The K-theory associated to a finite field: I

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One of the by-products of the ideas used in the proof of the Adams conjecture [3] is an argument showing what the groups $K_i \mathbb{F}_q$ for $i \geq 0$ be in the "correct" extension of the algebraic K-theory of Bass and Milnor to higher (positive) dimensions. In this series of papers I plan to develop this argument and give a definition of higher K-groups such that the formulas

$$K_{2i} \mathbb{F}_q = 0$$

$$K_{2i-1} \mathbb{F}_q = \mu_{q^i - 1}^{\otimes i}$$

hold for $i > 0$, where μ_m denotes the group of m-th roots of unity in an algebraic closure of $\mathbb{F}_q$.

In general given a ring A I shall construct an H-space B_A whose cohomology classes are essentially the same as characteristic classes of virtual representations of groups acting on finitely generated projective A-modules and whose homotopy group $\pi_i B_A$ will be the group $K_i A$ for $i \geq 1$. In the case $A = \mathbb{F}_q$, the lifting of a representation over $\mathbb{F}_q$ to a virtual complex representation furnished by the theory of Brauer provides a map of B_A to the space $E\Psi^q$ which is the fibre of the endomorphism $\psi^q - \mathrm{id}$ of BU in the sense of homotopy theory. By cohomology calculations I shall show that $B_{\mathbb{F}_q}$ and $E\Psi^q$ are thereby homotopy equivalent, and so obtain the above formulas.

This paper presents the part involving the cohomology of $B_{\mathbb{F}_q}$, or more precisely since this space is not defined here, it is devoted to studying the characteristic classes of representations over $\mathbb{F}_q$. The principal result

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is the computation of the mod ℓ cohomology of $GL_n \mathbb{F}_q$ where ℓ is a prime number not dividing q. Because the writing was easier, I have used where possible the elementary method introduced in the Adams conjecture paper for getting at this cohomology instead of the more powerful technique utilizing etale cohomology and the Lang isomorphism $G/G(\mathbb{F}_q) = G$ which will be presented in [4]. However, the elementary approach has the disadvantage of not working in the exceptional case $\ell = 2$ and $q \equiv 3 \pmod{4}$, so to finish the calculation for $\ell = 2$, I have had to appeal to the results of [4].

Section 1 contains the construction of certain characteristic classes in mod ℓ cohomology for representations over $\mathbb{F}_q$ which I call its arithmetic Chern classes. In part II of this series another definition of these classes will be given by applying certain universal cohomology classes of $E\mathbb{F}^q$ to the Brauer lifting of the representation, but here I have derived their existence from the general Chern classes for representations defined by Grothendieck [1]. In the second section the cohomology $H^*(GL_n \mathbb{F}_q, \mathbb{Z}/\ell)$ is computed by showing that it restricts injectively to the cohomology of a subgroup C^m analogous to the maximal torus, and then by proving that the subring of "Weyl group" invariants in the cohomology of C^m is generated by the restrictions of the arithmetic Chern classes of the canonical representation of the general linear group (ℓ odd). Needed for this is a (probably known) result presented in §3 determining the subring of elements of the de Rham complex $A[x_1, \ldots, x_n] \otimes \Lambda[dx_1, \ldots, dx_n]$ invariant under the action of the symmetric group.

Sections 4 and 5 are devoted to generalities about characteristic classes, which although trivial in some sense, furnish strong intuition for algebraic K-theory of higher dimensions. In particular it is clear that there should be two kinds of K-groups $K_i A$ and $K'_i A$, coinciding for $i = 0, 1$, depending on whether one works with the Grothendieck group $R_A(G)$ of virtual representations, which is generated by representations of G on finitely generated projective A-modules with relations coming from exact sequences, or with the larger Grothendieck group $R'_A(G)$ of fine virtual representations having the same generators with relations coming from direct sums. Formulas for

the groups $K_i A \otimes \mathbb{Q}$ and $K_i' A \otimes \mathbb{Q}$ in terms of characteristic classes in rational cohomology are given in 5.21. Finally, section 6 contains the results on the characteristic classes of representations over $\mathbb{F}_q$, most of which are easy consequences of the computation in §2. In particular we determine (6.9) the structure of the Hopf algebra $H_*(GL(\mathbb{F}_q), \mathbb{Z}/\ell)$.

§1. **Arithmetic Chern classes of representations over a finite field.** Let $\mathbb{F}_q$ denote a finite field with q elements, let $\overline{\mathbb{F}}_q$ be an algebraic closure of $\mathbb{F}_q$, and for m relatively prime to q let μ_m be the group of m-th roots of unity in $\overline{\mathbb{F}}_q$. Let ℓ denote a fixed prime number not dividing q and let r be the degree of the extension $\mathbb{F}_q(\mu_\ell)$; r is the least positive integer such that $q^r - 1$ is divisible by ℓ.

Let $H^*(G, M)$ be the cohomology of the group G with coefficients in the G-module M, and form the bigraded ring

$$(1.1) \qquad H^*(G, \mu_\ell^{\otimes *}) = \bigoplus_{i, j \geq 0} H^j(G, \mu_\ell^{\otimes i})$$

where $\mu_\ell^{\otimes i}$ is the i-fold tensor product of the abelian group μ_ℓ with itself endowed with the trivial action of G. The ring structure comes from the cup product and is anti-commutative with respect to the degree j. We denote by

$$(1.2) \qquad H^*(G, \mu_\ell^{\otimes *})[\varepsilon]$$

the ring obtained from 1.1 by adjoining an element ε satisfying

$$\varepsilon^2 = 0$$
$$\varepsilon a = (-1)^j a \varepsilon \quad \text{if} \quad a \in H^j(G, \mu_\ell^{\otimes i}) \ .$$

By a representation of G over $\mathbb{F}_q$ we mean a vector space E of finite dimension over $\mathbb{F}_q$ endowed with a linear action of G.

Theorem 1.3: To each representation E of a group G over $\mathbb{F}_q$ is canonically associated a collection of cohomology classes

$$c'_{jr}(E) \in H^{2jr}(G, \mu_\ell^{\otimes jr}) \quad \text{and} \quad c''_{jr}(E) \in H^{2jr-1}(G, \mu_\ell^{\otimes jr}) \qquad j \geq 1$$

with the following properties:

1.3.1: <u>These classes depend only on the isomorphism class of the representation and behave functorially for the restriction homomorphisms associated to a homomorphism of groups.</u>

1.3.2: $c'_{jr}(E) = 0 = c''_{jr}(E)$ <u>if</u> $jr >$ <u>the dimension of</u> E.

1.3.3: (<u>Product formula.</u>) <u>If</u>

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

is an exact sequence of representations, then $c(E) = c(E')c(E'')$ <u>where</u>

(1.4)
$$c(E) = \sum_{j \geq 0} c'_{jr}(E) + c''_{jr}(E)\varepsilon \in H^*(G, \mu_\ell^{\otimes *})[\varepsilon]$$

<u>with</u> $c'_0(E) = 1$ <u>and</u> $c''_0(E) = 0$. <u>In other words</u>

(1.5)
$$c'_{jr}(E) = \sum_{a+b=j} c'_{ar}(E')c'_{br}(E'') \qquad a, b \geq 0$$

$$c''_{jr}(E) = \sum_{a+b=j} c'_{ar}(E')c''_{br}(E'') + c''_{ar}(E')c'_{br}(E'') \ .$$

1.3.4. (<u>Normalization.</u>) <u>Let</u> C <u>be the cyclic group</u> $\mu_{q^r-1} = \mathbb{F}_q(\mu_\ell)^*$, <u>and let</u> W <u>be the natural representation of</u> C <u>on</u> $\mathbb{F}_q(\mu_\ell)$ <u>given by multiplication, but viewed as a representation of dimension</u> r <u>over</u> $\mathbb{F}_q$. <u>Then</u>

$$c(W) = 1 + (-1)^{r-1}(u^r + u^{r-1}v\varepsilon)$$

<u>where</u> $u \in H^2(C, \mu_\ell)$ <u>is the class corresponding to the extension</u>

(1.6)
$$0 \longrightarrow \mu_\ell \longrightarrow \mu_{\ell(q^r-1)} \overset{\ell}{\longrightarrow} \mu_{q^r-1} \longrightarrow 0$$

<u>and where</u> $v \in H^1(C, \mu_\ell)$ <u>is the class corresponding to the homomorphism</u>

(1.7)
$$(1-q^r)/\ell : \mu_{q^r-1} \longrightarrow \mu_\ell .$$

(It is an easy consequence of theorem 2.2 (or 6.8) below that the char-acteristic classes c'_{jr} and c''_{jr} are uniquely determined by properties 1.3.1, 1.3.3, and 1.3.4.)

We shall call the classes $c'_{jr}(E)$ and $c''_{jr}(E)$ the <u>arithmetic Chern classes</u> of the representation E, for as we shall see in the proof of this theorem they are special cases of the very general Chern classes introduced by Grothendieck [1]. (The adjective "arithmetic" is used as in [1, §5]; strictly speaking only the classes $c''_{jr}(E)$ are of arithmetic nature since the classes $c'_{jr}(E)$ do not change if the scalars are extended to a larger field.) Another construction of these classes will be given in Part II.

We turn now to the proof of the theorem. As the groups $GL_n(\mathbb{F}_q)$ are finite, it suffices to restrict attention to representations of finite groups G. We recall that Grothendieck [1, §5], using the etale cohomology of the projective scheme $\mathbb{P}E$ with G as operators, has defined (mod ℓ) Chern classes

(1.8)
$$c_i(E) \in H^{2i}(G \times \pi, \mu_\ell^{\otimes i})$$

where π is the Galois group of $\overline{\mathbb{F}}_q$ over $\mathbb{F}_q$ and the cohomology is the con-tinuous cohomology of the profinite group $G \times \pi$ with G acting trivially on μ_ℓ. Now π is a free profinite group with distinguished generator given by the Frobenius automorphism $\varphi : x \longmapsto x^q$, hence one knows that

$$H^j(\pi, \mu_\ell^{\otimes i}) = 0 \quad \text{if either } j \geq 2 \text{ or if } i \not\equiv 0 \pmod{r}$$

and that if i is divisible by r then

$$H^0(\pi, \mu_\ell^{\otimes i}) = \mu_\ell^{\otimes i}$$
$$H^1(\pi, \mu_\ell^{\otimes i}) = \mu_\ell^{\otimes i}.\varepsilon$$

where $z \longmapsto z.\varepsilon$ is the isomorphism of H^0 and H^1 given by cup product with

the element ε of $H^1(\pi, \mathbb{Z}/\ell)$ corresponding to the homomorphism $\pi \longrightarrow \mathbb{Z}/\ell$ sending φ to $1 \pmod{\ell}$. Using the Kunneth formula

$$H^j(G \times \pi, \mu_\ell^{\otimes i}) = \overset{j}{\underset{a=0}{\oplus}} H^{j-a}(G, H^a(\pi, \mu_\ell^{\otimes i}))$$

we find that

$$H^j(G \times \pi, \mu_\ell^{\otimes i}) = 0 \quad \text{if} \quad i \not\equiv 0 \pmod{r}$$

and that if i is divisible by r then there is an isomorphism

$$H^j(G, \mu_\ell^{\otimes i}) \oplus H^{j-1}(G, \mu_\ell^{\otimes i}) \overset{\sim}{\longrightarrow} H^j(G \times \pi, \mu_\ell^{\otimes i})$$

$$(a, b) \longmapsto pr_1^* a + pr_1^* b \cdot pr_2^* \varepsilon \quad .$$

Therefore we conclude that

$$c_i(E) = 0 \quad \text{if} \quad i \not\equiv 0 \pmod{r}$$

and that there are unique elements $c'_{jr}(E) \in H^{2jr}(G, \mu_\ell^{\otimes jr})$ and $c''_{jr}(E) \in H^{2jr-1}(G, \mu_\ell^{\otimes jr})$ such that

(1.9)
$$c_{jr}(E) = pr_1^* c'_{jr}(E) + pr_1^* c''_{jr}(E) \cdot pr_2^* \varepsilon \quad .$$

It is completely straighforward to verify properties 1.3.1 - 1.3.3 using the corresponding properties for the Chern classes 1.8 [1, 2.3].

To prove the normalization property 1.3.4 will require some computation. We first consider the case where ℓ divides $q-1$, hence $r = 1$, and we compute the class $c_1(W)$ where W is the representation of $C = \mu_{q-1}$ on $\mathbb{F}_q$ given by multiplication. According to the general setup of [1], we consider the topos of etale sheaves with C-action over the scheme $\text{Spec } \mathbb{F}_q$ and associate to W an invertible sheaf of modules $\mathbb{L}$ whose first Chern class $(\bmod \ell)$ is $c_1(W)$ by definition. This topos may be identified with the category of sets on which $C \times \pi$ operates continuously, and the structural ring $\mathcal{O}$ of the topos is $\overline{\mathbb{F}}_q$ with trivial C-action and

the obvious π-action. The invertible sheaf $\mathbb{L}$ associated to W is $\overline{\mathbb{F}}_q$ with the obvious π-action, but with C acting by multiplication. The torsor for $\mathcal{O}^*$ associated to $\mathbb{L}$ is the abelian group $\overline{\mathbb{F}}_q^*$ with $(g, a) \in C \times \pi$ acting by $(g, a)\lambda = g \cdot a(\lambda)$, and the class of $\mathbb{L}$, $cl(\mathbb{L}) \in H^1(C \times \pi, \overline{\mathbb{F}}_q^*)$, is represented by the crossed homomorphism

$$h(g, a) = g \cdot a(\lambda)\lambda^{-1}$$

for any $\lambda \in \overline{\mathbb{F}}_q^*$. Taking $\lambda = 1$, we see that $cl(\mathbb{L})$ is represented by the homomorphism $h : C \times \pi \longrightarrow \overline{\mathbb{F}}_q^*$ given by $h(g, a) = g$.

By definition $c_1(W) = \partial cl(\mathbb{L})$ where ∂ is the coboundary operator for the exact sequence of $C \times \pi$ modules

$$0 \longrightarrow \mu_\ell \longrightarrow \overline{\mathbb{F}}_q^* \overset{\ell}{\longrightarrow} \overline{\mathbb{F}}_q^* \longrightarrow 0$$

where C acts trivially. Recall that ∂ is computed on the cochain level by lifting the crossed homomorphism h to a continuous map $h' : C \times \pi \longrightarrow \overline{\mathbb{F}}_q^*$ such that $(h')^\ell = h$ and then taking the cocycle $\delta h'$ whose values lie in μ_ℓ. Choose a morphism of sets $s : C \longrightarrow \mu_{\ell(q-1)}$ with $s(g)^\ell = g$; then we can take $h'(g, a) = s(g)$, so $c_1(W)$ is represented by the 2-cocycle

$$(1.10) \quad \delta h'((g_1, a_1), (g_2, a_2)) = \{a_1 s(g_2)s(g_2)^{-1}\}[s(g_2)s(g_1 g_2)^{-1}s(g_1)] \quad .$$

The term in square brackets represents the inverse image under $pr_1 : C \times \pi \longrightarrow C$ of the extension 1.6, hence it represents the element $pr_1^* u$ of $H^2(C \times \pi, \mu_\ell)$.

Consider the cocycle t given by the term of 1.10 in curly brackets. It will be convenient in what follows to write the group operation of $\mu_{\ell(q-1)}$ additively; then t is the continuous function such that

$$t((g_1, \varphi^{m_1}), (g_2, \varphi^{m_2})) = (q^{m_1}-1)s(g_2)$$

if m_1 and m_2 are non-negative integers. This may be rewritten

$$(1.11) \qquad (1 + \ldots + q^{m_1 - 1})[(q-1)/\ell](\ell s(g_2)) = m_1[(q-1)/\ell \cdot g_2] \ .$$

Now the map $\varphi^m \longmapsto m$ (mod ℓ) extends by continuity to the homomorphism representing the element ε of $H^1(\pi, \mathbb{Z}/\ell)$, while $g \longmapsto (1-q)/\ell \cdot g$ is the homomorphism v (1.7), hence 1.11 shows that t is a 2-cocycle representing the cup product $- pr_2^* \varepsilon \cdot pr_1^* v$ and so we conclude that

$$(1.12) \qquad c_1(W) = pr_1^* u + pr_1^* v \cdot pr_2^* \varepsilon \ .$$

Thus $c_1'(W) = u$ and $c_1''(W) = v$, so 1.3.4 is proved in case ℓ divides $q-1$.

We consider now the general case where W denotes the representation of C over $\mathbb{F}_q$ obtained by restricting the scalars from the one-dimensional representation L given by the multiplication in $\mathbb{F}_q(\mu_\ell)$. Let π', φ', ε' be defined for the finite field $\mathbb{F}_q(\mu_\ell)$ in exactly the same way as π, φ, ε were defined for $\mathbb{F}_q$. Then π' is the profinite subgroup of π with generator $\varphi' = \varphi^r$ and

$$(1.13) \qquad res(\varepsilon) = r\varepsilon'$$

where from now on res denotes the restriction map on cohomology from π to π' (or from $C \times \pi$ to $C \times \pi'$). It is an immediate consequence of Grothendieck's definition of the Chern classes 1.8 that their formation is functorial with respect to extension of scalars, hence

$$res(c(W)) = c(W \otimes_{\mathbb{F}_q} \mathbb{F}_q(\mu_\ell))$$

$$= c(\bigoplus_{a=1}^{r} L^{\otimes q^a})$$

$$= \prod_{a=1}^{r} (1 + q^a c_1(L))$$

$$= 1 + (-1)^{r-1} c_1(L)^r$$

where c is the total Chern class and we have used the identity

$$X^r - 1 \equiv \prod_{a=1}^{r} (X - q^a) \qquad \pmod{\ell} .$$

By the case of 1.3.4 that has already been proved (1.12) we have

$$c_1(L) = u + v\varepsilon' \in H^2(C \times \pi', \mu_\ell)$$

where we have suppressed the pr_i^* to save writing. Hence

$$c_1(L)^r = u^r + ru^{r-1}v\varepsilon'$$

$$\text{res }(c_r(W)) = (-1)^{r-1}[u^r + u^{r-1}v(r\varepsilon')] .$$

Using 1.13 and the fact that res is injective because $[\pi : \pi'] = r$ is prime to ℓ, we obtain

$$c_r(W) = (-1)^{r-1}[u^r + u^{r-1}v\varepsilon]$$

which proves 1.3.4 and concludes the proof of the theorem.

§2. <u>Computation of</u> $H^*(GL_n(\mathbb{F}_q), \mathbb{Z}/\ell)$. To simplify the writing, we abbreviate $H^*(G, \mathbb{Z}/\ell)$ to $H^*(G)$ and we fix a generator ζ of μ_ℓ, using the cup product isomorphism

(2.1) $$. \zeta^{\otimes i} : H^*(G, \mathbb{Z}/\ell) \xrightarrow{\sim} H^*(G, \mu_\ell^{\otimes i})$$

to identify the arithmetic Chern classes $c'_{jr}(E)$, $c''_{jr}(E)$ of a representation E over $\mathbb{F}_q$ with elements of $H^*(G)$.

Let n be a fixed positive integer and denote by E_n the canonical representation of $GL_n(\mathbb{F}_q)$ on $\mathbb{F}_q^n$. Set $n = mr + e$ with $0 \leq e < r$.

<u>Theorem 2.2</u>: $H^*(GL_n(\mathbb{F}_q))$ <u>has a basis over</u> $\mathbb{Z}/\ell$ <u>consisting of the</u> <u>monomials</u>

(2.3) $$c'_r(E_n)^{\alpha_1} \ldots c'_{mr}(E_n)^{\alpha_m} c''_r(E_n)^{\beta_1} \ldots c''_{mr}(E_n)^{\beta_m} \qquad \alpha_i \geq 0, \ \beta_i = 0 \ \underline{or} \ 1 \ .$$

<u>If either</u> ℓ <u>is odd or</u> $\ell = 2$ <u>and</u> $q \equiv 1 \pmod 4$, <u>then</u> $H^*(GL_n(\mathbb{F}_q))$ <u>as a ring</u>

is the tensor product of the polynomial ring with generators $c'_{jr}(E_n)$, $1 \leq j \leq m$, and the exterior algebra with generators $c''_{jr}(E_n)$, $1 \leq j \leq m$.

(The last statement isn't true if $\ell = 2$ and $q \equiv 3 \pmod 4$, in fact in this case the ring $H^*(GL_n(\mathbb{F}_q))$ has no nilpotent elements. For example, when $n = 1$, the Sylow 2-subgroup of $\mathbb{F}_q^*$ is cyclic of order 2, so $H^*(\mathbb{F}_q^*)$ is a polynomial ring with one generator $c''_1(E_1)$.)

For the proof of 2.2 we will need a conjugacy class of subgroups of $GL_n(\mathbb{F}_q)$ which for the mod ℓ cohomology play a role similar to maximal tori in compact connected Lie groups. Let C be as in 1.3.4, and let $\pi_o \cong \mathbb{Z}/r$ denote the Galois group of the extension $\mathbb{F}_q(\mu_\ell)$ of $\mathbb{F}_q$. Combining the obvious actions of C and π_o on $\mathbb{F}_q(\mu_\ell)$ we obtain a representation over $\mathbb{F}_q$ of the semi-direct product $\pi_o \tilde{\times} C$. Let $V = \mathbb{F}_q(\mu_\ell)^m \oplus \mathbb{F}_q^e$; the symmetric group Σ_m acts naturally on the first summand and $GL_e(\mathbb{F}_q)$ acts on the second. Combining these with the action of $\pi_o \tilde{\times} C$ in each of the $\mathbb{F}_q(\mu_\ell)$ - factors, we obtain a representation of the group

$$(2.4) \qquad N = [\Sigma_m \tilde{\times} (\pi_o \tilde{\times} C)^m] \times GL_e(\mathbb{F}_q)$$

as $\mathbb{F}_q$ -linear transformations of V. This representation is faithful, so upon choosing a basis for V we can regard N and its normal subgroup C^m as subgroups of $GL_n(\mathbb{F}_q)$, well-defined up to conjugacy. It is easily verified that N is the normalizer of C^m in $GL_n(\mathbb{F}_q)$ and that the "Weyl group," i.e. the group of automorphisms of C^m produced by inner automorphisms in N is

$$(2.5) \qquad W = \Sigma_m \tilde{\times} \pi_o^m$$

acting in the obvious way on C^m. We denote by

$$(2.6) \qquad R : H^*(GL_n(\mathbb{F}_q)) \longrightarrow H^*(C^m)^W$$

the homomorphism induced by the restriction from $GL_n(\mathbb{F}_q)$ to C^m.

Theorem 2.7: The homomorphism R is injective. If ℓ is odd, then it is an isomorphism. (It is definitely not an isomorphism if $\ell = 2$

<u>and</u> $n \geq 2$.)

Remark 2.8: This theorem shows that the multiplicative structure in the exceptional case $\ell = 2$, $q \equiv 3$ (mod 4) can, at least in principle, be worked out by computing in $H^*(C^m)$; the same goes in general for the action of the Steenrod algebra.

We now take up the proofs of 2.2 and 2.7. It will be convenient to exclude the case $\ell = 2$ and $q \equiv 3$ (mod 4) until the end. We begin by showing that R is injective by the method used in [3].

Let N' denote the subgroup $\Sigma_m \tilde{\times} C^m$ of N and let ℓ^a be the highest power of ℓ dividing $q^r - 1$. As the mod ℓ cohomology of C is detected by abelian subgroups of exponent ℓ^a (i.e. a cohomology class is zero if it restricts to zero on all such subgroups), the same is true for the wreath product $\Sigma_m \tilde{\times} C^m$ by [3, 3.5]. But

$$|GL_n(\mathbb{F}_q)| = q^{n(n-1)/2} \prod_{j=1}^{n} (q^i - 1)$$

$$|N'| = m! \, (q^r - 1)^m$$

and q, $q^i - 1$ for $i \not\equiv 0$ (mod r), and $(q^{jr} - 1)/j(q^r - 1)$ are ℓ-adic units (here is where $\ell = 2$ and $q \equiv 3$ (mod 4) must be excluded); hence N' is of index prime to ℓ and so the mod ℓ cohomology of $GL_n(\mathbb{F}_q)$ is detected by N', hence by abelian subgroups of exponent ℓ^a. But any such subgroup A is conjugate to a subgroup of C^m as one sees immediately by using the fact that any non-trivial irreducible representation of A over $\mathbb{F}_q$ is the restriction of scalars of a one-dimensional representation over $\mathbb{F}_q(\mu_\ell)$. Therefore C^m detects the mod ℓ cohomology of the general linear group and R is injective.

Next we find formulas for the images of the arithmetic Chern classes of E_n under R. Let W, u, and v be as in 1.3.4 except that u and v are to be regarded as elements of $H^*(C)$ by means of 2.1. The cohomology of the cyclic group C is known to be

$$(2.9) \qquad H^*(C) = S[u] \otimes \Lambda[v] \ .$$

The group π_o is generated by the Frobenius automorphism φ which acts on $H^*(C)$ by $\varphi^*(u) = qu$, $\varphi^*(v) = qv$, hence

$$H^*(C)^{\pi_o} = S[x] \otimes \Lambda[y] \qquad \text{where}$$

$$x = c'_r(W) = (-1)^{r-1} u^r$$

$$y = c''_r(W) = (-1)^{r-1} u^{r-1} v \ .$$

Denoting by a subscript i the inverse image under the i-th projection $pr_i : C^m \longrightarrow C$, we have

$$(2.10) \qquad H^*(C^m)^{\pi_o^m} = S[x_1, \ldots, x_m] \otimes \Lambda[y_1, \ldots, y_m] \ ,$$

so the right side of 2.6 is the ring of invariants of this for the natural action of Σ_m. The restriction of E_n to C^m is isomorphic to the direct sum of the W_i and a trivial representation of rank e, hence by the product formula 1.3.3

$$R(c(E_n)) = \prod_1^m c(W_i) = \prod_1^m (1 + x_i + y_i \varepsilon) \qquad \text{so}$$

$$(2.11) \qquad R(c'_{jr}(E_n)) = \sum_{i_1 < \ldots < i_j} x_{i_1} \ldots x_{i_j}$$

$$R(c''_{jr}(E_n)) = \sum_{\substack{i_1 < \ldots < i_j \\ 1 \le k \le j}} x_{i_1} \ldots \hat{x}_{i_k} \ldots x_{i_j} y_{i_k}$$

where the roof means the factor is omitted.

Let c'_{jr} (resp. c''_{jr}) be indeterminates of degrees $2jr$ (resp. $2jr-1$) for $1 \le j \le m$ and let

$$(2.12) \qquad F : S[c'_r, \ldots, c'_{mr}] \otimes \Lambda[c''_r, \ldots, c''_{mr}] \longrightarrow H^*(GL_n(\mathbb{F}_q))$$

be the map of vector space over $\mathbb{Z}/\ell$ which sends the monomial

(2.13)
$$(c'_r)^{\alpha_1} \ldots (c'_{mr})^{\alpha_m}(c''_r)^{\beta_1} \ldots (c''_{mr})^{\beta_m}$$

to the monomial 2.3. Let us consider the right side of 2.10 as the de Rham complex of $S[x_1, \ldots, x_m]$ with $dx_i = y_i$, and the left side of 2.12 as a de Rham complex with $dc'_{jr} = c''_{jr}$. From 2.11 we have that $d[RF(c'_{jr})] = RF(c''_{jr})$, hence the composition

(2.14) $RF : S[c'_r, \ldots, c'_{mr}] \otimes \Lambda[c''_r, \ldots, c''_{mr}] \longrightarrow \{S[x_1, \ldots, x_m] \otimes \Lambda[y_1, \ldots, y_m]\}^{\Sigma_m}$

is essentially the homomorphism of de Rham complexes associated to the inclusion of the symmetric polynomials in $S[x_1, \ldots, x_m]$. Since R is injective and RF is a ring homomorphism it follows that F is a ring homomorphism. In the next section we shall show (3.2) that RF is injective and even an isomorphism if ℓ is odd. Consequently when ℓ is odd both R and F must be ring isomorphisms, which proves 2.2 and 2.7 in this case. We have also proved that when $\ell = 2$ and $q \equiv 1 \pmod 4$, then F is an injective ring homomorphism.

In the case $\ell = 2$ and $q \equiv 3 \pmod 4$ formula 2.10 must be replaced by

$$H^*(C^m) = S[y_1, \ldots, y_m]$$

and the formulas 2.11 are still valid with $x_i = y_i^2$. Filtering by the powers of the ideal $(x_1, \ldots, x_m)$ the associated graded ring is

$$gr\, H^*(C^m) = S[x_1, \ldots, x_m] \otimes \Lambda[y_1, \ldots, y_m]$$

and the images of the elements $RF(c'_{jr})$ and $RF(c''_{jr})$ in the associated graded ring are again given by the formulas 2.11. Thus the images of the monomials 2.13 in $H^*(C^m)$ are independent since they remain independent in the associated graded ring by what we have already proved. Therefore for $\ell = 2$ and any q we know that RF is injective.

To finish the proof we appeal to the results of [4] which show that there is a spectral sequence converging to $H^*(GL_n(\mathbb{F}_q))$ whose E_2 term is of the form $S[c_1, \ldots, c_n] \otimes \Lambda[e_1, \ldots, e_n]$ where c_i has degree 2i and e_i has

degree $2i-1$. This spectral sequence shows that the cohomology of the general linear group is at most as big as the source of F, so F is bijective and R is injective, proving 2.2 and 2.7.

For the sake of completeness we sketch the construction of the spectral sequence just mentioned. Let G be the algebraic group GL_n over $\overline{\mathbb{F}}_q$ and regard $G(\mathbb{F}_q) = GL_n(\mathbb{F}_q)$ as being the fixed points of the Frobenius endomorphism σ of G which raises the coordinates of a matrix to the q-th power. Let B_N be the Grassmannian variety over $\overline{\mathbb{F}}_q$ of n-planes in N-space, and let $f_N : P_N \longrightarrow B_N$ be the principal G-bundle furnished by the n-frames in N-space. One can show that the mod ℓ cohomology of P_N, in the sense of the etale topology, is zero in a range of dimensions increasing with N, hence from the spectral sequence of the covering $P_N \longrightarrow P_N/G(\mathbb{F}_q)$ one finds

$$H^*(G(\mathbb{F}_q)) = \varinjlim_N H^*(P_N/G(\mathbb{F}_q)) \ .$$

By Lang's theorem there is an isomorphism of varieties

$$G/G(\mathbb{F}_q) \simeq G$$
$$xG(\mathbb{F}_q) \longmapsto x(\sigma x)^{-1}$$

hence $P_N/G(\mathbb{F}_q) \longrightarrow B_N$ is a locally trivial fibre bundle with fibre G. The associated Leray spectral sequence can be shown to be of the form

$$E_2^{ab} = H^a(B_N) \otimes H^b(G) \Longrightarrow H^*(P_N/G(\mathbb{F}_q))$$

so letting N go to infinity and using the standard formulas for the cohomology of G and the Grassmannians, one obtains the required spectral sequence

$$E_2 = S[c_1, \ldots, c_n] \otimes \Lambda[e_1, \ldots, e_n] \Longrightarrow H^*(GL_n(\mathbb{F}_q)) \ .$$

§3. Appendix. Symmetric invariants in the de Rham complex. Let A be a commutative ring, let $C = A[x_1, \ldots, x_n]$ be a polynomial ring over A,

and let the symmetric group Σ_n act on C by permuting the x_i. Let B be the subring of invariant elements of C for this action; it is well-known that B is a polynomial ring: $B = A[c_1, \ldots, c_n]$, where c_i is the i-th elementary symmetric function of the x_i. The inclusion $u : B \longrightarrow C$ induces a map of de Rham complexes relative to A

$$(3.1) \quad U : A[c_1, \ldots, c_n] \otimes \Lambda[dc_1, \ldots, dc_n] \longrightarrow \{A[x_1, \ldots, x_n] \otimes \Lambda[dx_1, \ldots, dx_n]\}^{\Sigma_n}$$

where $\Lambda[e_1, \ldots, e_n]$ denotes the exterior algebra over A with generators $e_1, \ldots, e_n$. For example

$$u(c_1) = \Sigma x_i \qquad\qquad U(dc_1) = \Sigma dx_i$$

$$u(c_2) = \sum_{i<j} x_i x_j \qquad\qquad U(dc_2) = \sum_{i \neq j} x_i dx_j \ .$$

Proposition 3.2: The homomorphism U is injective. If 2 is not a zero-divisor in A, then U is an isomorphism.

Remark 3.3: If 2 is a zero-divisor in A, then U will not be an isomorphism for any $n \geq 2$.

The rest of this section will be concerned with the proof of 3.2. Let J be the Jacobian of the c_i with respect to the x_i; by using unique factorization in $\mathbb{Z}[x_1, \ldots, x_n]$ one sees easily that

$$J = \pm \prod_{i<j} (x_i - x_j)$$

hence $J^2 \in B$. Set $C' = C[J^{-1}]$ and $B' = B[(J^2)^{-1}]$ and let $u' : B' \longrightarrow C'$ be induced by u; then u' (or more precisely the associated map of spectra) is a principal covering with Galois group Σ_n [2, SGA 1960-61, V, 2.7]. As u' is etale one has

$$C' \otimes_{B'} \Omega^{\cdot}_{B'/S} \cong \Omega^{\cdot}_{C'/A}$$

where $\Omega^{\cdot}$ denotes the de Rham complex, and by Galois descent [2, SGA 1960-61, VIII] one knows for any B'-module M that

$$M \cong (C' \otimes_{B'} M)^{\Sigma_n} \quad .$$

Combining these we have that u' induces an isomorphism

$$\Omega^{\cdot}_{B'}/A \cong (\Omega^{\cdot}_{C'}/A)^{\Sigma_n} \quad .$$

But this map is what one obtains by localizing U with respect to J^2, hence we have shown

Lemma 3.4: U becomes an isomorphism after localizing with respect to J^2.

As J^2 is a non-zero divisor (it is a product of the non-zero divisors $x_i - x_j$ in C), it follows that U is injective.

Let $\omega = f(x)dx_1 \ldots dx_n$ be an invariant form of highest degree; then f is an anti-invariant polynomial, i.e. $\sigma^*(f) = (-1)^\sigma f$ for $\sigma \epsilon \Sigma_n$. Let V be the set of orbits of the symmetric group on the set of monomials $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. Then we can write $f = \Sigma f_v$, $v \epsilon V$, where f_v is the sum of those terms of the polynomial f involving the monomials in the orbit v. Clearly f_v is anti-invariant. Fix a $v \epsilon V$ such that $f_v \neq 0$, let x^α be a monomial in the orbit v, and let H_α be the subgroup of the symmetric group fixing x^α. If s is the coefficient of x^α in f_v, then $s = (-1)^\sigma s$ if $\sigma \epsilon H_\alpha$ and

$$f_v = \Sigma s(-1)^\tau x^{\tau(\alpha)}$$

where τ runs over left coset representatives for H_α in Σ_n. So now assume that 2 is a non-zero divisor in A. Then the function $\sigma \longmapsto (-1)^\sigma$ is identically one on H_α, hence $f = \Sigma s_v g_v$ with $s_v \epsilon A$, where g_v is an anti-invariant polynomial with ± 1 coefficients. But using unique factorization in $\mathbb{Z}[x_1, \ldots, x_n]$, one sees that $g_v = h_v J$ where h_v is a symmetric polynomial. Therefore with $b = \Sigma s_v h_v$ we have

$$\omega = bJdx_1 \ldots dx_n = U(bdc_1 \ldots dc_n)$$

proving that U is surjective, hence an isomorphism, on forms of highest degree.

Suppose next that ω is an invariant p-form. By 3.4 there is an integer N such that

$$(3.5) \qquad J^{2N}\omega = \Sigma\, U(b_I dc_I) \qquad\qquad b_I \in B$$

where I runs over the set of sequences $1 \le i_1 < \ldots < i_p \le n$ and

$$dc_I = dc_{i_1} \ldots dc_{i_p} \ .$$

Suppose N is the least non-negative integer such that a formula of this kind holds. If I' denotes the sequence of length n-p complementary to I, then

$$J^{2N}\omega U(dc_{I'}) = \pm\, U(b_I dc_1 \ldots dc_n)$$

and on the other hand $\omega U(dc_{I'})$ is an invariant n-form, hence by what we have just proved there is a g_I in B such that $\omega U(dc_{I'}) = U(g_I dc_1 \ldots dc_n)$; thus $J^{2N} g_I = \pm\, b_I$ for all I. Putting this back into 3.5 and using that J is a non-zero divisor, we see that N must be zero by its minimality, hence ω is in the image of U. Thus U is surjective and the proof of 3.2 is complete.

§4. Some generalities about characteristic classes with field coefficients.

In this section and the next $M. = (M_i)_{i>0}$ will denote a non-negatively graded P-module where P is a field (we use the dot notation to avoid an epidemic of stars). If G is a group let H.G be its homology with coefficients in P, and set

$$(4.1) \qquad H^o(G,\ M.) = \prod_{i \ge 0} H^i(G,\ M_i) \ .$$

By the universal coefficient formula there is a canonical isomorphism

$$(4.2) \qquad H^o(G,\ M.) = \mathrm{Hom}_P(H.G,\ M.)$$

where on the right is the set of degree zero homomorphisms of graded P-modules. This isomorphism is determined by the element

(4.3)
$$w_G \in H^o(G, H.G)$$

corresponding to the identity homomorphism of H. G.

The Kunneth formula gives a canonical isomorphism

(4.4)
$$H.G \otimes H.G' = H.(G \times G')$$

in terms of which one defines the cup product

(4.5)
$$H^o(G, M.) \times H^o(G', M.') \longrightarrow H^o(G \times G', M. \otimes M.')$$

by associating to a pair (u, u') the composition of $u \otimes u'$ and 4.4.

Let C be the category of "groups up to inner automorphisms"; it is the category having groups for its objects with a morphism from G to G' defined to be an orbit of the set of group homomorphisms from G to G' under the conjugation action of G'. The homology and cohomology of a group are naturally functors on C with values in P-modules. Let $C^\wedge$ be the category of functors $R : C^o \longrightarrow$ Sets, and identify C with a full subcategory of $C^\wedge$ by associating to X the functor $\text{Hom}_C (?, X)$. (We leave to the reader to supply the necessary modifications of this language to avoid set theory paradoxes, e. g. by restricting to functors R having a generating set; this will be true for the functors R appearing in the applications.) We extend the cohomology functor to $C^\wedge$ by setting

(4.6)
$$H^o(R, M.) = \text{Hom}_{C^\wedge} (R, H^o(?, M.)) \ .$$

Such a natural transformation will be called a <u>characteristic class</u> for R with coefficients in M.. The homology H.R is defined so as to generalize 4.2; thus recall that

$$R(G) = \varinjlim_{\overrightarrow{C}/R} \text{Hom}(G, X)$$

where the inductive limit is taken over the category of arrows $X \longrightarrow R$ in $C^\wedge$

with X an object of $\mathcal{C}$, hence

(4.7) $\qquad H^o(R, M.) = \varprojlim_X \operatorname{Hom}_P(H.X, M.) = \operatorname{Hom}_P(H.R, M.)$

where

$$H.R = \varinjlim_{\overrightarrow{\mathcal{C}/R}} H.X \quad .$$

Put another way the homology $H.R$ comes with a canonical natural transformation

(4.8) $\qquad w_R : R(G) \longrightarrow H^o(G, H.R)$

which is a universal characteristic class in the sense that any characteristic class for R with coefficients in $M.$ is induced from w_R by a unique homomorphism $H.R \longrightarrow M.$ of graded P-modules.

Given two classes $u \in H^o(R, M.)$ and $u' \in H^o(R', M.')$, we define their cup product to be the composition

$$R(G) \times R'(G) \xrightarrow{\ u \times u'\ } H^o(G, M.) \times H^o(G, M.') \xrightarrow{\ 4.5\ } H^o(G \times G, M. \otimes M.')$$

$$\xrightarrow{\ \Delta^*\ } H^o(G, M. \otimes M.') \quad .$$

By the universal property of w_R there is a canonical homomorphism

(4.9) $\qquad H.(R \times R') \longrightarrow H.R \otimes H.R'$

inducing this cup product.

Lemma 4.10: The map 4.9 is an isomorphism.

Proof. A homomorphism

(4.11) $\qquad H.R \otimes H.R' \longrightarrow M.$

is the "same" as a homomorphism

$$H.R \longrightarrow \operatorname{Hom}_P(H.R', M.)_. ,$$

where

$$\text{Hom}_P(M., M.')_j = \prod_{i \geq 0} \text{Hom}_P(M_i, M'_{i+j}) \; .$$

This is the same as a natural transformation

$$R(G) \longrightarrow \text{Hom}_P(H.G, \text{Hom}_P(H.R', M.).) = \text{Hom}_P(H.R', \text{Hom}_P(H.G, M.).)$$

with variable G, or again the same as a natural transformation

$$R(G) \times R'(G') \longrightarrow \text{Hom}_P(H.G', \text{Hom}_P(H.G, M.).)$$

$$= \text{Hom}_P(H.G \otimes H.G', M.) = \text{Hom}_P(H.(G \times G'), M.)$$

with variable G and G', where we have used the Kunneth isomorphism 4.4. But the latter is easily seen to be the same as a natural transformation

$$R(G) \times R'(G) \longrightarrow \text{Hom}_P(H.G, M.)$$

which is the same as a homomorphism

$$(4.12) \qquad\qquad\qquad H.(R \times R') \longrightarrow M. \quad ,$$

so we have established a bijection between homomorphisms 4.11 and 4.12, proving the lemma.

Suppose now that R is endowed with a commutative monoid structure. Using the lemma, the operation $R \times R \longrightarrow R$ gives rise to a product on $H.R$ making it a graded anti-commutative algebra. One sees immediately from the definitions that the canonical map 4.8 is a homomorphism of monoids, where the right side is endowed with the operation given by cup product. In general if $S.$ is a graded P-algebra (anti-commutative is to be understood unless stated otherwise) then by an <u>exponential characteristic class</u> for R with coefficients in $S.$ (exponential class for short), we mean a natural transformation $v : R \longrightarrow H^o(?, S.)$ which is a monoid homomorphism for the

cup product operation on the right, i. e.

$$v(0) = 1$$

(4. 13)

$$v(x + y) = v(x)v(y) \quad .$$

The following results immediately from the definition of the algebra structure on H. R and the universal property of w_R.

Proposition 4. 14: <u>The algebra structure of</u> H. R <u>is characterized by the requirement that</u> w_R (4. 8) <u>be an exponential characteristic class</u>. <u>Moreover</u> w_R <u>is a universal exponential class in the sense that any other with coefficients in</u> S. <u>is induced from</u> w_R <u>by a unique homomorphism</u> H. R $\longrightarrow$ S. <u>of graded</u> P-<u>algebras</u>.

If M. is a graded P-module then by an <u>additive characteristic class for</u> R with coefficients in M. (additive class for short) we mean a natural transformation $a : R \longrightarrow H^o(?, M.)$ which is a monoid homomorphism for the addition operation on cohomology. Clearly such a natural transformation is additive if and only if $1 + a : R \longrightarrow H^o(?, P \oplus M.)$ is an exponential class, where P $\oplus$ M. is the algebra with $(M.)^2 = 0$. Let $\overline{H}.R$ be the augmentation ideal of H. R and let

(4. 15) $$p : H. R \longrightarrow \mathcal{Q} H. R = \overline{H}. R/(\overline{H}. R)^2$$

be the projection onto the augmentation ideal followed by the canonical map to the space of indecomposables. Since algebra homomorphisms from H. R to P $\oplus$ M. are in one-one correspondence with module homomorphisms from $\mathcal{Q}$ H. R to M., we obtain

Corollary 4. 16: <u>The composition</u>

$$R \xrightarrow{w_R} H^o(?, H. R) \xrightarrow{p_*} H^o(?, \mathcal{Q} H. R)$$

<u>is a universal additive characteristic class with coefficients in a graded</u> P-<u>module</u>.

Let $i : M. \longrightarrow \mathrm{Sym}(M.)$ be a universal arrow to the underlying module of a graded anti-commutative algebra; clearly the composite

$$(4.17) \qquad R \xrightarrow{\quad P_* w_R \quad} H^o(?, H.R) \xrightarrow{\quad i_* \quad} H^o(?, \mathrm{Sym}(\mathcal{Q} H.R))$$

is a universal additive class with coefficients in (the underlying module of) a P-algebra. Suppose that P is of <u>characteristic zero</u> and that R is <u>connected</u>, by which we mean that $H_o R = P$, or equivalently that $R(e) = 0$ where e is the single element group. If v is an exponential class with coefficients in S., then the series

$$\log v = \underset{n \geq 1}{\Sigma} (-1)^{n-1}(v-1)^n/n$$

converges in the natural topology of the product 4.1 and is an additive class; one obtains in this way a one-one correspondence between exponential and additive classes, the inverse being given by the exponential series. Consequently the homomorphism which induces the additive class $\log w_R$ from 4.17

$$(4.18) \qquad \mathrm{Sym}(\mathcal{Q} H.R) \xrightarrow{\quad\quad} H.R$$

is an algebra isomorphism; in fact it is a Hopf algebra isomorphism as log is a homomorphism for the abelian group structures on the sets of exponential and additive classes. (This isomorphism is of course well-known in the theory of Hopf algebras [5].)

<u>Remark</u> 4.19: We have not used the explicit nature of $\mathcal{C}$, in fact the above discussion is completely formal once given the functor H. and the Kunneth isomorphism 4.4.

§5. <u>Characteristic classes of virtual and fine virtual representations.</u>
We continue with the notations of the preceding section. Fix a ring A, which need not be commutative; by a <u>representation</u> of a group G over A we mean a projective finitely generated A-module endowed with an A-linear action of G. The Grothendieck group $R_A(G)$ (resp. $R'_A(G)$) is defined as the target of

a universal arrow, denoted $E \longmapsto [E]$, from the set of isomorphism classes of representations of G over A to an abelian group such that the relation $[E] = [E'] + [E'']$ holds whenever

$$(5.1) \qquad 0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

is an exact sequence of representations (resp. a split exact sequence). There is an evident surjection

$$(5.2) \qquad R'_A(G) \longrightarrow R_A(G) \ .$$

Elements of $R_A(G)$ and $R'_A(G)$ will be called <u>virtual</u> and <u>fine virtual represen-</u><u>tations</u> of G over A, respectively.

Note that the group homomorphisms $e \longrightarrow G \longrightarrow e$, where e is the single element group, give rise to splittings

$$(5.3) \qquad \begin{aligned} R_A(G) &= K_o A \oplus \overline{R}_A(G) \\ R'_A(G) &= K_o A \oplus \overline{R}'_A(G) \end{aligned}$$

where $K_o A$ is the Grothendieck group of projective finitely generated A-modules. Elements of the barred groups will be called <u>reduced</u> virtual and fine virtual representations. For the rest of the section all representations are to be understood as being over A unless mentioned otherwise, and we abbreviate R_A to R, etc.

Let $GL_n(A)$ be the group of automorphisms of the A-module A^n and let $GL(A)$ be the union of these under the inclusion maps furnished by the canonical isomorphisms $A^n \oplus A = A^{n+1}$. (We abbreviate these groups by GL_n and GL when no confusion is possible.) Let $I_n G$ denote the set of isomorphism classes of representations E of G whose underlying A-module is isomorphic to A^n, so that there is a functorial isomorphism

$$(5.4) \qquad I_n G = \mathrm{Hom}_C (G, GL_n) \ .$$

The direct sum of representations gives morphisms of functors

$$(5.5) \qquad\qquad I_m \times I_n \longrightarrow I_{m+n}$$

and taking the direct sum with the trivial representation $\mathbb{1}$ of G on A furnishes maps $I_n \longrightarrow I_{n+1}$, permitting us to form the inductive limit

$$IG = \varinjlim I_n G$$

whose elements will be called <u>stable representations</u> of G. From 5.5, IG inherits a natural commutative monoid structure, and there is a functorial monoid homomorphism

$$(5.6) \qquad\qquad IG \longrightarrow \overline{R}'(G)$$

sending an element E of $I_n G$ to $[E] - n[\mathbb{1}]$.

Lemma 5.7: $\overline{R}'(G)$ <u>is the abelian group generated by</u> IG, <u>i.e. the arrow 5.6 is a universal homomorphism from</u> IG <u>to the underlying monoid of an abelian group.</u>

Proof. One sees easily that $R'(G)$ is isomorphic to the abelian group generated by the monoid of all isomorphism classes of representations of G, hence we may identify elements of $R'(G)$ with equivalence classes of pairs (E, F) of representations with the equivalence relation

$$(5.8) \qquad (E, F) = (E', F') \text{ iff } E + F' + Q = E' + F + Q$$

for some representation Q. The subgroup $\overline{R}'(G)$ consists of classes of pairs (E, F) such that the underlying A-modules of E and F are isomorphic. On the other hand, elements of the abelian group U generated by IG may be identified with equivalence classes of pairs (E, F) where E and F are representations on isomorphic free A-modules with the same kind of equivalence relation 5.8, but with Q free as an A-module. Now the map from U to $R'(G)$ induced by 5.6 is surjective because given any pair

(E, F) representing an element of $\overline{R}'(G)$, there is a projective finitely generated A-module Q with trivial G-action such that $E + Q$ and $F + Q$ are isomorphic free A-modules. Similarly one can assume that the Q in 5.8 is free over A, so the map $U \longrightarrow \overline{R}'(G)$ is injective and the lemma is proved.

We shall now discuss the characteristic classes of stable, virtual, and fine virtual representations with coefficients graded modules and algebras over the field P. Because of the splittings 5.3 and lemma 4.10, there are Hopf algebra isomorphisms

$$
\begin{aligned}
H.R &= P[K_oA] \otimes H.\overline{R} \\
H.R' &= P[K_oA] \otimes H.\overline{R}'
\end{aligned}
$$

(5.9)

where $P[K_oA]$ is the group algebra of K_oA over P. In other words any exponential class u for R can be uniquely written as a product $u = u_0 u_1$ where u_0 depends only on the underlying A-module of a representation (i.e. it factors through the projection of R onto K_oA), and where u_1 is an exponential class for $\overline{R}$. Moreover 5.9 shows that all characteristic classes for R can be expressed in terms of those for $\overline{R}$; similar comments hold for R' and $\overline{R}'$.

For stable representations we have

$$
IG = \varinjlim_{n} Hom_{C} (G, GL_n)
$$

hence

(5.10)
$$
H.I = \varinjlim_{n} H.GL_n = H.GL .
$$

More precisely, there is a unique natural transformation

(5.11)
$$
w : IG \longrightarrow H^o(G, H.GL)
$$

such that if E is a stable representation and $G \longrightarrow GL$ is the associated

map, then $w(E)$ is the element corresponding under the isomorphism 4.2 to the induced map on homology; moreover w is a universal characteristic class for I. Since I is a monoid-valued functor, we know by 4.14 that H.GL has a unique algebra structure such that w is an exponential class. Now I is connected, hence any exponential class for I with coefficients in S. takes its values in the group of units of $H^o(?, S.)$ reducing to 1 in $H^o(?, S_o)$, and so by lemma 5.7 such a class extends uniquely to $\bar{R}'$. Therefore exponential classes for I and $\bar{R}'$ are the same, i.e. 5.6 induces a Hopf algebra isomorphism

$$(5.12) \qquad\qquad H.GL \overset{\sim}{\longrightarrow} H.\bar{R}' \quad .$$

As characteristic classes are just linear maps from homology, this proves the following:

Proposition 5.13: Any characteristic class for stable representations with coefficients in M. (this may be identified with an element of $H^o(GL, M.)$) extends uniquely to a characteristic class for $\bar{R}'$.

Applied to the transformation w we find:

Proposition 5.14: There is a unique natural transformation

$$\theta' : \bar{R}'(G) \longrightarrow H^o(G, H.GL)$$

such that if E is a representation of G on a free A-module and $f : G \longrightarrow GL$ is the associated map in $\mathcal{C}$, then $\theta'[E]$ is the element corresponding to the map on homology induced by f under the isomorphism 4.2. Moreover

1) θ' is a universal characteristic class for $\bar{R}'$ with coefficients in a graded P-module.

2) There is a unique algebra structure on H.GL such that θ' is an exponential class, and with respect to this algebra structure θ' is a universal exponential characteristic class with coefficients in a graded P-algebra.

Finally we consider the characteristic classes of reduced virtual representations. Let $J_{mn}G$ be the set of isomorphism classes of exact

sequences 5.1 of representations of G where E' (resp. E'') is isomorphic to A^m (resp. A^n); then

(5.15)
$$J_{mn} G = \text{Hom}_C (G, GL_{m,n}(A))$$

where $GL_{m,n}(A)$ is the group of automorphisms of $A^m \oplus A^n$ preserving the first summand. The direct sum of exact sequences gives natural transformations $J_{mn} \times J_{m'n'} \longrightarrow J_{m+n, m'+n'}$; taking the direct sum with the exact sequences of trivial representations

$$0 \longrightarrow A \xrightarrow{\text{id}} A \longrightarrow 0 \longrightarrow 0$$

$$0 \longrightarrow 0 \longrightarrow A \xrightarrow{\text{id}} A \longrightarrow 0$$

gives rise to transformations $J_{mn} \longrightarrow J_{m+1, n}$ and $J_{mn} \longrightarrow J_{m, n+1}$, so we can form the set of stable isomorphism classes of exact sequences

$$JG = \varinjlim_{m,n} J_{mn} G$$

and endow it with a monoid structure in the natural way. By passage to the limit we have

(5.16)
$$H. J = H. GL^*(A)$$

where $GL^*(A)$ (GL^* for short) denotes the group of automorphisms of $A^\infty \oplus A^\infty$ preserving the first summand whose matrix with respect to the obvious basis is almost everywhere equal to the identity matrix. Define natural transformations of monoid functors

(5.17)
$$J \; \underset{v}{\overset{u}{\rightrightarrows}} \; I \xrightarrow{t} \bar{R}$$

such that $tu = tv$, where t is the composition of 5.6 and 5.2, and where u (resp. v) associates to an exact sequence such as 5.1 the representation E (resp. $E' \oplus E''$). Passing to homology we get morphisms of Hopf algebras

$$(5.18) \qquad H.GL^* \; \overset{u_*}{\underset{v_*}{\rightrightarrows}} \; H.GL \; \overset{t_*}{\longrightarrow} \; H.\overline{R} \; .$$

Proposition 5.19: The diagram 5.18 is exact in the category of P-modules, hence also in the categories of algebras and Hopf algebras over P.

Exactness here means that t_* is a categorical cokernel for the pair u_*, v_*. The exactness of 5.18 in the categories of algebras or Hopf algebras is an immediate consequence of the characterization of these algebras in terms of exponential classes (4.14) and the fact that the diagram of abelian groups associated to 5.17 is exact, which results immediately from the definitions. Let $h : I \longrightarrow J$ be the natural transformation which associates to a representation E the exact sequence

$$0 \longrightarrow E \overset{id}{\longrightarrow} E \longrightarrow 0 \longrightarrow 0 \; ;$$

then h induces an algebra homomorphism $h_* : H.GL \longrightarrow H.GL^*$ satisfying $u_* h_* = v_* h_* = id$, from which one sees easily that the image of $u_* - v_*$ is an ideal in $H.GL$. This shows that the cokernels of u, v in the categories of P-modules and P-algebras coincide, completing the proof of the proposition.

Corollary 5.20: A characteristic class θ for stable representations extends to $\overline{R}$ if and only if $\theta(E) = \theta(E' \oplus E'')$ for any exact sequence of representations.

Immediate, since characteristic classes are linear maps from homology.

Remark 5.21: In another paper we plan to produce connected H-spaces B_A and B'_A whose homology Hopf algebras with coefficients in P are canonically isomorphic to the Hopf algebras $H.\overline{R}_A$ and $H.\overline{R}'_A$ respectively, and whose homotopy groups, denoted by

$$K_i A = \pi_i B_A \qquad i \geq 1$$

$$K'_i A = \pi_i B'_A \qquad '' \quad ,$$

are natural candidates for higher dimensional groups in algebraic K-theory.

If $P = \mathbb{Q}$, then by a theorem of Milnor and Moore [5, appendix]

$$K_i A \otimes \mathbb{Q} = \mathscr{P} H_i \overline{R}_A \qquad\qquad i \geq 1$$

$$K_i' A \otimes \mathbb{Q} = \mathscr{P} H_i \overline{R}'_A = \mathscr{P} H_i GL(A) \qquad \text{''}$$

where $\mathscr{P}$ denotes the primitive subspace, which is canonically isomorphic to the indecomposable space. Consequently if we set $K_o' A = K_o A$ as in 5.3, then the rational K-groups are determined by the fact that they come with canonical natural transformations

$$R_A(G) \longrightarrow H^o(G, K_.A \otimes \mathbb{Q})$$

$$R'_A(G) \longrightarrow H^o(G, K'_.A \otimes \mathbb{Q})$$

which are universal additive characteristic classes with coefficients in a graded module over $\mathbb{Q}$.

§6. <u>Characteristic classes of virtual representations over $\mathbb{F}_q$</u>. In this section we consider only representation of groups over a given finite field $\mathbb{F}_q$ of characteristic p and we abbreviate $R_{\mathbb{F}_q}$, $GL_n(\mathbb{F}_q)$ to R, GL_n, etc., where convenient. We study characteristic classes of virtual and fine virtual representations, i.e. natural transformations from R and R' to $H^*(?, M)$ where M is an abelian group. As the groups GL_n are finite, any element of $R(G)$ or $R'(G)$ comes from a finite quotient group of G, so for the study of characteristic classes we can restrict attention to finite groups. Since the rational cohomology of a finite group is trivial, it is clear that any characteristic class with coefficients in a $\mathbb{Q}$-module is trivial, i.e. it depends only on the dimension of the virtual or fine virtual representation.

<u>Proposition 6.1</u>: <u>If</u> M <u>is a module over</u> $\mathbb{Z}_{(p)}$, <u>the local ring of the integers at the prime ideal</u> (p), <u>then</u> ~~all characteristic classes of virtual~~ ~~representations over $\mathbb{F}_q$ with coefficients in M are trivial, i.e.~~

$$\text{Hom}_{C^{\wedge}}(\overline{R}, \ H^i(?, \ M)) = \begin{cases} 0 & i > 0 \\ M & i = 0 \end{cases} .$$

It suffices to show that if u is such a characteristic class and if $x \in \overline{R}(G)$, then $u(x) = u(0)$. If G_p is a Sylow p-subgroup of G, then the restriction homomorphism

$$H^i(G, \ M) \longrightarrow H^i(G_p, \ M)$$

is injective by transfer theory, so we may suppose G is a p-group. But in this case $\overline{R}(G) = 0$, since irreducible representations of a p-group in characteristic p are trivial, so the proposition follows.

Remark 6.2: I don't know whether there are any non-trivial characteristic classes for <u>fine</u> virtual representations with coefficients mod p, i.e. whether $H^i(GL(\mathbb{F}_q), \ \mathbb{Z}/p) \neq 0$ for some $i > 0$.

<u>Proposition 6.3</u>: <u>If</u> M <u>is a</u> $\mathbb{Z}[p^{-1}]$-<u>module, then any natural transformation</u> $u : R' \longrightarrow H^i(?, \ M)$ <u>extends uniquely to</u> R.

Since R' maps surjectively to R, it suffices to show that if x and y are two elements of $R'(G)$ which become equal in $R(G)$, then $u(x) = u(y)$. For each prime number $\ell \neq p$ let G_ℓ be a Sylow ℓ-subgroup of G; then by transfer theory the map given by the restriction homomorphisms

$$H^i(G, \ M) \longrightarrow \prod_{\ell \neq p} H^i(G_\ell, \ M)$$

is injective, so it suffices to consider the case where G is an ℓ-group. But in this case $R'(G) = R(G)$ as representations are completely reducible (theorem of Maschke), so the proposition follows.

In the rest of the section we shall use the results of §2 to describe the characteristic classes with coefficients in the prime field $P = \mathbb{Z}/\ell$, $\ell \neq p$. We restrict ourselves to the reduced groups $\overline{R}$ and $\overline{R}'$; by 6.3 natural transformations from $\overline{R}$ and $\overline{R}'$ to $H^i(?, \ P)$ are the same, and by 5.13 they may be identified with elements of $H^i(GL, \ P)$. By theorem 2.2 and

letting n go to infinity, there is a ring isomorphism

$$(6.4) \qquad H^*(GL, P) = S[c'_r, c'_{2r}, \ldots] \otimes \Lambda[c''_r, c''_{2r}, \ldots]$$

except when $\ell = 2$, $q \equiv 3 \pmod 4$ when it is only an additive isomorphism. (In this isomorphism the indeterminates c'_{jr} and c''_{jr} correspond to the characteristic classes which associate to a virtual representation x its arithmetic Chern classes $c'_{jr}(x)$ and $c''_{jr}(x)$ regarded as elements of the cohomology in P via the isomorphisms 2.1.) The cohomology 6.4 is a Hopf algebra with coproduct obtained from the product formula 1.5

$$(6.5) \qquad \begin{aligned} \Delta c'_{jr} &= \sum_{a+b=j} c'_{ar} \otimes c'_{br} \\[2mm] \Delta c''_{jr} &= \sum_{a+b=j} c'_{ar} \otimes c''_{br} + c''_{ar} \otimes c'_{br} \end{aligned}$$

and we now describe the structure of the dual Hopf algebra.

By 2.2, $H^*(GL_r, P)$ has the basis 1, x^j, $x^{j-1}y$ for $j \geq 1$ where $x = c'_r(E_r)$ and $y = c''_r(E_r)$, hence given a characteristic class with coefficients in a graded P-module M.

$$t : \overline{R}(G) \longrightarrow H^o(G, M.)$$

we may define elements $\xi_j(t) \in M_{2jr}$ and $\eta_j(t) \in M_{2jr-1}$ for $j \geq 1$ by the formula

$$(6.6) \qquad t[E_r] = 1 + \sum_{j \geq 1} x^j . \xi_j(t) + x^{j-1}y . \eta_j(t)$$

where the dot denotes the cup product homomorphism (isomorphism for G finite)

$$H^i(G, P) \otimes M_i \longrightarrow H^i(G, M_i) \ .$$

Let

(6.7) $$\theta : \overline{R}(G) \longrightarrow H^o(G, H.GL)$$

be the universal characteristic class of 5.14, where 6.3 has been used to replace $\overline{R}'$ by $\overline{R}$.

Theorem 6.8: Associating to an exponential characteristic class t the family $(\xi_j(t), \eta_j(t))_{j \geq 1}$ gives a bijection of the set of exponential characteristic classes of $\overline{R}$ with coefficients in S. and the set of sequences $(\xi_j, \eta_j)_{j \geq 1}$ with $\xi_j \epsilon S_{2jr}$ and $\eta_j \epsilon S_{2jr-1}$ such that $(\eta_j)^2 = 0$.

Corollary 6.9: The Hopf algebra $H.GL(\mathbb{F}_q)$ is the tensor product of the polynomial ring with generators $\xi_j = \xi_j(\theta)$ of degree $2jr$ and the exterior algebra with generators $\eta_j = \eta_j(\theta)$ of degree $2jr-1$ for each $j \geq 1$, with co-product given by

(6.10)
$$\Delta \xi_j = \sum_{a+b=j} \xi_a \otimes \xi_b \qquad\qquad (a, b \geq 0, \xi_o = 1, \eta_o = 0)$$

$$\Delta \eta_j = \sum_{a+b=j} \xi_a \otimes \eta_b + \eta_a \otimes \xi_b \qquad\qquad ''$$

except if $\ell = 2$ and $q \equiv 3$ (mod 4) when the first formula must be replaced by

(6.11)
$$\Delta \xi_j = \sum_{a+b=j} \xi_a \otimes \xi_b + \sum_{a+b=j+1} \eta_a \otimes \eta_b .$$

This follows immediately from the theorem using the universal property of H.GL as an algebra. The coproduct formulas express how the family associated to a product $t't''$ of exponential classes is computed from the families associated to t' and t''; the difference in the exceptional case results from the fact that $y^2 = x$ instead of $y^2 = 0$.

Corollary 6.12: Given elements m'_j (resp. m''_j) of degree $2jr$ (resp. $2jr-1$) of a graded P-module M. for each $j \geq 1$, there is a unique additive characteristic class $t : \overline{R} \longrightarrow H^o(?, M.)$ such that $\xi_j(t) = m'_j$ and $\eta_j(t) = m''_j$.

This is straightforward using 4.16.

We now turn to the proof of the theorem.

Lemma 6.13: If t is an exponential class, then $\eta_j(t)^2 = 0$.

If ℓ is odd this follows immediately from the anti-commutativity of the algebra of coefficients of t, since $\eta_j(t)$ is of odd degree; the proof for $\ell = 2$ will be given below.

According to the lemma there is a graded ring homomorphism

$$T : S[\xi_1, \xi_2, \ldots] \otimes \Lambda[\eta_1, \eta_2, \ldots] \longrightarrow H.GL$$

sending the indeterminates ξ_j and η_j of degrees $2jr$ and $2jr-1$ respectively to $\xi_j(\theta)$ and $\eta_j(\theta)$ respectively. To prove the theorem it suffices to show T is an isomorphism, and in fact only that T is surjective since the target of T has the same Poincaré series as the source by 6.4. For this it is enough to prove that the augmentation ideal of $H.GL$ is generated by the $\xi_j(\theta)$ and $\eta_j(\theta)$, or equivalently that any exponential class t for $\overline{R}$ such that $t[E_r] = 1$ satisfies $t[E_n] = 1$ for all n. But the restriction homomorphism on cohomology from GL_n to C^m is <u>injective</u> (2.7) and E_n restricted to C^m is a direct sum of r-dimensional representations and a trivial representation, so this is clear and the theorem is proved.

It remains to prove the lemma when $\ell = 2$, hence $r = 1$. Let R denote the restriction homomorphism from GL_2 to C^2 as in section 2. Then

$$
\begin{aligned}
R(t[E_2]) &= t[L_1]t[L_2] \\
&= (\sum_i x_1^i \xi_i + x_1^{i-1} y_1 \eta_i)(\sum_j x_2^j \xi_j + x_2^{j-1} y_2 \eta_j) \\
&= \sum_{i<j} (x_1^i x_2^j + x_1^j x_2^i)\xi_i \xi_j + \sum_j (x_1 x_2)^j \xi_j^2 \\
&\quad + \sum_{i,j} (x_1^i x_2^{j-1} y_2 + x_2^i x_1^{j-1} y_1)\xi_i \eta_j \\
&\quad + \sum_{i<j} (x_1^{i-1} x_2^{j-1} + x_2^{i-1} x_1^{j-1}) y_1 y_2 \eta_i \eta_j \\
&\quad + \sum_j (x_1 x_2)^{j-1} y_1 y_2 \eta_j^2
\end{aligned}
$$

(6.14)

where $i, j \geq 0$ and $\xi_o = 1$, $\eta_o = 0$ and we have abbreviated $\xi_j(t)$ to ξ_j, etc. We know that this sum lies in the subring

$$\prod_{i \geq 0} H^i(GL_2, S_i) \subset \prod_{i \geq 0} H^i(C^2, S_i)$$

where S_i is the coefficient ring of t, and also that $H^*(GL_2)$ is the subring of $H^*(C^2)$ generated by the elements

(6.15)
$$c_1' = x_1 + x_2 \qquad\qquad c_1'' = y_1 + y_2$$
$$c_2' = x_1 x_2 \qquad\qquad c_2'' = x_1 y_2 + x_2 y_1 \ .$$

Consider first the case $q \equiv 1 \pmod 4$ and let A be the composition of the unique $S[x_1, x_2]$-module homomorphism

$$H^*(C^2) = S[x_1, x_2] \otimes \Lambda[y_1, y_2] \longrightarrow S[x_1, x_2]$$

sending 1, y_1, and y_2 to 0 and $y_1 y_2$ to 1, followed by the ring homomorphism $S[x_1, x_2] \longrightarrow S[x]$ sending x_1 and x_2 to x. It is easy to check that A kills the image of $H^*(GL_2)$, hence applying A to the equation 6.14 we obtain

$$0 = \sum_j x^{2j-2} \eta_j^2 \quad \text{in} \quad \prod_{i \geq 0} x^i S_{2i}$$

showing that $\eta_j^2 = 0$.

In the case $q \equiv 3 \pmod 4$, let A be the ring homomorphism

$$H^*(C^2) = S[y_1, y_2] \longrightarrow S[y]$$

sending y_1 and y_2 to y; then $A(c_1') = A(c_1'') = A(c_2'') = 0$ and $A(c_2') = y^4$, so A carries the cohomology of GL_2 into the subring of polynomials in y^4. Applying A to 6.14 yields

$$\sum_j y^{4j} \xi_j^2 + \sum_j y^{4j-2} \eta_j^2 \in \prod_{j \geq 0} y^{4j} S_{4j}$$

showing that $\eta_j^2 = 0$. Therefore the lemma and the theorem are proved.

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