P, \alpha)_* \in Aut(F(P)).$$

$$Aut_{\partial G}(P) = Aut(P) \times Aut(P).$$

$$Aut(P)^2 = Aut_{\partial G}(P) \longrightarrow Aut(F(P)) \simeq Aut(F(0)).$$

$$\begin{tikzcd}
P \arrow[rr, "(P \underset{\theta}{\overset{\theta}{\leftrightarrows}} P\text{, } \alpha)"] & & P \\
& 0 \arrow[ul, "(0 \leftrightarrows P\text{, } id_P)"] \arrow[ur, "(0 \leftrightarrows P\text{, } \alpha)"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& (0,0) \arrow[dl, "\beta"'] \arrow[dr, "\theta_2\beta\theta_1^{-1}"] & \\
(P,Q) \arrow[rr, "(\theta_1,\theta_2)"'] & & (P',Q') \\
& (0,0) \arrow[ul, "\alpha"] \arrow[ur, "\theta_2\alpha\theta_1^{-1}"'] &
\end{tikzcd}$$

$\alpha^{-1}\beta : P \to P$ $\qquad$ $(\theta_2\alpha\theta_1^{-1})^{-1}\theta_2\beta\theta_1^{-1}$

$\qquad\qquad\qquad\qquad$ $\theta_1\alpha^{-1}\beta\theta_1^{-1}$

Actually it seems preferable to use pair notation:

$$\begin{tikzcd}
(P,P) \arrow[rr, "{(\theta_1,\theta_2)}"] & & (Q,Q) \\
& (0,0) \arrow[ul, "P \overset{\alpha}{\simeq} P"] \arrow[ur, "Q \overset{\theta_2\alpha\theta_1^{-1}}{\simeq} Q"'] &
\end{tikzcd}$$

This is a commutative triangle in the diagonal subcat. Thus if F is morphism-inverting we get a comm. Δ

$$\begin{tikzcd}
F(P) \arrow[rr, "{(P \underset{\theta_1}{\overset{\theta_1^{-1}}{\rightleftarrows}} P,\ \theta_2\theta_1^{-1})_*}", "\sim"'] & & F(P) \\
& F(0) \arrow[ul, "{(0 \leftrightarrows P, \alpha)_*}", "\wr"'] \arrow[ur, "\wr", "{(0 \leftrightarrows P,\ \theta_2\alpha\theta_1^{-1})_*}"'] &
\end{tikzcd}$$

so what I want next is to see that ~~...~~ there is ~~...~~.

$\overline{(P,P)}$ $\qquad P \xrightarrow{\alpha} P \qquad \boxed{\text{no}}$

~~$(P,P) \xrightarrow{(\theta_1,\theta_2)} (Q,Q)$~~

$$\begin{tikzcd}
(P,Q) \\
(0,0) \arrow[u, shift left, "\alpha : P \simeq Q"] \arrow[u, shift right, "\beta : P \simeq Q"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
(P,Q) \arrow[rr, "{(\theta_1,\theta_2)}"] & & (P,Q) \\
& (0,0) \arrow[ul, "\alpha : P \simeq Q"] \arrow[ur, "\theta_2\alpha\theta_1^{-1} : P \to Q"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
P \arrow[rr, "(P \underset{\theta}{\overset{\theta^{-1}}{\leftrightarrows}} P,\ \beta)"] & & P \\
& 0 \arrow[ul, "(0 \leftrightarrows P,\ \alpha)"] \arrow[ur, swap, "(0 \leftrightarrows P,\ \beta\theta\alpha\theta^{-1})"] &
\end{tikzcd}$$

$$\begin{tikzcd}
(P,P) \arrow[rr, "(\theta,\beta\theta)"] & & (P,P) \\
& (0,0) \arrow[ul, "P \overset{\alpha}{\simeq} P"] \arrow[ur, swap, "P \xrightarrow{\beta\theta\alpha\theta^{-1}} P"] &
\end{tikzcd}$$

observe that the Δ

$$\begin{tikzcd}
(P,P) \arrow[rr, "(\theta_1,\theta_2)"] & & (P,P) \\
& (0,0) \arrow[ul, "P \overset{\alpha}{\simeq} P"] \arrow[ur, swap, "P \xrightarrow[\sim]{\theta_2\alpha\theta_1^{-1}} P"] &
\end{tikzcd}$$

shows that

$$(\theta_1,\theta_2)_* : F(P) \longrightarrow F(P)$$

will be the iff $\exists\ \alpha \ni \quad \theta_2\alpha\theta_1^{-1} = \alpha$

ie. $\theta_2\alpha = \alpha\theta_1$

example, this implies it is the identity if $\alpha = id$ and $\theta_2 = \theta_1$

or if $\theta_2 = \alpha\theta_1\alpha^{-1}$.

are conjugate. $\therefore$ the quotient is abelian

$g_1^{-1}(g_1g_2)g_1 = g_2g_1 \qquad (g_1g_2, g_2g_1)_* = 1.$

The point to observe probably is that if we have

$$0 \to P \to E \to V \to 0$$

and an auto Θ of E inducing the identity on P and on V, then I can conclude by means of the isom.

$$E \times_V E = E \times P$$

that $\Theta \times_V \Theta$ is conjugate to $\Theta \oplus id_P$

$$\begin{tikzcd} & (E,E) \\ (0,0) \arrow[ur, "\theta"] \end{tikzcd} \qquad (E \times_V E,\ E \oplus P)$$

$$\begin{tikzcd} (E,E) \\ (0,0) \arrow[u, "\theta"] \end{tikzcd} \qquad E \oplus P,\ E$$

$$\begin{tikzcd} (0,0) \arrow[r, shift left, "\theta"] \arrow[r, shift right, "id_E"'] & (E,E) \arrow[r] & (\quad) \end{tikzcd}$$

- basic fact: a map $\gamma : P \to Q$

$$\gamma = \left(\cancel{P\oplus R, Q \in A}\ P \oplus R \simeq Q,\ \Theta \in Aut(Q) \right)$$

and we observe that any autom. of Q

maybe we should try the functor

$$R \longmapsto \mathrm{Iso}(P\oplus R,\, Q\oplus R) \times^{\mathrm{Aut}(Q\oplus R)} K_1$$

<u>Question:</u> Why is an ~~elem.~~ elem. auto trivial?

$$\begin{tikzcd}[column sep=small] & (P,P) & \\ (0,0) \arrow[ur, "\theta"] & & (0,0) \arrow[ul, "id"'] \end{tikzcd}$$

$$P = P_1 \oplus P_2$$

$$\theta = id + in_1 \tau pr_2 \qquad \tau : P_2 \longrightarrow P_1$$

~~is it possible that~~

$$0 \to P_2 \longrightarrow P \longrightarrow P_1 \to 0$$

$$\begin{tikzcd} (P,P) \arrow[d, bend left] \\ (0,0) \arrow[u, bend left] \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] (0,0) \arrow[dr, "id"] & \\ & (P,P) \\ (0,0) \arrow[ur, "\theta"] & \end{tikzcd}$$

~~identity of P.~~

Thus the basic problem is clear. Why in the diagonal category does an elementary auto go to zero?

?? ~~then~~

~~(0,0) ... (P,P)~~

In any case what happens is ~~clear.~~

$$
\begin{tikzcd}
 & & (P\oplus R, P\oplus R) & \\
 & & (P,P) \arrow[u] & \\
(0,0) \arrow[uurr, "\alpha\oplus id"] \arrow[urr, "\alpha"] & & & (0,0) \arrow[ul, "id"] \arrow[uul, "id"']
\end{tikzcd}
$$

The category consists of (P,α) with a map $(P,\alpha) \longrightarrow (P',\alpha')$

Given (P,Q) with $d\,P = d\,Q$ ~~in~~ in K_0 I ~~[illegible]~~ was interested in paths in $\mathcal{IS}$ ~~[illegible]~~ of the form

$$(P,Q) \longrightarrow (P',Q') \longleftarrow (0,0)$$

These form the objects of a category in which the maps are

$$
\begin{tikzcd}
(P,Q) \arrow[r] \arrow[dr] & (\ ) \arrow[d] & (0) \arrow[l] \arrow[dl] \\
 & (\ ) &
\end{tikzcd}
$$

commutative. Alternately objects are pairs $(R, P\oplus R \xrightarrow{\sim} Q\oplus R)$ and a morphism ~~is an iso~~ $(R,\theta) \longrightarrow (R',\theta')$ is an iso class of an isom

$$R\oplus S \xrightarrow{\sim} R' \quad \ni$$

$$
\begin{array}{ccc}
P\oplus R\oplus S & \xrightarrow{\sim} & P\oplus R' \\
\wr\big| \theta\oplus id_S & & \wr\big| \theta' \\
Q\oplus R\oplus S & \xrightarrow{\sim} & Q\oplus R'
\end{array}
$$

commutes.

Thus we have the cofibred category over $\mathcal{I}$ with discrete fibres defined by the functor

$$R \longmapsto \mathrm{Iso}(P\oplus R, Q\oplus R).$$

If P,Q are 0. Then we have the functor

$$R \longrightarrow \mathrm{Aut}(R)$$

<u>not reasonable</u>.

Thus before I can get my hands on the univ. covering of $\mathcal{IS}_0(0)$ I need to select a max. tree, unless of course there is a natural candidate for the universal covering. It might ~~as~~ assoc. to a pair P, Q the set of ~~solutions of the~~ equivalence classes of stable isos. of P and Q.

$$(P,Q) \qquad\qquad \mathrm{Iso}(P\oplus R,\ Q\oplus R)$$

Thus we will consider paths in $\mathcal{IS}$ of the form.

$$\begin{tikzcd}
 & (\ , \ ) & \\
(P,Q) \arrow[ur] & & (0,0) \arrow[ul]
\end{tikzcd}$$

Is it possible to put a reasonable equivalence relation on these? We are considering ~~pairs of~~ ~~map~~ objects of $\mathcal{IS}$ which are under (P,Q) and $(0,0)$.

Question: Is a component of this category what we want?

e.g. take $\qquad (P,P) \longrightarrow (\quad) \longleftarrow (0,0)$

suppose we have $\alpha: P\oplus R \simeq P\oplus R$

better suppose we have $\alpha: P \xrightarrow{\sim} P$

becoming 0 in the K_1A.

already a reasonable question to ask

$$\begin{tikzcd}
 & (P,Q) & \\
(0,0) \arrow[ur, "\alpha"] & & (0,0) \arrow[ul, "\beta"']
\end{tikzcd}$$

$\alpha: P \longrightarrow Q$
$\beta \qquad \longrightarrow$

$\beta^{-1}\alpha \in \mathrm{Aut}(P).$

The program:

We have defined $\mathcal{IS}$ cons. of pairs (P,Q) and determined its homology. I want now to compute its homotopy groups with basepoint at 0.

Since the 0-comp $\mathcal{IS}$ is an H-space we known

$$\pi_1(\mathcal{IS}^0) \simeq H_1(\mathcal{IS}^0) = \varinjlim_P H_1(\mathrm{Aut}\, P) = K_1 \mathcal{P}.$$

I should give the isom. Enough to give a functor

$$\mathcal{IS}^0 \longrightarrow K_1\mathcal{P}$$

$$P,Q \longmapsto \cdot$$

$$(P\oplus R, Q\oplus R) \simeq (P',Q') \longmapsto$$

so you would have to first of all give a maximal tree, i.e. join (P,Q) to $(0,0)$, and this amounts to giving a stable isom

$$P\oplus S \underset{\beta}{\cong} Q\oplus S$$

$$H_*(\mathcal{S}) = \bigoplus_{S} H_*(\mathrm{Aut}\, P_s)$$

$$H_*(\mathcal{S})[S^{-1}] = \mathbb{Z}[S] \otimes H_*(\mathcal{S}) \Big/ S$$

a/s $\qquad a \in H_*(\mathrm{Aut}\, P_t)$

$s \in S$

$t - s = \alpha.$

~~Let $G: \mathcal{I} \longrightarrow$ groups be the functor with $G(P) = \mathrm{Aut}(P)$~~

Let G be the functor from $\mathcal{I}$ to groups such that

$$G(P) = \mathrm{Aut}(P)$$

and such that if (i,p) denotes the morphism from P to P' given by the pair

$$P \underset{i}{\overset{p}{\rightleftarrows}} P'$$

then $(i,p)_* : G(P) \longrightarrow G(P')$ is the homomorphism given by

$$((i,p)_*\alpha)\, i = i\alpha$$
$$(i,p)_*\alpha = id \quad \text{on } \mathrm{Ker}(p).$$

Associated to this functor is a cofibred category over $\mathcal{I}$, which will be denoted $G\mathcal{I}$, whose fibre over P is the group $G(P)$ regarded as a category. The category $G\mathcal{I}$ has the same objects as $\mathcal{P}$, and a morphism from P to P' in $G\mathcal{P}$ is a pair $(\alpha; i, p)$ consisting of a morphism $(i,p): P \longrightarrow P'$ in $\mathcal{I}$ and an element α of $G(P)$. Composition is defined by the formula

$$(\alpha; i', p')(\beta; i, p) = (\alpha \circ (i',p')_*(\beta);\ i'i,\ p'p).$$

We may identify $G\mathcal{I}$ with the full subcat of $\mathcal{S}\mathcal{I}$ consisting of the pairs (P,P) as follows. Associate to P the pair (P,P) and to $(\alpha; i, p): P \longrightarrow P'$

the triple consisting of the pair

$$P \underset{i\theta}{\overset{p\theta^{-1}}{\rightleftarrows}} P' \quad , \quad P \underset{i}{\overset{p}{\rightleftarrows}} P'$$

and the isomorphism

$$\operatorname{Ker}(p\theta^{-1}) \simeq \operatorname{Ker} p$$

induced by θ. We note that this second description shows that $G\mathcal{I}$ is a connected h-space.

We now wish to determine the fundamental group $\pi_1(G\mathcal{I}, 0)$.

Let $G : \mathcal{S} \to$ Groups be the functor with

$$G(P) = \mathrm{Aut}(P)$$

and where if u is a morphism from P to Q in $\mathcal{S}$ given by $\theta : P \oplus R \simeq Q$, ~~is a morphism in $\mathcal{S}$~~ then $u_* : G(P) \to G(Q)$ is the composition

$$\mathrm{Aut}(P) \longrightarrow \mathrm{Aut}(P \oplus R) \xrightarrow{\sim} \mathrm{Aut}(Q)$$

$$\alpha \longmapsto \alpha \oplus id_R \longmapsto \theta(\alpha \oplus id_R)\theta^{-1}$$

Associated to this functor is a cofibred cat, over $\mathcal{S}$, to be denoted $\mathcal{S}G$, ~~We let $\mathcal{S}G$ denote the cofibred category over $\mathcal{S}$~~ with fibre the group $G(P)$ over P. The objects of $\mathcal{S}G$ are the same as the objects of $\mathcal{S}$ and a morphism from P to Q in $\mathcal{S}G$ is a pair (u, α) consisting of a morphism $u : P \to Q$ in $\mathcal{S}$ and an $\alpha \in \mathrm{Aut}(P)$ with composition defined by

$$(v, \beta)(u, \alpha) = (vu, \beta v_*(\alpha))$$

We may identify $\mathcal{S}G$ with the full subcat of $\mathcal{S}\mathcal{S}$ consisting of the pairs (P, P) as follows. Associate to P the pair (P, P) and to $(u, \alpha) : P \to Q$, where u is det. by $\theta : P \oplus R \xrightarrow{\sim} Q$, the pair of iso.

$$\theta : P \oplus R \xrightarrow{\sim} Q, \quad \alpha\theta : P \oplus R \xrightarrow{\sim} Q.$$

This second description shows us that $\mathcal{S}G$ has a assoc. comm. unitary operation. Thus $\mathcal{S}G$ is a connected H-space.

Lichtenbaum letter:

GOAL:

Theorem: A = ring of integers in a number field F
$\Rightarrow K_i A$ finitely generated for all i.

Recall new definition of groups $K_i A$, A arbitrary:

$\mathcal{P}_A$ = proj. f.g. A-modules

$Q(\mathcal{P}_A)$

$BQ(\mathcal{P}_A)$ a classifying space for $Q(\mathcal{P}_A)$

Defn: $K_i A = \pi_{i+1}(BQ(\mathcal{P}_A))$.

Thm: Equivalent to the old ones.

By classical results of Serre one only has to show that $H_i(BQ(\mathcal{P}_A), \mathbb{Z})$ is finitely gen. for i. I should mention that Serre's results apply even though $BQ(\mathcal{P}_A)$ is not simply-connected, because it is a connected H-space. In fact, the direct sum operation on projective modules makes $Q(\mathcal{P}_A)$ into a "permutative" category, hence by the Segal-Anderson theory, $BQ(\mathcal{P}_A)$ is an infinite loop space.

$H_*(BQ(\mathcal{P}_A), \mathbb{Z})$ = derived functors of $\varinjlim$ on $\mathrm{Hom}(Q(\mathcal{P}_A), ab)$

$F_n Q(\mathcal{P}_A)$ = full subcategory of proj modules of rank $\leq n$.

Fix n, set

$$\mathcal{C} = F_n Q(\mathcal{P}_A)$$

$$\mathcal{C}' = F_{n-1} Q(\mathcal{P}_A)$$

and let $f : \mathcal{C}' \to \mathcal{C}$ be the inclusion.
We consider the spectral sequence

$$E^2_{pq} = H_p(\mathcal{C}, L_q f_! \mathbb{Z}) \Longrightarrow H_{p+q}(\mathcal{C}', \mathbb{Z}).$$

$$(L_q f_! \mathbb{Z})(M) = H_q(f/M, \mathbb{Z})$$

where f/M is the fibred category over $\mathcal{C}'$ whose objects are pairs (N, u), N in $\mathcal{C}'$, $u : fN \to M$.

Identify f/M with the ordered set of admissible layers (M_0, M_1) of M such that M_1/M_0 has rank $< n$.

Question: To understand L. conjectures.
curves over finite fields
how to relate orders homotopy groups
with Euler characteristics.

Possibility: Deligne's idea of recovering the ζ-function using the symmetric products

$$\frac{1}{\det(1-A)} = 1 + \mathrm{tr}A + \mathrm{tr}(S^2A) + \cdots$$

$$= e^{\sum \frac{1}{k}\mathrm{tr}(A^k)}$$

~~Goal: Prooof of Mather's theorem and the splitting theorem.~~

Acyclicity of the map $f : \underline{\underline{J}} \longrightarrow \Gamma$:

Given a topological category C such that the source map ArC → ObC is etale, we can form the standard resolution of the final object of $C^{\#}$ ($\#$ = a roof, $C^{\#}$ is the category of sheaves over ObC with right ArC-action, right C-objects):

$$Ar_2C \underset{\mu}{\overset{pr_2}{\rightrightarrows}} ArC \longrightarrow ObC$$

Alternative description: Consider the inclusion functor i from the subcategory I(C) of C with same objects and only identity morphisms; then the above simplicial object is the standard resolution associated to the pair of adjoint functors $(i_!, i^*)$:

$$Ar_{n+1}C = (i_! i^*)^{n+1}(ObC).$$

~~Claim this resolution can be used to compute cohomology of f*L for any Γ-sheaf L in the sense that Z[Ar_pC]_x is~~

Claim for any injective Γ-sheaf L that

$$H^*(J;f^*L) = H^*(n \longrightarrow Hom_J((i_{!ab}i^*)^{n+1}Z;f^*L))$$

$$= H^*(n \longrightarrow Hom_\Gamma(f_{!ab}(i_{!ab}i^*)^{n+1}Z;L))$$

To establish the first isomorphism we must know that

$$Ext^+_J((i_{!ab}i^*)^{n+1}Z, f^*L) = 0$$

But $i_{!ab}$ is exact, hence i^* carries injectives to injectives, so

$$Ext^q_J(i_!A; B) = Ext^q_{ObJ}(A ; i^*B)$$

and in the case where $B = f^*L$, we have that i^*f^*L is just the pullback of L over Ob Γ^* to L over Ob J; as this pullback is evidently injective, done.

Now $f_{!ab}(i_{!ab}i^*)^{p+1}Z = Z[Ar_pJ \times_{Ob\Gamma} Ar\Gamma^*]$, of the sort that the complex $n \longrightarrow f_{!ab}(i_{!ab}i^*)^{n+1}Z$ has for its stalk over y the chains on the nerve of the category of pairs (x,u) where x is an object of J and $u: y \longrightarrow f(x)$. Now Mather's axiom implies that this category is equivalent to the fibre category over y, so we are reduced to proving the nerve of the fibre category has trivial homology.

Acyclicity of the nerve of the fibre category.

The homology of the nerve is the same as the homology of the category, defined as derived functors of the ind.lim. functor on the category $\mathrm{HOm}(\mathcal{C}^o, Ab)$:

$$L_p \operatorname{lim.ind.}_{C^o} F = H_p(n \longmapsto F \otimes \mathbb{Z}Ar_{n+1}C)$$

Suffices by universal coefficients formulas to show this is zero when F is the constant functor with value a field k.

Because $\mathcal{C}_x$ is fibred over $\angle$, we have a spectral sequence

$$E^2_{pq} = H_p(\angle, n \longmapsto H_q(J_{xn}, F_n)) \Longrightarrow H_{p+q}(J_x;F)$$

Take $F = k$.

J_{xn} is the category with objects $(x,a_o,\dots,a_n)$/ x fixed where a morphism from this to another with primes is a diffeomorphism of $/x,a_n/ \longrightarrow /x,a_n'/$ which is the identity near x and near a_j is the translation carrying a_j to a_j'. It is clear that J_{xn} is isomorphic to the product

$$I_1^* \times I_1 \times \dots I_1 \qquad J_{nx}$$

where I_1^* is the full subcategory of I_1 consisting of the non-degenerate intervals. The map sends $(0,b),(0,a_1),\dots,(0,a_n)$ into

$$(-1, b-1, b-1+a_1, \dots) \qquad \text{assuming } x = -1.$$

Thus $H.(J_{nx}) = H.(I_1^*) \otimes R^{\otimes n}$. Identify the E^1 term with the bar resolution of k over R tensored over R with $H.(I_1^*)$. Then getting

$$E^2 = \mathrm{Tor}_p^R(H.(I_1^*),k)_q .$$

Next recall that

$$R = k \oplus H.(BG) \qquad \pi_o(I_1) = \{1,e\}$$

where e is in the center, and is an idempotent. Thus $H.(I_1^*) = Re$ is a projective R-module, and we see the Tor is zero except for k in degree 00.

Outline of the proof of the cohomological Mather theorem.

Simplicial/topological categories $\mathcal{J}$ and $\mathcal{W}$, and the maps

$$\overline{\mathcal{T}}/\mathcal{W} \xrightarrow{\gamma} \overline{\mathcal{J}}/\mathcal{J} \longrightarrow \Gamma$$

Acyclicity of the first map for constant coefficients: Need

$$E_2 = H^p(n \mapsto H^q(\overline{\mathcal{J}}_n, F_n)) \Longrightarrow H^{p+q}(\overline{\mathcal{J}}, F)$$

and corresponding one for $\overline{\mathcal{W}}$ to reduce to showing that $\overline{\mathcal{J}}_n \xrightarrow{\gamma_n} \overline{\mathcal{W}}_n$ is acyclic for constant coefficients. Using equivalent subcategories one sees that $\overline{\mathcal{J}}_n$ is equivalent to $\mathcal{W}_n \times \mathcal{K}$ where $\mathcal{K}$ is the ~~category~~ topological groupoid with objects $(x,0)$. Need somekind of argument to reduce to showing $\mathcal{K}$ acyclic, and another argument to show this follows from the contractibility of the $Ar_n Q$.

Acyclicity of the second map for all Γ-sheaves: Denote second map by f. To prove $R^+ f_* f^* = 0$ enough to do so after pulling back over $Ar\,\Gamma$, i.e. after forgetting the Γ-action. Identify the pullback: G-sheaves/$Ar\,\Gamma$ $\longrightarrow$ Γ-sheaves/$Ar\,\Gamma$ with G'-sheaves $\longrightarrow$ sheaves over R. G'-sheaves same as C-sheaves for a suitable category object in the topos of sheaves over R, hence by your notes it suffices to show that the fibre categories are acyclic, i.e. have trivial homology. One identifies any fibre category with the simplicial category $B(M',M)$ where $M = W_1$ and M' is M minus the degenerate object. Then one uses the descent spectral sequence in homology plus computation of the Tor term to prove the acyclicity of the fibres.

Use $\underline{\underline{I}}^{\sim}$ to denote I-sheaves

New notation: $\underline{\underline{I}}$ will be the simplicial category withobjects $(a_0,a_1,\dots,a_n)$ monotone sequences in ~~NNNN~~ with $a_0 = 0$. Thus $\underline{\underline{I}}$ is the ~~classifying~~ nerve of the monoid category $\underline{\underline{A}}^*$ with objects the intervals $/0,n/$ for each n $\mathbb{N}$.

$\underline{\underline{J}}$ be be the simplicial topological ~~sat~~ groupoid with objects $(x,a_0,\dots,a_n)$.

Γ will be as before.

Use letters u,v for monotone maps.

$f : \underline{\underline{J}} \to \Gamma$ functor sending $(x,a_0,\dots,a_n)$ to x, f_n its restriction to $\underline{\underline{J}}_n$.

~~XXXXXX~~ $\underline{\underline{J}}'$ the subcategory of $\underline{\underline{J}}$ with same objects ~~by...with~~ whose morphisms are the morphisms in $\underline{\underline{J}}$ which become identity morphisms in Γ. ~~Then~~ $f' : \underline{\underline{J}}' \to R\_$ the ~~induced~~ morphism induced by f ($R\_$ viewed as a category with only the identity morphisms).

$g : \underline{\underline{J}} \to \underline{\underline{I}}$ functor sending $(x,a_0,\dots,a_n)$ to $(0,a_1-a_0,\dots,a_n-a_0)$, ~~XXXXXX~~ $g_n : \underline{\underline{J}}_n \to \underline{\underline{I}}_n$ itseffect in degree n.

This seems to include all of the data in the proof. Next we need to understand the ~~proof~~ details. First identify the sheaves, which are essentially contravariant functors on the categories. Thus a $\underline{\underline{J}}$-sheaf consists of a family of sheaves $F_{a_0..a_n}$ on $R\_$ for each monotone sequence in N with action data expressible as follows. Denote by $F_{xa_0..a_n}$ the stalk of the sheaf $F_{a_0..a_n}$ at x; then given a monotone map $u : (0,..,m) \to (0,..,n)$ and a diffeo. germ $h : (x,a_{u(0)},\dots,a_{u(m)}) \to (x',a_0',\dots,a_m')$, one has a map

$$(u,h)^*: F_{x'a_0'..a_m'} \to F_{xa_0..a_n}.$$

I recall we have defined the topological groupoid G_n as follows. Its objects are sequences $(z, a_0, \ldots, a_n)$ where z is a real number less than 0 and $a_0 \ a_1 \quad a_n$ are in N. Morphisms are germs of diffeomorphisms $h: /z, a_n/ — /z', a_n'/$ which for each j, $0 \quad j \quad n$, coincide in a neighborhood of a_j with the translation carrying a_j to a_j'. We topologize the objects by taking a basic neighborhood of $(z, a_0, \ldots, a_n)$ to consist of the points $(z', a_0, \ldots, a_n)$ with z' running over a neighborhood of z in R. We topologize the morphisms by taking a basic neighborhood of $h : (z, a_0, \ldots, a_n) — (z', \ldots \quad)$ to consist of the germs Θ restricted to $/x, a_n/$ where Θ is a diffeomorphism representing h and x runs over a neighborhood of z. Given h :etc., let Θ ; be a diffeomorphism of an open interval containing $/z, a_n/$ with an open interval containing the target $/z', a_n'/$ representing the germ h. Then the set of germs $\Theta_x : /x, a_n/ — /\Theta(x), a_n/$ with x running over a nbd of z in the domain of Θ represented by Θ form a basic neighborhood for h in Ar G_n.

Claim G_n is a topological groupoid and that the source and target maps are etale.

Next we have the functor from G_n to W_n sending $(z, a_0, \ldots, a_n)$ to $(0, a_1 - a_0, \quad)$. I want to show this map induces isomorphisms on cohomology with locally constant coefficients. W_n is equivalent to its full subcategory consisting of the objects $(0, a_1, \quad a_n)$ with . G_n is equivalent to the full subcategory consisting of objects $(z, a_0, \ldots, a_n)$ with $a_0 = 0$, and this subcategory is the product of the category $\underline{\underline{V}}$ with objects $(z, 0)$ and W_n. W_n is in turn equivalent to its full subcategory consisting of the objects $(0, a_1, \ldots, a_n)$ with $a_n \quad 2n$, and this subcategory is the n-fold direct product of the category $pt // \underline{\underline{G}}$, $\underline{\underline{G}}_1$ denoting the category defined by the group G of diffeomorphsisms of $/0,1/$ with support in the interior. Thus we have equivalences

$$G_n \qquad \underline{\underline{V}} \times (pt // G)^n$$

$$W_n \qquad (pt // G)^n$$

hence to prove the claim it suffices to show that $\underline{\underline{V}}$ has no cohomology with constant coefficients.

We do this by showing each of the spaces $Ar_n\underline{V}$ are contractible and using the spectral sequence (of Cech type)

$$E_2^{pq} = H^p(n \to H^q(Ar_n V;F)) \Longrightarrow H^{p+q}(\underline{V},F)$$

~~Given a sequence $(h_1,\dots,h_n)$ with source h_{i-1} = target of h_i, we define a real number $r(h_1,\dots,h_n)$ to be the least x greater than or equal to h_n such that $h_j(z) = z$ for all j and z greater than x. Thus $r(\underline{h})$ is the upper bound for the support of of the family $\underline{h}$. Claim that $r(\underline{h})$ is continuous in $\underline{h}$. Indeed let θ_i be diffeos. representing the germs h_i. Thus if $h_i: /x_{i-1},0/ \text{---} /x_i,0/$ want θ_i to be a diffeo. of open intervals containing these closed intervals. a sequence $(\underline{h}')$ near $\underline{h}$ consists of the germs $h_i': /y_{i-1},0/ \text{--} /y_i,0/$ represented by θ_i where y_i is near x_i. There are two cases. First of all suppose $r(\underline{h})$ is greater than all of the x_i. Then $r(\underline{h}') = r(h)$ provided all y_i are less than $r(\underline{h})$. Secondly, can have max $x_i = r(h)$. Then have max y_i lessor=r(h') lessor= max$(y_i, r(h))$~~

Recall convention

Let $\underline{h} = (h_1,\dots,h_n)$ be an element of $Ar_n\underline{V}$ consisting of germs $h_i: /x_{i-1},0/ \leftarrow /x_i,0/$. Let $r(\underline{h})$ be the least $x \geq \max\{x_i\}$ such that $h_i(z) = z$ for all $z > x$ and all i; $r(\underline{h})$ is the upper bound of the support of the family $\underline{h}$. Claim $r(\underline{h})$ is a continuous function of $\underline{h}$. Indeed, let θ_i be a diffeomorphism from an open interval containing $/x_{i-1},0/$ to an open interval containing $/x_i,0/$ which represents the germ h_i. Then any element of Ar_nV near to $\underline{h}$ is of the form $\underline{h}' = (h_1',\dots,h_n')$ where $h_i':/y_{i-1},0/ \leftarrow /y_i,0/$ is the germ represented by θ_i and y_i is near x_i. ~~Then a neighborhood of h in Ar_nV consists of sequences of the form $\underline{h}' = (h_1',\dots,h_n')$, where for each i $h_i': /y_{i-1},0/ \text{--} /y_i,0/$ is the germ represented by θ_i, where y_i runs over a neighborhood of~~ If $\max\{x_i\} < r(\underline{h})$, then $r(\underline{h}') = r(\underline{h})$ provided $\max\{y_i\} < r(\underline{h})$. On the other hand, if $\max\{x_i\} = r(\underline{h})$, then

$$\max\{y_i\} \leq r(\underline{h}') \leq \max\{y_i, r(\underline{h})\},$$

so in either case we have that $r(\underline{h}')$ tends to $r(\underline{h})$ as $\underline{h}'$ tends to $\underline{h}$.

~~Set F($\underline{h}$, t) =~~ restriction of

Given $\underline{h}$ in $Ar_n V$, ~~setX(~~ let ~~d(h)~~ $/d(\underline{h}),0/$ be the source of h_n, i.e. ~~X~~. If ~~0xxxxxd(h)xxdenotexby~~ $0 \ x \ d(\underline{h})$, ~~let~~ define the restriction of $\underline{h}$ to the interval $/x,0/$, denoted $\underline{h}|/x,0/$, to be the sequence of germs $h_i' : /x_i,0/ \text{---} /x_{i-1},0/$ (where $x_0=x$, $x_{i-1}= h_i(x_i)$) induced by h_i on the smaller intervals. Set

$$F(\underline{h},t) = \begin{cases} \underline{h}\,|\,/tr(\underline{h}) + (1-2t)d(\underline{h}),0/ & 0 \leq t \leq \frac{1}{2} \\ (id_{I(t)},\dots,id_{I(t)}) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

where $I(t) = /(1-t)r(\underline{h}) + (2t-1)d(\underline{h}),0/$.

Claim $F : Ar_n V \times I \text{---} Ar_n V$ is continuous. Clear.

This gives a homotopy of the composite

$$Ar_n V \xrightarrow{\ d\ } R_- \xrightarrow{\ id^n\ } Ar_n V$$

with the identity map of $Ar_n V$ showing that $Ar_n V$ is contractible.

Mather's theorem: Logical structure.

If n $\mathbb{N}$, let $\underline{\underline{G}}_n$ be the topological groupoid with objects ~~denoted~~ sequences $(x,a_0,..,a_n)$ where x is a real no. less than 0 and $a_o - .. - a_n$ are in $\mathbb{N}$. Make these into the space

$$ObG_n = \text{disjoint union} \quad \mathbb{R}_- \quad .$$
$$a_o \qquad a_n$$

A morphism in $\underline{\underline{G}}_n$ from $(x,a_o,..,a_n)$ to $(x',a_o',..,a_n')$ consists of a germ/of diffeomorphism from $/x,a_n/$ to $/x',a_n'/$ such that

$$u(z) = z - a_i + a_i' \quad \text{for } z \text{ near } a_i \quad 0 \; i \; n.$$

One topologizes the morphisms ~~into~~ so as to be an etale ~~spac~~ space over the objects.

This is a simplicial topological groupoid: Given a monotone map ϕ: 0,..,m -- 0,..,n one defines a functor $\phi^* : \underline{\underline{G}}_n - \underline{\underline{G}}_m$ by

$$\phi^*(x,a_o,..,a_m) = (x,a_{\phi(0)},..,\phi(m))$$

and in the obvious way for the morphisms.

Evident ~~augmentation~~ augmentation to Γ: sending $(x,a_o,..,a_n)$ to x. (In this one ~~uses~~ ~~that~~ takes the objects of Γ to be $\mathbb{R}_-$.)

Functor to the simplicial category $\underline{\underline{A}}_n = (\underline{\underline{A}})^n$. Here $\underline{\underline{A}}$ is the category ~~with~~ one object ~~xxx0xxx~~ for each n $\mathbb{N}$ and ~~diffeomorphisms between the intervals~~ a morphism from m to n = a diffeo of 0,m with 0,n /having the standard identity behavior at the endpoints. The functor from $\underline{\underline{G}}_n$ to $\underline{\underline{A}}_n$ sends $(x,a_o,..,a_n)$ to the object $(a_1-a_o,..,a_n-a_{n-1})$ of $\underline{\underline{A}}^n$ ~~transpre~~ translating diffeos. in the evident way- $u:(x, a_o,..,a_n) -- (x',a_o',..,a_n')$ goes to the diffeo from $(a_1-a_o,..,a_n-a_o)$ to $(a_1'-a_o',..,a_n'-a_o')$ given by

$$z -- u(z+a_o) - a_o' .$$

Claim: $\underline{\underline{G}}_n$ equivalent to the disjoint union of the full subcategories consisting of $(x,a_0,..,a_n)$ with a_n n and $a_o = 0$. i.e. $\underline{\underline{G}}_n$ has 2^n components. ~~What this involves is producing~~. Enough to consider the case $n = 0$ to get the idea. The problem is to assign to (x,a_o) an isomorphism with $(x,0)$, ~~continuous~~ continuously for all x. Therefore I ask for a diffeom. ~~XXXXXXXXXXXXXXXXXXXXXXXXXX~~

$$u_x : \mathbb{R} - \mathbb{R}$$

~~such that~~ ~~$u_x(z)$~~

~~I have now defined~~. Assume

Lemma (?): Let P be a $\underline{\underline{G}}$-torsor over X such that the map $P \to \mathrm{Ob}G$ is acyclic ($R^q f_* . f^* = 0$ q 0 and $=$ id for $q = 0$), in particular if this map is a fibre~~wise~~ homotopy equivalent to the identity map of ObG. Then

$$H^*(\underline{\mathscr{G}}^-, F) = H^*(B, P \times^{\underline{\underline{G}}-} F).$$

(Eve rything here involves sheaf cohomology so there should be no problem.)

Lemma (?): There exists a $\underline{\underline{\mathscr{G}}}^-$-torsor P over a CW complex B satisfying the conditions of the above lemma. Moreover $\underline{\underline{G}}$ can be ~~chosen in a functorial fa~~ constructed functorially from $\underline{\underline{G}}$. (DENOTE by $B^*\underline{\underline{G}}$)

~~[illegible]~~

Lemma (?): (Dold's lemma) Let $P \to X$ be a map. Assume that there is a numerable covering of X such that the map when restricted to each finite intersetion of members of the covering is a homotopy equivalence. Then the map is a homotopy equivalence.

(This should have as consequence ~~theorem~~ that if $P \to$ OBG is a fibre equivalence, then B/ classifies homotopy bundles over numerable coverings, hence over all paracompact spaces.)

~~Thenxx~~ Consequences of the first two lemmas:

i) There is a ~~universal~~ CW complex BG which is universal among all paracompact spaces.

ii) The cohomology of the BG is the same as the G-cohomology with coefficients in any G-sheaf.

iii) One has a homotopy equivalence of BBG and $B/^-$ in the Mather situation, because we can take our basic diagram of topological groupoids and make it into a diagram of CW complexes, and then the isos. on cohomology which have been established by sheaf theory will give by the Whitehead theorem actual homotopy equivalences.

The only thing that might not be true is the second lemma; one must use ~~special~~ something special about the groupoid G, such as the fact that Ob G is a nice space.

A G-sheaf consists of a family of sheaves $F_{a_o..a_n}$ on R_- for each (weakly) monotone sequence of elements of N, together with for each monotone map u $0,..,m$ $0,..,n$ ~~a map~~ and germ $h : (x,a_{u(o)},...,a_{u(m)})$ —— $(x',a_o',...,a_m')$

$$(u,h)_* : F_{xa_o..a_m} \text{ — } F_{x'a_o'..a_n'}$$

CONFUSED

Given a monotone map $u: 0,..,m - 0,..,n$ there is a functor from G_n to G_m sending $(x,a_o,...,a_n)$ to $(x,a_{u(o)},...,a_{u(m)})$. And this leads as we know to a map

$$(u,id?) : F_{x,a_{u(o)},...,a_{u(m)}} \text{ -- } F_{x,a_o,...,a_n}$$

combined with a morphism h : $x'a_o'..a_m' \longleftarrow xa_{u(o)}..a_{u(m)}$ one obtains a map

$$(u,h)^* : F_{x'a_o'..a_m'} \text{ —— } F_{xa_o..a_n}$$

(Thus it seems we want to ~~consider~~ contravariant functors on categories, On one hand so that the category of G-sheaves appears as functors on a suitable topologica category, on the other hand to include things like the / - sheaves of functions and forms which are natumrally contravariant functors with respect to diffeomorphisms.)

Thus G is the topological category with objects $xa_o..a_n$ and ~~morphisms~~ in which a morphism from $xa_o,..a_n$ to $x'a_o'..a_m'$ consists of a monotone map $u :(0,..,m)$ -- $(0,..n)$ and a germ of diffeomorphisms h : $/x,a_{u(m)}/$ -- $/x',a_m'/$ such that for $j = 0,..,m$ h coincides near $a_{u(j)}$ with the translation carrying $a_{u(j)}$ to a_j'. G-sheaves are like contravariant functors on this category.

Need the Deligne descent spectral sequence

$$E_2^{pq} = H^p(n \text{ -- } H^q(G_n,F_n)) \Longrightarrow H^{p+q}(G,F)$$

Actually we use the map of this spectral sequence to the one for the category W, because the contractibility argument will show that G_n -- W_n induces an isomorphism on cohomology with constant coefficients.

I also want the Deligne spectral sequence in homology for the fibre of G over x

$$E^2_{pq} = H_p(n \text{ -- } H_q(W_n)) \Longrightarrow H.(W)$$

Problem: Suppose M = disjoint union of the nerves of GL_n n N; M is a simplicial monoid and we can consider the simplicial category (M,M) of M acting an itself to the right. I have a desire to think of this simplicial category as being the same as the category with objects ~~freexmo~~ E ~~X~~ free module and where a morphism is a direct injection $i:E \text{---} E'$ together with a choice of complement, i.e. the category of ~~ixxx~~ pairs (Q,θ), $\theta:E+Q=E'$ is equivalent to a category with only identity morphisms. Similarly the simplicial category $(M\times M,M)$ I would like to think of as being the same as the category with objects E and with a morphism from E to E' consisting of an isomorphism class of triples (Q,u_1,u_2) where $u_1:E+Q = E'$ and $u_2:E+Q = E'$ are two isomorphisms. Again the category of such triples is equivalent to the ~~full~~ subcategory consisting of those triples with Q a complement for the image of u_1, and only identity morphisms. This category looks suspiciously like the cofibred category constructed over the category of pairs (i,Q)/ i direct injection, Q complement, from the functor $E \text{---} Aut(E)$. In fact they are the same. Therefore it would be nice to show that the simplicial catgeory (M,M) and the category of (i,Q) have the same homotopy type.

First a functor is needed. So I recall that in simplicial degree n, we are considering the category of sequences $(V_o,..,V_n)$ of projfg modules, i.e. M^{n+1}. Suupose that I have an n-simplex in the category of (i,Q), i.e. I have a vector space ~~xxxxx~~ written as a direct sum $Q_0+Q_1+\ldots+Q_n$.

Go back to the original category ~~xfxpxxxxxxx~~ D with objects E and morphisms pairs consisting of a direct injection $i : E \text{---} E'$ and a choice Q of complement for the image. Then exists an equivalent full subcategory consisting of the free modules $E = R^n$. So a morphism from R^n to R^m consists of ~~xx~~ a direct injection $i:R^n \text{--} R^m$ and a choice for the complement, i.e. a projection operator backwards. Now take a simplex in the nerve of this category; ~~itx~~ it consists of a diagram

$$R^{a_o} \qquad R^{a_1} \qquad\qquad R^{a_n}$$

with direct injection and projection operators. Now it is clear how to obtain an element of the (M,M) simplicial category, namely, take the kernels of the various projection operators.

The point somehow is this: Take the category consisting of the n-simplices in

There are many things to be understood. & = sigma. $E = sphere bundle of E. At the moment I don't understand why $B\&_{\infty}^{+}$ should have anything to do with loop spaces of the spheres. There is a basic KADL- action, and one can look at the orbit of a critical point.

Back to Mather's theorem: I have defined now a simplicial topological groupoid $G_\cdot$ with augmentation to $\bar{\Gamma}$ and a map $G_\cdot \to A_\cdot$ where $A_\cdot$ is the nerve of the monoid categroy equivalent to the union of a point to the category BG. Now there are three things to prove.

1) $G_\cdot \longrightarrow \Gamma$ is acyclic, i.e. $F = \underline{\underline{R}}f_* f^*(F)$ for all Γ-sheaves F (locally?)

2) $G_n \dashrightarrow A_n$ is acyclic for/constant coefficients.

The above two give me an isomorphism of $H^*(\Gamma,A) = H^*(BBG,A)$ for all abelian groups A. Now to finish Mather's theorem I need

3) $H^*(\Gamma,A) = H^*(B\Gamma,A)$ for all abelian groups A.

In fact what I want to prove is that for the Milnor classifying spaces the singular cohomology equals the sheaf cohomology. Precisely, the Milnor construction gives us a space $B\underline{\underline{G}}$ for any topological category $\underline{\underline{G}}$, and I hope it is not difficult to show this space has the correct/weak homotopy type, i.e. it classifies homotopy classes of G-torsors over paracompact spaces. Now I want to know that singular and sheaf cohomology for $B\underline{\underline{G}}$ are the same, because if I then replace BG by the realization of its singular complex, then I have not changed ~~the singular cohomology,~~ hence the sheaf cohomology remains the same.

The logical point is that because the space $B\underline{\underline{G}}$ is so nasty, it is not a priori clear that it doesn't have cohomology classes which die when pulled back to any finite complex, or that there are no characteristic classes for G-torsors over finite complexes that do not come from classes computed on the category of Γ-sheaves. So a more precise version of what you want is:

We have defined a map $H^*(\Gamma,F) \longrightarrow H^*(X,P\times^{\Gamma} F)$ for any Γ sheaf F. Show that this map is an isomorphism when X is a CW complex which is a $B\Gamma$. It suffices to show therefore that there is a CW complex B endowed with a G-structur