October 1, 1970

Let G be a finite group acting on a complex analytic manifold X preserving the structure. Then there is an index map

$$f_! : K_G(X) \longrightarrow K_G(pt)$$

where $f: X \to pt$, defined by taking an equivariant bundle E to $E \otimes (\bar\partial\text{-symbol})$ making a G-elliptic operator symbol and taking index. It has the property that if E is an equivariant holomorphic bundle, then

$$f_![E] = \sum (-1)^i [H^i(X, \underline{E})]$$

where $\underline{E}$ denotes the sheaf of holomorphic fns.

Now we want a Riemann-Roch formula for the orbit variety X/G, stick to complex analytic case. That is, a formula for the map

$$K_{hol}(X/G) \xrightarrow{\chi} \mathbb{Z}$$

$$E \longmapsto \chi(X/G, \underline{E})$$

in terms of the characteristic classes of E. So we make some observations:

1) $$H^*(X/G, \underline{E}) = H^*(X, \underline{\pi^* E})^G \qquad \text{where } \pi: X \to X/G.$$

Indeed $\pi_*(\underline{\pi^*E})^G = \underline{E}$ (local on X/G, hence can assume $\underline{E} = \mathcal{O}$, whence it says that holomorphic functions on X/G = invariant holomorphic functions on X) and the two spectral sequences for equivariant cohomology degenerate because the coefficients are $\mathbb{Q}$ and

π is finite.

Hence the map we want is

$$\begin{tikzcd}
K(X/G)_{hol} \arrow[r, "\chi"] \arrow[d, "\pi^*"] & \mathbb{Z} \\
K_G(X)_{hol} \arrow[r, "f_!"] & R(G) \arrow[u, "\text{inner product with trivial rep} = \int_G ."']
\end{tikzcd}$$

Thus we can extend χ to $K(X/G)$

2) $H^*(X/G) = H^*(X)^G$, hence the formula we want is of the form

satisfies Poincaré duality

$$\chi(X/G, E) = \left(ch(E) \cdot \text{Todd}\right)[X/G]$$
$$= ch(E) \cap \{\text{Todd} \cap [X/G]\}$$

So thus the problem is to find the homology class

$$\text{Todd} \cap [X/G] \subset H_*(X/G)$$ (rational coeffs.)

Our problem thus becomes the following. We have the map χ

$$K(X/G) \xrightarrow{\pi^*} K_G(X) \xrightarrow{f_!} R(G) \xrightarrow{\int_G} \mathbb{Z}$$

and want to express it in the form

$$\chi(E) = ch(E) \cap \alpha \quad \text{for some } \alpha \in H_{ev}(X/G).$$

$$
\begin{tikzcd}
K(X/G) \arrow[r, "\pi^*"] \arrow[d, "ch"] & K_G(X) \arrow[r, "f_!"] & R(G) \arrow[d, "\int_G"] \\
H^{ev}(X/G) \arrow[rr, "\cap \alpha"] & & \mathbb{Q}
\end{tikzcd}
$$

Example: Take X to be the projective space PV, V a representation of G. Then

$$H^{ev}(X/G) \xrightarrow{\sim} H^{ev}_G(X) \xrightarrow{\sim} H^{ev}(X) = \mathbb{Q}[H]/H^d \qquad d = \dim V$$

$H = c_1(\mathcal{O}(1))$. Now note that a G-vector bundle E' on X is of the form $\pi^* E$ iff the isotropy representations are trivial. Hence $\mathcal{O}(n)$ comes from X/G where n is the exponent of G. Now however one knows that $ch(\mathcal{O}(n)-1) = e^{nH}-1$ so the elements

$$ch\,(\mathcal{O}(n)-1)^i \qquad i = 0, \ldots, d-1$$

form a $\mathbb{Q}$-basis for $H^{ev}(X/G)$. It follows therefore that we ought to be able to compute α, because α is completely determined by the formulas

$$e^{inH} \cap \alpha = \dim \{ S_{in}(V^\vee)^G \}$$

for all $i \geq 0$.

Suppose take G to be a cyclic group of order p and V a representation with the generator having given eigenvalues. Computation should be manageable

October 3, 1970. Sullivan's Stiefel-Whitney classes.

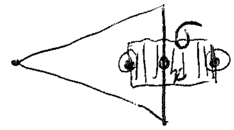
Let X be a polyhedron. A function on X will be called constructible if it is constant on each open simplex of some linear subdivision of X, and similarly for a sheaf. If we consider constructible sheaves of A-modules A a field say, then the Grothendieck ring of these sheaves may be identified with the ring of constructible integer-valued functions, the map assigning to a sheaf F, the function $x \mapsto \operatorname{rank} F_x$. Let $R(X)$ be this ring.

There is a natural linear function $R(X) \longrightarrow \mathbb{Z}$ which associates to F the Euler characteristic $\chi(X,F)$. In terms of functions it assigns the value $(-1)^i$ to the characteristic function of an open i-simplex. More generally given a map $f: X \longrightarrow Y$ of polyhedra (piecewise-linear on some subdivision of X) then we have $f_!: R(X) \longrightarrow R(Y)$ defined as usual.

We think of $F \mapsto \chi(X,F)$ as a intrinsic "measure" on constructible functions. Call a function $f = \boxed{F}$ <u>harmonic</u> if for each point x

$$\sum (-1)^i \dim H^i_{\{x\}}(X,F) = f(x).$$

Suppose f constant on a given subdivision and let σ be the open simplex to which x belongs. Then a neighborhood of x is the product of σ and the cone on the link $L(\sigma)$ of σ and F constant on the σ factor, and so $H^i_{\{x\}}(X,F)$ is the same as



~~$\tilde{F}_\tau = F_{\sigma\cup\tau}$~~

~~$H^{i-d}_{\{x\}}(\text{Cone } L(\sigma), \tilde{F})$, $d = \dim\sigma$. Using exact sequence~~

~~$$\longrightarrow H^j_{\{x\}}(\text{Cone } L(\sigma), \tilde{F}) \longrightarrow H^j(\text{Cone } L(\sigma), \tilde{F}) \longrightarrow H^j(L(\sigma)\times I, \tilde{F})$$~~

~~together with the fact that $\tilde{F}$ is constant on the generators of the cone we see that~~

~~$$\chi(L(\sigma), \tilde{F}) = \tilde{F}(\sigma)$$~~

$H^{i-d}_{\{x\}}(\text{Cone } L(\sigma), F)$, where $d = \dim\sigma$,
Now use the exact sequence

$$H^j_{\{x\}}(\text{Cone } L(\sigma), F) \longrightarrow H^j(\text{Cone } L(\sigma), F) \longrightarrow H^j(L(\sigma)\times[0,1), F)$$

together with fact that F constant along the generatrices of the cone and one finds that

$$f(x) = \chi(L(\sigma), F) + \sum(-1)^{i}\, \dim H^i_{\{x\}}(\text{Cone } L(\sigma), F)$$

$$\boxed{\dim F_x = \chi(L(\sigma), F) + (-1)^d\, \chi(H^*_{\{x\}}(X, F))}$$

For example in an $n-$dimensional manifold with the function 1 and a vertex, we have

$$1 = \chi(S^{n-1}) + (-1)^n$$

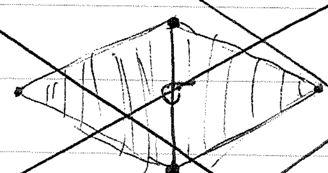
which is OKAY. So we see that mod 2 harmonic is equivalent with the link of every simplex having Euler characteristic zero. Note that the function 1 on a manifold is harmonic iff the manifold has even dimension.

Let σ be a simplex and let U_σ be its star, i.e. points with positive coordinates at each vertex of σ. Then ~~the~~

$$H^*(U_\sigma, F) = \begin{cases} 0 & * > 0 \\ F_\sigma & * = 0 \end{cases}$$

~~and~~

$U_\sigma - \sigma$ homeo $L(\sigma) \times (0,1) \times \sigma$



while U_σ homeo. to $\overset{\text{open}}{\text{Cone}}\, L(\sigma) \times \sigma$. I want $H_c^*(U_\sigma, F)$.

$$H_c^i(\overset{\text{open}}{\text{Cone}}\, L(\sigma) \times \sigma, F) = H_c^{i-d}(\overset{\text{open}}{\text{Cone}}\, L(\sigma), F)$$

$$H_c^*(\text{open cone}) \longrightarrow H^*(\text{cone}) \longrightarrow H^*(L(\sigma))$$

~~so~~ $$\chi_c(U_\sigma, F) = (-1)^d \left[F_\sigma - \chi(L(\sigma), F) \right]$$

$$= (-1)^d \left[(-1)^d \chi\left(H^*_{\{\text{baryc.}\ \sigma\}}(X, F)\right) \right]$$

$=$

and $\qquad H_c^*(U_\sigma, F) = H^*_{\{\text{barycenter } \sigma\}}(X, F)$. By hypothesis then

$$\chi(U_\sigma, F) = \chi_c(U_\sigma, F).$$

Now given a subcomplex A of X, consider the covering by U_σ where σ runs over vertices. Then

$$\chi_A(X, F) = \chi_c(\mathrm{star}(A), F)$$

$$= \sum_{\sigma \text{ in } A} (-1)^{\dim \sigma} \chi_c(U_\sigma, F)$$

and similarly

$$\chi(A, F) = \chi(\mathrm{star}(A), F)$$

$$= \sum (-1)^{\dim \sigma} \chi(U_\sigma, F)$$

(these are both consequences of Mayer-Vietoris)

Thus $\quad F$ harmonic $\Rightarrow$

$$\boxed{\chi(A, F) = \chi_A(X, F) \qquad A \text{ closed}}$$

$$\begin{array}{cc} \| & \| \\ \chi(X,F) - \chi_c(X-A, F) & \chi(X,F) - \chi(X-A, F) \end{array}$$

hence

$$\boxed{\chi(U, F) = \chi_c(U, F) \qquad U \text{ open}}$$

(immediate and simpler from Mayer-Vietoris relation.)

October 4, 1970

The relation $\chi(U,F) = \chi_c(U,F)$ for a harmonic function is very reasonable if you think of the constant function 1 on an even dimensional manifold. Then one knows by duality that $H_c^*(U)$ and $H^*(U)$ have a non-singular pairing into $H_c^{2n}(X) = \mathbb{R}$ and hence have same Euler characteristic

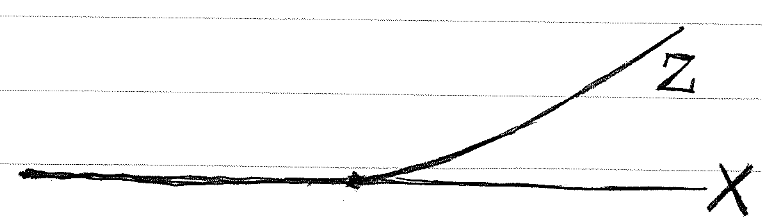
If $f: X \longrightarrow Y$, then $f_!$ carries harmonic functions to harmonic functions, because

$$\chi(V, f_! F) = \chi(f^{-1}V, F)$$

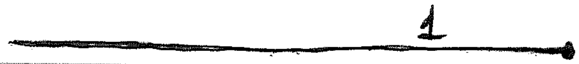
$$\chi_c(V, f_! F) = \chi_c(f^{-1}V, F)$$

by Leray spectral sequence.

However harmonic functions are not closed under inverse image even when X, Y are manifolds. Indeed suppose in the plane $= Y$ we take F to be the constant function on the graph of a function 0 for $x \leq 0$



and $X =$ x-axis. Then pull-back is the function



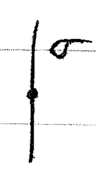
which is clearly not harmonic (mod 2 Euler chars.) as the link is wrong. Just as a check we compute the (local) cohomology of the sheaf $\mathbb{Z}_{(-\infty,0]}$ at 0.

$$\begin{array}{ccccc} H^*_{\{0\}}(\mathbb{Z}_{(-\infty,0]}) & \longrightarrow & H^*(\mathbb{Z}_{(-\infty,0]}) & \longrightarrow & H^*(\mathbb{R}-0, \mathbb{Z}_{(-\infty,0]}) \\ & & \| & & \| \\ & & H^*((-\infty,0], \mathbb{Z}) & \longrightarrow & H^*((-\infty,0), \mathbb{Z}) \end{array}$$

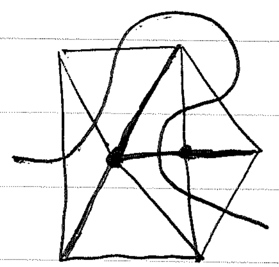
Thus the local cohomology at the point 0 is $\equiv 0$ so the $\chi \neq$ value at zero.

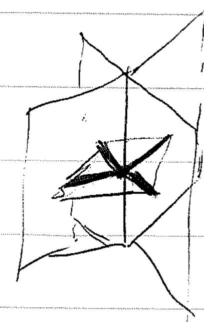
So what we want to prove is that the inverse image of a harmonic function F by a map $f: X \longrightarrow Y$ is again harmonic provided f and F are transversal in a sense to be made precise.

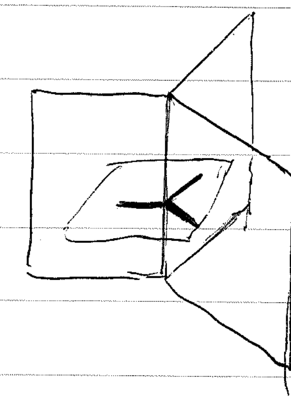
Suppose f is a smooth map of manifolds, and F is constant on the open simplices of a triangulation of Y. Suppose f is transversal to the simplices of the triangulation. Then there is an induced stratification of X and I claim that the inverse image of F is harmonic on X. It's enough to worry about an embedding. Take $x \in X$ and suppose $f(x) \in \sigma$.



By transversality locally near x, Y is $L(\sigma) \times \sigma$ and X is $L(\sigma) \times (X \cap \sigma)$ and F is constant in the σ direction. Thus the link condition remains the same and so $F|X$ is harmonic. (The point to remember is that the the normal structure of the strata around $X \cap \sigma$ is exactly the same as the link of σ.







)

Of course there is a dimension shift, so when working with integral valued harmonic functions it is necessary to require that the relative dimension of f be even.

Suppose now that F is a constructible function on X transversal to a submanifold Y of codimension 1 and that Y has an interior Z, $\partial Z = Y$. Then

$$\int_Y F = 0$$

because it is the difference of $\chi(Z,F)$ and $\chi_c(Z,F)$, which are the same by harmonicity. (To doing anything cobordism-theoretically we have to work mod 2. In effect you want to start with the function 1 on an even dimensional manifold restrict to odd dimensional submanifolds ~~[illegible]~~ (so ~~[illegible]~~ cobordism can be used) and then take χ which of course gives zero.)

Now work mod 2 and suppose F is a harmonic function on a ~~[illegible]~~ manifold X. Define now a map

$$\mathfrak{N}_*(X) \longrightarrow \mathbb{Z}_2$$

by $$[Z \xrightarrow{f} X] \longmapsto \int_Z f^*(F) \qquad f \text{ trans. to } F$$

As we've proved that transversal inverse images are ~~[illegible]~~ harmonic and ~~[illegible]~~ that integral $\underbrace{}_{\text{of a harmonic fn.}}$ vanishes on a boundary, it follows that this map is well-defined. On the other hand ~~[illegible]~~ if M is a compact manifold, then this map associates to $[M \times Z \xrightarrow{f pr_2} X] = [M] \cdot [Z \longrightarrow X]$ the number $\chi(M) \cdot \int_Z f^*(F)$ so that it factors

$$(\mathbb{Z}_2)_\chi \underset{\mathfrak{N}_*}{\otimes} \mathfrak{N}_*(X) \longrightarrow \mathbb{Z}_2$$

Now the ~~[illegible]~~ significance of Deligne's question becomes apparent because ~~[illegible]~~ he wanted to check that there is an isomorphism

$$(\mathbb{Z}_2)_\chi \otimes_{\mathfrak{N}_*} \mathfrak{N}_*(X) \xrightarrow{\sim} H_*(X)$$

October 9, 1970: Localization at a maximal A.

Let A be elementary abelian subgroup of G (compact Lie gp.), N its normalizer, $\mathfrak{p}_A \subset H^*_G$ and $\mathfrak{q}_A \subset H^*_N$ the associated prime ideals. Claim that if A is maximal among [p]-gps fixing points of X, then

$$(1) \qquad H^*_G(X)_{\mathfrak{p}_A} \xrightarrow{\sim} H^*_N(X^A)_{\mathfrak{q}_A}.$$

First of all the localization theorem say

$$H^*_G(X)_{\mathfrak{p}_A} \xrightarrow{\sim} H^*_G(GX^A)_{\mathfrak{p}_A}$$

because the third side of the triangle is a module over $H^*_G(X-GX^A)_{\mathfrak{p}_A} = 0$ since A fixes no points of $X-GX^A$ hence the spectrum of this ring is ϕ. Hence

$$\boxed{\underset{\text{(not. nec. maximal)}}{A \triangleleft G} \implies H^*_G(X)_{\mathfrak{p}_A} = H^*_G(X^A)_{\mathfrak{p}_A}}$$

This special case applied to N shows that

$$H^*_N(X^A)_{\mathfrak{q}_A} = H^*_N(X)_{\mathfrak{q}_A}$$

hence the right side of (1) will give an exact sequence when applied to the diagram

$$X \longleftarrow X \times F \leftleftarrows X \times F \times F.$$

So this reduces us to the case where the isotropy groups are all [p]-gps, $X = GX^A$. It follows that

$$G \times^N X^A \xrightarrow{\sim} X$$

and N/A acts freely on X^A. (Any isotropy groups will contain a conjugate xAx^{-1} and as it is a [p]-groups and A is maximal it will follow that it coincides with this conjugate.) Then

$$H^*_G(X) \xrightarrow{\sim} H^*_N(X^A)$$

If $B = xAx^{-1} \subset N$, $x \in G$, and $B \neq A$, then

$$H^*_N(X^A)_{q_B} = H^*_N(N \cdot (X^A)^B)_{q_B} = 0$$

because $AB > A$ is not a [p]-grp. Hence the only prime in the support of $H^*_N(X^A)$ over p_A is q_A. So

$$H^*_G(X)_{p_A} = H^*_N(X^A)_{p_A} = H^*_N(X^A)_{q_A}$$

which finishes the proof of [redacted].

Theorem: If A maximal [p]-gp with normalizer N, and A determines p_A in H^*_G and q_A in H^*_N, then

$$H^*_G(X)_{p_A} = H^*_N(X^A)_{q_A}.$$

The fundamental problem is to attach an Euler characteristic to $H^*_G(X)_{\mathfrak{p}_A}$. The above thm. reduces this problem to A normal in G. The following example shows that the value of the Poincaré series at $t=-1$ is not additive.

$G=\mathbb{Z}/p^2\mathbb{Z}$, $A=\mathbb{Z}/p\mathbb{Z}$, replace the map $G/A \to pt$ by an inclusion $Y=G/A \longrightarrow$ disk in a faithful repn. of $G/A = X$. Then have

$$\begin{array}{ccccccc}
\xrightarrow{\delta} & H^*_G(X,Y) & \longrightarrow & H^*_G(X) & \longrightarrow & H^*_G(Y) & \longrightarrow \\
 & \shortparallel & & \shortparallel & & \shortparallel & \\
 & & \longrightarrow & H^*_G & \xrightarrow{res} & H^*_A &
\end{array}$$

$$\begin{array}{ccccc}
0 & \longrightarrow & k & \xrightarrow{\sim} & k \\
k & \longrightarrow & k & \xrightarrow{0} & k \\
k & \longrightarrow & k & \xrightarrow{\sim} & k \\
k & \longrightarrow & k & & 0
\end{array}$$

so

$$\text{P.S. } H^*_G(X) = \frac{1}{1-t} \qquad = \frac{1}{2} \quad \text{at } t=-1$$

$$\text{P.S. } H^*_G(Y) = \frac{1}{1-t} \qquad = \frac{1}{2} \quad \text{at } t=-1$$

$$\text{P.S. } H^*_G(X,Y) = \frac{t}{1-t} \qquad = -\frac{1}{2} \quad \text{"}$$

and this isn't additive.

When $A \triangleleft G$, and $X = X^A$, then we have Hochschild-Serre

$$E_2 = H^*(G/A, H^*_A \otimes H^*(X)) \Longrightarrow H^*_G(X)$$

If A maximal, then $H^*(G/A) \longrightarrow H^*_G$ is zero in large degree, hence for r large one expects E_r to be bounded horizontally. It perhaps is reasonable to conjecture that $H^*_G(X)$ might be a free module over a subring Γ of H^*_G such that H^*_A is also free over Γ, say possibly after localizing. If so one might then be able to define ~~$\chi(H^*(X))$ as the quotient of~~

$$\chi\{H^*_G(X) : H^*_A\} = \frac{\chi\{H^*_G(X) : \Gamma\}}{\chi\{H^*_A : \Gamma\}}$$

e.g. Γ might be like $(H^*_A)^{\mathrm{Aut}\,A}$

It seems that one always has a spectral sequence

$$E_2^{pq} = R^p \varprojlim_A \{H^q_A \text{[redacted]}\} \Longrightarrow H^{p+q}_G \text{[redacted]}$$

but there doesn't seem to be any reason for $E_2^{pq} = 0$ for $p \geq N$. This spectral sequence arises from composite functor

$$\varprojlim_A M^A = M^G .$$

October 13, 1970: On Thom's theorem realizing rational classes.

X manifold, $x \in H^q(X)$.
Thom's theorem asserts that $n \cdot x$ ~~can~~ be realized by an oriented submanifold of codimension q for some n

In terms of his realizability criterion, this means that $MSO_q \longrightarrow K(\mathbb{Z},q)$ given by Thom class admits a section in the rational homotopy category. NOT QUITE - see below If q is odd this is trivial as $S^q \simeq K(\mathbb{Z},q)$, so one can realize x by a framed submanifold. If $q = 2p$ we show $MU_p \longrightarrow K(\mathbb{Z},2p)$ admits a section

~~point is that~~

$$\begin{array}{ccc} H^*(K(\mathbb{Z},p)) & \longrightarrow & H^*(MU_p) \\ \| & & \\ \mathbb{Q}[u] & \longrightarrow & \mathbb{Q} \oplus \mathbb{Q}[c_1,\dots,c_n]u \end{array}$$

~~is a free map consequently~~ ~~in the spectral sequence of $MU_p \to K(\mathbb{Z},2p)$ the fibre~~ ~~is tnhz.~~ in the rational homotopy category. But

$$BU_p \xrightarrow{(c_i)} \prod_{i=1}^{p} K(\mathbb{Z},2i)$$

is a rational equivalence and

$$\begin{tikzcd} BU_p \arrow[r, "\text{zero section}"] \arrow[dr, "c_p"'] & MU_p \arrow[d, "u"] \\ & K(\mathbb{Z},2p) \end{tikzcd}$$

so it's clear.

Actually we must be careful of Mumford's objection — all we get this way is a map $X \to MU_{2p} \otimes \mathbb{Q}$. So what must be proved is that when dimension of X is bdd, n can be found $\ni$ dotted arrow exists in

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, dotted] & K(n\mathbb{Z}, 2p) \arrow[d] \\ BU_p \arrow[r] & K(\mathbb{Z}, 2p). \end{tikzcd}$$

More precisely given k want to find n $\ni$ section

$$\begin{tikzcd} & BU_p \arrow[d] \\ K(n\mathbb{Z}, 2p)^{(k)} \arrow[ur, dotted] \arrow[r] & K(\mathbb{Z}, 2p) \end{tikzcd}$$

If F is the fibre of $BU_p \to K(\mathbb{Z}, 2p)$, consider the Postnikov system of the map.

Better work with the map

$$BU_p \longrightarrow \prod_{i=1}^{p} K(\mathbb{Z}, 2i)$$

and try for

$$\begin{tikzcd} & BU_p \arrow[d] \\ \prod_{i=1}^{p} K(n\mathbb{Z}, 2i)^{(k)} \arrow[ur, dotted] \arrow[r] & \prod_{i=1}^{p} K(\mathbb{Z}, 2i) \end{tikzcd}$$

Now the homotopy groups of the fibre F are finite, so

$$\begin{tikzcd}
 & BU_p \arrow[d] & \\
 & Z_{(m)} \arrow[r] \arrow[d] & K(\pi_{m+1}F, m+2) \\
 & Z_{(m-1)} \arrow[r] \arrow[d] & K(\pi_m F, m+1) \\
 & \vdots \arrow[d] & \\
\prod_{i=1}^{p} K(n\mathbb{Z}, 2i) \arrow[r] \arrow[uur] & \prod_{i=1}^{p} K(\mathbb{Z}, 2i) &
\end{tikzcd}$$

so what one needs to know is

<u>Lemma</u>: For any finite abelian group A

$$\varinjlim_n H^m\left(\prod_{i=1}^{p} K(n\mathbb{Z}, 2i), A\right) = 0 \qquad m > 0$$

<u>Proof</u>: By dévissage can assume $A = \mathbb{Z}/p\mathbb{Z}$, by Kunneth can worry about $\{K(n\mathbb{Z}, j)\}_n$ and then by the spectral sequence can use induction on j. For $j = 1$

$$H^*(K(n\mathbb{Z}, 1), \mathbb{Z}_p) \qquad \underline{\text{OKAY.}}$$

In geometrical terms what we have just proved is that given $u \in H^{2p}(X, \mathbb{Z})$, then $\exists$ cx. vector bundle E (of dim. p) such that $c_p(E) = n \cdot u$, n universal depending on dim X.

<u>October 13, 1970:</u>

Need to understand the exponential map for $GL_n(\mathbb{C})$.

$$A \longmapsto e^A = \sum_{n\geq 0} \frac{A^n}{n!}$$

$$\exp : \mathfrak{gl}_n \longrightarrow GL_n .$$

We begin by finding the singular values of exp.

$$\frac{d}{dt} e^{(A+\varepsilon B)t} = (A+\varepsilon B) e^{(A+\varepsilon B)t}$$

$$= A e^{(A+\varepsilon B)t} + \varepsilon B e^{At} \qquad \varepsilon^2 = 0$$

$$\frac{d}{dt}\left\{ e^{-At} e^{(A+\varepsilon B)t} \right\} = \varepsilon\, e^{-At} B e^{At}$$

$$= \varepsilon \sum_{n\geq 0} (\mathrm{ad}(-A))^n \cdot B \, \frac{t^n}{n!}$$

$\Big(\frac{d}{dt}(e^{-At} B e^{At}) = [-A, e^{-At} B e^{At}]$

so if $e^{-At} B e^{At} = \sum t^n \alpha_n$

$$n \alpha_n = [-A, \alpha_{n-1}] \Rightarrow \qquad \alpha_n = \frac{1}{n!}(\mathrm{ad}-A)^n \Big).$$

Then integrating

$$e^{-At} e^{(A+\varepsilon B)t} = I + \varepsilon \sum_{n\geq 0} (\mathrm{ad}\,-A)^n \cdot B \, \frac{t^{n+1}}{(n+1)!}$$

so

$$\boxed{e^{A+\varepsilon B} = e^A + \varepsilon\, e^A \sum_{n\geq 0} \frac{(\mathrm{ad}\,-A)^n B}{(n+1)!} \qquad \varepsilon^2 = 0}$$

$$\therefore\ e^{-A}de^{A} = \sum_{n\geq 0} \frac{(-1)^n}{(n+1)!} \underbrace{[A[A\cdots[AB]\cdots]}_{n \text{ times}} \qquad B = dA.$$

$$\operatorname{tr} e^{-A}de^{A} = \operatorname{tr} dA = d(\operatorname{tr} A)$$

which agrees with formulae

$$e^{\operatorname{tr} A} = \det(e^{A})$$

$$\cancel{e^{\operatorname{tr}A}} \cdot \operatorname{tr} dA = \operatorname{tr}(e^{-A}de^{A}) \cdot \cancel{\det A}.$$

For what values of A is this transformation singular? Suppose A diagonal eigenvalues $\{\lambda_i\}_{i=1}^{n}$, then

$$[A, e_{ij}] = (\lambda_i - \lambda_j)\, e_{ij}$$

$$\sum_{n\geq 0} \frac{(-1)^n}{(n+1)!} (\operatorname{ad} A)^n (e_{ij}) = \frac{1 - e^{\lambda_j - \lambda_i}}{\lambda_i - \lambda_j}\, e_{ij}$$

with understanding that the coefficient is 1 if $\lambda_i = \lambda_j$

$$\therefore\ d e^{\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix} + \varepsilon e_{ij}} = \varepsilon\, \frac{e^{\lambda_i} - e^{\lambda_j}}{\lambda_i - \lambda_j}\, e_{ij}$$

Therefore the exponential map is singular at a diagonal matrix iff two eigenvalues differ by $2\pi i n$ where $n \in \mathbb{Z}$ and $n \neq 0$.

October 13, 1970.

We want to start with a Bott cocycle and reconstruct the bundle it came from. Take a 2-cocycle. If we work with SL_2-bundles then the Bott 4-cocycle should determine the bundle up to torsion.

So we are given

$$h_{UVW} \in \Gamma(U \cap V \cap W, \Omega^2) \qquad \text{alternating}$$

$$k_{UV} \in \Gamma(U \cap V, \Omega^3) \qquad \text{"}$$

$$\ni \quad \delta h = 0, \quad dh = \delta k, \quad dk = 0.$$

So if the covering consists of two elements U, V then all we have is a single form $\omega \in \Omega^3(U \cap V)$ which we want to put in the form

$$\omega = tr(A^{-1}dA)^3$$

where $A : U \cap V \longrightarrow SU_2 = S^3$. But $tr(A^{-1}dA)^3$ is a closed form on S^3, the invariant volume and so our problem is to construct a map $A : U \cap V \longrightarrow S^3$ such that $\omega = A^*(\text{volume})$. Since any volume is locally $= dx_1 \wedge dx_2 \wedge dx_3$, it's clear that this can't always be done, since there exist indecomposable closed 3 forms

$$\dim \, Grass_3(\mathbb{R}^n) = 3(n-3)$$

$$\dim \, \mathbb{P}(\Lambda_3(\mathbb{R}^n)) = \binom{n}{3} - 1$$

So unlike line bundles the form must be modified, and we see that the critical case is to understand the map

$$[X, S^3] \longrightarrow H^3_{DR}(X),$$

which we know induces an isomorphism

$$[X, S^3] \otimes \mathbb{C} \xrightarrow{\ \sim\ } H^3_{DR}(X)$$

($\otimes$ in the sense of Malcev, actually the non-abelian-ness is small since $\pi_6(S^3) = \mathbb{Z}_{12}$.)

[It seems reasonable to consider more generally the map

$$[X, U(n)] \longrightarrow \prod_{i=1}^{n} H^{2i-1}_{DR}(X)$$

given by the map

$$A \longmapsto \operatorname{tr}(A^{-1}dA)^{2i-1}$$
]

OBSOLETE

October 14, 1970: funny unitary groups (problem of Sullivan.)

Let q be a power of p and l a prime $\neq p$. Then I want to compute

$$\varprojlim_{\nu} H^*(GL_n(\mathbb{F}_{q^{l^\nu}}), \mathbb{Z}/l\mathbb{Z})$$

Let $r \mid l-1$ be the order of q in $(\mathbb{Z}/l\mathbb{Z})^*$. Then r is the same for q^{l^ν} since l^ν prime to $l-1$. We know that

$$H^*(GL_n(\mathbb{F}_{q^{l^\nu}}), \mathbb{Z}/l\mathbb{Z}) \hookrightarrow H^*(\{\mathbb{F}_{q^{l^\nu}}(\mu_l)^*\}^m, \mathbb{Z}/l\mathbb{Z})$$

$$m = [\tfrac{n}{r}].$$

Now

$$\mathbb{F}_q(\mu_l)^* \longrightarrow \mathbb{F}_{q^l}(\mu_l)^* \longrightarrow \mathbb{F}_{q^{l^2}}(\mu_l)^* \dashrightarrow$$

cyclic order	cyclic order	cyclic order
$q^{l-1}-1$	$q^{l(l-1)}-1$	$q^{l^2(l-1)}-1$

and

$$\nu_l(q^{l^\nu(l-1)}-1) = \nu_l(l^\nu) + \nu_l(q^{l-1}-1)$$

since

$$\nu_l(q^{l-1}-1) \geq 1$$

and say $l \neq 2$.
OKAY once you take q^2.

So it seems then that

$$\varprojlim_{\nu} H^*(\{\mathbb{F}_{q^{l^\nu}}(\mu_l)^*\}^m, \mathbb{Z}/l\mathbb{Z}) = \mathbb{Z}/l\mathbb{Z}[x_1, \ldots, x_m]$$

whence all the $c''_{j(l-1)}$ disappear in the limit.

<u>October 14, 1970</u>

Still want to understand

$$[X, S^3] \longrightarrow H^3_{DR}(X).$$

The result is that if $\omega \in \Omega^3(X)$ is closed with integral periods, then for some n, $n\cdot\omega - d\eta = A^*\nu$ where $A : X \to S^3$ and ν is the invariant volume element on S^3.

On S^3 there are three ~~left~~ right invariant forms $\omega_1, \omega_2, \omega_3$ which satisfy the Maurer-Cartan formulas

$$d\omega_i = \sum c^i_{jk}\,\omega_j \wedge \omega_k$$

where c^i_{jk} are the structural constants for the Lie algebra. So the map $A : X \to S^3$ gives three one-forms on X whose product is $n\omega - d\eta$ and which satisfy the Maurer-Cartan formulas.

Conversely given $\lambda_i \in \Omega^1(X)$ $i = 1,2,3$ satisfying M-C relations we consider $X \times S^3$ and the ideal $\mathfrak{g}$ in the exterior algebra generated by $pr_1^*(\lambda_i) - pr_2^*(\omega_i)$. This ideal will be stable under d so defines a codimension 3 foliation of $X \times S^3$, which is etale over X as the ω_i span the cotangent space of S^3 at each point. Note that the foliation is right invariant under S^3 multiplication. Consequently an integral leaf will be a covering space of X mapping to S^3.

<u>Conclusion:</u> If $\pi_1(X) = 0$, then $\mathrm{Map}(X, S^3)/S^3$ right mult. is same as forms $\lambda_1, \lambda_2, \lambda_3 \in \Omega^1(X)$ satisfying M-C formulas.

Need to understand non-commutative integration a bit. Suppose G is a nilpotent Lie group simply-connected. Then I claim that there are natural maps

$$G * \cdots * G \longrightarrow G$$

right equivariant which assigns to $\sum t_i g_i$ the appropriate center of gravity. Indeed by induction using exact sequence

$$0 \longrightarrow \mathbb{R}^n \longrightarrow G \xrightarrow{\pi} G' \longrightarrow 0$$

and we have

$$G' * \cdots * G' \longrightarrow G'$$

which saying fixing $g_1 \cdots g_n$ gives us $\sum t_i \pi(g_i)$. Now have to check

$$\begin{tikzcd}
\{0, \ldots, n\} \arrow[r] \arrow[dr] & E \arrow[r] \arrow[d, "f"] & G \arrow[d] \\
& \Delta(n) \arrow[r] & G'
\end{tikzcd}$$

that if f is an "affine" $\mathbb{R}^n$-bundle over $\Delta(n)$ and if you give liftings of the vertices, then there is a canonical section.

need that transition functions are constant affine transformations.

~~[illegible]~~

Problem: To find out what is happening in the proof that $ch: K(X)\otimes\mathbb{Q} \xrightarrow{\cong} H^{ev}(X,\mathbb{Q})$.

For example start with formula

$$\begin{array}{ccc} [X,\ \mathbb{C}P^\infty] & \xrightarrow{\simeq} & H^2(X,\mathbb{Z}) \\ \| & & \\ [X,\ SP^\infty(\mathbb{C}P^2)] & & \end{array}$$

What makes this result true? For example suppose we have a complex-analytic manifold X. Then ~~[illegible]~~ an analytic map $X \longrightarrow SP^n(\mathbb{C}P^2) = \mathbb{C}P^n$ is a line bundle together with $n+1$ generating sections. Thus the proof of this formula requires something about C^∞ functions.

A better understanding is ~~[illegible]~~ achieved by use of sheaf theory. Thus one looks at

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{\exp 2\pi i} \mathcal{O}^*_X \longrightarrow 0$$

and gets a long exact sequence

$$H^1(X,\mathcal{O}_X) \longrightarrow H^1(X,\mathcal{O}^*_X) \xrightarrow{\delta} H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathcal{O})$$

where the ends vanish by partitions of unity. This proves the isomorphism for $X = C^\infty$ and of course it works to do the analytic case also.

∞ Symmetric products

<u>The basic idea:</u> $\quad H^0(X, C_\cdot(Y)) = [X, SP^\infty(Y)]$
ignoring basepoint

Barry's formulation: An element of

$$[X, SP^\infty(Y)] \otimes \mathbb{Q} = \prod_i \mathrm{Hom}(H_i(X), H_i(Y))$$

map of degree zero from X to Y.

Can you algebraically define a map from

$$H^*(Y) \longrightarrow H^*(X)$$

for each map $X \longrightarrow SP^\infty(Y)$

From the rational point of view this is easy because

$$H^*(SP^\infty(Y)) = S\{H^*(Y)\} \qquad \text{as Hopf algebras}$$

~~So now the great problem arises namely~~

$$H^*(X) \xleftarrow{\text{ring hom}} S\{H^*(Y)\} \longleftarrow H^*(Y)$$

so the correspondence is $\boxed{\text{fairly clear.}}$

D. Quillen

October 16, 1970: Bott's formula for Chern classes.

Bott has produced a formula for the Chern classes of a vector bundle in terms of the transition functions for the bundle, which I want to understand.

The idea: Start with $E \longrightarrow X$ complex bundle over a manifold or scheme. Then form the bundle Y over X whose sections are connections, i.e. the bundle of splittings of

$$0 \longrightarrow E \otimes T^* \longrightarrow J_1(E) \longrightarrow E \longrightarrow 0$$

Let $f : Y \longrightarrow X$ be the canonical map. Then $f^*(E)$ has a canonical connection and so global De Rham classes representing the Chern classes. Now if we are given local trivializations

$$s_{\mathcal{U}} : \mathcal{U} \times \mathbb{C}^n \xrightarrow{\sim} E$$

$\mathcal{U} \in$ some covering $\mathcal{U}$, then over each $\mathcal{U}$ we have a canonical section $D_{\mathcal{U}}$ of Y, hence can pull back these classes. For example $\mathrm{tr}\{K^n\}$ pulls back to give an element in $C^0(\mathcal{U}, \Omega^{2n}_X)$ (which is zero because $D_{\mathcal{U}}$ is flat). On $\mathcal{U} \cap V$, then we have a family $t_0 D_{\mathcal{U}} + t_1 D_V$ $t_0 + t_1 = 1$ of connections, hence a formula

$$\mathrm{tr}\{K^n_V\} - \mathrm{tr}\{K^n_{\mathcal{U}}\} = d \int_0^1 dt_1 \{\sim\} = d\, h^{(n)}_{\mathcal{U}V}$$

and the $h^{(n)}_{UV}$ define an element of $C^1(\mathcal{U}, \Omega^{2n-1}_X)$. On $U \cap V \cap W$ we get the family $t_0 D_U + t_1 D_V + t_2 D_W$ of connections which should produce an [redacted] element $h^{(n)}_{UVW} \in \Gamma(U \cap V \cap W, \Omega^{2n-2}_X)$ such that

$$d\, h^{(n)}_{UVW} = h^{(n)}_{VW} - h^{(n)}_{UW} + h^{(n)}_{UV}$$

In general one gets by this process a [redacted] Čech cochain $h^{(n)}_i = \{ h^{(n)}_{U_0 \cdots U_i} \} \in C^i(\mathcal{U}, \Omega^{2n-i}_X)$.

satisfying

$$d h^{(n)}_i = \delta h^{(n)}_{i-1} .$$

It's more or less clear that [still need a proof to be sure]

$$h^{(n)}_{U_0 \cdots U_i} = \int_{[redacted]\, t_1 + \cdots + t_i \leq 1} [redacted] \{ tr(K_t^n) \}$$

where

$$K_t = d\theta_t + \theta_t \theta_t$$

$$\theta_t = \sum_{j=1}^{i} t_j\, g^{-1}_{u_j u_0}\, d g_{u_j u_0}$$

(Here I recall that $s_U = s_V g_{VU}$ and that the ~~connection~~ connection D_V is given ~~by~~ relative to the connection D_U by the form $\theta_U^{D_V}$ determined by

$$D_V s_U = s_U \theta_U^{D_V}$$

$$\parallel$$

$$D_V(s_V g_{VU}) = s_V \, dg_{VU} = s_U g_{VU}^{-1} dg_{VU}$$

i.e.

$$\theta_U^{D_V} = g_{VU}^{-1} dg_{VU}$$

so that family D_t joining the D_{U_j} is relative to D_{U_0} given by the connection form

$$\theta_t = \sum_{j=1}^{i} t_j \, g_{U_j U_0}^{-1} dg_{U_j U_0}).$$

Note that $h_i^{(n)} = 0$ if $i > n$ because in K_t^n cannot get $dt_1 \cdots dt_i$. This says that the n-Chern class takes its values in

$$H^n(X, \Omega^n \to \Omega^{n+1} \to \cdots)$$

The component $h_n^{(n)} \in C^n(\mathcal{U}, \Omega^n)$ should be a δ-cocycle and represent the Atiyah-Hodge class in $H^n(X, \Omega^n)$

Computations for $n=1, 2$ ignoring signs.

$$h^{(1)}_{uv} = tr\{A^{-1}dA\} \qquad A = g_{vu}$$

$$\begin{cases} h^{(2)}_{uv} = \frac{1}{3} tr\,(A^{-1}dA)^3 & A = g_{vu} \\ h^{(2)}_{uvw} = tr\{A^{-1}dA \cdot B^{-1}dB\} & A = g_{vu},\ B = g_{wu} \end{cases}$$

$$g_{wv} = BA^{-1}$$

$$dh^{(2)}_{uv} = -tr\,(A^{-1}dA)^4 = 0$$

$$\delta(h^{(2)}_{1})_{uvw} = h^{(2)}_{vw} - h^{(2)}_{uw} + h^{(2)}_{uv}$$

$$= \frac{1}{3}\left[tr\left((BA^{-1})^{-1}d(BA^{-1})\right)^3\right] - \frac{1}{3} tr(B^{-1}dB)^3 + \frac{1}{3} tr(A^{-1}dA)^3$$

$(BA^{-1})^{-1}d(BA^{-1}) = A[B^{-1}dB - A^{-1}dA]A^{-1}$

$$= \frac{1}{3} tr\,(B^{-1}dB - A^{-1}dA)^3$$

$$= tr\left(B^{-1}dB \cdot (A^{-1}dA)^2\right) - tr\left((B^{-1}dB)^2 A^{-1}dA\right)$$

$$d\,h^{(2)}_{uvw} = tr\left((A^{-1}dA)^2(B^{-1}dB)\right) - tr\left((A^{-1}dA)(B^{-1}dB)^2\right)$$

$$\delta(h^{(2)}_{2})_{u_0u_1u_2u_3} \qquad A = g_{u_1u_0},\ B = g_{u_2u_0},\ C = g_{u_3u_0}$$

$$\cancel{\pm tr\left[(BA^{-1})^{-1}d(BA^{-1}) \cdot (CA^{-1})^{-1}d(CA^{-1})\right] - tr\left[B^{-1}dB \cdot (CB^{-1})^{-1}d(CB^{-1})\right]}$$

$$\cancel{\pm tr\left[A^{-1}dA \cdot (CA^{-1})^{-1}d(CA^{-1})\right] - tr\left[A^{-1}dA \cdot (BA^{-1})^{-1}d(BA^{-1})\right]}$$

$$A\left(B^{-1}dB - A^{-1}dA\right)\left(C^{-1}dC - A^{-1}dA\right)A^{-1}$$

$$\delta h_2^{(2)} = tr\left[(BA^{-1})^{-1}d(BA^{-1}) \cdot (CA^{-1})^{-1}d(CA^{-1})\right]$$
$$- tr\left[B^{-1}dB \cdot C^{-1}dC\right]$$
$$+ tr\left[A^{-1}dA \cdot C^{-1}dC\right]$$
$$- tr\left[A^{-1}dA \cdot B^{-1}dB\right] = 0$$

The above shows that normalized groups cocycles might be very ugly.

However in unnormalized term we have associated to the two simplex (A_0, A_1, A_2) of PG the element

$$tr\left[(A_1A_0^{-1})^{-1}d(A_1A_0^{-1}) \cdot (A_2A_0^{-1})^{-1}d(A_2A_0^{-1})\right]$$

$$= tr\left[\left(A_1^{-1}dA_1 - A_0^{-1}dA_0\right)\left(A_2^{-1}dA_2 - A_0^{-1}dA_0\right)\right]$$

$$= tr\left(A_1^{-1}dA_1 \cdot A_2^{-1}dA_2\right) - tr\left(A_0^{-1}dA_0 \cdot A_2^{-1}dA_2\right)$$
$$+ tr\left(A_0^{-1}dA_0 \cdot A_2^{-1}dA_2\right)$$

The first formula makes visible the right invariance, and the last the fact it is a cocycle

Quite generally in an exterior algebra we have the identity

$$(z_1 - z_0)\wedge\cdots\wedge(z_n - z_0) = \sum_{i=0}^{n}(-1)^i\, z_0\wedge\cdots\wedge\hat{z}_i\wedge\cdots\wedge z_n$$

(induction on n:
$$\sum_{i=0}^{n}(-1)^i\, z_0\wedge\cdots\wedge\hat{z}_i\wedge\cdots\wedge z_n\wedge(z_{n+1}-z_0)$$
$$\sum_{i=0}^{n}(-1)^i\, z_0\cdots\hat{z}_i\cdots z_{n+1} + (-1)(-1)^n z_0\wedge\cdots\wedge z_n)$$

Hence denoting by

$$\varphi_n(z_1,\ldots,z_n) = \sum_{\sigma\in\Sigma_n}(-1)^\sigma \operatorname{tr}(z_{\sigma_1}\cdots z_{\sigma_n})$$

we have

$$\varphi_n\left((A_1A_0^{-1})^{-1}d(A_1A_0^{-1}),\ldots,(A_nA_0^{-1})^{-1}d(A_nA_0^{-1})\right)$$
$$= \varphi_n\left((A_1^{-1}dA_1 - A_0^{-1}dA_0),\ldots,(A_n^{-1}dA_n - A_0^{-1}dA_0)\right)$$
$$= \sum_{i=0}^{n}(-1)^i\,\varphi_n\left(A_0^{-1}dA_0,\ldots,\widehat{A_i^{-1}dA_i},\ldots,A_n^{-1}dA_n\right)$$

(Note that φ is a fn. on exterior algebra because it vanishes if two z_i are equal). The first formula shows that φ_n is invariant under right multiplication and the last one shows it is a cocycle.

Now this gives the Hodge components (up to scalars) of ch_n. The rest must involve something similar using different symmetrizations e.g. the $h_1^{(n)}$ component is

$$tr\left(\left((A_1A_0^{-1})^{-1}\, d(A_1A_0^{-1})\right)^{2n-1}\right) = tr\left((A_1^{-1}dA_1 - A_0^{-1}dA_0)^{2n-1}\right).$$

Now the reason for looking at these formulas is that for any ring R they represent classes

$$ch_i \in H^i(GL(R), \Omega_R^i \to \Omega_R^{i+1} \to \cdots) \qquad \Omega^*_{R/\mathbb{Z}} = \Omega^*_R$$

and hence maps

$$K_a(R) \longrightarrow H^{i-a}(\Omega_R^i \to \Omega_R^{i+1} \to \cdots)$$

$$= \begin{cases} H_{DR}^{2i-a}(R/\mathbb{Z}) & 0 \leq a < i \\ Z_R^i = \mathrm{Ker}\{\Omega_R^i \to \Omega_R^{i+1}\} & a = i. \end{cases}$$

too naive

The conjecture to make is that for R over $\mathbb{Q}$ these maps are essentially isomorphisms, i.e.

$$ch^{\#} : K_a(R) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \bigoplus_{i \geq 0} H^{i-a}(\Omega_R^i \to \Omega_R^{i+1} \to \cdots)$$

is an isomorphism.

<u>Question</u>: Does there exist a relative K-group of R over k similar to $\Omega^*_{R/k}$? The idea somehow is

to realize algebraically the ~~[illegible]~~ idea of the topology on $GL_n(R)$ forcing one to use a different kinds of classifying space. So instead of thinking of $K_*(R)$ as related to $K(S \times R) \longrightarrow H^*(S)$ where S is a variable topos we want to ~~[illegible]~~ allow S to be an arbitrary k-scheme.

October 24, 1970: On symmetric products

Recall the Dold–Thom theorem: Let X be a connected space with basepoint. Then

$$\pi_i\, SP^\infty(X) \cong \tilde{H}_i(X;\mathbb{Z})$$

They prove this by showing that given a cofibration

$$Y \longrightarrow X \longrightarrow X/Y$$

(both Y, X are pointed & connected) then

$$SP^\infty(X) \longrightarrow SP^\infty(X/Y)$$

is a quasi-fibration with fiber $SP^\infty(Y)$, hence one gets a long exact sequence

$$\longrightarrow \pi_i\, SP^\infty(Y) \longrightarrow \pi_i(SP^\infty(X)) \longrightarrow \pi_i(SP^\infty(X/Y)) \xrightarrow{\partial} \cdots$$

Thus the functor $F_*(X) = \pi_*(SP^\infty(X))$ for pointed connected spaces is a generalized homology theory and the only thing left is to identify $SP^\infty(S^1)$. But S^1 being a topological abelian group one knows there are maps

$$S^1 \longrightarrow SP^\infty(S^1) \longrightarrow S^1$$

which one would like to know are homotopy equivalences. Doesn't seem to be entirely trivial, however $SP^n(\mathbb{C}^*)$ can be identified with monic polynomials $z^n + a_1 z^{n-1} \cdots + a_n$ where a_n is a unit. This gives a fibration $SP^n(\mathbb{C}^*) \longrightarrow \mathbb{C}^*$

whose fiber is a vector bundle of dimension $n-1$.
So now everything is clear.

Next I want to see that

$$\boxed{[Y; SP^\infty X]_0 = \mathrm{Hom}_{D(ab)}(\tilde{C}.(Y), \tilde{C}.(X))}$$

Enough to define the map really and that works this way

$$[Y; SP^\infty X]_0 \longrightarrow \mathrm{Hom}_{D(ab)}(\tilde{C}.(Y), \tilde{C}.(SP^\infty X))$$

so we need a map

$$\tilde{C}.(SP^\infty X) \longrightarrow \tilde{C}.(X).$$

But semi-simplicially this is obvious, namely you have dimension-wise a map

$$SP^\infty(X) \longrightarrow \tilde{\mathbb{Z}}X \qquad (\tilde{\mathbb{Z}}X = \mathbb{Z}X/\mathbb{Z}*)$$

which extends to

$$\tilde{\mathbb{Z}}\, SP^\infty(X) \longrightarrow \tilde{\mathbb{Z}}X$$

in a canonical way.

To see if this can be understood geometrically. Thus if X is a space I want to define a map

$$\tilde{H}_*(SP^n X) \longrightarrow \tilde{H}_*(X)$$

This must be something like the transfer in the Borel books. I recall this:

Suppose G finite acts on X Hausdorff. Then for F on X/G we have

$$\left(f_* f^* F\right)_y = \prod_{x \in f^{-1}\{y\}} F_y \qquad f : X \to X/G$$

and we want to define a $\overbrace{\text{map}}^{\text{trace}}$

$$f_* f^* F \longrightarrow F.$$

The obvious thing to try is the sum map

$$\prod_{x \in f^{-1}\{y\}} F_y \longrightarrow F_y$$

$$(a_x) \longmapsto \sum a_x$$

Unfortunately if $F = \mathbb{Z}$, then we have that the composite map

$$\mathbb{Z} \longrightarrow f_* f^* \mathbb{Z} \longrightarrow \mathbb{Z}$$

is multiplication by card $f^{-1}\{y\}$ on fibers over y which won't be locally constant. Hence we need a multiplicity function $x \longmapsto m(x)$ which ~~[illegible]~~ gives the multiplicity of x ~~[illegible]~~ in the fiber $f^{-1}f(x)$.

Thus I need to have a ~~[illegible]~~ continuous map

$$X/G \longrightarrow SP^n(X)$$

$$y \longmapsto f^{-1}\{y\} \text{ counted with multiplicity}$$

which assigns to $y \in X/G$ the divisor

$$\sum_{x \in f^{-1}\{y\}} m(x)$$

But if $n = |G|$ then the obvious multiplicity function is

$$m(x) = \text{card } G_x$$

but the most efficient multiplicity function it appears is when

$$n = \text{l.c.m.} \{\text{card } f^{-1}\{y\} ; y \in Y\}.$$

and then

$$m(x) = \frac{\text{card } G_x}{\text{g.c.d.}\{\text{card } G_x\}}$$

Thus when X has one orbit type G/H we can take all $m(x) = 1$.

So now given a multiplicity fn. $m(x)$ we define the trace

$$f_* f^* F \xrightarrow{tr} F$$

by

$$\prod_{x \in f^{-1}\{y\}} F_y \longrightarrow F_y$$

$$(a_x) \longmapsto \sum m(x) a_x .$$

To see this is well-defined we have to show that it maps continuous sections of $f_* f^* F$ to continuous sections of F. Work near y_0 and suppose we have elements $a_x \in F_{y_0}$ for $x \in f^{-1}\{y_0\}$. Then over some nbd. U of y_0 we get sections $s_x \in \Gamma(U, F) \ni s_x(y_0) = a_x$ for $x \in f^{-1}\{y_0\}$. I want to show that

$$\sum_{x' \in f^{-1}\{y'\}} m(x')\, s_x(y')$$

is continuous for y' near y_0. However if y' is really close to y_0, then each x' is closed to only one x and then

$$\sum_{x' \text{ close to } x} m(x') = m(x)$$

so this is all clear.

<u>Conclusion</u>: ~~Suppose~~ Whenever a **finite group** G acts on a (separated) space X there is a natural trace map on cohomology

$$f_* : H^i(X;\Lambda) \longrightarrow H^i(X/G;\Lambda) \qquad \Lambda \text{ arbitrary.}$$

satisfying $f_* f^* = |G|$, $f^* f_* = \sum_{g \in G} g^*$. The universal situation is to define

$$H^i(X;\Lambda) \longrightarrow H^i(SP^n(X);\Lambda)$$

which is the equivariant sum $u \mapsto \sum_i pr_i^* u$

<u>Definition:</u> Recall that

$$F \xrightarrow{\sim} (f_* f^* F)^G$$

and summing over the group defines a map

$$f_* f^* F \xrightarrow{\sigma} (f_* f^* F)^G = F.$$

Thus one gets the required map

$$\begin{tikzcd}
H^i(X; f^*F) \arrow[drr, bend right, "f_*"'] & & H^i(X/G, f_* f^* F) \arrow[ll, "\sim"'] \arrow[d, "\sigma"] \\
& & H^i(X/G, F)
\end{tikzcd}$$

(Leray s.s. degenerate)