January 25, 1970: $H^*(B\Sigma_n, \mathbb{Z}_p)$ p odd.

Form the ring

$$R = \bigoplus_{n\geq 0} H_*(B\Sigma_n) \qquad \text{mod } p \text{ coefficients}$$

with multiplication

$$H_*(B\Sigma_i) \otimes H_*(B\Sigma_j) \xrightarrow{\sim} H_*(B(\Sigma_i \times \Sigma_j)) \longrightarrow H_*(B\Sigma_{i+j})$$

where the second map is restriction for the homomorphism $\Sigma_i \times \Sigma_j \to \Sigma_{i+j}$. It's useful to note that we can also form a ring

$$R(X) = \bigoplus_{n\geq 0} H_*(E\Sigma_n \times_{\Sigma_n} X^n) \qquad (X \text{ can also be a pair of spaces.})$$

the same way. ($\Sigma_n = \{e\}$ if $n = 0, 1$; it thus is the autos. of a set with n elements. $E\Sigma_n \times_{\Sigma_n} X^n = *$ if $n = 0$.) (Alternative notations to be decided upon later:

$$E\Sigma_n \times_{\Sigma_n} X^n = E\Sigma_n S X$$

$$H_*(E\Sigma_n \times_{\Sigma_n} X^n) = H_*^{\Sigma_n}(X^n).)$$

Note that if $X = (S^1, pt)$, then $R(X)$ gives the homology of Σ_n with twisted coefficients modulo p for all n.

Let B be a $\mathbb{Z}_2$-graded ring $B = B^+ \oplus B^-$ (alternative notation $B^{ev} \oplus B^{odd}$.) Then a ring hom. $R(X) \to B$ ~~is a collection of elements~~ respecting the grading is a collection of elements:

$$\alpha_k^+ : H_+^{\Sigma_k}(X^k) \longrightarrow B^+ \quad , \quad \alpha_k^- : H_-^{\Sigma_k}(X^k) \longrightarrow B^-$$

or equivalently ~~elements~~ elements

$$\alpha_k \in \left(H^*_{\Sigma_k}(X^k) \otimes B \right)^{ev}$$

such that

$$\mathrm{res}^{\Sigma_{i+j}}_{\Sigma_i \times \Sigma_j} \alpha_{i+j} = \alpha_i \otimes \alpha_j .$$

with the convention that $\alpha_0 = 1$

We denote by

$$\prod_{k \geq 0}{}' H^{ev}_{\Sigma_k}(X^k; B)$$

the set of such families $(\alpha_k)_{k \geq 0}$. As we work always with elements of even degree there should be no problem with signs so this should form a ring natural in B. Thus $R(X)$ should be an affine ring scheme in the $\mathbb{Z}_2$-graded framework.

(This is all fine except that we would like the formula

$$\mathrm{Hom}_{srgs}(R(X), B) \cong \mathrm{Hom}_{srgs}(R, B) \otimes H^*(X)$$

where srgs is the category of skew-rings. In the mod 2 case we proved this formula by means of the ~~[illegible]~~ map

$$
\begin{tikzcd}
\mathrm{Hom}_{rgs}(R(X), B) = \prod_k{}' H_{\Sigma_k}(X^k, B) & \prod_k{}' H_{\Sigma_k}(pt, B) \otimes H(X) \arrow[l, "\cong"'] \\
(\alpha_k) \cdot (Q_k x) & (\alpha_k) \otimes x \arrow[l, mapsto]
\end{tikzcd}
$$

and the fact that we had an isomorphism is because the isom. fitted into

$$
\begin{tikzcd}
\prod_k{}' H_{\Sigma_k}(X^k, B) \arrow[d, phantom, "\cong\ \text{Nakaoka iso.}"] & \\
\prod_k{}' H(B\Sigma_k, H(X^k) \otimes B) \arrow[r, phantom, "\cong"] & \prod_k{}' H(B\Sigma_k, B) \otimes H(X) \arrow[ul]
\end{tikzcd}
$$

since $\prod_k{}' H(B\Sigma_k, V^{\otimes k} \otimes B)$ additive in V.

Somehow this all has to be generalized, which means ultimately that $R(X)$ has to be enlarged to include twisted coefficients. Before doing this we should work out the operations corresponding to $\mathbb{Z}_{p^n} \subset \Sigma_{p^n}$.)

Start with

$$P : H^{ev}(X) \longrightarrow H^{ev}(B\mathbb{Z}_p \times X)^N$$

where N is the normalizer of $\mathbb{Z}_p$ in Σ_p, in this case $N = \mathbb{Z}_p^* \cong \mathbb{Z}_{p-1}$. Now P is a ring homomorphism since after inverting $v = e(\eta)$ it is and since v is a non-zero divisor in ~~$H^*(B\mathbb{Z}_p) = \mathbb{Z}_p[u, v]$ where u generates $H^1(B\mathbb{Z}_p)$~~

$$H^*(B\mathbb{Z}_p) \cong \Lambda[u] \otimes S[v]$$

Here u denotes the generator of $H^1(B\mathbb{Z}_p)$ with $\beta u = v$.
Now if $i \in \mathbb{Z}_p^*$ then $i^*(u) = i^{-1} \cdot u$ so

$$H^*(B\mathbb{Z}_p)^N \cong \Lambda[dw] \otimes S[w]$$

where $w = c_{p-1}(\mathrm{reg}\,\mathbb{Z}_p) = -v^{p-1}$ (Wilson's theorem $(p-1)! \equiv -1 \ (p)$)
and dw is an abusive notation for $-uv^{p-2}$, so that $\beta dw = w$.
So set

$$\cancel{Px = \sum_{i \geq 0} v^i P_i x + u v^i P'_i x}$$

~~where if~~ $\cancel{x \in H^{2q}(X)}$ ~~then~~

$$\cancel{P_i x \in H^{2pq-2i}(X)}$$

$$\cancel{P'_i x \in H^{2pq-2i+1}(X)}$$

$$Px = \sum_{i \geq 0} (w^i\, St_i\, x) + \sum_{i \geq 1} (dw \cdot w^{i-1}\, St'_i\, x)$$

where if $x \in H^{2q}(X)$, then

$$\deg(St_i\, x) = 2pq - 2i(p-1) = 2q + 2(p-1)(q-i)$$

$$\deg(St'_i\, x) = 2pq - 2i(p-1) + 1 = 2q + 2(p-1)(q-i) + 1$$

Now these aren't stable operations and the correct stable operations are

$$w^{-q} Px = \sum_{i \geq 0} w^{-q+i} St_i x + \sum_{i \geq 1} dw \cdot w^{-q+i-1} St_i' x$$

$$\cancel{w^{-q} Px = \sum w^{-j} St^j x + \quad St}$$

$$w^{-q} Px = \sum_{j \geq 0} w^{-j} St^j x + \sum_{j \geq 0} w^{-j-1} \cdot dw \cdot St^{j\prime} x$$

total degree

$2pq - q \cdot 2(p-1) = 2q$

Not clear yet, but perhaps will become so when the Milnor generators are introduced.

Suppose fixed therefore elements dw, w such that

$$H^*(B\mathbb{Z}_p)^N = \Lambda[dw] \otimes S[w]$$

Then I get a basis $w^i, dw \cdot w^i$ for $H^*(B\mathbb{Z}_p)^N$ and I let $\delta_i, \varepsilon_i \in H_*(B\mathbb{Z}_p)_N \xrightarrow{\sim} H_*(B\Sigma_p)$ be the dual basis. Then

$$\deg(\varepsilon_i) = 2i(p-1) - 1 \qquad i \geq 1$$
$$\deg(\delta_i) = 2i(p-1) \qquad i \geq 0$$

and we have the formula

$$Px = \sum_{i \geq 0} w^i \underset{\substack{\parallel \\ St_i x}}{\langle \delta_i, Px \rangle} + \sum_{i \geq 1} dw \cdot w^{i-1} \underset{St_i' x}{\langle \varepsilon_i, Px \rangle}$$

Up to ~~[illegible]~~ elements of $\mathbb{Z}_p^*$ we have that for $x \in H^{2q}(X)$

$$St_i x = P^{q-i} x$$

$$St_i' x = \beta P^{q-i} x$$

January 26, 1970.

The following example arose in conversation with Scharlau and shows that Norm$_f$ for a double covering is a new operation not expressible in terms of f_*.

Scharlau asked whether there was a simple formula for $w_t(f_* E)$, where $f: X \longrightarrow Y$ is a double covering, E a real vector bundle on X, in terms of the characteristic classes of E and the covering and f_*.

If E is a line bundle, then we know that

$$w_1(f_* E) = f_*(w_1 E) + r \qquad r = \text{char. class of the covering } f$$

$$w_2(f_* E) = \mathrm{Norm}_f\, w_1(E)$$

Suppose that $w_2(f_* E)$ could be expressed in terms of r, f_*, $w_1(E)$. The relevant monomials are

$$f_*((w_1 E)^2), \quad (f_*(w_1 E))^2, \quad r f_*(w_1(E)), \quad r^2$$

and these are the only way to get elements of degree 2 in the base. Now apply f^*. Then

$$f^* w_2(f_* E) = w_2(f^* f_* E) = w_2(E + \sigma E) = w_1(E) \cdot \sigma w_1(E)$$

where σ is the $\mathbb{Z}_2$-action on X.

$$f^* f_* w_1(E)^2 = w_1(E)^2 + (\sigma w_1(E))^2$$

$$f^*(f_*(w_1(E)))^2 = (w_1 E + \sigma w_1(E))^2 = (w_1 E)^2 + (\sigma w_1 E)^2$$

$$f^*(r f_* w_1(E)) = r \cdot (w_1 E + \sigma w_1(E)) \qquad f^* r^2 = r^2$$

and you see that you don't get the cross term

$$\omega_1(E) \cdot \sigma \omega_1(E)$$

from the expressions.

January, 26, 1970 (Invariants in $\Lambda V^* \otimes SV^*$).

Let l be an odd prime and work over Z_l. I propose to complete the symmetric **invariants** in $\Lambda[x_1, \ldots, x_n] \otimes S[y_1, \ldots, y_n]$. We regard this as the ~~[illegible]~~ algebraic de Rham complex of affine n-space with $dy_i = x_i$. Let c_i be the ith symmetric function of $y_1, \ldots, y_n$. I claim that the natural map

$$(*) \qquad \boxed{\begin{array}{l}\Lambda[dc_1, \ldots, dc_n] \otimes S[c_1, \ldots, c_n] \xrightarrow{\cong} \{\Lambda[x_1, \ldots, x_n] \otimes S[y_1, \ldots, y_n]\}^{\Sigma_n} \\ \text{if } \tfrac{1}{2} \text{ exists.}\end{array}}$$

is an isomorphism. Now we know this map is injective since

$$dc_1 \wedge \cdots \wedge dc_n = \operatorname{Jac}\left(\frac{c_1, \ldots, c_n}{y_1, \ldots, y_n}\right) dy_1 \wedge \cdots \wedge dy_n$$

and the Jacobian is non-zero since the map $A_n \longrightarrow A_n$ ~~[illegible]~~ given by the c_i is generically etale (invariants of a group acting on a field gives a separable extension). Actually

$$\operatorname{Jac}\left(\frac{c_1, \ldots, c_n}{y_1, \ldots, y_n}\right) = \pm \prod_{i<j} (y_i - y_j)$$

since the right-side divides the left and the degrees are equal $\left(\frac{n(n-1)}{2}\right)$. This we did before.

The new idea is to note that for a Galois covering $X \xrightarrow{f} Y$ $\Omega^{\cdot}_Y = f_*(\Omega^{\cdot}_X)^G$. In fact for any vector bundle E on Y one has $(f_* f^* E)^G = E$ and $f^* \Omega^i_Y = \Omega^i_X$ as f is étale. Now this means that on inverting the discriminant

$$\Delta = \prod_{i<j} (y_i - y_j)^2,$$

the map $(*)$ becomes an isomorphism. Suppose that $\lambda = a\, dy_1 \wedge \cdots \wedge dy_n$

here's where char $\neq 2$ is used to see that $a \equiv 0 \ (y_i - y_j)$

is an invariant n-form. Then a is skew-invariant, hence divisible by J and so $a = f(c_1, \ldots, c_n) J$, so $\lambda = f(c) dc_1 \wedge \cdots \wedge dc_n$.
Suppose λ is an invariant q-form. By the above argument $\exists N$ $\ni$

$$\Delta^N \cdot \lambda = \sum_{i_1 < \cdots < i_q} f_{i_1, \ldots, i_q} \, dc_{i_1} \wedge \cdots \wedge dc_{i_q}$$

where $f_{i_1, \ldots, i_q}$ are elements of $S[c]$. Assume N least such that this holds. Let $I' = j_1, \ldots, j_{n-q}$ be complementary to $i_1, \ldots, i_q = I$ (Hörmander's notation), then

$$\Delta^N \cdot \lambda \cdot dc_{I'} = f_I \, dc_1 \wedge \cdots \wedge dc_n$$

But $\lambda \cdot dc_{I'}$ is invariant hence of the form $g(c) dc_1 \wedge \cdots \wedge dc_n$, so

$$\Delta^N \cdot g = f_I$$

for all I showing that $N = 0$ by minimality. Thus $(*)$ is proved. (This argument generalizes to other Lie groups, see page 6)

Can the above argument be generalized to Dickson's theorem? ~~The situation is more complicated but runs as follows~~ Let V be a vector space of dimension n over $\mathbb{F}_q$. We set

$$\prod_{\lambda \in V^*} (X + \lambda) = X^{q^n} + c_{q^n - q^{n-1}} X^{q^{n-1}} + \cdots + c_{q^n - 1} X$$

Then the c's are the analogues of the elementary symmetric functions and we know that

$$S(V^*)^{Gl(V)} = \mathbb{F}_q[c_{q^n-q^{n-1}}, \dots, c_{q^n-1}]$$

Consider the mapping $\Phi : V_\Omega \longrightarrow \Omega^n$ given by the functions c_j. If the action of $Gl(V)$ at a geometric point ξ is not free, say $g\xi = \xi$ where $g \neq 1$, then as $\operatorname{Ker}(g-1)$ is defined over $\mathbb{F}_q$ it follows that $\xi \in \bigcup W_\Omega$ where W runs over the codimension 1 subspaces of V. Thus the bad set of Φ is this union which is where $c_{q^n-1} = 0$. Therefore as Φ is a Galois covering on the complement we have that

$$(*) \qquad \Lambda[dc_{q^n-q^{n-1}}, \dots, dc_{q^n-1}] \otimes S[c_{q^n-q^{n-1}}, \dots, c_{q^n-1}] \longrightarrow (\Lambda V^* \otimes SV^*)^{Gl(V)}$$

is injective and becomes an isomorphism after c_{q^n-1} is inverted.

Suppose that $\lambda \in \Lambda^n V^* \otimes SV^*$ is an invariant n-form. Choose a basis $e_1, \dots, e_n$ and let $y_1, \dots, y_n$ be the dual basis of V^*. Then

$$\lambda = a\, dy_1 \cdots dy_n$$

where since

$$g^*(dy_1 \cdots dy_n) = (\det g)(dy_1 \cdots dy_n)$$

we have

$$g^*(a) = (\det g)^{-1} a.$$

Let $S \subset V^* - 0$ be a set of representatives for the lines in V^* and set

$$\gamma_{\frac{q^n-1}{q-1}} = \prod_{\lambda \in S} \lambda$$

Note that

$$c_{q^n-1} = \prod_{\lambda \in S} \prod_{z \in \mathbb{F}_q^*} z\lambda = (-1)^{n} \gamma^{q-1}$$

and that γ divides any element of $S(V^*)$ which vanishes on $\bigcup W_\Omega$. It follows that γ is a semi-invariant ie

$$g^*\gamma = (\det g)^a \gamma \quad \text{for some } a \in \mathbb{Z}_{q-1}$$

To determine a take g to be a scalar matrix $\overbrace{}^{z\cdot I}$ whence

$$g^*\gamma = z^{\frac{q^n-1}{q-1}} \cdot \gamma = z^{na} \cdot \gamma$$

for all $z \in \mathbb{F}_q^*$, hence

$$n \equiv \frac{q^n-1}{q-1} \equiv na \quad \text{mod } q-1$$

which tends to suggest but doesn't prove that $a = 1$ unless n is prime to $q-1$. To do it correctly use embedding of Gl_1 in Gl_n. In fact since we've chosen a basis $y_1, \dots, y_n$ for V^* it follows that we can take

$$\gamma = \prod_{(z_2,\dots,z_n)\in\mathbb{F}_q^{n-1}} (y_1 + z_2 y_2 + \dots + z_n y_n) \prod_{(z_3,\dots,z_n)\in\mathbb{F}_q^{n-2}} (y_2 + z_3 y_3 + \dots + z_n y_n) \dots$$

$$= \prod_{\mu\in W^*} (y_1 + \mu) \prod_{\lambda\in S(W)} \lambda$$

where W is the subspace $y_1 = 0$. Thus if $g = \begin{pmatrix} z & & \\ & 1 & \\ & & 1 \end{pmatrix}$ we have

$$g^*\gamma = \prod_{\mu\in W^*} (zy_1 + \mu) \prod_{\lambda\in S(W)} \lambda$$

$$= \prod_{\mu\in W^*} (zy_1 + z\mu) \prod_{\lambda\in S(W)} \lambda$$

$$= z^{q^{n-1}} \gamma = z\gamma \qquad \text{since } q^{n-1} \equiv 1 \mod q-1$$

Thus

$$\boxed{g^*\gamma = (\det g)\,\gamma \qquad \text{all } g\in Gl(V)}$$

Let $f\in S(V^*)$ be a semi-invariant not an invariant, i.e. $g^*f = (\det g)^{\varepsilon} f$ where $0<\varepsilon< q-1$. I claim that f is divisible by γ. It suffices to show that f vanishes on a hyperplane W. Take $W: y_1=0$ and let g be $\begin{pmatrix} z & & \\ & 1 & \\ & & \ddots \end{pmatrix}$ matrix. Then $(g^*f)(w) = f(gw) = f(w)$ and $(g^*f)(w) = (\det g)^{\varepsilon}\cdot f(w) = z^{\varepsilon}\cdot f(w)$. Thus $f(w)=0$. So we have proved

<u>Lemma:</u> Let $f\in S(V^*)$ satisfy $g^*f = (\det g)^{\varepsilon}\cdot f$ for all $g\in Gl(V)$ where ε is an integer $0\le\varepsilon< q-1$. Then

$$f = \gamma^{\varepsilon}\cdot f_1$$

where $f_1\in S(V^*)^{Gl(V)}$.

As an application consider

$$dc_{q^n-q^{n-1}}\cdots dc_{q^n-1} = \mathrm{Jac}\left[\frac{c_{q^n-q^{n-1}},\ldots,c_{q^n-1}}{y_1,\ldots,y_n}\right] dy_1\cdots dy_n$$

Then the Jacobian J has degree

$$(q^n-q^{n-1}-1)+\cdots+(q^n-1-1) = n(q^n-1)-\frac{q^n-1}{q-1}$$

so

$$\mathrm{Jac}\left[\frac{c_{q^n-q^{n-1}},\ldots,c_{q^n-1}}{y_1,\ldots,y_n}\right] = d\cdot\gamma^{q-2}\cdot c_{q^n-1}^{\,n-1} = d'\cdot\gamma^{nq-n-1}$$

where d,d' are non-zero constants.

Consequently the map $(*)$ page 3 is not an isomorphism on n-forms since $\gamma^{q-2}\,dy_1\wedge\cdots\wedge dy_n$ is invariant yet not a multiple of $\mathrm{Jac}\cdot dy_1\wedge\cdots\wedge dy_n$

~~I yet don't have a conjecture as to what the invariant forms are and propose now to compute for $n=2$. Let $V^* = \mathbb{F}_q y_1 \oplus \mathbb{F}_q y_2$. Then~~

$$\prod_{z_1,z_2}(X+z_1y_1+z_2y_2) = \prod_{z_1}\left[(X+z_1y_1)^q-(X+z_1y_1)y_2^{q-1}\right]$$

$$= \prod_{z_1}(X^q-Xy_2^{q-1})+z_1(y_1^q-y_1y_2^{q-1})$$

$$= (X^q-Xy_2^{q-1})^q-(X^q-Xy_2^{q-1})(y_1^q-y_1y_2^{q-1})^{q-1}$$

$$= X^{q^2}-X^q\left(y_2^{q^2-q}+(y_1^q-y_1y_2^{q-1})^{q-1}\right)$$

$$+X(y_2y_1^q-y_1y_2^q)^{q-1}$$

Borel has pointed out ~~[illegible]~~ for a semi-simple Lie algebra (maybe even reductive) that the Jacobian $\mathrm{Jac}\left[\dfrac{c_1,\ldots,c_\ell}{y_1,\ldots,y_\ell}\right]$ has the same degree as Δ^2 where Δ is the basic anti-invariant (product of the positive roots) at least in char. 0. Thus if mod p $H(BT)\to H(G/T)$ is onto, then

$$H(BG)=H(BT)^W \;\&\; p\neq 2 \implies H(BG)\otimes H(G) \xrightarrow{\sim} (H(BT)\otimes H(T))^W$$

because the argument given on pages 1+2 generalizes.

<u>proof of 2nd theorem:</u> G algebraic group ~~which one understands~~ l-good prime for G, A maximal elementary abelian l-subgroup of G^{σ}. (If l is odd then A is unique; hopefully if $l=2$, then all such A have the same rank.) Z centralizer of A.

<u>Lemma:</u> l odd $\Longrightarrow$ Z connected & l good for Z

<u>Proof:</u> Let A act on G by conjugation. Then in the spectral sequence

$$H_A^* \otimes H^*(G) \Longrightarrow H_A^*(G)$$

the fibre is thick so $H_A^*(G)$ is a free H_A^*-module. Similarly for $G \times G$, and

$$H_A^*(G) \otimes_{H_A^*} H_A^*(G) \xrightarrow{\sim} H_A^*(G \times G)$$

since both terms are free of same rank and the map reduces to an isomorphism modulo the augmentation ideal of H_A^*. It follows that $H_A^*(G)$ is a Hopf algebra over H_A^*. On the other hand take generators for $H^*(G)$ and lift them to $H_A^*(G)$. Since l is odd, they generate an exterior algebras. Thus $H_A^*(G) \simeq H_A^* \otimes H^*(G)$ as algebras

By the localization theorem

$$H_A^*(G)[w^{-1}] \cong H_A^*(Z)[w^{-1}] = H_A^*[w^{-1}] \otimes H^*(Z)$$

~~Now it is clear that $H_A^*[w^{-1}] \otimes H^*(G)$ has no idempotents,~~ hence Z is connected.

~~By Borel's~~

Proof: Consider $H_A^*(G)$ where A acts on G by conjugation. Since ℓ is good for G we know that the fibre is thick in the spectral sequence

$$H_A^* \otimes H^*(G) \Longrightarrow H_A^*(G)$$

so $H_A^*(G)$ is a free H_A^* module with

$$H_A^*(G) \otimes_{H_A^*} \mathbb{Z}_\ell \cong H^*(G).$$

Let e_i be a basis for $PH^*(G)$ and lift them to elements $\tilde{e}_i$ of $H_A^*(G)$. Since ℓ is odd the $\tilde{e}_i$ generate an exterior subalgebras and so there is an algebra isomorphism

$$(*) \qquad H_A^*(G) \cong S(A^*) \otimes \Lambda A^* \otimes H^*(G)$$

which shows that $H_A^*(G)$ is an exterior algebra over $S(A^*)$ with odd degree generators.

By the localization thm. there is an isomorphism

$$H_A^*(G)[w^{-1}] \cong H_A^*(Z)[w^{-1}],$$

and as the latter is alg. isom. to $S(A^*)[w^{-1}] \otimes \Lambda A^* \otimes H^*(Z)$, it follows

By looking at the odd-even grading & using $(*)$ that $H^*(Z)$ is generated as an algebra over $\mathbb{Z}_\ell$ by its elements of odd degree. Since ℓ is odd and

$H^*(Z)$ is a Hopf algebra, $H^*(Z)$ is an exterior algebra by Borel's theorem. ~~More precisely~~

Thus Z is connected and l is good for Z.

<u>January 27, 1970</u> $H^*(B\Sigma_n, \mathbb{Z}_p)$ where p is odd (cont.)

A review of the Milnor description of the dual of the Steenrod algebra:

Following Grothendieck for any (anti-) commutative graded $\mathbb{Z}_p$-algebra S introduce the base extension $H^*(X)\otimes S$ and consider the group of multiplicative stable S-automorphisms of $H^*(X)\otimes S$. This gives a functor from S to groups which is represented by the dual of the Steenrod algebra, $H_*(K(\mathbb{Z}_p,\infty)) = a_*$. Thus any multiplicative stable operation

$$H^*(X) \longrightarrow S \otimes H^*(X)$$

is obtained from the universal one

$$\theta : H^*(X) \longrightarrow a_* \otimes H^*(X)$$

by composition with a ring homomorphism $a_* \longrightarrow S$. According to Milnor if $\eta \in H^1(X)$, then

$$\theta(\eta) = 1 \otimes \eta + \sum_{i \geq 0} \tau_i \otimes (\beta\eta)^{p^i} \qquad \tau_i \in a_{2p^i-1}$$

$$\theta(\beta\eta) = \sum_{i \geq 0} \xi_i \otimes (\beta\eta)^{p^i} \qquad \xi_i \in a_{2p^i-2}, \ \xi_0 = 1$$

and moreover

$$\Lambda[\tau_1, \tau_2, \ldots] \otimes S[\xi_1, \ldots] \xrightarrow{\sim} a_*$$

Now I want to apply these results to analyze the

cohomology operations obtained from Klein groups.
Fix an integer r and consider the operation

$$H^{ev}(X) \xrightarrow{P} H^{ev}(BZ_{p^r} \times X).$$

Then we know geometrically that

$$P\, e(L) = e(\rho \otimes L) \qquad \rho = \text{reg } Z_{p^r}$$
$$= \sum_{i=0}^{r} c_{p^r - p^i}\, e(L)^{p^i}$$

If σ_2 is the generator of $\tilde{H}^2(S^2)$, then $P\sigma_2 = c_{p^r-1}\sigma$ showing that if we invert c_{p^r-1}, we obtain a stable operation. As c_{p^r-1} is a non-zero divisor in $H^*(BZ_{p^r})$, it follows that P must be additive before inverting c_{p^r-1}. Let R be the stable operation obtained from P:

$$Rx = (c_{p^r-1})^{-q} Px \qquad \text{if } x \in H^{2q}(X)$$

Then R extends to all degrees

$$R: H^*(X) \longrightarrow H^*(BZ_{p^r})[c_{p^r-1}^{-1}] \otimes H^*(X)$$

and hence there is a canonical ring homomorphism

$$\Phi: \mathcal{a}_* \longrightarrow H^*(BZ_{p^r})[c_{p^r-1}^{-1}]$$

which I will now very carefully calculate. According to Milnor we must see what happens to the ξ_i, τ_i, or equivalently

what happens to one-dimensional classes and their Bocksteins.
As $\beta\eta$ is an Euler class I know that

$$R(\beta\eta) = c_{p^r-1}^{-1} \sum_{i=0}^{r} c_{p^r-p^i} (\beta\eta)^{p^i}$$

so

$$\boxed{\Psi(\xi_i) = \frac{c_{p^r-p^i}}{c_{p^r-1}} \qquad i \geq 0}$$

we go back to operations associated to $\mathbb{Z}_{p^a}$
Thus we get a map

$$\begin{tikzcd}
H^{ev}(X) \arrow[r] \arrow[drr, dashed, "\text{conjecture}"'] & H^{ev}(B\mathbb{Z}_{p^a} \times X)^{Gl} \arrow[r] & (\Lambda V^* \otimes S V^*)^{Gl} \otimes H^*(X) \\
 & & \Lambda[dc_{q^n-q^{n-1}}, \dots, dc_{q^n-1}] \otimes S[c_{q^n-q^{n-1}}, \dots, c_{q^n-1}] \otimes H^*(X) \arrow[u]
\end{tikzcd}$$

~~Now after~~. Idea is that you have basic operation

$$H^*(X) \longrightarrow A \otimes H^*(X)$$

$A =$ dual of St alg.

universal for stable operations, so add in $A[t, t^{-1}] \otimes H^*(X)$
Thus when you construct your map

$$H^{ev}(X) \longrightarrow \{R \otimes H^*(X)\}^{ev}$$

where $R = \left(H(B\mathbb{Z}_{p^a})[c_{p^a-1}^{-1}]\right)^{Gl_n(\mathbb{Z}_p)}$ you get a homomorphism

$$A[t, t^{-1}] \longrightarrow R$$

<u>so here's the situation:</u>

$$H^{ev}(X) \xrightarrow{P} H^{ev}(B\mathbb{Z}_{p^a} \times X)[c_{p^a-1}^{-1}]$$

gives a basic homomorphism from $A[t, t^{-1}] \longrightarrow H^*(B\mathbb{Z}_{p^a})$
which I want to understand. According to Milnor, there is a nice way of describing A.

$$\begin{array}{ccc} & G\times G & & G \\ & \downarrow & & \downarrow \\ & EG\times G & \longrightarrow & EG \\ & \downarrow & & \downarrow \\ & BG & \longrightarrow & BG \end{array}$$

more significant is

$$\begin{array}{c} G\times G \\ \downarrow \\ EG \times EG \\ \downarrow \\ BG\times BG \end{array}$$

~~the idea is to introduce~~
The idea is to embed your ~~pairing~~ map as a diagonal of sorts mimicking the Bott–Samelson pairing

in the spectral sequence

$$\cancel{E_2^{pq} = H^p(BG)\otimes H^q(G) \longrightarrow H^{p+q}(*)}$$

~~one knows~~

$$E^2 = H_*(BG)\otimes H_*(G) \longrightarrow H_*(pt)$$

one knows that E^r_{pq} admits a module structure over E^r_{pq}

$$\ni \quad d_r(x\cdot y) = d_r x\cdot y + (-1)^{\deg x}\, x\cdot d_r y .$$

~~[illegible]~~ What is the nature of this pairing?

now intuitively we have a map

$$\underbrace{H_*(BG)\otimes H_*(G)}$$

January 29, 1970. On the Steenrod operations for p odd.

1) Why the "norm" is equivalent to P_{ext}:

Recall how we define the norm map for a finite covering of degree k, $f: X \to Y$, in terms of P_{ext}.

$$U^{2q}(X) \xrightarrow{P_{ext}} U^{2kq}_{\Sigma_k}(X^k) \xrightarrow{res} U^{2kq}_{\Sigma_k}((X/Y)^k_{reg}) \underset{descent}{\cong} U^{2kq}(Y)$$

where $(X/Y)^k_{reg}$ is the subset of $(X/Y)^k$ consisting of k-tuples $(x_1,\ldots,x_k)$ where the x_i are distinct. (Alternatively if we think of Σ_k as the automorphisms of a set S, then

$$(X/Y)^k_{reg} = \operatorname{Iso}_Y(Y\times S, X)$$

is the principal bundle describing the covering. Call this P. Then on lifting to P we get a tautological isom. of $Y\times S$ with X but not equivariant for Σ_k.)

Here's how to define P_{ext} in terms of the norm. Let S have k elements and let Σ be Aut S. Then have maps

$$\begin{array}{ccccc} X & \xleftarrow{ev} & X^S \times S & \xrightarrow{pr_1} & X^S \\ f(s) & \longleftarrow\!\shortmid & (f, s) & & \end{array}$$

and P_{ext} is the composition

$$U^{2q}(X) \xrightarrow{inf} U^{2q}_{\Sigma}(X) \xrightarrow{(ev)^*} U^{2q}_{\Sigma}(X^S\times S) \xrightarrow{Norm_{pr_1}} U^{2qk}_{\Sigma}(X^S).$$

This definition of P_{ext} yields the same P in virtue of the commutative diagram

$$\begin{tikzcd}
\mathcal{U}^{2q}_{\Sigma}(X) \arrow[r, "(ev)^*"] \arrow[dr, "pr_1"'] & \mathcal{U}^{2q}_{\Sigma}(X^S \times S) \arrow[r, "\mathrm{Norm}_{pr_1}"] \arrow[d, "(\Delta\times id)^*"] & \mathcal{U}^{2qk}_{\Sigma}(X^S) \arrow[d, "\Delta^*"] \\
& \mathcal{U}^{2q}_{\Sigma}(X \times S) \arrow[r, "\mathrm{Norm}_{pr_1}"] & \mathcal{U}^{2qk}_{\Sigma}(X)
\end{tikzcd}$$

2) Steenrod operations for sheaves:

Given a finite covering $f: X \longrightarrow Y$ and a sheaf F of abelian groups on X we have the sheaf

$$(f_* F)_y = \prod_{x\in f^{-1}\{y\}} F_x = \bigoplus_{x\in f^{-1}\{y\}} F_x$$

and also the analogues of the other elementary symmetric functions

$$(\mathrm{Norm}_f F)_y = \bigotimes_{x\in f^{-1}\{y\}} F_x$$

$$(\sigma_j F)_y = \bigoplus_{\substack{I\subset f^{-1}\{y\} \\ \mathrm{card}\, I = j}} \ \bigotimes_{x\in I} F_x$$

(it would seem that this makes sense for an arbitrary covering)

Thus

$$\bigoplus_{j\geq 0} \sigma_j F = \mathrm{Norm}_f (\mathbb{Z}\oplus F).$$

Corresponding to the map (in fact isomorphism)

$$H^0(X, F) \longrightarrow H^0(Y, f_* F)$$

there is the map

$$H^0(X, F) \longrightarrow H^0(Y, \mathrm{Norm}_f F)$$

$$s \longmapsto \left(y \longmapsto \bigotimes_{x \in f^{-1}\{y\}} s_x = (\mathrm{Norm}_f\, s)_y \right).$$

The question now is how to extend the norm of a section to higher cohomology classes.

The first step is to note that the norm map extends to complexes of sheaves. Thus if $F^\cdot$ is a complex of sheaves (bounded below?) I can define

$$(\mathrm{Norm}_f F^\cdot)_y = \bigotimes_{x \in f^{-1}\{y\}} F^\cdot_x$$

(How to write this up precisely will be a real mess, however the simplest approach appears to take $P = (X/Y)^k_{reg}$ where k is the degree of f so that

$$\begin{tikzcd}
P \times \{1, \dots, k\} \arrow[r] \arrow[d] & X \arrow[d] \\
P \arrow[r] & Y
\end{tikzcd}$$

where the horizontal arrows are principal Σ_k bundles. Then I have maps $p_i : P \longrightarrow X$ $i = 1, \dots, k$ and I form the tensor product

$$\bigotimes_{1 \le i \le k} p_i^* F^\cdot$$

Now using the natural associativity and commutativity isomorphisms for the tensor product of complexes of sheaves, I see this tensor product has a natural Σ_k action and so descends to a complex of sheaves on Y which is denoted

$$\mathrm{Norm}_f\, F^\cdot$$)

Suppose for simplicity that I am working with sheaves of $\mathbb{Z}_p$-modules. Then $F^\cdot \longrightarrow \mathrm{Norm}_f F^\cdot$ carries quasi-isos. to quasi-isos. and hence passes to the derived category. Next given a class $u \in H^q(X, F^\cdot)$ where $F^\cdot$ is a flask complex we may identify u with a homomorphism $u\colon \Sigma^{q} \longrightarrow F^\cdot$ whence we get a map

$$\cancel{\mathrm{Norm}_f(u) : H^{qk}(Y, \mathrm{Norm}_f(\Sigma^q)) \longrightarrow H^{qk}(Y, \mathrm{Norm}_f F^\cdot)}$$

$$\mathrm{Norm}_f(\Sigma^q) \longrightarrow \mathrm{Norm}_f F^\cdot .$$

Now $\Sigma^q = \mathbb{Z}_p[q]$ and one computes easily that

$$\mathrm{Norm}_f(\Sigma^q) = \mathcal{O}_f^{\otimes q}[qk]$$

where $\mathcal{O}_f$ is the sign (or orientation) bundle of the covering, that is, the sheaf $P \times_{\Sigma_d} \mathbb{Z}_p(\mathrm{sgn})$ where Σ_d acts by the sign representation on $\mathbb{Z}_p(\mathrm{sgn})$. Therefore finally we get a map

$$\mathrm{Norm}_f : H^q(X, F^\cdot) \longrightarrow H^{qk}(Y, \mathrm{Norm}_f F^\cdot \otimes \mathcal{O}_f^{\otimes q}),$$

well-defined on $D^+(X, \mathbb{Z}_p)$.

January 30, 1970 (groggy again)

Multiplicative property of the norm: Suppose given $f: X \longrightarrow Y$ a covering of degree d and $F^\cdot, G^\cdot \in D^+(X, \mathbb{Z}_p)$ and $u \in H^p(X, F^\cdot)$, $v \in H^q(X, G^\cdot)$. Then I have the element $u \cdot v \in H^{p+q}(X, F^\cdot \otimes G^\cdot)$ defined as follows. ~~Repre~~ Identify u with a map $\mathbb{Z}_p[p] \longrightarrow F^\cdot$ and v with a map $\mathbb{Z}_p[q] \longrightarrow G^\cdot$, then $u \cdot v$ is the composition

$$\mathbb{Z}_p[p+q] \cong \mathbb{Z}_p[p] \otimes \mathbb{Z}_p[q] \xrightarrow{u \otimes v} F^\cdot \otimes G^\cdot$$

Now I want to show why

$$\mathrm{Norm}_f (u \cdot v) = (-1)^{\frac{d(d-1)}{2} \deg u \cdot \deg v}\, \mathrm{Norm}_f u \cdot \mathrm{Norm}_f v$$

see page 8.

and keep things sign-consistent.

First to understand multiplicativity of Norm on sheaves.

$$\mathrm{Norm}_f (F \otimes G)_y = \bigotimes_{x \in f^{-1}y} (F \otimes G)_x \cong \bigotimes_{x \in f^{-1}y} F_x \otimes G_x$$

$$\cong \bigotimes_{x \in f^{-1}y} F_x \otimes \bigotimes_{x \in f^{-1}y} G_x$$

$$\cong \mathrm{Norm}_f F \otimes \mathrm{Norm}_f G.$$

Thus there is a canonical isomorphism

$$\mathrm{Norm}_f (F \otimes G) \cong (\mathrm{Norm}_f F) \otimes (\mathrm{Norm}_f G)$$

such that if $s \in H^0(X, F)$, $t \in H^0(X, G)$, then under this isom

$$\mathrm{Norm}(s\otimes t) = \mathrm{Norm}(s)\otimes\mathrm{Norm}(t).$$

The next stage is to understand the norm for complexes but again there is no problem. There is a canonical isom

$$(*)\qquad \mathrm{Norm}_f(F^\bullet\otimes G^\bullet) \cong (\mathrm{Norm}_f F^\bullet)\otimes(\mathrm{Norm}_f G^\bullet).$$

~~[illegible]~~ (What we are using here is that we have a fibred category over spaces with a good tensor product and Galois descent. Thus the Norm map can be defined for vector bundles, fibre spaces, etc.)

Now suppose given a cohomology class $u \in H^q(X, F^\bullet)$. First for $q=0$ we note that we have a canonical map

$$(**)\qquad \mathrm{Norm}_f : Z^0(X, F^\bullet) \longrightarrow Z^0(Y, \mathrm{Norm}(F^\bullet))$$

which is compatible with the tensor product isomorphism $(*)$. This because $F^\bullet \mapsto \mathrm{Norm}_f(F^\bullet)$ is a $\otimes$-functor and because $\mathrm{Norm}_f\,\mathbb{Z}$ is canonically isomorphic to $\mathbb{Z}$.

Next point is to ask whether the functor

$$\begin{array}{ccc} C^+(X) & \xrightarrow{\text{Norm}} & C^+(Y) \\ \downarrow & & \downarrow \\ D^+(X) & & D^+(Y) \end{array}$$

has a derived functor of some sort. Now my instincts tell me that I want the left-derived functor because the norm somehow is like an $f_!$. But to keep things simple suppose we are over a field K. Then it is clear that $F^\cdot \longrightarrow \mathrm{Norm}\, F^\cdot$ preserves quasi-isomorphisms ~~and in fact~~ since

$$\mathcal{H}^\cdot(\mathrm{Norm}\, F^\cdot)_y = \bigotimes_{x \in f^{-1}y} \mathcal{H}^\cdot(F^\cdot_x) = (\mathrm{Norm}\, \mathcal{H}^\cdot(F^\cdot))_y$$

where $\mathcal{H}^\cdot$ denotes the homology of a complex. Thus Norm extends to the derived categories and we get a map

$$\mathrm{Norm}_f : H^0(X, F^\cdot) \longrightarrow H^0(X, \mathrm{Norm}_f F^\cdot)$$

For higher cohomology we use the suspension isomorphism

$$H^q(X, F^\cdot) \cong H^0(X, \Sigma^{+q} F^\cdot)$$

where

$$\Sigma^q F^\cdot = \Sigma^q \otimes F^\cdot$$

and Σ^q is the complex with generator σ_q ~~[illegible]~~ of homological degree q and $d\sigma_q = 0$. Then we have

$$\begin{array}{ccc} H^q(X, F^\cdot) & \cong & H^0(X, \Sigma^q \otimes F^\cdot) \\ & & \downarrow \mathrm{Norm}_f \\ \Big\downarrow {\scriptstyle \text{defn. of } \mathrm{Norm}_f \text{ on } H^q} & & H^0(Y, \mathrm{Norm}_f\, \Sigma^q \otimes F^\cdot) \\ & & \wr\| \\ H^{qd}(Y, \mathcal{O}_f^{\otimes q} \otimes \mathrm{Norm}_f F^\cdot) & \cong & H^0(Y, \mathrm{Norm}_f \Sigma^q \otimes \mathrm{Norm}_f F^\cdot) \end{array}$$

where we have ~~[illegible]~~ used an isomorphism

$$\mathrm{Norm}_f\, \Sigma^{g} \cong \Sigma^{gd} \otimes \mathcal{O}_f^{\otimes g}$$

whose specific properties will determine the sign behavior. The formula that we want is

$$\boxed{\mathrm{Norm}_f(u \cdot v) = (-1)^{\frac{d(d-1)}{2} \deg u \cdot \deg v}\, \mathrm{Norm}_f u \cdot \mathrm{Norm}_f v}$$

~~There are two ways of seeing that this is the only possibility (interchanging ... causes the shift to~~

To see that this is correct we note that for the trivial covering $X \times \{1,\dots,k\}$ we want $\mathrm{Norm}_f f^* y = y^d$ and that the sign difference between $(y_1 y_2)^d$ and $y_1^d y_2^d$ is as above. This formula forces us to use the isomorphism of

$$\mathrm{Norm}_f\, \Sigma^{g} = P \times_{\Sigma_d} K(\underbrace{\sigma_g \otimes \cdots \otimes \sigma_g}_{d \cdot \text{times}}) \qquad \text{and}$$

$$\Sigma^{gd} \otimes \mathcal{O}_f^{\otimes g} = P \times_{\Sigma_d} \left(K\sigma_{gd} \otimes (\mathrm{sgn}) \right)$$

which identifies

$$\sigma_g \otimes \cdots \otimes \sigma_g \longleftrightarrow (\sigma_{gd} \otimes 1^{g}) \cdot (-1)^{\frac{d(d-1)}{2} \cdot \frac{g(g-1)}{2}}$$

The way to see the sign is to use that $\sigma_g = \sigma_1{}^{g}$ and that

$$\sigma_g{}^{\otimes d} = (\sigma_1 \sigma_1{}^{g-1})^{\otimes d} = (-1)^{\frac{d(d-1)}{2}(g-1)} \sigma_1{}^{\otimes d} (\sigma_1{}^{g-1})^{\otimes d}$$

$$= (-1)^{\frac{d(d-1)}{2}[(g-1)+(g-2)+\cdots+1]} (\sigma_1{}^{\otimes d})^{g}$$

$$= (-1)^{\frac{d(d-1)}{2}\,\frac{g(g-1)}{2}} \left(\sigma_d \otimes 1\right)^g$$

$$= (-1)^{\frac{d(d-1)}{2}\cdot\frac{g(g-1)}{2}} \left(\sigma_{gd} \otimes 1^g\right)$$

<u>Conclusion:</u> If $u \in H^g(X, F^\bullet)$, then we can define

$$\mathrm{Norm}_f(u) \in H^{dg}\left(X, \theta_f^{\otimes g \otimes}\,\mathrm{Norm}_f(F^\bullet)\right)$$

by

$$\begin{tikzcd}
H^g(X, F^\bullet) \arrow[r, "\simeq", "u \,\mapsto\, \sigma_g \cdot u"'] \arrow[dddd, "u \,\mapsto\, \mathrm{Norm}_f u"'] & H^0(X, \sigma_g \otimes F^\bullet) \arrow[d, "\mathrm{Norm}"] \\
 & H^0(Y, \mathrm{Norm}_f(\sigma_g \otimes F^\bullet)) \arrow[d, "\cong", "\text{canonical isom.}"'] \\
 & H^0(Y, \mathrm{Norm}_f\, \mathbb{Z}\sigma_g \otimes \mathrm{Norm}_f F^\bullet) \arrow[d, "\cong", "\alpha"'] \\
 & H^0(Y, \sigma_{gd} \otimes \theta_f^{\otimes g} \otimes \mathrm{Norm}_f F^\bullet) \\
H^{dg}(Y, \theta_f^{\otimes g} \otimes \mathrm{Norm}_f F^\bullet) \arrow[r, "\simeq", "z \,\mapsto\, \sigma_{gd} \cdot z"'] & H^0(Y, \sigma_{gd} \otimes \theta_f^{\otimes g} \otimes \mathrm{Norm}_f F^\bullet)
\end{tikzcd}$$

where $\alpha\left(\sigma_g^{\otimes d}\right) = (-1)^{\frac{d(d-1)}{2}\cdot\frac{g(g-1)}{2}} \sigma_{gd} \otimes 1^g.$

Now consider the case where the covering is orientable, by which I mean that the action of $\pi_1(Y, y_0)$ on the fibre $f^{-1}y_0$ has a trivial sign representation. Then the sign representation is trivial. Choosing an orientation we get an oriented covering, i.e. $f_*(\mathbb{R})$ is an oriented bundle. Thus for an oriented covering we have an isomorphism $O_f \simeq \mathbb{Z}$ and so the norm map

$$\mathrm{Norm}_f : H^q(X, F^\cdot) \longrightarrow H^{qd}(Y, \mathrm{Norm}_f F^\cdot)$$

is defined. [redacted]

Now we are interested in the case where f [redacted] is the trivial covering ~~$X \times A$~~ $pr_1 : X \times A \longrightarrow X$ where A is an elementary abelian p-group acting trivially on X. Then for any sheaf F on X we have

$$\mathrm{Norm}_f\, f^*(F) = \bigotimes_{\alpha \in A} F$$

coefficients mod p; the real regular representation of A oriented.

and there is the Steenrod operation

$$u \longmapsto \mathrm{Norm}_f(f^* u) \quad : \quad H^q(X, F) \longrightarrow H_A^{qd}(X, \otimes_A F) \qquad d = |A|$$

So if F is a commutative ring we can compose this with $\otimes F \longrightarrow F$ to get the usual Steenrod map.

So take $F = \mathbb{Z}_p$ and we have defined

$$P : H^q(X) \longrightarrow H_A^{qd}(X) \qquad \text{for all } q$$

(depends on a choice of orientation for o_f) satisfying

$$P(xy) = (-1)^{\frac{p(p-1)}{2}(\deg x)(\deg y)} Px \cdot Py$$

$$P(x+y) = Px + Py \qquad \text{(this requires proof).}$$

January 31, 1970 (groggy).

Here's how to do things for p-odd. Recall that we have defined Steenrod operations

$$P : H^{g}(X) \longrightarrow H^{gd}_{G}(X)$$

where $G \to \Sigma_d$ is an oriented representation. The problem now is to compute the effect of P on $x, \beta x$ where x is a 1-dimensional class. First we consider the case where $G = \mathbb{Z}_p \hookrightarrow \mathrm{Aut}_{sets}(\mathbb{Z}_p)$ is oriented by the ordering $0, 1, \dots, p-1$ of $\mathbb{Z}_p$.

Recall that $H^1_{\mathbb{Z}_p}(pt, \mathbb{Z}_p)$ has a canonical generator x corresponding to the covering $\mathbb{Z}_p \longrightarrow pt$ in the topos of (left) $\mathbb{Z}_p$-sets where $\mathbb{Z}_p$ acts on the right. (More generally in $H^1_G(pt, G)$ there is a canonical element.) I claim that $\beta x = c_1(\eta)$ where η is representation of $\mathbb{Z}_p$ which sends 1 to $\exp(2\pi i/p)$. Indeed we have a map of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "p"] \arrow[d, equal] & \mathbb{Z} \arrow[r] \arrow[d, "\frac{1}{p}"] & \mathbb{Z}_p \arrow[r] \arrow[d, "\eta"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "1"] & \mathbb{C} \arrow[r, "z \mapsto \exp(2\pi i z)"] & \mathbb{C}^* \arrow[r] & 0
\end{tikzcd}$$

which gives rise to commutative triangle

$$\begin{tikzcd}
H^1(X, \mathbb{Z}_p) \arrow[dr, "\delta"] \arrow[dd] & \\
& H^1(X, \mathbb{Z}) \\
H^1(X, \underline{\mathbb{C}}^*) \arrow[ur, "\delta'"'] &
\end{tikzcd}$$

for any space X.

By definition $\delta x = \beta x$ and $\delta' Q = c_1(L)$ if Q is a principal $\mathbb{C}^*$-bundle over X with associated line bundle L. The vertical arrow associates to a principal $\mathbb{Z}_p$-bundle $Y \to X$ the ~~associated~~ principal $\mathbb{C}^*$ bundle $Y \times_{\mathbb{Z}_p, \eta} \mathbb{C}^*$, so the formula $\beta x = c_1(L)$ is clear. Note this formula is independent of how δ is defined.

Thus we have canonical generators

$$H^*_{\mathbb{Z}_p}(pt) = \mathbb{Z}_p[x, \beta x] \qquad (p\text{-odd})$$

Moreover if we let $\mathbb{Z}_p^*$ act on $\mathbb{Z}_p$ by multiplication, then we have

$$i^*(x) = ix \qquad i \in \mathbb{Z}_p^*$$

where i^* is the map on cohomology induced by $i : \mathbb{Z}_p \to \mathbb{Z}_p$.

Now I want to determine the map

$$P : H^*_{\mathbb{Z}_p} \longrightarrow H^*_{\mathbb{Z}_p \times \mathbb{Z}_p}$$

as ~~p~~ P is almost a ring homomorphism, it's enough to know Px and $P(\beta x)$. Since $\beta x = e(L)$ we can use geometry. Thus ~~working with an principal~~ let $i : X \to L$ be the zero section with Thom class ~~[illegible]~~ $i_* 1 \in H^2(L, L-X)$. Then $P(i_* 1) \in H^{2p}_{\mathbb{Z}_p}(\rho \otimes L, \rho \otimes L - X)$ must be the Thom class of $\rho \otimes L$ since it has the correct restriction and so taking i^* we find

$$P(e(L)) = e(\rho \otimes L) = \prod_{i=0}^{p} e(\eta^i \otimes L) = \prod_{i=0}^{p} (e(L) + i\, e(\eta))$$

$$= e(L)^p - (\beta y)^{p-1} e(L) \quad = (\beta x)^p - (\beta y)^{p-1} \beta x$$

where to keep from getting lost we let y (resp. x) be the generator of the first (resp. 2nd) factor of $H^1_{\mathbb{Z}_p\times\mathbb{Z}_p}$. Thus for an arbitrary $x\in H^1(X)$ we have the formula

$$\boxed{P(\beta x) = (\beta x)^p - (\beta y)^{p-1}\cdot\beta x.}$$

Now $Px \in H^p_{\mathbb{Z}_p}(X)$ is well-defined after the orientation is chosen, hence is invariant under the action of $(\mathbb{Z}_p^*)^2 \subset \mathbb{Z}_p^*$. Thus we see there are constants $\alpha_1, \alpha_2 \in \mathbb{Z}_p$ such that

(*) $$Px = \alpha_1(\beta y)^{\frac{p-1}{2}}\, x + \alpha_2\, y(\beta y)^{\frac{p-1}{2}-1}\beta x$$

To evaluate α_1 take ~~$x \in H^1_c(\mathbb{R})$ be the generator~~ x to be the generator σ of $H^1_c(\mathbb{R})$. Then $P_{ext}x$ is the Thom class of the representation of $\mathbb{Z}_p$ on $\mathbb{R}^p$. We want to restrict to the diagonal subbundle $\mathbb{R} \xhookrightarrow{\Delta} \mathbb{R}^p$. Now write

$$\mathbb{R}^p \cong \Delta(\mathbb{R}) \oplus \overline{\mathbb{R}^p}$$

whence the Thom class of the representation of $\mathbb{R}^p$ is the product of the two Thom classes

$$\begin{tikzcd}
\mathbb{R}^p \arrow[r, "p_2"] \arrow[d, shift left, "p_1"] & \overline{\mathbb{R}^p} \\
\mathbb{R} \arrow[u, shift left, "\Delta"] \arrow[r] & pt \arrow[u]
\end{tikzcd}$$

$$U_{\mathbb{R}^p} = p_1^* U_{\mathbb{R}} \cdot p_2^* U_{\overline{\mathbb{R}^p}}$$

$P\sigma = \Delta^*(U_{\mathbb{R}^p}) = \sigma\cdot e(\overline{\mathbb{R}^p})$ where the last has to be computed. Now $\overline{\mathbb{R}^p} \cong \eta \oplus \cdots \oplus \eta^{\frac{p-1}{2}}$ as real representations so orienting η^i by means of its complex structure we have

$$\pm e(\overline{\mathbb{R}^p}) = \prod_{i=1}^{\frac{p-1}{2}} e(\eta^i) = \left(\tfrac{p-1}{2}\right)! \, (\beta y)^{\frac{p-1}{2}} .$$

The sign comes from whether or not the orientations agree. If the minus sign appears choose the other orientation for the covering $\mathbb{Z}_p \to pt$; this choice can be made to define P in odd dimensions at the beginning so as to make the formulas simpler. With this convention we see that

$$\boxed{P\sigma = \left(\tfrac{p-1}{2}\right)! \, (\beta y)^{\frac{p-1}{2}} \sigma \qquad \text{if } \beta\sigma = 0}$$

Next note that

$$\boxed{\beta(Pu) = 0}$$

This is the same kind of result as the additivity of P and is proved in Steenrod's book by noting that $\beta(P_{ext} u) = \operatorname{ind}_{1 \to \mathbb{Z}_p}(\beta u \otimes \underbrace{u \otimes \cdots \otimes u}_{p-1})$ restricts to zero on the diagonal. A good proof would involve producing the Pontryagin–Thomas operations

$$H^*(X, \mathbb{Z}_{p^k}) \longrightarrow H^*(X, \mathbb{Z}_{p^{k+1}})$$

which I don't yet understand.

Assuming this we see from * page 13 that $\alpha_1 + \alpha_2 = 0$ whence

$$\boxed{x \in H^{q}(X) \implies Px = \left(\tfrac{p-1}{2}\right)!\,(\beta y)^{\frac{p-1}{2}}x - \left(\tfrac{p-1}{2}\right)!\; y(\beta y)^{\frac{p-1}{2}-1}\beta x}$$

Now we can begin to use induction to determine the map

$$P^{(r)} : H^*(X) \longrightarrow H^*_{\mathbb{Z}_p^r}(X)$$

which is the iterate

$$H^*(X) \xrightarrow{P^{(1)}} H^*(B\mathbb{Z}_p \times X) \xrightarrow{P^{(1)}} H^*(B\mathbb{Z}_p^2 \times X) \longrightarrow \cdots\cdots$$

To see this note that $P^{(r)}u = u^{\otimes p^r}$ as an equivariant class under $\mathbb{Z}_p^r$ acting on itself by translations and that $(u^{\otimes p^{r-1}})^{\otimes p} = P^{(1)}P^{(r-1)}u$ with $\mathbb{Z}_p \times \mathbb{Z}_p^{r-1}$ action. These are the same clearly.

February 2, 1970. Steenrod operations for p odd. (cont.)

If $u \in H^2(X)$, then $P^{(1)}u \in H^{2p}(B\mathbb{Z}_p \times X)^{\mathbb{Z}_p^*}$ and hence there is an expansion

$$P^{(1)}u = 1 \otimes \alpha(u)_{2p} + dc_{p-1} \otimes \alpha(u)_3 + c_{p-1} \otimes \alpha(u)_2$$

where $\alpha(u)_i$ represents a class $\in H^i(X)$ depending naturally on u. Forgetting the $\mathbb{Z}_p$ action we see that $\alpha(u)_{2p} = u^p$. Now we know that

$$P^{(1)}u = 1 \otimes u^p + c_{p-1} \otimes u$$

if $u \in H^2(X, \mathbb{Z})$ since then $u = e(L)$ for some complex line bundle L. Apply β to both sides

$$0 = \beta P^{(1)} u = 0 + \beta dc_{p-1} \otimes \alpha(u)_3 - dc_{p-1} \otimes \beta\alpha(u)_3$$
$$+ c_{p-1} \otimes \beta\alpha(u)_2$$

Recall d is the derivation of degree -1 of $H^*(B\mathbb{Z}_p)$ inverse to β on elements of degree 1, so

$$(d\beta + \beta d)\, c_{p-1} = \beta d c_{p-1} = (p-1)\, c_{p-1} = -c_{p-1}$$

Thus we find that

$$\alpha(u)_3 = \beta\alpha(u)_2$$

$$\beta\alpha(u)_3 = 0. \qquad \text{(clear from } \beta^2 = 0)$$

I claim that $\alpha(u)_2 = u$. Indeed the only zero-degree cohomology operations are multiplications by elements of $\mathbb{Z}_p$ (this follows from $H^n(K(\mathbb{Z}_p,n),\mathbb{Z}_p) = p$ (by Hurewicz); alternatively $H^n(X) \hookrightarrow H^n(sk_n(X))$ and any class of $H^n(sk_n X)$ is induced by map to a sphere. Thus can assume u = can. elt of $H^2(S^2)$ and so $\alpha(u)_2 = u$. Note similarity between this and the proof that $P^0 = id.$)

Thus we obtain the formula

$$\boxed{P^{(1)} u = u^p + dc_{p-1}\cdot\beta u + c_{p-1}\cdot u \qquad \text{if } u \in H^2(X)}$$

which for generalization to higher rank elementary abelian p-groups should be written

$$P^{(1)}(\beta x) = (\beta x)^p + c_{p-1}(\beta x)$$

$$P^{(1)}(\sigma x) = -dc_{p-1}\cdot\sigma(\beta x) + c_{p-1}\,\sigma x$$

$$= \sigma\left[dc_{p-1}\cdot\beta x + c_{p-1}\cdot x\right]$$

The ~~conjecture~~ result is that these formulas generalize to

1) $$P^A(\beta x) = \sum_{i=0}^{n} c_{p^n-p^i}(\mathrm{reg}\,A)\cdot\beta x^{p^i}$$

2) $$P^A(\sigma x) = \sigma\left[\sum_{i=0}^{n-1} dc_{p^n-p^i}(\mathrm{reg}\,A)\cdot\beta x^{p^i} + c_{p^n-1}(\mathrm{reg}\,A)\cdot x\right]$$

To prove these formulas we use induction on the rank of A, writing $A = B \times \mathbb{Z}_p$ where y is the canonical generator of $H^1(\mathbb{Z}_p)$. Then (recall $e(\eta) = \beta y$)

$$e_t(\operatorname{reg} A) = \prod_{i=0}^{p-1} e_t(\operatorname{reg} B \otimes \eta^i) = \prod_{i=0}^{p-1} e_{t+i\beta y}(\operatorname{reg} B)$$

By induction hypothesis

$$e_t(\operatorname{reg} B) = \sum_{i=0}^{r-1} c_{p^{r-1}-p^i}(\operatorname{reg} B)\, t^{p^i}$$

is an additive function of t, hence

$$e_t(\operatorname{reg} A) = \prod_{i=0}^{p-1} \{ e_t(\operatorname{reg} B) + i\, e_{\beta y}(\operatorname{reg} B) \}$$

$$= e_t(\operatorname{reg} B)^p - e_{\beta y}(\operatorname{reg} B)^{p-1} e_t(\operatorname{reg} B),$$

proving $e_t(\operatorname{reg} A)$ is additive in t.

We now prove formula 1) by induction assuming it is true for B. Then

$$P^A(\beta x) = P^B(P^{(1)}(\beta x)) = P^B\{ (\beta x)^p - (\beta y)^{p-1} \beta x \}$$

$$= (P^B(\beta x))^p - (P^B(\beta y))^{p-1} P^B(\beta x)$$

$$= e_{\beta x}(\operatorname{reg} B)^p - e_{\beta y}(\operatorname{reg} B)^{p-1} e_{\beta x}(\operatorname{reg} B)$$

$$= e_{\beta x}(\operatorname{reg} A)$$

by the above. Now we prove formula 2) starting with

$$P^{(1)}(\sigma x) = \sigma\left[\, y(\beta y)^{p-2}\cdot\beta x - (\beta y)^{p-1}\cdot x\,\right]$$
$$= (\beta y)^{p-2}\{\sigma y\cdot\beta x - \sigma x\cdot\beta y\}$$

so

$$P^A(\sigma x) = \left(P^B(\beta y)\right)^{p-2}\{P^B(\sigma y)\cdot P^B(\beta x) - P^B(\sigma x)\,P^B(\beta y)\}$$

$$= e_{\beta y}(\mathrm{reg}\,B)^{p-2}\sigma\cdot\Big\{\sum_{i=0}^{r-1}\sum_{j=0}^{r-1} dc_{p^{r-1}-p^i}(\mathrm{reg}\,B)\,c_{p^{r-1}-p^j}(\mathrm{reg}\,B)\left[(\beta y)^{p^i}(\beta x)^{p^j} - (\beta x)^{p^i}(\beta y)^{p^j}\right]$$
$$+ c_{p^{r-1}-1}(\mathrm{reg}\,B)\left[y\,e_{\beta x}(\mathrm{reg}\,B) - x\,e_{\beta y}(\mathrm{reg}\,B)\right]\Big\}$$

using induction hypothesis. ~~Then~~ But

$$\sum_{i=0}^{r} dc_{p^r-p^i}(\mathrm{reg}\,A)(\beta x)^{p^i} + c_{p^r-1}(\mathrm{reg}\,A)\,x = d\,e_{\beta x}(\mathrm{reg}\,A) + c_{p^r-1}(\mathrm{reg}\,A)\cdot x$$

$$= d\{e_{\beta x}(\mathrm{reg}\,B)^p - e_{\beta y}(\mathrm{reg}\,B)^{p-1}e_{\beta x}(\mathrm{reg}\,B)\} + c_{p^r-1}(\mathrm{reg}\,A)\,x,$$

$-e_{\beta y}(\mathrm{reg}\,B)^{p-1}c_{p^r-1}(\mathrm{reg}\,B)$ $\|$

$$= e_{\beta y}(\mathrm{reg}\,B)^{p-2}\{d\,e_{\beta y}(\mathrm{reg}\,B)\cdot e_{\beta x}(\mathrm{reg}\,B) - e_{\beta y}(\mathrm{reg}\,B)\,d\,e_{\beta x}(\mathrm{reg}\,B)\} + c_{p^r-1}(\mathrm{reg}\,A)\cdot x$$

$$= e_{\beta y}(\mathrm{reg}\,B)^{p-2}\Big\{\sum_{i,j=0}^{r-1} dc_{p^{r-1}-p^i}(\mathrm{reg}\,B)\cdot\beta y^{p^i}\,c_{p^{r-1}-p^i}(\mathrm{reg}\,B)\cdot\beta x^{p^j} + c_{p^{r-1}-1}(\mathrm{reg}\,B)\,\cancel{y}\cdot e_{\beta x}(\mathrm{reg}\,B)$$
$$- \sum_{i,j=0}^{r-1} c_{p^{r-1}-p^j}(\mathrm{reg}\,B)(\beta y)^{p^j}\,dc_{p^{r-1}-p^i}(\mathrm{reg}\,B)\cdot\beta x^{p^i} - c_{p^{r-1}-1}(\mathrm{reg}\,B)\,x\cdot e_{\beta y}(\mathrm{reg}\,B)\Big\}$$

so comparing we get 2) for A.

February 3, 1970:

From char. 2 we are led to the following conjectures

1) $\operatorname{Im}\{H^*(B\Sigma_{p^r}) \longrightarrow H^*(B\mathbb{Z}_p^r)\} = \Lambda[dc_{p^r-p^i}] \otimes S[c_{p^r-p^i}]_{i=0}^{r-1}$

2) $\operatorname{Im}\{\mathcal{P} H^*(B\Sigma_{p^r}) \hookrightarrow H^*(B\mathbb{Z}_p^r)\} = (\Lambda[dc_{..}] \otimes S[c_{..}])(c_{p^r-1}, dc_{p^r-1})$

We know that the inclusions $\supseteq$ are valid because the elements $dc_{..}, c_{..}$ come from the standard representation of Σ_{p^r} on $(\mathbb{F}_q)^{p^r}$ where $\mathbb{F}_q$ is a finite field $\ni$ $v_p(q-1)=1$. The point is that we have a commutative diagram

$$\begin{tikzcd}
H(\mathbb{Z}_p^r) & & H(B\Sigma_{p^r}) \arrow[ll] \\
H(B\mathbb{Z}_p^{p^r}) \arrow[u, "\phi"] & & H(BGl_{p^r}(\mathbb{F}_q)) \arrow[ll] \arrow[u]
\end{tikzcd} \cong \Lambda[dc_1,\dots,dc_{p^r}] \otimes S[c_1,\dots,c_{p^r}]$$

and the arrow ϕ commutes with d. Unlike char. 2 we cannot conclude equality because $\Lambda[dc_{..}] \otimes S[c_{..}]$ is **not** the subring of invariants in $H^*(B\Sigma_{p^r})$, so we have to find another method.

Idea: First prove conjecture

3) $\exists$ a universal multiplicative (unstable) natural transformation

$$\gamma:\ H(X)^{ev} \longrightarrow \{R \otimes H(X)\}^{ev}$$

Moreover $R = \mathbb{Z}_p[\xi_i, \tau_i]_{i \geq 0}$ where

$$\gamma(\beta x) = \sum_{i\geq 0} \xi_i (\beta x)^{p^i}$$

$$\gamma(\sigma x) = \sigma\left[\sum_{i\geq 0} \tau_i (\beta x)^{p^i} + \xi_0 x\right]$$

Here's how to construct such an operation γ. I claim that by induction on the rank of A, we can show that

$$\rho^A : H^{ev}(X) \longrightarrow \left\{\Lambda[dc_{p^r-p^i}(\mathrm{reg}\,A)]_{i=0}^{r-1} \otimes S[c_{p^r-p^i}(\mathrm{reg}\,A)]_{i=0}^{r-1} \otimes H(X)\right\}^{ev}$$

For $r=1$ it's true for the image is contained in the $Gl(A)$-invariants.