August 11, 1970

The immeuble for Σ_n: Recall that the immeuble for $GL_n(k)$ is the simplicial complex associated to the partially ordered set of proper subspaces of k^n. Thus an i-simplex is an increasing family $0 < V_0 < \cdots < V_i < k^n$ of subspaces. This complex is of dimension $n-2$ and Serre, I believe, told me that it has the homotopy type of a bouquet of $(n-2)$-spheres.

The analogous simplicial complex for Σ_n has for its i-simplices ~~an ordered subset~~ a chain $\phi < V_0 < \cdots < V_i < \{1, \ldots, n\}$ of proper subsets. Thus this simplicial complex is the barycentric subdivision of the simplicial complex of proper subsets of $\{1, \ldots, n\}$ and so it is $\Delta(n-1)^{\circ}$ which has homotopy type of S^{n-2}.

August 16, 1970:

Segal claims to prove that an invertible H-space with traces B_0 extends to a connected Ω-spectrum $\{B_\nu\}$, but he doesn't show that the two categories are equivalent. The first question is whether $\{B_\nu\} \mapsto B_0$ is faithful, or equivalently given a map $\{B_\nu\} \longrightarrow \{B'_\nu\}$ of connected Ω-spectra such that $B_0 \longrightarrow B'_0$ is the zero map, does it follow that $B_\nu \longrightarrow B'_\nu$ is zero for all ν?

Related question: Let X be a complex which begins in dimension ν ($(\nu-1)$-connected). Then is $\underset{\rightarrow}{S}X = \{S^N X\}$ a retract of $\underset{\rightarrow}{S} S^\nu Y = \{S^{N+\nu} Y\}$ for some Y?

August 22, 1970: The Tits complex of $GL_n(k)$.

Tits(V) Let V be an n-diml. vector space over k, $n\geq 2$. The Tits complex is the simplicial complex of dim $n-2$ whose vertices are the proper subspaces of V and whose i-simplices are chains $0<W_0<\cdots<W_i<V$ of $i+1$ proper subspaces. In other words $\mathrm{Tits}(V)$ is the simplicial complex associated to the partially-ordered set of proper subspaces of V.

Claim $\mathrm{Tits}(V)$ has the homotopy type of a bouquet of $(n-2)$-spheres. Assume this and let's compute the number for k finite, card $(k)=q$, using Euler characteristics. We break up the simplices according to orbits under action of $\mathrm{Aut}(V)$. One gets an orbit for each increasing ~~sequence~~

$$0<j_0<j_1<\cdots j_i<n$$

and the stabilizer of this is

$$\begin{bmatrix} \boxed{j_0} & * & * & * \\ & \boxed{j_1-j_0} & * & * \\ & & \boxed{j_2-j_1} & * \\ & & & \boxed{n-j_i} \end{bmatrix}$$

$s_0 = j_0$

$s_1 = j_1 - j_0$

$s_i = j_i - j_{i-1}$

s_i

which has order

$$\prod_{\substack{1\leq a\leq s_b \\ 0\leq b\leq i+1}} (q^a-1) \quad q^{\frac{n(n-1)}{2}}$$

So the Euler characteristic is something like

$$\sum_{d=1}^{n-1} (-1)^{d+1} \sum_{\substack{s_0,\dots,s_d>0 \\ \sum_0^d s_a = n}} \frac{\prod_{b=1}^{n} q^b - 1}{\prod_{a=0}^{d} \prod_{b=1}^{s_a} (q^b - 1)}$$

e.g.

$$n=2 \qquad \chi = q+1$$
$$n=3 \qquad \chi = -q^3+1$$
$$n=4 \qquad \chi = q^6+1$$

Therefore we want to prove that Tits(V) is of the homotopy type of a bouquet of $q^{\frac{n(n-1)}{2}}$ $(n-2)$-spheres.

We use induction on n. Let us write $V = L \oplus V'$. Let $Z \subset$ Tits(V) be the subcomplex with vertices

$$Z : \{W \mid W+L/L \text{ proper subspace of } V/L\}$$

I claim that Z is of the homotopy type of Tits(V'). Indeed there is an evident embedding of Tits$(V') \longrightarrow Z$ and a retraction given by the projection of V on V'. I claim this is a deformation retraction. Indeed given a vertex W of Z it can be joined to $W+L$ which is joined to $f(W)$ where $f(W) \subset V'$ is $f(W) = (W+L) \cap V'$. Moreover a simplex $\{W_0, \ldots, W_i\}$ is homotopic to the simplex $\{W_0 + L, \ldots W_i + L\}$ which is homotopic to $\{f(W_0), \ldots, f(W_i)\}$. More precisely define

$$Z \times [0,1] \xrightarrow{h} Z \quad \cancel{\{W \in Z \mid L \subset W\}}$$
$$\{W, 0\} \longmapsto W$$
$$\{W, 1\} \longmapsto W + L$$

and follow this homotopy by

$$Z' \times [0,1] \longrightarrow Z$$
$$\{W, 0\} \longmapsto W \cap V'$$
$$\{W, 1\} \longmapsto W$$

where $Z' = \{W \in Z \mid W \supset L\}$.

It follows that the inclusion

$$\mathrm{Cone}\ \mathrm{Tits}(V') \longrightarrow Z * \{V'\} = Z \underset{\mathrm{Tits}\, V'}{\cup} \mathrm{Cone}\{\mathrm{Tits}\, V'\}$$

is a homotopy equivalence, hence $Z * \{V'\}$ is contractible. Now to go from $Z * \{V'\}$ we must add the vertices

L and all hyperplanes $\neq V'$.

Let H be a hyperplane $\neq V'$

Then for any $Z * \{V'\} \subset \Gamma \subset Z$

$$\left(\overset{\Gamma}{\cancel{Z * \{V'\}}}\right) * \{H\} = \left(\overset{\Gamma}{\cancel{Z * \{V'\}}}\right) \cup \underset{\mathrm{Tits}(H)}{\mathrm{Cone\,Tits}(H)}$$

So in adding a vertex H we are attaching the cone on a wedge of S^{n-3}'s. So adding one H at a time inductively one sees that Γ is a wedge of S^{n-2}'s. (Γ always $n-3$ connected). So we get to $Z - \{L\}$, but adding L is like putting the cone on $\mathrm{Tits}(V/L)$ so again add more S^{n-2}'s. If Total number of such spheres is σ_n then

$$\sigma_n = \left\{\mathrm{card}\, H + \underset{L}{1}\right\} \sigma_{n-1}$$

$$= \left\{q^{n-1} - 1 + 1\right\} \sigma_{n-1}$$

so

$$\sigma_n = q^{\frac{n(n-1)}{2}}$$

as it should be.