inconclusive

July 5, 1970: Cohomology of the Sylow 2-subgroup of $GL_3\mathbb{F}_{2^d}$.

Let $N(\mathbb{F}_2) = \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \subset GL_3\mathbb{F}_{2^d}$. Then

$$\left(\begin{bmatrix} 1 & a & \\ & 1 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ & 1 & b \\ & & 1 \end{bmatrix} \right) = \left(e_{12}^{a}, e_{23}^{b} \right) = e_{13}^{ab}$$

$$= \begin{bmatrix} 1 & 0 & ab \\ & 1 & 0 \\ & & 1 \end{bmatrix}.$$

Thus $N(\mathbb{F}_2)$ is as Heisenberg 2-gp. and I know that

$$H^*(N(\mathbb{F}_2)) = \mathbb{Z}/2\,[t_1, t_2]/(t_1 t_2)$$

I can use the same method to compute $H^* N(\mathbb{F}_{2^d})$. Indeed consider Hochschild-Serre for

$$1 \longrightarrow \mathbb{F}_{2^d} \longrightarrow N(\mathbb{F}_{2^d}) \longrightarrow (\mathbb{F}_{2^d})^2 \longrightarrow 1$$

Set $A = \mathbb{F}_{2^d}$, regarded as an algebra over $\mathbb{F}_2$. Then

$$E_2 = S(A^* \oplus A^*) \otimes S(A^*)$$

with $d_2 : A^* \longrightarrow A^* \otimes A^*$ dual to the multiplication. Now one knows

$$A \otimes \overline{\mathbb{F}}_2 \cong \overline{\mathbb{F}}_2^{\,d} \quad \text{as } \overline{\mathbb{F}}_2 \text{ algebras}$$

hence $\{d_2\lambda_i\}$, λ_i basis for A^*, is a regular sequence in $S(A^*\oplus A^*)$, and the spectral sequence has

$$E_3 = S(A^*\oplus A^*)/d_2(A^*) \otimes S((A^*)^{(2)})$$

and so we want to show that ~~the ...~~ $(E_2^{01})^2 \subset E_2^{02} = S_2A^*$ consists of infinite cycles. Thus given $\lambda \in A \to \mathbb{F}_2$ must product an extension of $N(A)$ whose restriction to the center is classified by λ^2. It's enough to worry about a homomorphism $\lambda: A \longrightarrow \mathbb{F}_2$ by "linearity", so we arrive at this question: Can you produce an extension of $N(A)$ by A which induces the extension of A by A classified by the identity map? ~~Probably another formulation is whether the canonical ext. extension $(\mathbb{Z}_2\times\mathbb{Z}_2)\times^k\mathbb{Z}_4$ of $(\mathbb{Z}_2\times\mathbb{Z}_2)\times^k\mathbb{Z}_2$ is algebraic? But this should be clear, because~~

See what happens when $A = \mathbb{Z}_2$. Then the extension of the Heisenberg group $(\mathbb{Z}_2\times\mathbb{Z}_2)\times^k\mathbb{Z}_2$ comes from the Euler class of the Heisenberg representation.

Suppose $[K:\mathbb{Q}_p] < \infty$, A, k as usual, and that G is a reductive grp. scheme over Spec A. Then I want to explain the relation between $G(k)$ and the equalizer of id and the auto. of BG_K coming from monodromy. So form the situation over A_{nr} with residue field $\bar{k}$. Better you want to relate the Frobenius endo. of $G_{\bar{k}}$ to the monodromy auto.

Suppose G is a torus T with character group M which is a finitely gen. free abel. gp on which $\hat{\mathbb{Z}} = \pi_1 \operatorname{Spec} A = \operatorname{Aut}(A_{nr}/A)$ acts. (For example $T = \mathbb{G}_m$ $M = \mathbb{Z}$ with trivial action) Now the endo. of $G_{\bar{k}}$ "geometric Frobenius" is the composite of the auto of $T_{\bar{k}}$ produced by the generator of $\hat{\mathbb{Z}}$ and the map $t \longmapsto t^q$, I think.

Without twisting and just for $\mathbb{G}_m$ we have two things to compare

(a) effect of Galois group of $\mathbb{C}/K$ on $BG_{\mathbb{C},et} = (BG_{\mathbb{C},cl})^\wedge$

(b) effect of Frobenius endo. of $G_{\bar{k}}$.

Thus for $\mathbb{G}_m$ we have the effect of Galois on $(B\mathbb{C}^*)^\wedge = (B\mu)^\wedge$ and we have the effect of $z \longmapsto z^q$. Similar, but not on elements of $\mathbb{C}^*$. Thus on units in $\mathbb{Z}(\zeta)$ $\zeta = \exp(2\pi i/m)$ we have no relation between qth power and .

$$\sum a_i \zeta^i \longmapsto \sum a_i \zeta^{iq}$$

July 10, 1970

Cohomology of $N(\mathbb{F}_{2^d})$ where $N(A) = \begin{bmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{bmatrix}$.

Set $k = \mathbb{F}_{2^d}$; then have a central extension

$$(*) \qquad 0 \longrightarrow k \longrightarrow N(k) \longrightarrow k \times k \longrightarrow 0$$

giving rise to Hochschild-Serre spec. sequence

$$(**) \qquad E_2 = S(k^* \oplus k^*) \otimes S(k^*) \qquad k^* = \mathrm{Hom}_{\mathbb{F}_2}(k, \mathbb{F}_2).$$

The quadratic function giving the extension is computed by lifting + squaring

$$\begin{bmatrix} 1 & a_1 & 0 \\ & 1 & a_2 \\ & & 1 \end{bmatrix}\begin{bmatrix} 1 & a_1 & 0 \\ & 1 & a_2 \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a_1a_2 \\ & 1 & 0 \\ & & 1 \end{bmatrix}$$

hence $Q(a_1, a_2) = a_1 a_2$, and so

$$\begin{array}{ccc} d_2 : E_2^{01} & \longrightarrow & E_2^{20} \\ \| & & \| \\ k^* & & S_2(k^* \oplus k^*) \\ & \searrow^{\mu^t} & \uparrow \\ & & k^* \otimes k^* \end{array}$$

where $\mu : k \times k \to k$ is the multiplication. Claim:

(i) μ^t is injective and its image generates a regular ideal

? (ii) this ideal is stable under action of Steenrod algebra.

To prove these we can extend the base from $\mathbb{F}_2$ to $\overline{\mathbb{F}}_2$. Then

$$k^* \otimes_{\mathbb{F}_2} \overline{\mathbb{F}}_2 = \mathrm{Hom}_{\mathbb{F}_2}(k, \overline{\mathbb{F}}_2)$$

has a basis consisting of the set s homomorphisms $\sigma : k \to \overline{\mathbb{F}}_2$. Then $\mu^t(\sigma) = \sigma \otimes \sigma$ since

$$\begin{tikzcd} k \arrow[d, "\sigma"] & k \otimes k \arrow[l, "\mu"'] \arrow[d, "\sigma\otimes\sigma"] \\ \overline{\mathbb{F}}_2 & \overline{\mathbb{F}}_2 \otimes \overline{\mathbb{F}}_2 \arrow[l] \end{tikzcd}$$

commutes, hence the ideal generated by $d_2(k^*)$ (after base extension) is the ideal in $\overline{\mathbb{F}}_2[x_1, \cdots, x_d; y_1, \cdots, y_d]$, where $x_i = \sigma_i \otimes 1$, $y_i = 1 \otimes \sigma_i$ generated by the elements

$$x_i y_i ,$$

hence the ideal is regular. But also

? $$Sq_{\underline{t}}(x_i y_i) = \Big(\sum_{j\geq 0} t_j x_i^{2^j}\Big)\Big(\sum_{j\geq 0} t_j y_i^{2^j}\Big)$$

$$= x_i y_i \Big(\sum_{j\geq 0} t_j x_i^{2^j-1}\Big)\Big(\sum_{j\geq 0} t_j y_i^{2^j-1}\Big)$$

so this ideal is closed under action of Steenrod algebra. (i) implies that

$$E_3 = \big[S(k^* \otimes k^*)/\mu^t(k^*)\big] \otimes S(k^*)^2$$

Now $Sq^1 : k^* \xrightarrow{\cong} (k^*)^2$ and $d_3(Sq^1 x) = Sq^1(d_2 x) = 0$ because ideal is closed under Sq^1. Thus $E_3 = E_\infty$.

7 (NO) Conclusion: $H^*(N(k), \mathbb{F}_2) \cong S(k^* \oplus k^*)/\mu^t(k^*) \otimes S[(k^*)^2]$, ring isomorphism.

(NO) Better formula: $H^*(N(k), k) \cong k[x_\sigma, y_\sigma, e_\sigma]_{\sigma \in Gal(k/\mathbb{F}_2)} / (x_\sigma y_\sigma)$ where $x_\sigma, y_\sigma \in H^1 = Hom(N(k), k)$ are the homomorphisms $N(k) \rightrightarrows k \xrightarrow{\sigma} k$, and e_σ is of degree 2.

Question: Canonical choices for e_σ?

It seems that you might have made a mistake in proving (ii). Thus if $x \in H^1(N(k), \bar{\mathbb{F}}_2)$ we do not have $Sq_{\underline{t}}(x) = \sum_{j \geq 0} t_j x^{2^j}$, unless x belongs to $H^1(N(k), \bar{\mathbb{F}}_2)$.

Disaster: the variety of $S(k^* \oplus k^*)/(\mu^t(k^*))$ is a union of 2^d-subspaces of $\bar{\mathbb{F}}_2^{\,2d}$ of dimension d. These subspaces are not rational over $\mathbb{F}_2$, only 2 of them are so the ideal generated by $\mu^t(k^*)$ is not closed under Steenrod algebra. Hence $d_3 : (k^*)^2 \longrightarrow E_3^{30}$ is non-zero and the ideal of relations for $E_2^{*0} \longrightarrow E_\infty^{*0}$ is not regular. Seems impossible to get a simple formula

July 19, 1970

The following seems to be what's involved in Sullivan's theorem: $BG = BImJ \times$ (another H-space) (at a prime l).

Fix a prime l and a finite field $\mathbb{F}_q$. Define

$$GL_n\mathbb{F}_q \longrightarrow \Sigma_{q^n}$$

by considering the natural action of $GL_n\mathbb{F}_q$ on $\mathbb{F}_q^n$. The composition with the embedding of $\Sigma_{q^n} \to GL_{q^n}\mathbb{F}_q$ induces the map $V \longrightarrow \mathbb{F}_q[V]$ on representations which is exponential. Taking the standard filtration of the group ring we see that for for a one-dimensional representation L we have

$$[\mathbb{F}_q[L]] = 1 + L + \cdots + L^{q-1}$$

$$(\text{in fact } gr\, \mathbb{F}_q[V] = SV/(V^{q}).)$$

so this operation is ρ_q. Thus if we recall that

$$B^{E\Psi^q} = \tilde{B}\left(\coprod_n GL_n\mathbb{F}_q\right)$$

$$BG[q^{-1}] = B\tilde{B}\mathcal{C} \quad \text{where } \mathcal{C} = \text{category of finite sets of order } q^n_{n\geq 0} \text{ with products for operations.}$$

then we have defined a map

$$B E\Psi^q \longrightarrow BG[q^{-1}]$$

whose composition with the map

$$BG \longrightarrow B\left((1+E\Psi^q)^{\otimes}\right)[q^{-1}]$$

is ρ_q. To be on the really ~~[illegible]~~ safe side I prefer to work with the maps

$$\begin{tikzcd} E\Psi^q \arrow[r] \arrow[rr, bend right, "\rho_q"'] & \left(1+(\Omega^\infty S^\infty)_0\right)^{\times} \arrow[r] & (1+E\Psi^q)^{\otimes}[q^{-1}] \end{tikzcd}$$

To prove the splitting theorem it therefore seems advisable to understand how ρ_q acts on $\pi_*(E\Psi^q)$ at the prime l, and this leads to Bernouilli nos. Supposedly one gets an isomorphism on the l component when q generates $\mathbb{Z}_l^*$.

July 28, 1970: λ-ring structure on $K^0(X, A)$.

Let A be a commutative ring and

$$K^0(X,A) = [X, K_0A \times BGL(A)^+]$$

for any space X. I recall that there is a canonical map

$$\overline{R}_A(\pi_1 X) \longrightarrow [X, BGL(A)^+]_0 = \varinjlim_n [X, BGL_n A^+]_0$$

if X is a pointed connected finite complex which is a universal map of $\overline{R}_A(\pi_1 X)$ to a "representable" functor.

From this universal property it follows that the functor $\tilde{K}(X,A)$ is a (reduced) λ-ring since $\overline{R}_A$ is Indeed the product $R_A \times R_A \longrightarrow R_A \longrightarrow \tilde{K}_A$ induces a product $\overline{R}_A \times \overline{R}_A$ and similarly for λ-operations. The various identities amongst the λ operations + sums + product also hold. Should be no problem adding on K_0A (ugh!)

Lemma: If $x \in \operatorname{Im}\{[X, BGL_nA^+]_0 \to [X, BGL(A)^+]_0\}$, then $\gamma^i x = 0$ for $i > n$.

Claiming that $[X, BGL_n A^+] \longrightarrow [X, BGL(A)^+] \xrightarrow{\gamma^i} [X, BGL(A)^+]$ is zero for X finite complex. Enough to worry about finite skeleton of $BGL_n(A)$. So $x = [E]-[n]$, E repn.

$$\gamma^t([E]-n) =$$

$$\lambda_{\frac{t}{1-t}}(E) \Big/ \left(\frac{1}{1-t}\right)^n = \sum_{i=0}^{n} t^i (1-t)^{n-i} \lambda_i E.$$

so its clear.

So now we can proceed as Atiyah does and conclude that the monomials

$$0 = \gamma^{i_1}(x) \cdots \gamma^{i_k}(x) \qquad \sum i_j > N$$

where N depends upon the integers $n \ni x, -x$ **comes from** dimension n. A **corollary** of is **that** the eigenspaces of Ψ^k on rational K-theory have eigenvalues k^j for various j. Indeed there is a map

$$\mathbb{Z}[\gamma_1, \dots, \gamma_n] = K(BU_n) \longrightarrow K(X, A)$$

$$[E_n] \longmapsto x$$

and in fact

$$K(\mathrm{Grass}_{n,m}) \longrightarrow K(X, A)$$

so you can argue universally.

Thus we can decompose $K(X,A) \otimes \mathbb{Q} \cong \oplus H^i(X, K_i A \otimes \mathbb{Q})$ (Dold) into eigenspaces under Ψ^k's, in fact it's enough to do this for each $K_i A \otimes \mathbb{Q}$. Can speak of these weight spaces.

The conjectured stability theorem shows that every element of $[S^i, BGL(A)^+]_0$ comes from dimension $i+d$ where

$d = \dim \mathrm{Max}(A)$, hence only the eigenvalues k^j $1 \leq j \leq i+d$ should occur (smallest Ψ-stable subspace of $K_i^{\circ}(A)$ containing x is spanned by $\Psi^j x$ $1 \leq j \leq i+d$.) On the other hand the etale Chern character (A f.t./$\mathbb{Z}$)

$$ch_j^{\#} = (-1)^{j-1}(j-1)!\, c_j^{\#} : K_i A \longrightarrow H^{2j-i}(A)(j)$$

might product an eigenvalue k^j with

$$i < 2j \leq d+i \qquad (A \text{ affine}).$$

(since $H^0(A)(j)$ $j>0$ has no invariants)

This tends to suggest that in the affine situation at hand the etale Chern character might be rather unfaithful even for $K_0 A$. (Can you give an example of a non-singular affine variety of dim d whose Chow ring is non-trivial in dimension $> \frac{d}{2}$? Yes take an affine non-singular curve such that $\mathrm{Pic} \neq 0$ i.e. $g \geq 1$ minus a pt.)

Example: $K_1 A = A^* \oplus SK_1(A)$ and Ψ^k acts as k on the part belonging to A^*.

For a finite field $K_{2t-1}\mathbb{F}_q = \mu_{q^t-1}^{\otimes t}$ and Ψ^k acts as k^t

Conjecture: $K_i A$ should contain Ψ^k-eigenvalues k^j only with $\boxed{\frac{i}{2} \leq j \leq i+d.}$ For a finite field OK because

$$\frac{2t-1}{2} \leq t \leq 2t-1 \qquad \text{for } t \geq 1$$

OK also for $K_2 F$ has Ψ^k-eigenvalue k^2 $\quad \frac{2}{2} \leq 2 \leq 2$

What kind of structure can we produce for stable homotopy theory. We know that

$$[X, B\Sigma^{+}]_0 = \tilde{\pi}_s^0(X)$$

is the universal "representable" functor on pointed connected finite complexes receiving a map from the monoid

$$\varinjlim_{n} [X, B\Sigma_n] = I(X)$$

of isomorphism classes of finite coverings of X. Thus on G-sets we have sum, product, symmetric product, deleted symmetric product. In particular we see that the Adams operations act as the identity, e.g.

NO

$$S^2X - \Lambda^2 X = X.$$

Question: Are the Adams operations trivial on $\mathbb{Z}$-K-theory, more generally does $\Psi^k = id$ on $R_{\mathbb{Z}}(G)$?

Serre shows that for G finite

$$\bigoplus_p R_{\mathbb{Z}_p}(G) \longrightarrow R_{\mathbb{Z}}(G) \longrightarrow R_{\mathbb{Q}}(G) \longrightarrow 0$$

is exact, and one knows that Ψ^k acts trivially if k is prime to the order of the group by the Galois theory interpretation. So if we take k to be the exponent of the group, then Ψ^k is non-trivial on $R_{\mathbb{Q}}(G)$ hence non-trivial on $R_{\mathbb{Z}}(G)$.

<u>Paradox</u>: On $K_1\mathbb{Z} = \mathbb{Z}^* = \mathbb{Z}_2$, Ψ^2 is not the identity but this element is in the image of $\pi_1^s \longrightarrow K_1\mathbb{Z}$?

Given a group G we consider the **Grothendieck** group of finite G-sets, kG and the homomorphism

$$kG \longrightarrow R_{\mathbb{Z}}(G)$$

which associates to a G-set S, the G-module $\mathbb{Z}[S]$. This map is a ring homomorphism and if I set $S_2(S) = S\times S/\mathbb{Z}_2$, then

$$\mathbb{Z}[S_2(S)] = S_2(\mathbb{Z}[S]).$$

so ψ^2.

Mistake occurs in identifying $\mathbb{Z}[S_2(S) - S]$ with $\Lambda_2(\mathbb{Z}S)$

<u>Question</u>: Does kG become a λ-ring by means of the symmetric powers? If not what is the structure obtained and does one get interesting operations on $\pi_S^0 X$?

Power operations: $kG \longrightarrow k(\Sigma_n \times G)$ induced by $S \mapsto S^n$

A ring of dim Max $A = 1$. Then $K_1 A$ generated by $GL_2 A$ according to Bass. $K_1 A = A^* \oplus SK_1(A)$. If E is a 2-dimensional representation

$$\lambda_t(E-2) = \frac{1 + tE + t^2\lambda^2 E}{(1+t)^2}$$

Assume $\lambda^2 E = 1$. To compute $\Psi^k E$.

$$\lambda_t(E-2) = \frac{(1+2t+t^2)+t(E-2)}{(1+t)^2} = 1 + \frac{t}{(1+t)^2}(E-2) = 1 - \sum_{k\geq 1} k(-t)^k(E-2)$$

$$\Uparrow$$

$$\frac{1}{1+t} = \sum_{k\geq 0}(-t)^k \Rightarrow \frac{+t}{(1+t)^2} = \sum_{k\geq 1} -k(-t)^k$$

$$\Psi^t = \Psi^0 - t\frac{d}{dt}\log \lambda_{-t} \qquad \Psi^t L = 1 + \frac{tL}{1-tL} = \frac{1}{1-tL} \checkmark$$

$$\log \lambda_{-t}(E-2) = \log\Big(1 - \sum_{k\geq 1} k t^k (E-2)\Big)$$

$$= -\sum_{k\geq 1} k t^k (E-2)$$ since $(E-2)^2 = 0$ over a sphere

$$\Psi^t(E-2) = +\sum k^2 t^k (E-2)$$

Thus $\Psi^k(E-2) = k^2(E-2)$, so in the case at hand

$$\boxed{\Psi^k(x) = x^2 \quad \text{if } x \in SK_1(A) \quad \text{dim max } A = 1}$$