$$\bigoplus_{n\geq 0} H_*(BU(n)) = \mathbb{Z}[t_0, t_1, t_2, \cdots]$$

where $H_*(BU(1)) = \mathbb{Z}t_0 + \mathbb{Z}t_1 + \cdots\cdots$

~~and we set $t_0 = 1$.~~

does $\exists$ an additive Δ?

now the usual diagonal is the ~~cato~~ one with

$$\Delta_m t_n = \sum_{i+j=n}' t_i \otimes t_j$$

given by the Δ on $H_*(BU(1))$

thus

$$\mathrm{Hom}_{rgs}\left(\bigoplus_{n\geq 0} H_*(BU(n)), R\right) = H^*(BU(1), R)$$
$$= R[[c]]$$

so the additive Δ would appear to be

$$\Delta_{add}\, t_n = t_n \otimes 1 + 1 \otimes t_n .$$

~~in fact necessarily this because it should go use maps~~

thus we construct maps $H_*(BU(n) \times BU(m)) \longleftarrow H_*(BU(n+m))$

$$(t\otimes 1 + 1 \otimes t)^\alpha \qquad t^\alpha$$

$$\sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\,\gamma!}\, t^\beta \otimes t^\gamma$$

$$H \underset{i}{\overset{\pi}{\rightleftarrows}} G$$

$\pi^*(\iota^* x)$

$\iota_*(\iota^* x) = \underbrace{\iota_* 1}\cdot x$

$\iota_* 1 = [G : H]$

Review Milnor's dual ~~a~~ for $\mathcal{A}(p)$ p odd.

$\eta \qquad \beta\eta$

$$H^*(X) \longrightarrow H^*(X) \otimes \mathcal{A}_*$$

apply operation and you find

$$\eta \longmapsto \eta \otimes 1 + \sum (\beta\eta)^{p^i} \otimes \tau_i \qquad ?$$

$$\beta\eta \longmapsto \sum_{i \geq 0} (\beta\eta)^{p^i} \otimes \xi_i \qquad \xi_0 = 1. \qquad ?$$

this is what you need.

What do I do

try cyclic group operation

$$H^*(B\Sigma_p) \xrightarrow{\sim} H^*(B\mathbb{Z}_p)^{\mu} = \mathbb{Z}_p[z, \beta z]$$

where z generates $H^{2p-3}(B\Sigma_p)$

and $\beta z = \underbrace{c_{p-1}(\cancel{\mathrm{reg}}\ \Sigma_p)}$

when written out it is

$$\prod_{i=1}^{p-1} c_1(\eta^i) = (p-1)!\, c_1(\eta)^{p-1}$$
$$= -c_1(\eta)^{p-1}$$

by Wilson's thm.

and this gives rise to a ring operation

$$H^*(X) \longrightarrow H^*(B\mathbb{Z}_p)^N \otimes H^*(X)$$

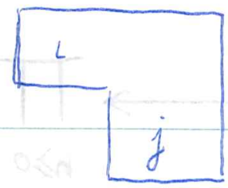
with

$$e(L) \longmapsto e(\mathrm{reg}\,\mathbb{Z}_p \otimes L)$$
$$\| $$
$$\boxed{c_{p-1}(\mathrm{reg}\,\mathbb{Z}_p)\, e(L) + e(L)^p}$$

you figure out what happens to a 1-diml. class!

now $H = \Gamma_{ij}$

$K = \Gamma_{ab}$



stabilizes i |

j 0

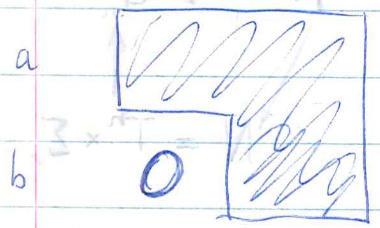
$g_{a'a''b'b''} H g^{-1}$ —— stab

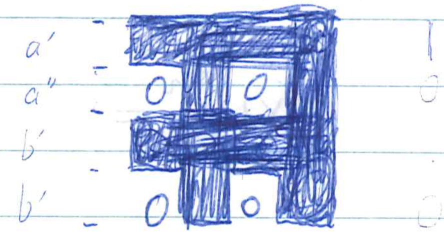
a' |

a'' 0

b' |

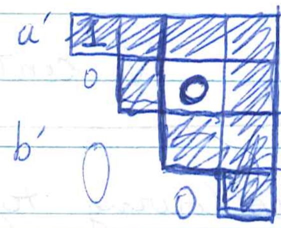
0





therefore

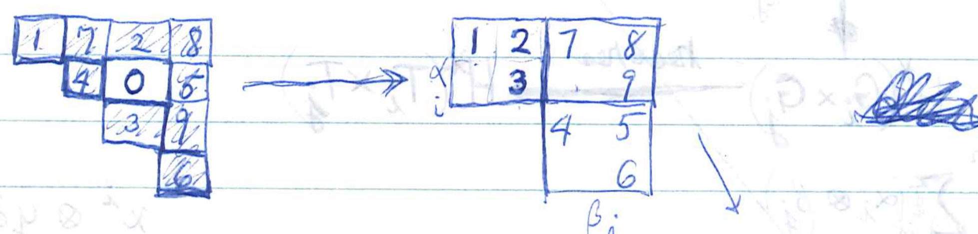
$K \cap gHg^{-1}$



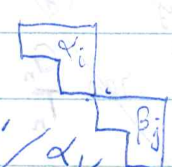
|

Its difficult to calculate j_g^* which is conj. with g.

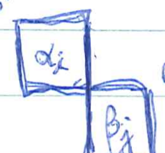
I think j_g is prob. the inclusion



By assumption

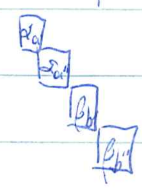


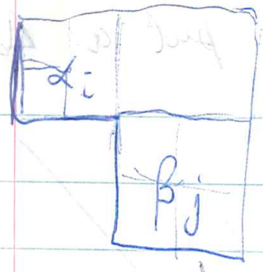
$$\mathrm{res}^{G_i}_{\Gamma_{a'b'}} \alpha_i = \mathrm{inf}^{G_{a'}}_{\Gamma_{a'b'}} \alpha_{a'} \cdot \mathrm{inf}^{G_{b'}}_{\Gamma_{a'b'}} \alpha_{b'}$$



hope is that

$$\mathrm{res}^{G_n}_{\Gamma_{ab}} \mathrm{ind}^{G_n}_{\Gamma_{ij}} \alpha_i \otimes \beta_j = \sum_{a'}$$





$$H^*(V \times_\sigma G) \hookrightarrow H^*(V \times_\sigma P)$$

$$\underline{\text{H-S.}}$$

$$H^*(G, H^*(V))$$

$$H^*(G, S(V^*))$$

---

mod p.

$$\mathrm{res}^{G_{i+j}}_{\Gamma_{i,j}} \alpha_{i+j} = \pi_1^* \alpha_i \cdot \pi_2^* \alpha_j$$

~~[illegible]~~

$$\bigoplus_{n \geq 0} H_*(T_n)/\Sigma_n \longrightarrow \bigoplus_{n \geq 0} H_*(G_n)$$

to show injective I must show that no primitive element $\xi_i \in H_{2i}(T_1)$ is ℓ power nilpotent, ~~[illegible]~~ i.e. have to produce an element in $H^*(G_{\ell^\nu})$ whose inner product with ξ_i^ν is $\neq 0$. Start with

$$z = y^i \in H^{2i}(G_1) = H^{2i}(T_1)$$

then form the wreath product element

$$Q_{\ell^\nu}(z) \in H^{2i\ell^\nu}(N_{\ell^\nu})$$

and induce up to $H^{2i\ell^\nu}(G_{\ell^\nu})$

---

then ~~[illegible]~~ you want to calculate

$$\left\langle \xi_i^{\ell^\nu}, \mathrm{res}^{G_{\ell^\nu}}_{T_{\ell^\nu}} \mathrm{ind}^{G_{\ell^\nu}}_{N_{\ell^\nu}} Q_{\ell^\nu}(z) \right\rangle = [G_{\ell^\nu} : N_{\ell^\nu}] \left\langle \xi_i^{\ell^\nu}, \overbrace{\mathrm{res}^{N_{\ell^\nu}}_{T_{\ell^\nu}} Q_{\ell^\nu}(z)}^{z^{\ell^\nu}} \right\rangle$$

~~what are coset reps. for G_a~~

$G_n / G_i \times G_j$ is almost the Grassmannian of i planes

possibly necessary to modify addition law so that we use the comp.

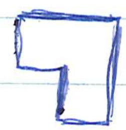
$G_i \times G_j \longleftarrow$  $\xrightarrow{\text{end}}$

since the nilpotent block has powers of 2.

~~Need to look at a class of representations~~

consider the situation of an i plane and an a plane and it looks like it might work out.

we should be working in char $p \neq 2$. with mod 2 coh. then the unipotent block doesn't affect things

$G_i \times G_j \longleftarrow$  $= \Gamma_{ij}$

the map $\Gamma_{ij} \longrightarrow G_i \times G_j$ is intrinsic and any two sections are conjugate in Γ_{ij}. Consequently the maps

$$H^*(\Gamma_{ij}) \rightleftarrows H^*(G_i \times G_j)$$

are intrinsic, i.e. indep. of choices. The idea is to ~~prove that E_{ij}~~ write your condition as

$$\operatorname{res}^{G_{i+j}}_{\Gamma_{ij}} \alpha_{i+j} = \alpha_i \otimes \alpha_j$$

however this needed by same as a ring hom. So to keep things simple assume $2 \neq p$ & $Gl(n, \mathbb{F}_{p^a})$

$$\operatorname{res}^{G_n}_{\Gamma_{ab}} \sum_{i+j=n} \operatorname{ind}^{G_n}_{\Gamma_{ij}} \alpha_i \otimes \beta_j$$

V $\qquad$ Γ_{ij} subgrp fixing $\underline{\mathbb{F}_q^{\,i}} \subset \mathbb{F}_q^{\,n}$

G_n/Γ_{ij} $\qquad$ $\operatorname{Grass}_i(\mathbb{F}_q^n)$

Mackey formula. $\qquad$ $G/H \simeq \coprod_{g \in S} K g H/H$

$$\begin{tikzcd} K \cap gHg^{-1} \arrow[r, "\text{mult by } g^{-1}"] \arrow[d, "i_g"', "\text{inc}"] & H \arrow[d, "i"] \\ K \arrow[r, "j"] & G \end{tikzcd}$$

$$j^* i_* = \sum_{g \in S} (i_g)_* j_g^*$$

Γ_{ab} fixed $\quad U = \mathbb{F}_q^{\,a}$

~~repres~~

$\cancel{V}$

all W of dim i

$\dim(U \cap W)$ is the invariant

$\overline{a'}\ \ \overline{b'}$

$\underset{U\ a}{\underline{\qquad}}\ \underset{b}{\underline{\qquad}}$

so $g_{a'a''b'b''}$ is the map $\quad \underbrace{\underline{a'}\ \underline{b'}}_{i}\ \underbrace{\underline{a''}\ \underline{b''}}_{j} \mapsto \underline{a'}\ \underline{a''}\ \underline{b'}\ \underline{b''}$

represents the plane $\quad \underline{\ \ }_{a'} \cdot \underline{\ \ }_{b'}$

$H^*(Gl_n(\mathbb{F}_q), \mathbb{Z}_\ell)$, $(\ell, q) = 1$

January 17, 1970 (groggy again)

obselete except for putting Δ_{add} on $\oplus H_*(BGl_n)$, see page 14

I am interested in determining the mod ℓ cohomology of $Gl_n(\mathbb{F}_q)$ where $\ell \nmid q$ in analogy with the symmetric groups. So I set $G_n = Gl_n(\mathbb{F}_q)$ and form the ring

$$R = \bigoplus_{n \geq 0} H_*(BG_n).$$

Then for any $\mathbb{Z}/\ell$ algebra A, I have

$$\mathrm{Hom}_{rgs}(R, A) = \prod_{n \geq 0}{}' H^*(BG_n, A)$$

where $'$ denotes the subset of $(\alpha_n)_{n \geq 0}$ such that

$$\mathrm{res}^{G_{i+j}}_{G_i \times G_j} \alpha_{i+j} = \alpha_i \otimes \alpha_j \qquad \alpha_0 = 1.$$

Suppose that $\ell \mid q-1$, i.e. $\mu_\ell \subset \mathbb{F}_q^*$. Let A be an elementary abelian ℓ-subgroup of G_n so we have a faithful representation of A on $\mathbb{F}_q^n$. We break it up into a sum of irreducible repns. which is possible since $(\ell, q) = 1$. Each of these is 1-dimensional since A is abelian and every element has its eigenvalues rational over $\mathbb{F}_q$. Thus A is conjugate to a subgroup of ${}_\ell T_n$ where $T_n \subset G_n$ is the group of diagonal matrices. So we see there is a unique maximal elementary abelian ℓ-subgroup up to conjugacy. The normalizer of ${}_\ell T_n$ is clearly $N_n = T_n \times_\sigma \Sigma_n = \mathbb{F}_q^* \wr \Sigma_n$.

I claim that N_n contains ~~the~~ a sylow ℓ-subgroup of G_n for any ℓ prime to q? This is presumably a kind of Blichtfeldt's

except $\ell = 2$, [illegible] $q \equiv 3$ (4)

theorem, but to prove it we shall compute the index

$$|G_n| = \prod_{i=1}^{n} q^n - q^i = q^{\frac{n(n-1)}{2}} \prod_{i=1}^{n} q^i - 1$$

$$|N_n| = n!\,(q-1)^n$$

~~Suppose that~~ $\cancel{v_\ell(q-1) = a}$ ~~so that~~

$$\cancel{q = 1 + \varepsilon \ell^a} \qquad \cancel{(\varepsilon, p) = 1}$$

$$\cancel{q^i = 1 + i\varepsilon\ell^a + \frac{i(i-1)}{2}\varepsilon^2\ell^{2a} + \cdots}$$

~~Suppose~~ $\cancel{i = \eta\ell^b}$ $\cancel{(\eta, \ell) = 1}$. ~~Then~~

$$\cancel{q^i - 1 = \eta\varepsilon\ell^{b+a} +}$$

We have to determine $v_\ell(q^i - 1)$. The claim is that

$$v_\ell(q^i - 1) = v_\ell(i) + v_\ell(q-1) \qquad (\text{recall } \ell \mid q-1)$$

except when $\ell = 2$ and $q \equiv 3\ (4)$. In this case

$$v_2(q^{2j+1} - 1) = 1$$

$$v_2(q^{2j} - 1) = v_2(j) + v_2(q^2 - 1)$$

The idea is that $q \in (1 + \ell^a \mathbb{Z}_{\hat{\ell}})^\times$ which is $\cong \ell^a \mathbb{Z}_{\hat{\ell}}$ by the exponential function unless $\ell = 2$ and $a = 1$. Thus $q^i \in i\ell^a\mathbb{Z}_{\hat{\ell}}$. ($q = 31$ is an exception since $q-1$ has order 1 and $q^2 - 1 \equiv 0\ (32)$). So in the non-exceptional case we have

$$v_\ell\left(\frac{q^i - 1}{i(q-1)}\right) = 1 \qquad i = 1, \dots, n$$

and consequently $v_\ell |G_n| = v_\ell |N_n|$ and so N_n contains a Sylow ℓ-subgroup.

Let's now drop the assumption that $\ell \mid q-1$ and put $d = [\mathbb{F}_q(\mu_\ell) : \mathbb{F}_q]$. Then d is the least pos. int. $\ni q^d \equiv 1 \ (\ell)$ so $d \mid \ell - 1$. Suppose that A is an elementary abelian ℓ-subgroup of G_n and break the representation up into irreducibles. Let V be an irred. non-trivial rep. of A over $\mathbb{F}_q$. Then $\mathrm{Hom}_A(V,V)$ is a finite skew-field hence an extension field K of $\mathbb{F}_q$. Moreover $A \to K^*$ is non-zero so $\mu_\ell \subset K^*$ and $\therefore \mathbb{F}_q(\mu_\ell) \subset K$. Clearly V must be one dimensional over $\mathbb{F}_q(\mu_\ell)$ and so we see that V is isomorphic to $\mathbb{F}_q(\mu_\ell)$ with A acting through a homomorphism $A \to \mu_\ell$. Write $n = md + r$ $0 \le r < d$, choose an isomorphism of $\mathbb{F}_q^{\,n}$ with $\mathbb{F}_q(\mu_\ell)^m + \mathbb{F}_q^{\,r}$, and let T_n be the image of the obvious map $(\mathbb{F}_q(\mu_\ell)^*)^n \longrightarrow Gl_n(\mathbb{F}_q)$. We have just seen that ${}_\ell T_n$ is the unique maximal elem. abelian ℓ-subgroup of G_n. It is clear that the normalizer of ${}_\ell T_n$ consists of those $\mathbb{F}_q$-linear transformations permuting the blocks and on a $\mathbb{F}_q(\mu_\ell)$ block it can be conjugation by an element of $\mathrm{Gal}(\mathbb{F}_q(\mu_\ell) : \mathbb{F}_q)$. This is also the normalizer of T_n and will be denoted N_n. Thus

$$N_n \cong \left\{ \left((\mathbb{F}_q(\mu_\ell)^*) \times \mathbb{Z}_d \right)^m \times_{\text{semi}} \Sigma_m \right\} \times Gl_r(\mathbb{F}_q)$$

$$|N_n| = (q^d - 1)^m \cdot d^m \cdot m! \quad \underbrace{q^{\frac{r(r-1)}{2}} \prod_{i=1}^{r} q^i - 1}_{\text{prime to } \ell \quad \text{since } r<d.}$$

If $l \nmid q-1$ then $l \neq 2$ and so

$$v_l(q^i - 1) = 0 \qquad i \not\equiv 0 \ (d)$$

$$v_l(q^{jd} - 1) = v_l(j) + v_l(q^j - 1)$$

and of course $v_l(d) = 0$ since $d \mid l-1$. Thus there is no exceptional case here and so N_n contains a Sylow l-subgroup of G_n. As a result of this calculations we obtain the following ~~theorem~~.

Proposition: Let $\mathbb{F}_q$ be a finite field and let l be a prime number $l \neq \text{char} = p$. Then $Gl_n(\mathbb{F}_q)$ has a unique maximal elementary abelian l-subgroup up to conjugacy obtained as follows. Let $d = [\mathbb{F}_q(\mu_l) : \mathbb{F}_q]$ and $n = md + r$, $0 \le r < d$. Choosing a $\mathbb{F}_q$-vector space isomorphism ~~we get~~ $\mathbb{F}_q(\mu_l)^m + \mathbb{F}_q^r = \mathbb{F}_q^n$ we get a natural map $(\mathbb{F}_q(\mu_l)^*)^m \hookrightarrow Gl_n(\mathbb{F}_q)$, whose image we denote $T_n^{(l)}$ (l understood). Then ${}_lT_n$ is a maximal abelian l-subgroup.

Proposition: (a kind of Blichtfeld's theorem). Any l-subgroup of $Gl_n(\mathbb{F}_q)$ is conjugate to a subgroup of the normalizer of $T_n^{(l)}$ except when $l = 2$ and $X^2 + 1 = 0$ is irreducible in $\mathbb{F}_q$ (equivalently $q = p^{2j+1}$ and $p \equiv 3 \ (4)$). Equivalently any irreducible representation of an l-group G over $\mathbb{F}_q$ is induced from a homomorphism $H \longrightarrow \mathbb{F}_q(\mu_l)^*$.

Proof of 2nd proposition: The exceptional cases are where $l = 2$ and $q \equiv 3 \ (4) \Rightarrow q \equiv 3, 7 \ (8) \Longrightarrow v_2(q-1) = 1$ and $v_2(q^2-1) \geq 3$. Now if

$q = p^n \equiv 3\ (4)$, it must be that $p \equiv 3\ (4)$ and n is odd. X^2+1 irred. $\Longleftrightarrow \mu_4 \not\subset \mathbb{F}_q \Longleftrightarrow 4 \nmid q-1 \Longleftrightarrow$ $q \equiv 3\ (4)$ (since q is odd).

If V is a irreducible repn. of an l-group H over $\mathbb{F}_p$, look at the image H' of H in $Gl(V)$. We can suppose this normalizes a $T^{(l)}$ i.e. that $V \cong \mathbb{F}_q(\mu_l)^d$ and that H acts by permuting the axes and by multiplications by μ_{l^a} where $a = v_l(q-1)$. Thus V is induced from a character of a subgroup of H with values in $\mathbb{F}_q(\mu_l)^*$.

Example: Suppose $l=2$, $q=3$. Then $|Gl_2(\mathbb{F}_3)| = (9-1)(9-3) = 48 = 16 \cdot 3$, so the Sylow subgroup has order 16. But $|N_2| = |(\pm 1)^2 \times \Sigma_2| = 8$, so N_2 doesn't contain a Sylow subgroup.

Remark: Let $R(G,k)$ denote the representation ring of a finite group G with coefficients in a field k. Then if G is an l-group and p is a prime no. $\neq l$ we have just shown that

$$R(G, \mathbb{F}_p(\mu_l)) \xrightarrow{\cong} R(G, \mathbb{F}_p)$$

Recall that the action of frobenius on the right is the same as the effect of ψ^p, hence

$$\psi^{p^d} = id \quad \text{if} \quad d = [\mathbb{F}_p(\mu_l) : \mathbb{F}_p]$$

~~... least integer such that $\ell \mid p^d - 1$~~.

<u>Remark</u>: Let $R(G,k)$ denote the Grothendieck group of representations of a finite group with coefficients in a field k. Then we have just shown for an ℓ-group that

$$\text{Weil's rest. of scalars}: \quad R(G, \mathbb{F}_q(\mu_\ell)) \longrightarrow R(G, \mathbb{F}_q)$$

is surjective.

<u>Lemma</u>: Let $Y_i \longrightarrow X$ be a family of maps such that

$$H^*(X) \longrightarrow \prod_i H^*(Y_i)$$

is injective. Then the maps

$$\begin{tikzcd} B\mathbb{Z}_\ell \times Y_i \arrow[rr, "\Delta"] & & E\mathbb{Z}_\ell \times_{\mathbb{Z}_\ell} X^\ell \\ E\mathbb{Z}_\ell \times_{\mathbb{Z}_\ell} (Y_1 \amalg Y_2 \amalg \cdots)^\ell \arrow[urr] & & \end{tikzcd}$$

have the same property.

<u>Proof</u>: We have noted before that

$$\Delta^* \quad H^*_{\mathbb{Z}_\ell}(X^\ell)[w^{-1}] \xrightarrow{\sim} H^*_{\mathbb{Z}_\ell}(X)[w^{-1}]$$

and hence by the spectral sequence that the kernel of

$$H^*_{\mathbb{Z}_\ell}(X^\ell) \xrightarrow{\Delta^*} H_{\mathbb{Z}_\ell}(X)$$

is the kernel of mult. by w and is ~~[scribbled out]~~ contained in $\Gamma_\ell(H^*(X))$ which is detected by forgetting the $\mathbb{Z}_\ell$ action. In other words

$$H^*_{\mathbb{Z}_\ell}(X^\ell) \longrightarrow H^*(X^\ell) \times H^*_{\mathbb{Z}_\ell}(X)$$

is injective. However by the Künneth formula as $H^*(X) \longrightarrow H^*(Y)$ is injective, so is $H(X^\ell) \longrightarrow H(Y^\ell)$ and $H^*(B\mathbb{Z}_\ell \times X) \longrightarrow H^*(B\mathbb{Z}_\ell \times Y)$. where $Y = \cup Y_i$ so the lemma follows.

Corollary: If A_i is a family of subgroups of G which detect all cohomology classes mod ℓ, then the family consisting of the groups

$$A_i \times \mathbb{Z}_\ell \xrightarrow{(\Delta, id)} G \int \mathbb{Z}_\ell$$

$$A_{i_1} \times \cdots \times A_{i_\ell} \longrightarrow G \int \mathbb{Z}_\ell \qquad \text{all } i_1, \ldots, i_\ell$$

detects mod ℓ classes of $G \int \mathbb{Z}_\ell$.

Proposition: $H^*(G_n) \longrightarrow H^*(T_n^{(\ell)})$ is injective in the non-exceptional case.

Proof: I can replace G_n by any subgroup containing a sylow ℓ-subgroup which is $(\mu_{\ell^a}) \int \Sigma(n,\ell)$, hence I can assume that $n = \ell^b$ and $\mu_\ell \subset \mathbb{F}_q$ so that $d = 1$. Then we use induction on b. So assume that $T_{\ell^{b-1}}$ detects for $G_{\ell^{b-1}}$. Then by the corollary $(T_{\ell^{b-1}})^\ell = T_{\ell^b}$ and $T_{\ell^{b-1}} \times \mathbb{Z}_\ell$ detect for $G_{\ell^{b-1}} \int \mathbb{Z}_\ell$ which detects for G_{ℓ^b} since it contains a sylow subgroup. However $T_{\ell^{b-1}} \times \mathbb{Z}_\ell$ being an abelian subgroup of exponent dividing $q-1$ is conjugate to a subgroup of T_{ℓ^b}. QED.

Remark: The proposition might be true in the exceptional case. In any case the kernel is nilpotent.

January 18, 1970:

We want to determine the image of the map of the proposition. According to our general results the image consists up to F-isomorphism of all $u \in H^*(T_n)$ having the same restriction under any pair of maps $A \rightrightarrows T_n$ where A is an elementary abelian l-subgroup of G_n.

A choice of a $T^{(l)}$ in $Gl_q(V)$ is equivalent to writing

~~A Lemma: Let V be an n-dimensional representation of an elementary abelian l group A over $\mathbb{F}_q$. Then any two ways o~~

V as a direct sum of m 1-dimensional vector spaces L_i over $\mathbb{F}_q(\mu_l)$ plus $\mathbb{F}_q^r$ where $n = dm + r$. Thus if we are given a representation $\rho: A \longrightarrow Gl_q(V)$ to choose a $T^{(l)}$ containing $\rho(A)$ is the same thing as giving an isomorphism over $\mathbb{F}_q$ of V with a sum of 1-dimensional representations L_i over $\mathbb{F}_q(\mu_l)$ and a trivial rep. of A in $\mathbb{F}_q^r$. I claim that any two such extra structures on V are conjugate under $\mathrm{Hom}_A(V,V)^* =$ Cent of $\rho(A)$ in $Gl_q(V)$. This is reasonably clear, but we shall check it in detail: Intrinsically we can break up V into its eigenspaces $V = V_0 \oplus V_1 \oplus \cdots \oplus V_k$, where A acts trivially on V_0 and through a different cyclic quotient for each $i \geq 1$. A torus of $Gl(V)$ containing $\rho(A)$ must be equivalent to assigning a flag structure for V_i over $\mathbb{F}_q(\mu_l)$ stable under A for $i \geq 1$ and for V_0 a similar structure $\oplus \mathbb{F}_q^r$. Given two such flag structures we can by an A-linear autom. ~~so~~ make the two match up as direct sums; in the one dimensional lines we have two conjugate characters of A in $\mathbb{F}_q(\mu_\ell)$. So it's clear although we don't have the correct language yet.

Lemma: If $A \xrightarrow{\rho} Gl(V)$ is a representation of an elementary abelian ℓ-group over $\mathbb{F}_q$, then any two $T^{(\ell)}$ of $Gl(V)$ containing $\rho(A)$ are conjugate under $\mathrm{Cent}\,\rho(A)$.

Corollary: If ~~[illegible]~~ A and $xAx^{-1} \subset T$, then we can assume that $x \in N$.

Proof: $\exists y \in \mathrm{Cent}(A) \ni T = y^{-1}x^{-1}Txy$ by lemma so $xy \in N$ and $xy(A)(xy)^{-1} = xAx^{-1} \subset T$.

Conclusion: $H^*(G_n) \longrightarrow H^*(T_n)^{N_n}$ is an F-isomorphism.

Proof: By our general theorems we know that

$$H^*(G_n) \longrightarrow H^*({}_\ell T_n)^{N_n}$$

is an F-isomorphism. However $H^*(T_n) \longrightarrow H^*({}_\ell T_n)$ is an F-isomorphism and passage to invariants will not affect this, this being a special case of $R_i \to R_i'$ F-isom $\Longrightarrow \varprojlim R_i \to \varprojlim R_i'$ also an F-isom.

Now $T_d = \mathbb{F}_q(\mu_\ell)^*$ whose ℓ-primary component is μ_{ℓ^a}

and

(*) $$H^*(T_d) \cong \mathbb{Z}_\ell[x, y] \qquad \text{assume } \ell \text{ odd}$$

where

$$x \text{ generates } H^1(\mu_{\ell^a}, \mathbb{Z}_\ell) = \mathrm{Hom}(\mu_{\ell^a}, \mathbb{Z}_\ell) \cong \mu_\ell^*$$
$$y \text{ generates } H^2(\mu_{\ell^a}, \mathbb{Z}_\ell) = \mathrm{Ext}^1(\mu_{\ell^a}, \mathbb{Z}_\ell) \cong \mu_\ell^*$$

Thus if F is the Frobenius automorphism of $\mathbb{F}_q(\mu_\ell)$ $\quad Fz = z^q$ we have that

$$Fx = q^{-1}x$$
$$Fy = q^{-1}y.$$

Thus the invariants are

$$H^*(T_1)^{N_1} \cong \mathbb{Z}_\ell[x', y']$$

where $x' = xy^{d-1}$ generates $H^{2d-1}(T_d)$ and $y' = y^d$ generates $H^{2d}(T_1)$.

The above formulas also hold for $\ell = 2$ and the non-exceptional case since then T_d has 2-primary part μ_{2^a} with $a > 1$ so the ring structure is still given by (*). In either case we have

<u>Proposition:</u> $H_*(T_d)_{N_d}$ has a basis over $\mathbb{Z}_p$ consisting of elements σ, τ_i, ξ_i where

$$\deg \tau_i = 2di - 1 \qquad i = 1, 2, \ldots$$
$$\deg \xi_i = (2d)i \qquad i = 1, 2, \ldots$$

and $\sigma \in H_0(T_d)$ is the trivial element.

Remark: $H^*(G_d) \simeq H^*(T_d)^{N_d} \cong \mathbb{Z}_\ell[x', y']$ as above.

Proof: It's a general fact that when the Sylow ℓ-subgroup is abelian, then the (ℓ-primary) cohomology of the group is the invariants of the same for the Sylow subgroup under the normalizer of the Sylow subgroup. (Because if B and $xBx^{-1} \subset P$ then $P, x^{-1}Px$ are both Sylow subgroups of $\mathrm{Norm}(B)$, hence we can modify x to lie in $\mathrm{Norm}(P)$).

Now form the ring

$$R = \bigoplus_{n\geq 0} H_*(G_n)$$

Let $\sigma \in H_0(G_1)$ be the class of a point and let

$$\tau_i \in H_{2di-1}(G_d) \qquad\qquad i = 1, 2, \dots$$
$$\xi_i \in H_{2di}(G_d)$$

be generators. The conjecture is that

$$\mathbb{Z}_\ell[\sigma, \xi_i, \tau_i]_{i\geq 1} \longrightarrow R$$

is an isomorphism. I claim that it surjective by the proposition on page 7, (in the non-exceptional case) and I want to prove it is an isomorphism

First suppose $d = 1$. Recall that $N_n = T_1^n \times_{\text{semi}} \Sigma_n$ and let

$$S = \bigoplus_{n\geq 0} H_*(N_n)$$

Then S admits the structure of an affine ring scheme; in fact it is our old ring $R(X)$ where $X = BT_1$. I'm going to give R a ring scheme structure such that

$$S \longrightarrow R$$

makes Spec R a sub-ring scheme of Spec S. We seek a map Φ_{ij} such that

$$\begin{tikzcd} H_*(N_{i+j}) \arrow[r, "\mathrm{ind}"] \arrow[d, "(\varepsilon_{i+j})_*"] & H_*(N_i \times N_j) \arrow[d, "(\varepsilon_i \times \varepsilon_j)_*"] \\ H_*(G_{i+j}) \arrow[r, "\Phi_{ij}"] & H_*(G_i \times G_j) \end{tikzcd}$$

Commutes. Since

$$(\varepsilon_i \times \varepsilon_j)_* (\varepsilon_i \times \varepsilon_j)^* = [G_{i+j} : N_{i+j}]$$

we must have for $x \in H_*(G_{i+j})$ that

$$\cancel{[G_i:N_i][G_j:N_j]\,\Phi_{ij}\,x = (\varepsilon_i \times \varepsilon_j)_* ?}$$

$$\begin{aligned} [G_{i+j} : N_{i+j}]\,\Phi_{ij}\,x &= \Phi_{ij}(\varepsilon_{i+j})_*(\varepsilon_{i+j}^* x) \\ &= (\varepsilon_i \times \varepsilon_j)_*\,\mathrm{ind}\,(\varepsilon_{i+j}^* x) \\ &= (\varepsilon_i \times \varepsilon_j)_*\,(\varepsilon_i \times \varepsilon_j)^* \left(\mathrm{in}_{G_i \times G_j}^{G_{i+j}}\right)^* \\ &= [G_i : N_i][G_j : N_j] \left(\mathrm{ind}_{G_i \times G_j}^{G_{i+j}}\right)^* \end{aligned}$$

So I conclude that the formula must be

$$\Phi_{ij} = \frac{[G_i : N_i][G_j : N_j]}{[G_{i+j} : N_{i+j}]} \left(\mathrm{in}_{G_i \times G_j}^{G_{i+j}}\right)^*$$

By our previous computations we know that

$$(*) \qquad \frac{[G_i : N_i][G_j : N_j]}{[G_{i+j} : N_{i+j}]} = q^{-ij} \frac{\prod_{\nu=1}^{i} \frac{q^\nu - 1}{\nu(q-1)} \cdot \prod_{\nu=1}^{j} \frac{q^\nu - 1}{\nu(q-1)}}{\prod_{\nu=1}^{i+j} \frac{q^\nu - 1}{\nu(q-1)}}$$

is a unit modulo l in the non-exceptional case and hence Φ_{ij} is well-defined. In the exceptional case there is trouble, e.g. if $q=3$, $l=2$, $i=j=1$, then you have

$$3^{-1} \cdot \frac{1 \cdot 1}{1 \cdot \frac{8}{4}} = \frac{1}{6}$$

However the formula also indicates that there might be a suitable formula with $l | q$ provided one induces from the triangular subgroup . Another expression for the numerical factor $(*)$ is

$$(**) \qquad q^{-ij} \frac{\mathrm{card}\left\{\frac{\Sigma_{i+j}}{\Sigma_i \times \Sigma_j}\right\}}{\mathrm{card}\left\{\mathrm{Grass}_{ij}(\mathbb{F}_q)\right\}}$$

which is nicely independent of l dividing $q-1$. It is necessary to determine whether the factor changes when $l \nmid q-1$, but this is clear because the factor is not integral at other primes. For example take $i=j=1$, then the factor is

$$q^{-1}\frac{2}{q+1}$$

which is not integral when l is a prime divisor of $q+1$. This seems to indicate that it is not possible to put a Δ_{add} on $\bigoplus_{n\geq 0} H_*(G_n)$ in a uniform way independent of the coefficient field $\mathbb{Z}_l$.

To return to the case $l \mid q-1$, non-exceptional. Then we define

$$\Delta_{add} : \bigoplus_n H_*(G_n) \longrightarrow \bigoplus_{i,j} H_*(G_i \times G_j)$$

using the Φ_{ij}. It follows that $S \to R$ is a surjective ring homomorphism compatible with Δ_{add}, hence Δ_{add} is associative, commutative and compatible with Δ_{mult}. In terms of cohomology we have defined the sum of

$$(\alpha_n), (\beta_n) \in \prod_{n\geq 0}{}' H^*(G_n; A)$$

by

$$\gamma_n = \sum_{i+j=n} \lambda_{ij} \, \mathrm{ind}_{G_i \times G_j}^{G_{i+j}} \alpha_i \otimes \beta_j$$

where λ_{ij} is the numerical factor (**) page 13. Thus we have made R into a ring scheme.

Jan. 20, 1970: (groggy but better)

So now we can show that $S \longrightarrow R$ is an isomorphism. By Hopf algebra theory it suffices to show that each $\xi_i \in H_{2i}(G_1)$ is not nilpotent, at least when $l \neq 2$ so that the τ_i generate an exterior subalgebra. What we are going to do is to use the Chern classes

$$c_i(V) \in H^{2i}(Gl_n(\mathbb{F}_q), \mathbb{Z}_l)$$

obtained from the standard representation $Gl_n(\mathbb{F}_q) \longrightarrow \underline{Gl_n}$ (over $\mathbb{F}_q$) and an isomorphism chosen once & for all of $\mu_l \cong \mathbb{Z}_l$. Then

$$\mathrm{res}^{G_n}_{T_n} c_{\underline{t}}(V) = \prod_{j=1}^{n} c_{\underline{t}}(x_j) = \prod_{j=1}^{n} (1 + t_1 x_j + t_2 x_j^2 \cdots)$$

where $x \in H^2(T_1)$ is the canonical generator (relative to $\mu_l \cong \mathbb{Z}_l$) and $\langle \xi_i, x^i \rangle = 1$. Thus

$$\langle \xi_i^n, c_{\underline{t}}(V) \rangle_{G_n} = \langle \xi_i^n, \prod_j (1 + t_1 x_j + \cdots) \rangle_{T_n}$$

$$= t_i^n$$

and more generally

$$\langle \xi^\alpha, c_{\underline{t}}(V) \rangle = t^\alpha.$$

This shows that the ξ's and σ generate a polynomial subalgebra in R, so if $l \neq 2$ we are finished.

Now suppose $l = 2$ and $q \equiv 1$ (4). We want to show that $\tau_i^2 \in H_{2(2i-1)}(G_2)$ is zero. First take $i=1$ to see what's happening. Then we have in

$H_2(T_2)_{\Sigma_2}$ the basis $\sigma\cdot\xi_1$ and $\tau_1^{\,2}$. Also $H^2(T_2)^{\Sigma_2}$ has dual basis $y\otimes 1+1\otimes y$ and $x\otimes x$. Now we know that $y\otimes 1+1\otimes y$ is the image of c_1 in $H^2(G_2)$ and hence $\tau_1^{\,2}=0$ is equivalent to $x\otimes x$ not being in the image of $\mathrm{res}^{G_2}_{T_2}$. Here's how to prove that $(xy^{i-1})^{\otimes 2}$ is not in the image of $\mathrm{res}^{G_2}_{T_2}$. Suppose on the contrary that

$$(xy^{i-1})^{\otimes 2} = \mathrm{res}^{G_2}_{T_2}(\alpha) \qquad \alpha\in H^{4i-2}(G_2)$$

Recall $N_2 = T\wr\mathbb{Z}_2$ and compute the composite

$$H^*(G_2) \longrightarrow H^*(N_2) \xrightarrow{(\Delta\times id)^*} H^*(T\times\mathbb{Z}_2)$$

The point is that the representation of $T\times\mathbb{Z}_2$ given by the homomorphism $T\times\mathbb{Z}_2 \xrightarrow{\Delta, id} N_2 \longrightarrow G_2$ is $\chi\otimes\mathrm{reg}\,\mathbb{Z}_2$, where χ is the standard character of T, and this rep. is isomorphic to $\chi\oplus\chi\otimes\eta$. Thus the homomorphism is conjugate to the homomorphism

$$T\times\mathbb{Z}_2 \xrightarrow{id\times in} T\times T \xrightarrow{(id,\mu)} T\times T \hookrightarrow G_2$$

so

$$\mathrm{res}^{N_2}_{T\times\mathbb{Z}_2}(\mathrm{res}^{G_2}_{N_2}\alpha) = (pr_1,\mu)^*\,\mathrm{res}^{G_2}_{T_2}\,\alpha$$
$$= pr_1^*(xy^{i-1})\cdot\mu^*(xy^{i-1})$$

Now since $\mathbb{Z}_4\subset T$ we have that

$$\mathrm{res}^{T}_{\mathbb{Z}_2}\,x = 0 \qquad \mathrm{res}^{T}_{\mathbb{Z}_2}\,y = y$$

so that

$$\mu^*(xy^{i-1}) = (id\times in)^*(x\otimes 1+1\otimes x)(y\otimes 1+1\otimes y)^{i-1}$$
$$= (x\otimes 1)(y\otimes 1+1\otimes y)^{i-1}$$

and so therefore

$$\mathrm{res}^{N_2}_{T\times\mathbb{Z}_2}\left(\mathrm{res}^{G_2}_{N_2}\,\alpha\right) = (x^2y^{i-1}\otimes 1)(y\otimes 1 + 1\otimes y)^{i-1} = 0$$

since $x^2=0$. However we know that $\mathrm{Ker}\left(\mathrm{res}^{N_2}_{T\times\mathbb{Z}_2}\right)$ maps isomorphically $\Lambda_2 H(T) \subset \Gamma_2 H(T)$. Thus $\mathrm{res}^{G_2}_{T_2}\,\alpha \neq (xy^{i-1})^{\otimes 2}$ and we have a contradiction.

By Hopf algebra theory we know that the elements $\sigma_j^2\,\sigma\xi_i, \sigma\tau_i$ $\xi_i\tau_j$, $\xi_i\xi_j$ are independent and these shows that the image of $\mathrm{res}^{G_2}_{T_2}$ is at least $\Lambda_2 H(T)$. By the above argument, it can't be bigger so we have proved $\tau_i^2=0$, and we reach the following.

<u>Theorem:</u> In the non-exceptional $\underbrace{\text{case}}_{\text{and } \ell \mid q-1}$ $R = \bigoplus_{n\geq 0} H_*(G_n)$ is a polynomial ring over $\mathbb{Z}_\ell$ with generators $\sigma \in H_0(G_1)$, $\xi_i \in H_{2i}(G_1)$ tensored with an exterior algebra with generators $\tau_i \in H_{2i-1}(G_1)$, where $i=1,2,\ldots$. Moreover if $\ell \neq 2$, then

$$H^*(G_n) \cong H^*(T_n)^{\Sigma_n}$$

This statement ~~hasn't been adequately checked~~ follows because we have shown $H_*(G_n) =$ homogeneous degree n part of anti-symmetric alg. generated by $H_*(T_1)$ which when $\ell\neq 2$ is $H_*(T_1)^{\otimes n}/\Sigma_n$.

<u>How to compute $H^*(B\,G(\mathbb{F}_q), \mathbb{Z}_\ell)$ when G is a connected linear algebraic group defined over $\mathbb{F}_q$ and $(\ell, q) = 1$:</u>

We use the Leray spectral sequence for the fibration $G/G(\mathbb{F}_q) \longrightarrow B\,G(\mathbb{F}_q) \longrightarrow BG$ and étale cohomology. (I shall pretend this has meaning either by Deligne's simplicial model for BG or by approximating BG by actual varieties after first putting $G \subset Gl_n$). The point is that usually this fibration isn't of much good since one doesn't know the cohomology of $G/G(\mathbb{F}_q)$. However here

$$G/G(\mathbb{F}_q) \simeq G$$

as varieties. To see this let G act on itself by $x \cdot y = xy(Fx)^{-1}$ where F is the Frobenius endomorphism of G. Then look at the map $G \longrightarrow G$, $x \longmapsto x \cdot 1$. It is etale and surjective since G is connected and the stabilizer of 1 is $\{x \mid x = Fx\} = G(\mathbb{F}_q)$. So the spectral sequence reads

$$E_2 = H^*(BG) \otimes H^*(G) \Longrightarrow H^*(B\,G(\mathbb{F}_q))$$

since the base is simply-connected. In the good cases the spectral sequence will degenerate, $H^*(BG)$ and $H^*(G)$ will be a polynomial ring, resp exterior algebra with r generators (where r is the rational rank). The problem is to find the good theorem which guarantees this situation.

<u>Conjecture</u>: Suppose that G is a connected reductive algebraic group defined over $\mathbb{F}_q$ and that G has no l-torsion in the sense that the ~~[illegible]~~ Chow ring of G/B is generated ~~[illegible]~~ by the divisors associated to the characters of B modulo. Then the above spectral sequence degenerates. ~~[illegible]~~ (assume B defined over $\mathbb{F}_q$ too.) (should be by Lefschetz

Here's why this should be true: First choose a basic isomorphism (orientation ?!) $\hat{\mathbb{Z}}_\ell \to \varprojlim \mu_{\ell^n}$ ~~[illegible]~~ $\mu_{\ell^n} \subset \overline{\mathbb{F}}_q$. Now since there is no l-torsion

$$H^*(BG, \hat{\mathbb{Z}}_\ell) \simeq \hat{\mathbb{Z}}_\ell [c_1, \cdots, c_n]$$

and

$$H^*(G, \hat{\mathbb{Z}}_\ell) \simeq \bigwedge_{\hat{\mathbb{Z}}_\ell} [e_1, \cdots, e_n]$$

where in the basic spectral sequence

$$H^*(BG) \otimes_{\hat{\mathbb{Z}}_\ell} H^*(G) \Rightarrow \hat{\mathbb{Z}}_\ell$$

e_i is transgressive and τe_i is represented by c_i. The action of Galois is such that

$$F(c_i) = q^{d_i} c_i$$
$$F(e_i) = q^{d_i} e_i$$

here's where you use that B is defined over $\mathbb{F}_q$

where $2d_i$ is the degree of c_i. Moreover the elements e_i are primitive with respect to the ~~multiplicative~~ coproduct on $H^*(G, \hat{\mathbb{Z}}_\ell)$ coming from the multiplication.

Now we have a morphism of fibrations

$$\begin{tikzcd} & G \arrow[r] \arrow[d] \arrow[dl] & EG \arrow[r] \arrow[d] & BG \arrow[d, equal] \\ G \arrow[r, phantom, "\simeq"] & G/G(\mathbb{F}_q) \arrow[r] & BG(\mathbb{F}_q) \arrow[r] & BG \end{tikzcd}$$

So this gives use a map of spectral sequences

$$\varphi^* : H^*(BG) \otimes H^*(G) \longrightarrow H^*(BG) \otimes H^*(G)$$

over $\hat{\mathbb{Z}}_\ell$ such that
$$\varphi^* c_i = c_i$$
$$\varphi^*(e_i) = (1 - q^{d_i})\, e_i$$

false argument

The last equation is immediate from the behavior of primitive elements. ~~Now tensor the spectral sequences with~~ I claim this implies that in the first spectral sequence e_i is transgressive and moreover $\tau(e_i)$ is represented by $(1-q^{d_i})c_i$. In effect after tensoring with $\mathbb{Q}$ the map of spectral sequences becomes an isomorphism hence the first non-zero differential in the first spec. seq. must occur at the same place since everything is torsion free. Similarly

NO

$$\varphi^*(\tau e_i) = \tau \varphi^* e_i = \tau(1-q^{d_i})e_i = (1-q^{d_i})c_i$$
$$= \varphi^*\left[(1-q^{d_i})\, c_i\right]$$

so as φ^* is injective it's clear.

So now we know the spectral sequence over $\hat{\mathbb{Z}}_\ell$ and we can map it to the spectral sequence with coefficients $\mathbb{Z}_\ell$.

If $\ell | q-1$ it follows that the spectral sequence degenerates as claimed (mod ℓ). If $\ell \nmid q-1$ then only the c_i's and c_i's with $\ell | q^{d_i}-1$ survive so that after taking fixed spaces under Frobenius the spectral sequence degenerates.

Remark: 1) I should think that Lefschetz fixpoint formula for G/B (over $\bar{\mathbb{F}}_q$) implies the existence of many rational points.

2) If $\ell \neq 2$, then the degeneracy of the spectral sequence implies that $H^*(BGl(\mathbb{F}_q), \mathbb{Z}_\ell)$ = product of poly ring on c_{i_j}, exterior algebra on c_{i_j}

where the c_{i_j} run through the c_i with $\ell | q^{d_i}-1$.

3) At least here, there doesn't seem to be the exceptional case encounted before

January 21, 1970.

Suppose K is a field of char. $\neq 2$. I propose to calculate the invariants under the symmetric group in

$$\Lambda[x_1, \dots, x_n] \otimes S[y_1, \dots, y_n]$$

Proposition: The invariants are $\Lambda[e_1, \dots, e_n] \otimes S[c_1, \dots, c_n]$ where

$$c_i = \sum_{j_1 < \dots < j_i} y_{j_1} \cdots y_{j_i}$$

$$e_i = d c_i$$

where d is the differential on $\Lambda[x \dots] \otimes S[y \dots]$ such that $d y_i = x_i$.

Proof. It's clear that d commutes with the action of the symmetric group and that hence the e_i are invariants. I claim that $\Lambda[e \dots] \otimes S[c] \hookrightarrow \Lambda[x \dots] \otimes S[y \dots]$. To see this note that the kernel is an ideal and that if non-zero it contains an element of the form $f(c) \cdot e_1 \wedge \dots \wedge e_n$ where $f(c) \neq 0$. So it suffices to show that $e_1 \wedge \dots \wedge e_n \longmapsto dc_1 \wedge \dots \wedge dc_n = \text{Jac} \cdot x_1 \wedge \dots \wedge x_n$ where Jac is the Jacobian of the map $y_i \mapsto c_i$. We know that

$$\text{Jac} = \pm \prod_{i<j} (y_i - y_j);$$

in effect Jac vanishes whenever two y's are equal, so Jac is divisible by this product on the other hand it's degree is obviously $0 + 1 + \dots + n-1 = \frac{n(n-1)}{2}$. Thus Jac $\neq 0$ and the map is injective.

The other part consists of calculating the Poincaré series of the invariants. This we do by observing that

$$\Lambda[x_1,\ldots,x_n]\otimes S[y_1,\ldots,y_n] = V^{\otimes n}$$

where $V = \Lambda[x]\otimes S[y]$. ~~This has the Poincare se~~ Thus the invariants have the same P.S. as $(V^*)^{\otimes n}/\Sigma_n$. But

$$\bigoplus_{n\geq 0} (V^*)^{\otimes n}/\Sigma_n$$

is the free commutative ~~symmetric~~ algebra generated by V^*, hence is a polynomial ring on V^*_+ tensored with an exterior algebra on V^- (here is where we use char. $\neq 2$). Thus letting $V^* = \mathbb{Z}_\ell\sigma + \mathbb{Z}_\ell\tau_i + \mathbb{Z}_\ell\xi_i$ where τ_i generates V^*_{2i-1}, and ξ_i generates V^*_{2i}, and σ generates V^*_0, we see that $(V^*)^{\otimes n}/\Sigma_n$ has a basis consisting of

(*) $\qquad \sigma^{n-|\alpha|-|\beta|}\xi^\alpha\tau^\beta \qquad |\alpha| = \sum\alpha_i,\ |\beta| = \sum\beta_i.$

where $\alpha = (\alpha_1,\alpha_2,\ldots\ldots)$ $\alpha_i\geq 0$ and $\beta = (\beta_1,\beta_2,\ldots\ldots)$, $0\leq\beta_i\leq 1$. Thus the bigraded P.S. is

$$\sum_{n\geq 0} s^n\, P_t((V^*)^{\otimes n}/\Sigma_n) = \prod_{j=0}^{\infty}\frac{1+st^{2i+1}}{1-st^{2i}}$$

and so what we are trying to prove is the identity

$$\prod_{j=0}^{\infty}\frac{1+st^{2i+1}}{1-st^{2i}} = \sum_{n\geq 0} s^n\frac{(1+t)\cdots(1+t^{2n-1})}{(1-t^2)\cdots\cdots(1-t^{2n})} \quad !$$

Recall (?!) the famous partition identity

$$(*) \qquad \sum_{n\geq 0} \frac{x^n}{(1-t)\cdots(1-t^n)} = \prod_{n\geq 0} \frac{1}{1-xt^n}$$

which one may prove by noting

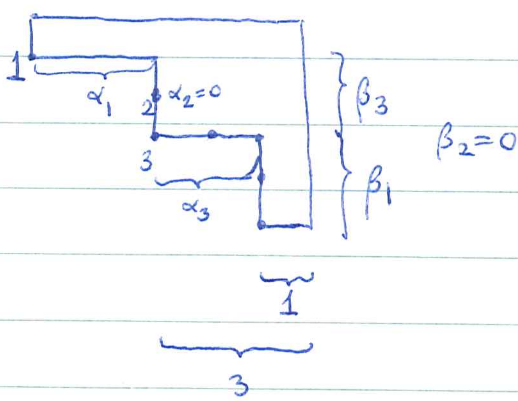
$$\bigoplus_{n\geq 0} H_*(BO(n)) = \mathbb{Z}_2[\sigma, t_1, t_2, \ldots\ldots] \xmapsto{P.S.} \prod_{n\geq 0} \frac{1}{1-xt^n}$$

$$H^*(BO(n)) = \mathbb{Z}_2[w_1, \ldots, w_n] \xmapsto{P.S.} x^n \cdot \frac{1}{(1-t)\cdots(1-t^n)}$$

The classical argument for proving $(*)$ consists of identifying sums

$$\left\{ (\alpha_i)_{i\geq 0} : \begin{array}{l} \sum_{i\geq 0} \alpha_i = n \\ \sum i\alpha_i = d \end{array} \right\} \longleftrightarrow \left\{ (\beta_j)_{1\leq j\leq n} : \sum j\beta_j = n \right\}$$

by the pictorial scheme



where d is the area of the block. There probably exists some analogous arrangement in the case of interest to us now, but it seems involved.

In any case our previous calculations show the proposition on page 1 is true. In effect for an odd l

~~[illegible]~~ choose q with $\ell | q-1$. Then we've proved that

$$\text{P.S. } H^*(Gl_n(\mathbb{F}_q), K) \ll \text{P.S}(\Lambda[e_1, \ldots, e_n] \otimes S[c_1, \ldots, c_n])$$

~~[illegible]~~ by means of the spectral sequence, and on the other hand we know by our direct calculation that

$$H^*(Gl_n(\mathbb{F}_q), K) \cong (\Lambda[x_1, \ldots, x_n] \otimes S[y_1, \ldots, y_n])^{\Sigma_n}$$

Unfortunately I was hoping to check the earlier results by directly proving the proposition.

this has to be checked since we don't know the s.s. degenerates.

<u>Remark:</u> ~~If~~ We know now that for a reductive connected group G ~~[illegible]~~ over $\mathbb{F}_q$ without ℓ-torsion and with $\ell | q-1$, ~~that~~ ℓ odd, that

$$H^*(B\mathbf{G}(\mathbb{F}_q), \mathbb{Z}_\ell) = \text{ext.} \otimes \text{symm. alg.}$$

One knows that the symmetric algebra part is $H^*(BT)^W$ and it has fairly canonical set c_i of generators of degree $2m_i$. One can conjecture and hope to prove by using G/T somehow that

$$H^*(BG(\mathbb{F}_q), \mathbb{Z}_\ell) = H^*(BT(\mathbb{F}_q))^W$$

$$\cong \Lambda[e_i] \otimes S[c_i]$$

where $c_i = de_i$ and d is the derivation of $H^*(BT(\mathbb{F}_q)) = \mathbb{Z}_\ell[x \ldots y \ldots]$ with $dy_i = x_i$.

January 24, 1970

Remarks on $H^*(BGl_n(\mathbb{F}_q), \mathbb{Z}_\ell)$:

1.) Actually we have shown that

$$H^*(BGl_n(\mathbb{F}_q), \mathbb{Z}_\ell) \cong \Lambda[e_1, \dots, e_n] \otimes S[c_1, \dots, c_n]$$

if ℓ is odd and $\ell \mid q-1$. By the spectral sequence

$$P.S.\{H^*(BGl_n(\mathbb{F}_q), \mathbb{Z}_\ell)\} \leq P.S.\{\Lambda[e \dots] \otimes S[c \dots]\}$$

and by our discriminant calculation

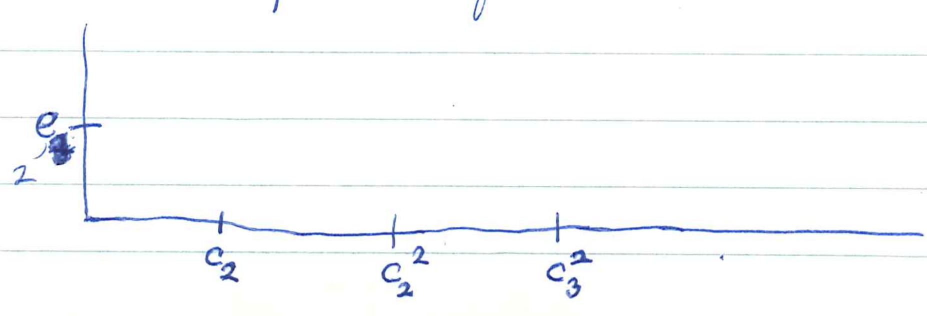
$$P.S.\{\Lambda[e] \otimes S[c \dots]\} \leq P.S.\{H^*(BT_n(\mathbb{F}_q), \mathbb{Z}_\ell)^{\Sigma_n}\}$$

and finally by our direct methods we know that

$$H^*(BGl_n(\mathbb{F}_q), \mathbb{Z}_\ell) = H^*(BT_n(\mathbb{F}_q), \mathbb{Z}_\ell)^{\Sigma_n}$$

so all of these inequalities are in fact equalities. In particular the spectral sequence degenerates.

2.) Here's an example to illustrate that the exceptional case is indeed exceptional. First take $Sl_2(\mathbb{F}_q) = Sp_2(\mathbb{F}_q)$. This has no torsion so the spectral sequence is



In this case there is no problem with e_2 being transgressive and so the spectral sequence over $\hat{\mathbb{Z}}_l$ has

$$\tau(e_2) = (q^2-1)\, c_2$$

Observe that modulo l, e_2 gives rise to an element of degree 3 in $H^*(BSl_2(\mathbb{F}_q))$ whose square is zero since there is nothing in dimension 6. Thus

$$H^*(BSl_2(\mathbb{F}_q), \mathbb{Z}_2) = \Lambda[e] \otimes S[c] \qquad \begin{cases} \dim c = 4 \\ \dim e = 3 \end{cases}$$

even for $l=2$.

Now one notes that

$$\frac{|Sl_2(\mathbb{F}_q)|}{|\text{norm. of torus}|} = \frac{q \cdot q^2-1}{(q-1)\cdot 2} = \frac{q(q+1)}{2}$$

is prime to l if l is odd and hence that the normalizer of the torus contains the Sylow l-subgroup. Thus for l odd or $l=2$ & $q \equiv 1$ (4), we have

$$H^*(BSl_2(\mathbb{F}_q), \mathbb{Z}_l) \hookrightarrow H^*(BT(\mathbb{F}_q), \mathbb{Z}_l)^{\mathbb{Z}_2}$$

In the case where $l=2$ and $q \equiv 3$ (4) however

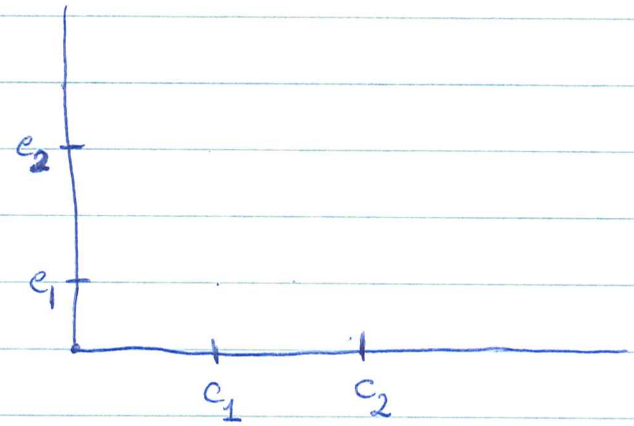
$$H^*(BT(\mathbb{F}_q), \mathbb{Z}_2) = \mathbb{Z}_2[w] \qquad \deg w = 1$$

has no nilpotent elements and therefore

$$H^*(BSl_2(\mathbb{F}_q), \mathbb{Z}_2) \longrightarrow H^*(BT(\mathbb{F}_q), \mathbb{Z}_2)$$

is not injective, since it kills the element of degree 3.

$Gl_2(\mathbb{F}_q)$:



For dimesional reasons e_1, e_2 are transgressive and by comparision with universal spectral sequence we have

$$\tau e_1 = (q-1)c_1$$
$$\tau e_2 \text{ is rep. by } (q^2-1)c_2$$

so modulo 2 we have no differentials and therefore

$$gr\left\{H^*(B\,Gl_2(\mathbb{F}_q), \mathbb{Z}_2)\right\} = \Lambda[e_1, e_2] \otimes S[c_1, c_2].$$

So I want to determine the multiplicative structure. Consider the map $Gl_2 \longrightarrow Gl_1$ given by the determinant. This gives a retraction

$$\begin{array}{ccccc} H^*(BT_1(\mathbb{F}_q)) & \longleftarrow & H^*(BGl_2(\mathbb{F}_q)) & \longleftarrow & H^*(B\mu_q) \\ \| & & e_1 \longleftarrow\!\shortmid\ w & & \| \\ \mathbb{Z}_2[w] & & c_1 \longleftarrow\!\shortmid\ w^2 & & \mathbb{Z}_2[w] \end{array} \qquad \left(\begin{array}{c} \ell = 2 \\ q \equiv 3\ (4) \end{array}\right)$$

which for dimensional reasons must be an isomorphism in dimensions 1 and 2. Thus we see that

$$e_1^2 = c_1 \quad \text{in} \quad H^*(BGl_2(\mathbb{F}_q)).$$

Now if we restrict to $Sl_2(\mathbb{F}_q)$, then e_1, c_1 go to zero and e_2, c_2

go into elements with the same name. Now the torus of Gl_2 can be viewed as the product of the torus in Sl_2 $\left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right\}$ and Gl_1 $\left\{\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}\right\}$. ~~this e_3~~ Note that $H^3(BGl_2(\mathbb{F}_8))$ is of dimension 2 with basis e_3, e_1c_1, so we can assume that e_3 chosen so that it goes to zero on $BGl_1(\mathbb{F}_8)$. ~~The~~ But we know it goes to zero in the torus of Sl_2 so

$$\cancel{H^*(BGl_2(\mathbb{F}_8)) \longrightarrow H^*(BT_2(\mathbb{F}_8))}$$

~~is not injective, since e_3 is contained in the kernel. The image is~~

$e_3 | T_2(\mathbb{F}_8)$ ~~if~~ vanishes on all non-zero lines in the points of order 2 so

$$e_3 | T_2(\mathbb{F}_8) = 0 \text{ or } x_1 x_2 (x_1 + x_2)$$

?

Question: According to Milgram if X is an infinite loop spaces one gets many kinds of operations on $H^*(X)$ namely Dyer-Lashof operations? How are these defined?

Candidate: Given X have sum $X^k \xrightarrow{\mu} X$ which equivariantly comes from a map

$$EG \times_G X^k \xrightarrow{\mu} X$$

composing with diagonal we get a fundamental map

$$BG \times X \longrightarrow EG \times_G X^k \longrightarrow X$$

In terms of cohomology one gets a map

$$H^*(X) \longrightarrow H^*(BG \times X) \cong H^*(BG) \otimes H^*(X)$$

and hence elements of $H_*(BG)$ give rise to operations on $H^*(X)$.

Relation with bundle theories: Suppose X is the universal bases space for a bundle theory with associated Thom spectrum M. Then $EG \times_G M^k$ is the Thom spectrum of the bundle over $EG \times_G X^k$ of the sort that the diagram

ρ = stand. rep of G

$$\begin{tikzcd}
H^*\{(\rho \otimes E)^+\} & H^*\{(EG \times_G E^k)^+\} \arrow[l] & H^*\{M\} \arrow[l] \\
H^*(BG \times X) \arrow[u, "\cong"] & H^*(EG \times_G X^k) \arrow[l] \arrow[u, "\cong"] & H^*(X) \arrow[l] \arrow[u, "\cong"]
\end{tikzcd}$$

must commute by naturality of the Thom isomorphism. I want to conclude that the D-L operations on $H^*(X)$ must be related to the power ops. on the Thom class. But now we see a difference: The power operation would be a map

$$H^*\{M\} \longrightarrow H^*\{BG \times M\}$$

whereas what we have constructed is an operation

$$H^*\{M\} \longrightarrow H^*\{(\rho \otimes E)^+\}.$$

More precisely let us take up real vector bundles again. Then suppose E is a vector bundle of dim n over X, form the bundle $\rho \otimes E$ over $BG \times X$ with Thom space $M(\rho \otimes E)$. Then we have Thom ism

$$\begin{array}{ccc} H^*(BG \times X) & \longrightarrow & \tilde{H}^{*+kn}(M(\rho \otimes E)) \\ \uparrow & & \uparrow \\ H^*(EG \times_G (X)^k) & \longrightarrow & \tilde{H}^{*+kn}(M(EG \times_G E^k)) \end{array}$$

Now before I somehow had identified

$$M(EG \times_G E^k) \qquad\qquad EG \times_G (ME, \infty)^k$$

which seems reasonable since both are 1-point compactifications of $EG \times_G E^k$. Your mistake was not at this stage but earlier where you ignore the difference between

$$BG \times \{ME, \infty\} \xrightarrow{\Delta} EG \times_G \{ME, \infty\}^k$$

which of course gives you just the power operations on the Thom class and the much more interesting

$$M\{\mathring{\rho}\otimes E\} \longrightarrow EG\times_G (ME)^k$$

Consider what happens for $p=2$. Then we get for the power operation

$$P(i_*1) = \sum v^i Sq_i(i_*1) \qquad v=e(\eta)$$
$$= \sum v^i Sq^{n-i}(i_*1)$$
$$= i_*\{\sum v^i w_{n-i}(E)\}$$

where $P: \tilde{H}^n(ME) \longrightarrow H^{2n}(B\mathbb{Z}_2\times ME, B\mathbb{Z}_2\times\{\infty\})$. Now consider the other direction

$P_{ext}(i_*1)$ the Thom class for $E\mathbb{Z}_2\times_{\mathbb{Z}_2}(E)^2$ over $B\mathbb{Z}_2\times_{\mathbb{Z}_2}X^2$

$\downarrow$

the Thom class for $\mathring{\rho}\otimes E$ over $B\mathbb{Z}_2\times X$

~~and therefore the resulting operation but~~

$$\cancel{\begin{array}{ccc} H^*(X) & \xrightarrow{D} & H^{*}(B\mathbb{Z}_2\times X) \\ i_*\downarrow\cong & & j_*\downarrow\cong \\ \tilde{H}^{*+n}(ME) & \xrightarrow{\delta P_{ext}} & \tilde{H}^{*+2n}(M(\mathring{\rho}\otimes E)) \end{array}}$$

~~send~~ $\cancel{j_* D x = \delta P_{ext}\, i_* x = \delta P_{ext}(x\cdot i_*1)}$

$\cancel{= \delta(P_{ext}x\cdot P_{ext}\, i_*1)}$

Let $\gamma \in H^*(BO_n)$. Then you get that the D-L operation carries γ to $\gamma(\rho \otimes E)$ over $B\mathbb{Z}_2 \times BO(kn)$. For example if $\gamma = e(E_n)$, then

$$\gamma(\rho \otimes E) = e(\rho \otimes E)$$
$$= e(E) \cdot e(\eta \otimes E).$$

<u>Conclusion</u>: The candidate we have for the Dyer-Lashof operations on BPL or $BTop$ is the induced map on cohomology corresponding to the operation

$$E \longmapsto \rho \otimes E$$

on bundles where ρ is a permutation representation of G.

Recall that $P : H^*(X) \longrightarrow H^*(BG \times X)$ satisfies

$$P\{e(E)\} = e(\rho \otimes E) \qquad \rho = \text{reg rep of } G.$$

Therefore if you start with PL-bundles and their cohomology Euler classes all you generate via Dyer-Lashof is the Wu classes.

You seem to have shown that the two maps

$$P : H^*(X) \longrightarrow H^*(BG \times X)$$

$$DL : H^*(X) \longrightarrow H^*(BG \times X) \qquad \text{induced by an honest map of spaces}$$

coincide in an interesting case. We examine this for BO more closely.

By our way of thinking DL is the map on cohomology induced by the map of spaces [struck-out text] $B\mathbb{Z}_2 \times BO \to BO$ given on bundles by $E_{univ} \longmapsto reg\,\mathbb{Z}_2 \otimes E_{univ}$. Thus

$$DL\,(w_i(E_{univ})) = w_i(reg\,\mathbb{Z}_2 \otimes E_{univ})$$

or put more neatly

$$DL\,(w_t(E_{univ})) = w_t(E_{univ})\,w_t(\eta \otimes E_{univ})$$

On the other hand P is the mult. map on cohomology which on 1-dimensional classes is given by

$$P\,e(L) = e(L)(v + e(L))$$

hence

$$P(1 + te(L)) = 1 + te(L)v + te(L)^2$$

$$\cancel{(1+te(L))(1+tv+te(L))}$$

If you restrict to a bundle $E = L_1 + \cdots + L_n - n$ then

$$w_t(E_{univ}) \longmapsto \prod_{j=1}^{n} 1 + te(L_j)$$

$$w_t(\eta \otimes E_{univ}) \longmapsto \prod_{j=1}^{n} \frac{1 + tv + te(L_j)}{1 + tv}$$

Thus

$$w_t(L)\,w_t(\eta \otimes (L-1)) = \frac{(1+te(L))(1+tv+te(L))}{1+tv} = \frac{1+t^2e(L)^2 + tv + t^2ve(L)}{1+tv}$$

and we see that these are not the same.

Summary of open problems related to work on Adams conjecture.

1.) Unstable implications.

$$BGl_n(\mathbb{F}_q) \longrightarrow BU_n[\tfrac{1}{p}] \underset{id}{\overset{F}{\rightrightarrows}} BU_n[\tfrac{1}{p}]$$

should be exact in the homotopy category. Actually

$$BGl_n(\overline{\mathbb{F}}_p) \longrightarrow BU_n[p^{-1}]$$

induces isomorphisms mod l. Hence $\widehat{BU_n[p^{-1}]}$ carries Frobenius which descends a la Sullivan ~~to~~ $BU_n[p^{-1}]$.

Sullivan proves existence of action of Galois $(\overline{\mathbb{Q}}/\mathbb{Q})$, but doesn't get ~~canonical~~ an action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})_{ab}$ on $\widehat{BU_n}[p^{-1}]$ although this seems to follow from your model

Question: Does inertia group act non-trivially on $(\widehat{BU_n})_p$?

2.) **Theorem** that for any gen. coh. theory h with finite coefficients and principal U_n bundle P over X we have

$$f^*: h(X) \longrightarrow h(P/N_n)$$

injective onto a direct summand. (Is this true in general?) Is there a canonical f_* map? Is there any method of reducing computation of $h(BU)$ to $h(BN) = h(QBT_1)$ $(Q = \Omega^\infty S^\infty)$?

$\underset{\mathbb{C}P^\infty}{\overset{\shortparallel}{BT_1}}$

3.) $H^*(BO_n(\mathbb{F}_q), \mathbb{Z}_2)$ } exact formulas for.
$H^*(BSp_n(\mathbb{F}_q), \mathbb{Z}_2)$ }

4.) Construct splitting ~~[illegible]~~ $\operatorname{Im} J \times \operatorname{Coker} J = G$ ~~by~~ using finite ~~[illegible]~~ general linear groups and the symmetric groups.

5.) Correct definitions of $RO_A(G)$ and $RSp_A(G)$ together with the decomposition homomorphisms. Perfect complexes with orthogonal or symplectic ~~structure~~?