<u>September 20, 1970:</u>

General problem: Let G be a group in a topos $\mathcal{T}$. Then under what conditions might the cohomology of the discrete group $\Gamma(G)$ be the same as the cohomology of $\mathcal{T}_G$.

Suppose for example G is a sheaf of groups over a topological space S. Then we have canonical map $\Gamma(S,G)_S \longrightarrow G$ of sheaves of groups over S, and this induces a morphism

$$H^*(S,G) \longrightarrow H^*(S,\Gamma(S,G)_S).$$

Necessary to explain this notation. Thus if Γ is a discrete group Γ_S means the constant group sheaf associated to Γ over S, i.e. with etale space $S\times\Gamma \to S$. If G is a group sheaf and if X is a G-sheaf, e.g. an open set of S with trivial action, then $H^*(X,G)$ is the cohomology (coeffs. mod ℓ) of $(Sh/S)_G$ over X. Thus one has the familiar E-M spectral sequence

$$E_2^{pq} = \dot{H}^p(\nu \mapsto H^q(G^\nu \times X)) \Longrightarrow \dot{H}^{p+q}(X,G).$$

Now under suitable conditions I expect a spectral sequence

$$H^p(S, s \mapsto H^q(BG_s)) \Rightarrow H^{p+q}(S, G)$$

which should be that of the composite functor: first $F \mapsto H^0(G;F)$ = subsheaf of invariant elements of F, composed with global sections over S. In particular for G constant $= \Gamma_S$ one expects a Kunneth formula

$$H^*(S) \otimes H^*(B\Gamma) = H^*(S, \Gamma_S)$$

under finiteness conditions.

Situation for a finite field k_0. Take G ~~to be~~ GL_n as part of the gross etale topos over k_0. Then working mod l (r least $\ni q^r \equiv 1 \mod l$, $q = \mathrm{card}\, k_0$) one can compute

$$H^*(\mathrm{Spec}\, k_0, GL_n; \mathbb{Z}/l)$$

using the spectral sequence over $\mathrm{Spec}\, k_0$. Thus one gets

$$\mathrm{gr}\, H^n(\mathrm{Spec}\, k_0, GL_n; \mathbb{Z}/l) = H^n(\mathrm{Spec}\, k, GL_n; \mathbb{Z}/l)^{Gal}$$
$$\oplus\, H^{n-1}(\mathrm{Spec}\, k, GL_n; \mathbb{Z}/l)_{Gal}$$

Unfortunately [redacted] as $Gal = \hat{\mathbb{Z}}$ acts on cohomology of $H^*(\mathrm{Spec}\, k, GL_n; \mathbb{Z}/\ell) = \mathbb{Z}/\ell[c_1, \dots, c_n]$ $\quad \sigma^*(c_i) = q^i c$
the invariants already are [redacted] too big to coincide with the mod ℓ cohomology of $GL_n(k_0)$, i.e.

$$H^*(\text{[redacted]}\ \mathrm{Spec}\, k, GL_n; \mathbb{Z}/\ell)^{Gal} \cong \text{[redacted]} \bigoplus_{j \geq 0} H^{2rj}(BU, \mathbb{Z}/\ell)$$

has too big a Poincaré series.

September 20, 1970. equivariant cohomology

Let us define $H^*_G(X,\Lambda)$, Λ a discrete G-module, by sheaf theory à la Grothendieck. I want to have the spectral sequence

$$E_2 = H^p(BG, H^q(X,\Lambda)) \Longrightarrow H^{p+q}_G(X,\Lambda)$$

under suitable conditions. Thus one uses the Leray spectral sequence for the map $f: X_G \to e_G = BG$ and has to prove the base change formula for Λ (which is constant) in diagram

$$\begin{tikzcd} G^\nu \times X \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ G^\nu \arrow[r, "g"'] & pt \end{tikzcd}$$

for all ν. Cases where this holds:

1) G discrete, X arbitrary

~~2) G locally contractible, X locally compact (Hausdorff): (where locally contractible means that $\forall x \in G$ $\forall U \ni x$ $\exists V \subset U$ $V \ni x$ $\ni$ V contracts to x in U.)~~

~~Proof: Since X is locally~~

2) G locally contractible locally compact, X loc. comp. (locally contractible here may be taken to mean each nbd U of x contains a smaller nbd V which contracts to

x in U.

<u>Proof:</u> One has to show
$$\varinjlim_{U \ni x} H^i(U \times X) = H^i(X).$$

For a locally compact space Y, the proper base change theorem for $Y \times I \to Y$ implies $H^i(Y \times I) = H^i(Y)$, hence if V contracts to x in U, then have diagram

$$\begin{tikzcd}
H^i(U \times X) \arrow[r, "h^*"] \arrow[dr, shift left, "res^x_V res^U_x"] \arrow[dr, shift right, swap, "res^U_V"] & H^i(I \times V \times X) \arrow[d, shift right, swap, "i_0^*"] \arrow[d, shift left, "i_1^*"] \\
& H^i(V \times X)
\end{tikzcd}$$

because $V \times X$ is loc. compact.

thus it follows that the maps $V \hookrightarrow U$ and $V \to x \hookrightarrow U$ have same effect, i.e. that the inductive system $U \mapsto H^i(U \times X)$ is essentially constant with limit $H^i(X)$.

---

3) G, X locally compact, $H^i(X)$ finite dimensional. In effect claim that $f : X \to pt$ has base change thm. for all compact $g : K \to pt$. Indeed by using proper base change for the map $K \times X \to X$ one gets a spectral sequence
$$E_2 = H^*(X, H^*(K)) \Longrightarrow H^*(K \times X)$$

which by finite dimensionality of $H^*(X)$ can be written

$$E_2 = H^*(X) \otimes H^*(K) \Longrightarrow H^*(K \times X).$$

This degenerates, so get $H^*(K \times X) \cong H^*(X) \otimes H^*(K)$. Now take limit as K goes to a point and you see $f : X \to pt$ is OKAY.

Next one wants to know when the orbit space spectral sequence

$$\text{(2)} \qquad E_2 = H^*(X/G,\ \mathcal{O} \mapsto H^*_G(\mathcal{O})) \Longrightarrow H^*_G(X)$$

holds. Actually in our proof, it will be necessary to know this only when the isotropy groups are finite.

The following two pages show this spectral sequence holds for G discrete when X is Hausdorff and the action is discontinuous.

Grothendieck ~~situation~~ : X Hausdorff, G discrete group, action discontinuous, i.e. $\forall x$ has a nbd $U_x \ni$

$$\{g \mid gU_x \cap U_x \neq \phi\} \text{ is finite}$$

(For example if G is finite) In this case one has the spectral sequence

$$① \qquad H^p(X/G, \mathcal{O} \mapsto H^q_G(\mathcal{O})) \Longrightarrow \underset{G}{H}^{p+q}(\cancel{X})$$

(this should be in Tohoku).

Proof of ②: One considers the equivariant cohomology defined using G-sheaves and obtains a spectral sequence like ② by considering $F \mapsto \Gamma(X,F)^G$ as the composite functor

$$F \longmapsto f_* F^G \longmapsto \Gamma(X/G, (f_*F)^G).$$

One must show that derived functors of $(f_*F)^G$ coincide with $\mathcal{O} \mapsto H^q_G(\mathcal{O}, F)$, or that

$$\varinjlim_{\substack{U \supset \mathcal{O} \\ U \text{ inv}}} H^q_G(U, F) \xrightarrow{\sim} H^q_G(\mathcal{O}, F).$$

Necessary to show (i) second coh. functor is effaceable (ii) coincide at $q=0$. ~~By assumption the~~ As this is local around $\mathcal{O}$ one can replace X by a U of form $G \times^{G_x} V$ where V is a nbd of $x \in \mathcal{O}$ (here use hypothesis Hausdorff + discont as well as the fact that $\mathcal{F} \mapsto \mathcal{F}/U$ preserves injectives as has an exact left adjoint $A \mapsto j_! A$, $j_!$ extension by 0). Now easy to see that we are reduced to case of finite G_x since G-sheaves on $G \times^{G_x} V$ same as G_x-sheaves on V. So

$Map(G, \mathcal{F}) = \prod_{g \in G} \mathcal{F}$

can assume G finite. But then have embedding $\mathcal{F} \longrightarrow$ which effaces $H^*_G(\mathcal{O}, F) = H^*(G_x, F_x)$ and one has also that

$$\varinjlim_{\substack{U \supset \mathcal{O} \\ U \text{ inv}}} \mathcal{F}(U) = \mathcal{F}(\mathcal{O})$$

as any neighborhood contains an invariant nbd. and as $\mathcal{O}$ is finite and X is Hausdorff.

Apply ② to the G-space $PG \times X$ and one finds that

$$H^*(X_G, f_* F^G) = H^*_G(X, F)$$

so the Grothendieck coh. coincides with sheaf coh. of X_G. Note that ① arises from composite functor

$$F \longmapsto \Gamma(X, F) \longmapsto \Gamma(X, F)^G.$$

If G is a compact Lie group and X is locally compact, then have orbit space spectral sequence always. Indeed if $P_n \to B_n$ is an inductive system of principal G-bundles with B_n compact locally contractible and P_n getting higher + higher connected, then one shows first that

$$H^i_G(X) \xrightarrow{\sim} H^i(P_n \times^G X)$$

for n large using the spectral sequence of type ① for the map $X \longleftarrow P_n \times X$ of locally-compact spaces. By proper base change for the map $P_n \times^G X \longrightarrow X/G$ one gets spec. sequences

$$E_2^{pq} = H^p(X/G, \mathcal{O} \longmapsto H^q_G(P_n \times \mathcal{O})) \Rightarrow H^{p+q}_G(P_n \times X)$$

and now one can use stability as n goes to infinity to get the desired spectral sequences.

I expect the orbit space spectral sequence to hold for G locally compact, X loc. compact, provided action is discontinuous in the sense that for any x the isotropy group H_x is compact and leaves invariant many compact nbds N of x and $\{g \in G \mid gN \cap N \neq \phi\}$ compact. (locally X is of the form $G \times^{H_x} N$, H_x compact, N compact)

September 24, 1970.

I want to establish the F-isomorphism theorem for a general compact G-space X by passage to the limit from the finite dimensional case.
It is necessary to reinterpret

$$(*) \qquad \varprojlim_{I_G(X)} S(A^\vee) = Q_G(X)$$

Suppose we let $\mathcal{J}$ be a set of representatives for the conjugacy classes of [el] subgps in G. Then there is an exact diagram

$$(**) \qquad \varprojlim_{I_G(X)} S(A^\vee) \longrightarrow \prod_{A \in \mathcal{J}} S(A^\vee)^{\pi_0(X^A)} \rightrightarrows \prod_{\substack{u: A \to B \\ A, B \in \mathcal{J}}} S(A^\vee)^{\Delta_u(X)}$$

where $\Delta_u(X) = \{ (\alpha, \beta) \mid \alpha \in X^A, \beta \in X^B \text{ and } u^*(\beta) = \alpha \}$

i.e. diagrams
$$\begin{tikzcd} G/A \arrow[d, "u"] \arrow[dr, "\alpha"] & \\ G/B \arrow[r, "\beta"'] & X \end{tikzcd}$$

Now when X is nice, the sets $\pi_0(X^A)$, $\Delta_u(X)$ are all finite, so the thing to do is to define $(*)$ by exactness of $(**)$ where for a general X $S(A^\vee)^{\pi_0 X^A}$ is the continuous maps from $\pi_0(X^A)$ to $S(A^\vee)$.

Thus an element of $Q(X)$ is a function assigning to each A a locally constant function on X^A

with values in $S(A^\vee)$,
in such a way as to be compatible with conjugation and restriction (i.e. if $A \subset B$, then $X^B \subset X^A$ and f_A restricts to $\mathrm{res}^B_A f_B$.)

Now we can also pass to the limit over G. Thus suppose G profinite. Then we have an F-isomorphism

$$H^*_{G_\nu}(pt) \longrightarrow Q^*_{G_\nu}(pt).$$

so take limit over ν.

$$\varinjlim_{\nu} Q_{G_\nu}(pt) = ?$$

First suppose G has no element of order p. Then want to show that

$$\varinjlim_{\nu} Q^+_{G_\nu}(pt) = 0.$$

But given an non-zero element f of this inductive limit, it is non-zero on some G_ν. Start with $f_\nu \in Q_{G_\nu}(pt)$, then $\exists$ a $\nu' > \nu$ such that all elementary abelian ℓ-subgroups of $G_{\nu'}$ go to 1 under the homomorphism $G_{\nu'} \to G_\nu$. Indeed for each ν' set

$$X_{\nu'} = \{ g \in G_{\nu'} \mid g^\ell = 1, \quad g \notin \mathrm{Ker}\, G_{\nu'} \to G_\nu \}$$

Then as $\varprojlim X_{\nu'} = \emptyset \Longrightarrow X_{\nu'} = \emptyset$ some large ν'.

Application to Thms. of Serre: A profinite group G having a closed subgroup G' of finite index with $H^*(G')$ f.d. and no element of order p has $H^*(G)$ f.d.

Indeed, the above argument $\Rightarrow$ every element in H^*_G killed by some power of Frobenius and, on the other hand, spectral sequence Hochschild–Serre implies that H^*_G f. g.

(doesn't seem possible to get Serre's result that $cd_\ell G' < \infty \Rightarrow cd_\ell G < \infty$ except by Serre's method $G/G' = \mathbb{Z}/\ell$ + periodicity

---

For the spectrum of a compact topological group we can use passage to the limit. Thus suppose that $\mathfrak{p}$ is a Steenrod invariant prime ideal $H^*(G) = \varinjlim_{\nu} H^*(G_\nu)$. Then $\mathfrak{p} \cap H^*(G_\nu)$ is a Steenrod invariant prime in $H^*(G_\nu)$ hence of form $\mathfrak{p}_{A_\nu}$ where A_ν is a $[\ell]$-subgroup of G_ν.

To each G_ν associate the compact space X_ν of conjugates of A_ν. Then for $\nu' > \nu$

$$\begin{tikzcd} H^*(G_\nu) \arrow[r] & H^*(G_{\nu'}) \arrow[d] \\ & H^*(A_{\nu'}) \end{tikzcd}$$

so $X_{\nu'}$ induces X_ν. Then $\varprojlim X_\nu$ will be non-empty so we get an $[\ell]$-subgroup of G unique up to conjugacy as desired.

Standard situation: $G' \triangleleft G$, $[G:G'] < \infty$ and $H^*(G')$ f.d. To show that the spectrum of $H^*(G)$ admits usual description, in particular that there exist finitely many conjugacy classes of $[\ell]$-subgroups.

One knows $H^*(G)$ fin gen. ring, so fin pres. which implies that $\exists\ \nu' > \nu \ni$

$$H^*(G) = \mathrm{Im}\left\{ H^*(G_\nu) \xrightarrow{\text{finite}} H^*(G_{\nu'}) \right\}$$

so $H^*(G_\nu) \twoheadrightarrow H^*(G)$ for large $\nu \Longrightarrow H^*(G)$ has only finitely many Steenrod invariant primes ideals $\Longrightarrow$ only finitely many conjugacy classes of $[\ell]$-subgroups.

OKAY for compact p-adic analytic groups.

Let G be a pro-ℓ-group $\ni$ $H^*(G)$ fin. gen.
Then for ν large

$$H^*(G_\nu) \twoheadrightarrow H^*(G),$$

hence $G' = \mathrm{Ker}\{G \to G_\nu\}$ is ℓ-torsion-free, because if $\exists\ \mathbb{Z}/\ell\mathbb{Z} \hookrightarrow G'$ one gets a prime ideal $\mathfrak{p}$ in $H^*(G)$ $\ni$ $\mathfrak{p} < H^+(G)$ while the inverse image of $\mathfrak{p}$ in $H^*(G_\nu)$ will be $H^+(G_\nu)$

$$\begin{array}{ccc} H^*(G_\nu) & \longrightarrow & H^*(G) \\ \downarrow & & \downarrow{\scriptstyle \text{finite}} \\ H^*(AG')^{\circ} & \longrightarrow & H^*(A) \neq 0 \end{array}$$

Moreover $H^*(G') = H^*(G, M)$ where $M = \mathrm{Map}_{G'}(G, \mathbb{Z}/\ell\mathbb{Z})$, and M will have composition quotients $\mathbb{Z}/\ell\mathbb{Z}$ so $H^*(G, M)$ is a f.g. $H^*(G)$ module any M. So $H^*(G')$ fin. gen. and every element will be nilpotent so $H^*(G')$ will be finite.

Conversely if G contains an open G' $\ni$ $H^*(G')$ is finite we know already $H^*(G)$ f.g. Thus have

<u>Proposition</u>: A pro-ℓ-group G has $H^*(G)$ fin. generated $\Longleftrightarrow$ it contains an open subgroup G' $\ni$ $H^*(G')$ finite.

<u>Corollary</u>: If G is profinite group of exponent ℓ^ν such that $H^*(G)$ is finitely generated, then G is finite.

Attempts to apply this to Burnside problem don't seem to work because $H^*(G)$ f.g. can't be proved in any obvious way.

<u>Question</u>: If $H^1(G)$ fixed can you obtain a bound on where all the other generators are valid for all finite groups?

~~Suppose G is a profinite group such that $H^*(G)$ fin. gen. Then again there is an open normal subgroup G' such that G' is ℓ-torsion free. Let $G'' \supset G'$ be $\ni$ G''/G' is a Sylow ℓ subgroup of G/G'. Then $H^*(G) \hookrightarrow H^*(G'')$ as $[G:G'']$ is prime to ℓ, and the Hilbert argument [h~~

September 27, 1970.

Relations between $H^*(X)$ and $H^*(X^A)$ in the good case. Assume $X_A \to BA$ has fibre tnhtz. Then

$$H_A^*(X) \otimes_{H_A^*} S(A^\vee) = \Gamma$$

is a graded anti-commutative algebra which is finite flat over $S(A^\vee)$. At the zero point $S(A^\vee) \longrightarrow k = \mathbb{Z}/p$. one has

$$\Gamma_0 \xrightarrow{\sim} H^*(X)$$

and at a point ξ of $A \otimes \Omega$ with $e(\xi) \neq 0$

$$\Gamma_\xi \xrightarrow{\sim} H^*(X^A) \otimes \Omega$$

Thus $H^*(X^A)$ is a deformation of $H^*(X)$ in some sense.

If p is odd, then $\Gamma = \Gamma^{ev} \oplus \Gamma^{odd}$ and the deformation preserves the grading. Thus $H^{odd}(X) = 0 \Leftrightarrow H^{odd}(X^A) = 0$.

Suppose A cyclic. Then we get an isom mod finite length modules over H_A^*:

$$H_A^*(X) \longrightarrow H_A^*(X^A)$$

hence an isomorphism mod finite length modules over $S(A^\vee)$

$$\Gamma^* \longrightarrow S(A^\vee) \otimes H^*(X^A)$$

hence

$$\Gamma^{2n} \xrightarrow{\sim} \bigoplus_{i=0}^{d(X^A)} S_{n-i}(A^\vee)\otimes H^{2i}(X^A) \qquad (p \text{ odd})$$

but

$$gr\,\Gamma^{2n} \simeq \bigoplus_{i=0}^{d} S_{n-i}(A^\vee)\otimes H^{2i}(X) \qquad ''$$

so one concludes that

$$\sum_{i\geq 0} \dim H^{2i}(X) = \sum_{i\geq 0} \dim H^{2i}(X^A).$$

similarly

$$\boxed{\left.\begin{aligned} \dim H^{ev}(X) &= \dim H^{ev}(X^A) \\ \dim H^{odd}(X) &= \dim H^{odd}(X^A) \\ \chi(X) &= \chi(X^A) \end{aligned}\right\} p \text{ odd}}$$

$$\boxed{\dim H^*(X) = \dim H^*(X^A) \qquad p=2}$$

Now for n large we have

$$H_A^n(X^A) = \bigoplus_i H_A^{n-i}\otimes H^i(X^A)$$

$$gr\, H_A^n(X) = \bigoplus_i H_A^{n-i}\otimes H^i(X)$$

and the map $H^n_A(X) \xrightarrow[\sim]{\phi} H^n_A(X^A)$ preserves filtration i.e.

$$\phi\left(\mathrm{Filt}_{n-q} H^n_A(X)\right) \subset \bigoplus_{i\le q} H_A^{n-i} \otimes H^i(X^A)$$

which implies that

$$\boxed{\sum_{i\le q} \dim H^i(X) \le \sum_{i\le q} \dim H^i(X^A). \quad \text{all } p}$$

Start with $H_A^*(X)$ graded algebra over H_A^* with filtration

$$\mathrm{Filt}^q H^n_A(X) = \mathrm{Filt}_{n-q} H^n_A(X)$$

This is an increasing filtration ∋

$$\mathrm{Filt}^q/\mathrm{Filt}^{q-1} \cong H_A^* \otimes H^q(X).$$

Similarly for X^A. Next for

$$\Gamma = H_A^*(X) \otimes_{H_A^*} S(A^\vee)$$

and you take induced filtration, for which it is again true that

$$\mathrm{Filt}^q(\Gamma)/\mathrm{Filt}^{q-1}(\Gamma) \simeq S(A^\vee)\otimes H^q(X)$$

(by some normal flatness argument). Similarly for X^A. Now you can split into odd & even and you get

$$\boxed{\begin{gathered}\sum_{i\leq q} \dim H^{2i}(X) \leq \sum_{i\leq q} \dim H^{2i}(X^A) \\ \sum_{i\leq q} \dim H^{2i+1}(X) \leq \sum_{i\leq q} \dim H^{2i+1}(X^A) \\ p \text{ odd}\end{gathered}}$$

These relations that I have derived must hold not just for A cyclic but in general, because reduces to a 1-parameter deformation by taking a generic specialization $S(A^\vee) \longrightarrow S(\Omega^\vee)$.

Difference of numerical nature between $p=2$ and p odd is that when the A-action on X is trhz

$$\chi(X) = \chi(X^A) \qquad p \text{ odd, but}$$

$$\chi(X) \equiv \chi(X^A) \quad \text{mod } 2 \quad \text{if } p=2$$

<u>Example</u>: Let $A < O_n$ be the diagonal matrices, and let it act on O_n by conjugation. One knows the action is trhz because already the conjugation action of O_n on itself is trhz, since suspension $H^*(BO_n) \longrightarrow H^*(O_n)$ is surjective. But

$$O_n^A = \text{cent. of } A \text{ in } O_n = A$$

hence have

$$\chi(O_n) = 0$$

$$\chi(O_n^A) = 2^n$$

as O_n is a Lie group of pos. dimension, hence has a everywhere non-zero vector field.

Contrast this with $A =$ points of order p on $T \subset U_n$.

Then

$$U_n^A = T_n \qquad \chi(U_n^A) = 0 \quad + \quad \chi(U_n) = 0.$$

<u>Euler characteristics:</u>

Work with finite groups + compactifiable G-manifolds.

Define
$$\chi(X,G) = \frac{\chi(X)}{\operatorname{card} G},$$
also define $\chi_c(X,G)$ using cohomology with compact support. These numbers for a manifold will be same up to sign by Poincaré duality.

Now define a constructible function on X/G by
$$f(\mathcal{O}) = \chi(\mathcal{O},G) = \frac{\operatorname{card} \mathcal{O}}{\operatorname{card} G}.$$

Integration of constructible functions is possible i.e.
$$\int_{X/G} \sum \lambda_i \overset{\text{char fn. of } Y_i}{Y_i} = \sum \lambda_i\, \chi(H_c^*(Y)).$$

Claim that
$$\boxed{\int_{X/G} \left(\mathcal{O} \mapsto \frac{\operatorname{card} \mathcal{O}}{\operatorname{card} G}\right) = \chi(X,G)}$$

Indeed work with G sheaves on X which are constructible and prove that
$$\int_{X/G} \left(\mathcal{O} \longmapsto \frac{\dim \Gamma(\mathcal{O},\mathcal{F})}{\operatorname{card} G}\right) = \chi(X,G;\mathcal{F}).$$

Both sides being additive one can reduce to $\mathcal{F}$ being concentrated on an orbit type components whence it's clear.

How to think about this integration formula. Following Grothendieck introduce the Grothendieck group of constructible $\text{mod } p$ G-sheaves on X; this should be the same as the ring of constructible sections of the sheaf on X/G which associates to each orbit $\mathcal{O}$ $=Gx$ the Grothendieck group of representations of the isotropy group G_x over $\mathbb{Z}/p\mathbb{Z}$. Call this $R(X,G)$. Observe that there is an integration map: $\int_* : R(X,G) \longrightarrow R(Y,G)$. Does there exist any kind of Riemann-Roch theorem, any natural transformation such as the character?

Möbius type formula: Let $X^{(H)}$ be the subspace whose points have isotropy group H. Then

$$X^H = \coprod_{H \subset H'} X^{(H')}$$

so

$$\chi_c(X^H) = \sum_{H \subset H'} \chi_c(X^{(H')})$$

and the Möbius inversion formula reads

$$\chi_c(X^{(H)}) = \sum_{H \subset H'} \mu(H,H')\, \chi_c(X^{H'})$$

where μ is the Möbius function defined by

$$\sum_{H'} \mu(H,H')\, \zeta(H',H'') = \delta_{H,H''}$$ (+ inverse possibly)

the ζ-function being

$$\zeta(H', H'') = \begin{cases} 1 & H' \subset H'' \\ 0 & \text{otherwise.} \end{cases}$$

This absurd formalism deserves to be understood for a category (e.g. elementary abelian p-subgroups of G).

<u>Example</u>: Suppose G is an elementary abelian p-group. Then the relevant inversion formula is

$$\chi_c(X^{(A)}) = \sum_{B \subset A} (-1)^{rg(B/A)} p^{\binom{rg(B/A)}{2}} \chi(X^B)$$

$$= \chi(X^A) - \sum_{rg(B/A)=1} \chi(X^B) + p \sum_{rg(B/A)=2} \chi(X^B) - p^3 \sum_{rg(B/A)=3} \chi(X^B) + \cdots$$

As a check take $X = pt$, rang $G = 3$ whence right side becomes

$$1 - (q^2+q+1) + q(q^2+q+1) - q^3 \quad = 0$$

and for rang $G = 4$ get

$$1 - (q^3+q^2+q+1) + q\frac{(q^4-1)(q^3-1)}{(q^2-1)(q-1)} - q^3(q^3+q^2+q+1) + q^6 \quad = 0$$

$$\begin{gathered} \| \\ q(q^2+1)(q^2+q+1) \\ \| \\ q^5+q^4+2q^3+q^2+q \end{gathered}$$

September 28, 1970.

What I was thinking of doing somehow is to try to fashion an Euler characteristic out of $H_G^*(X)$ say by using the Grothendieck group of H_G^*-modules, although a suitable subcategory taking into account the special nature of $H_G^*(X)$ should probably be required to get something interesting.

Then letting $\Psi(X)$ be this Euler characteristic $\underbrace{}_{}$ for $H_G^*(X)_c$ we have

$$\Psi(X) = \Psi(Y) + \Psi(X-Y)$$

and hence can extend Ψ to a homomorphism $\mathrm{Con}(X,G) \longrightarrow (\text{target } \Psi)$ where $\mathrm{Con}(X,G)$ is the ring of constructible function on X/G with integer valued. Thus we should think of Ψ as an Euler characteristic for (pairs) (X,F), F constructible G-sheaf on

Now by devissage one can break up F into locally constant sheaves supported on pieces of constant orbit type. I claim that in such a constant situation the only additive invariant compatible with homotopy equivalence (as $H_G^*(X)$ is) is a multiple of the Euler characteristic $\chi(X,F)$. Indeed a closed simplex must contribute same thing as a point so the open simplex contributes $(-1)^{\dim}$ pt.

If T = target of Ψ, then taking an orbit $G/H \overset{j}{\subset} X$ we get a map

$$\begin{aligned} R(pt, H) &\longrightarrow T \\ F &\longmapsto \Psi\big(j_*(G \times_H F)\big) \end{aligned}$$

This function depends only on the orbit type $\mathcal{O}$, call it

$$\nu_{\mathcal{O}} : R_G(\mathcal{O}) \longrightarrow T \qquad\qquad R_G(\mathcal{O}) \text{ equiv; } K\text{-discrete coeffs.}$$

$$\nu_{\mathcal{O}}(G \times_H F) = \Psi(j_* G \times_H F)$$

Then what we have shown is that

$$\Psi(X, G; F) = \int_{X/G} \nu([F]).$$

Meaning: $[F]$ denotes the section $\mathcal{O} \longmapsto [F] \in R_G(\mathcal{O})$; $\nu([F])$ is a T-valued constructible function on X/G and its integral is with respect to the Euler characteristic (Grothendieck measure).

<u>Example</u>: Suppose A elementary abelian p-group of rank r with p odd. Then $H_A^{ev}(X)$ and $H_A^{odd}(X)$ are finitely generated graded $S(A^\vee)$-modules and you want to consider some kind of **difference**

$$\begin{tikzcd}
H^{ev}(U) \arrow[r] & H^{ev}(X) \arrow[r] & H^{ev}(Y) \arrow[d] \\
H^{odd}(Y)[1] \arrow[u, "+2"] & H^{odd}(X)[1] \arrow[l] & H^{odd}(U)[1] \arrow[l]
\end{tikzcd}$$

so to get the correct additivity it seems necessary to have an additive function of the graded module M which is independent of shifts. However the Grothendieck group of the category of finitely-generated graded $S(A^\vee)$-modules

is clearly $\mathbb{Z}[T,T^{-1}]$ (T = class of $S(A^\vee)$-shifted down by one).
(Indeed associating to M the Laurent polynomial $Q_M(t)$ defined by

$$\sum t^n \dim M_n = \frac{Q_M(t)}{(1-t)^r}$$

is the universal additive class, as one sees using syzygies thm.)
Hence the only additive fn. φ such that $\varphi(M[1]) = \varphi(M)$ is $M \longmapsto Q_M(1)$, i.e. the generic rank:

$$M \longmapsto \dim_K M \otimes_S K$$

$S = S(A^\vee)$
K = q.f. of $S(A^\vee)$.

So by the localization theorem the only additive function obtained in this way is

$$(X,F) \longmapsto \chi(X^A, F).$$

Suppose A cyclic. Then in large dimensions I know that

$$H^n_A(X) \xrightarrow{\sim} H^n_A(X^A)$$

and in particular that the distinguished element $u \in H^1_A$ gives exact sequences

$$H^{n-1}_A(X) \xrightarrow{u} H^n_A(X) \xrightarrow{u} H^{n+1}_A(X).$$

So another additive function is ~~the~~ possibly

$$\varphi(X) = \sum_{k \geq 0} (-1)^k \dim \frac{\mathrm{Ker}\{u : H^k_A(X) \to H^{k+1}_A(X)\}}{\mathrm{Im}\{u : H^{k-1}_A(X) \to H^k_A(X)\}}$$

To see this is additive use long exact sequence

$$\begin{tikzcd}
H^k_A(X-X_A)_c \arrow[r] & H^k_A(X) \arrow[r] & H^k_A(X^A) \arrow[dll, bend left] \\
{} \arrow[r] & {} \arrow[r] & {}
\end{tikzcd}$$

and break up into short exact sequences

$$0 \longrightarrow I^k \longrightarrow H^k_A(X) \longrightarrow {}'I^k \longrightarrow 0$$

$$0 \longrightarrow {}'I^k \longrightarrow H^k_A(X^A) \longrightarrow {}''I^k \longrightarrow 0$$

$$0 \longrightarrow {}''I^k \longrightarrow H^{k+1}_A(X-X^A)_c \longrightarrow I^{k+1} \longrightarrow 0$$

~~These are exact sequences of graded modules with derivation and with trivial homology in high dimensions~~ Take homology long exact sequences with respect to differential given by u. Note that I and $''I$ are zero in large dimensions and $'I$ is acyclic in large dimensions, thus will get

$$\varphi(I) + \varphi('I) = \varphi(X)$$

$$- \quad \varphi('I) + \varphi(''I) = \varphi(X^A)$$

$$\cdot \quad \varphi(''I) + \varphi(I) = -\varphi(X - X^A)$$

$$\Longrightarrow \quad \varphi(X) = \varphi(X^A) + \varphi(X - X^A)$$

Now
$$\varphi(X^A) = 0$$
$$\varphi(X - X^A) = \chi\, H_c^*((X-X^A)/A) = \frac{1}{p}\, \chi_c(X - X^A).$$

(Eulerchar same for homology)

Thus

$$\boxed{\varphi(X) = \frac{1}{p}\left[\chi(X) - \chi(X^A)\right]}$$

showing φ is additive. Conclude that I can recover $\chi(X^A)$ and $\chi(X)$ from $H_A^*(X)$ when A is cyclic and p is odd.

Basic conjectures: Quite generally to each Steenrod-invariant prime ideal in H_G^* there should be associated an additive function of the type above, and hence some kind of "measure" on the orbit types. For example let A be a maximal elementary abelian p-subgroup of G and $\mathfrak{p}$ the associated minimal prime ideal of H_G^*. What is the "measure" associated to the additive function

$$\Psi(X,F) = \text{length}\ H_G^{ev}(X,F)_{\mathfrak{p}} - \text{length}\ H_G^{odd}(X,F)_{\mathfrak{p}} \ ?$$

For an elementary abelian p-group these measures should be complete, i.e. $\chi(X^B)$ all $B \subset A$ should be obtainable from $H_A^*(X)$. In general these measures should describe only that part of the Grothendieck ring of constructible functions $X/G \to R$ which is detectible by the mod p cohomology. One should get a different set of measures using equivariant K-theory.