January 5, 1970 (still groggy) (the following is all mod 2 to avoid signs.)

I want to understand Dyer-Lashof operations. If $X$ is an infinite loop space, (or equivalently an infinitely homotopy commutative H-space by Boardman-Vogt), then one gets maps

$$\mu_k : E\Sigma_k \times_{\Sigma_k} X^k \longrightarrow X$$

which express the higher commutativity of the operation on $X$. The Dyer-Lashof operations are defined in analogy with Steenrod operation but with $\mu_k$ taking the place of the external powers. In cohomology they appear as

$$H^*(X) \xrightarrow{\mu_k^*} H^*_{\Sigma_k}(X^k) \xrightarrow{\Delta^*} H^*_{\Sigma_k}(X)$$

but it seems better to regard them as homology operations since then they raise degrees and also can be fitted together over all $k$ in the following way:

Given a space $X$, let

$$R(X) = \mathbb{Z}_2 \oplus \bigoplus_{k \geq 1} H_*(E\Sigma_k \times_{\Sigma_k} X^k).$$

I claim that $R(X)$ has a natural structure as an (affine) ring scheme. The product is defined as the composition

$$H_*(E\Sigma_k \times_{\Sigma_k} X^k) \otimes H_*(E\Sigma_\ell \times_{\Sigma_\ell} X^\ell) \longrightarrow H_*(E\Sigma_k \times E\Sigma_\ell \times X^{k+\ell}/\Sigma_k \times \Sigma_\ell)$$
$$\xrightarrow{(*)} H_*(E\Sigma_{k+\ell} \times_{\Sigma_{k+\ell}} X^{k+\ell})$$

where $(*)$ is the natural map which one obtains from ~~a~~ $\Sigma_k \times \Sigma_\ell$-equivariant homotopy equivalence $E\Sigma_k \times E\Sigma_\ell \sim E\Sigma_{k+\ell}$. Hence

$$\operatorname{Hom}_{\text{rings}}(R(X), A) = \left\{ (\theta_k)_{k\geq 1} \;\middle|\; \theta_k \in H^*_{\Sigma_k}(X^k, A) \text{ and } \operatorname{res}^{\Sigma_{k+\ell}}_{\Sigma_k \times \Sigma_\ell} \theta_{k+\ell} = \theta_k \otimes \theta_\ell \right\}.$$

$$= \prod_{k\geq 1}{}' H^*_{\Sigma_k}(X^k, A),$$

which we know has a natural ring structure. Note that

$$\operatorname{Hom}_{\text{rings}}(R(X), A) \cong \prod_{k\geq 1}{}' H^*(B\Sigma_k, H^*(X)^{\otimes k} \otimes A)$$

$$\cong \operatorname{Hom}_{\text{rings}}(R, A) \otimes_{\mathbb{Z}_2} H^*(X) \qquad \text{(ring isos.)}$$

where this isomorphism is given by $P_{ext} : H^*(X) \longrightarrow \prod_{k\geq 1}' H^*_{\Sigma_k}(X^k)$. Consequently we see that the functor represented by $R(X)$ is the base extension of the functor represented by $R$ by the map $\mathbb{Z}_2 \longrightarrow H^*(X)$. Therefore $R(X)$ is a polynomial ring with generators in 1-1 correspondence with the product of the generators of $R$ and a basis for $H^*(X)$. (Can you detect these generators by means of homomorphisms obtained via

$$H(X\times Y) \longrightarrow \prod{}' H^*_{\Sigma_k}(X^k \times Y^k) \xrightarrow{\Delta_Y^*} \prod{}' H^*_{\Sigma_k}(X^k) \otimes H(Y)$$

$$\cong \operatorname{Hom}_{\text{rings}}(R(X), H(Y))?)$$

<u>Question:</u> Consider the map

$$H^*(X\times Y) \xrightarrow{P_{ext}} \prod_{k\geq 1}{}' H^*_{\Sigma_k}(X^k\times Y^k) \xrightarrow{\Delta_Y^*} \prod_{k\geq 1}{}' H^*_{\Sigma_k}(X^k\times Y)$$

$$\cong \prod_{k\geq 1}{}' H^*((E\Sigma_k \times_{\Sigma_k} X^k)\times Y)$$

$$\cong \prod_{k\geq 1}{}' H^*(E\Sigma_k \times_{\Sigma_k} X^k)\otimes H^*(Y)$$

$$\cong \prod_{k\geq 1}{}' H^*(B\Sigma_k, H^*(X)^{\otimes k})\otimes H^*(Y)$$

$$\cong H^*(X)\otimes \mathrm{Hom}_{rgs}(R, H^*(Y))$$

This is a ring homomorphism natural in $X$ and in $Y$. I think it is fairly clear that it sends $pr_1^* x$ to $x\otimes 1$. and that on $pr_2^* y$ it gives the total power operation. Can this be generalized?

Now suppose that $X$ is an infinitely commutative $H$-space, so that there are the maps $\mu_k$ of page 1. Then the product $\mu : X \times X \longrightarrow X$ defines a commutative algebra structure on $H_*(X)$ and there is a ring homomorphism

$$\oplus \mu_{k*} : R(X) \longrightarrow H_*(X)$$

In effect by definition of $\mu_k$ the diagram

$$\begin{tikzcd}
(E\Sigma_k \times_{\Sigma_k} X^k) \times (E\Sigma_\ell \times_{\Sigma_\ell} X^\ell) \arrow[r, "\mu_k \times \mu_\ell"] \arrow[d, "\simeq"] & X \times X \arrow[dd, "\mu"] \\
E(\Sigma_k \times \Sigma_\ell) \times_{\Sigma_k \times \Sigma_\ell} X^{k+\ell} \arrow[d] & \\
E\Sigma_{k+\ell} \times_{\Sigma_{k+\ell}} X^{k+\ell} \arrow[r, "\mu_{k+\ell}"] & X
\end{tikzcd}$$

is homotopy commutative. Thus we obtain an element in

$$(*) \qquad \mathrm{Hom}_{rgo}(R(X), H_*(X)) \underset{\substack{\uparrow \\ \text{ring isom.}}}{\cong} \mathrm{Hom}_{rgo}(R, H_*(X) \otimes H^*(X))$$

~~Note next that the multiplicative coproduct on $R(X)$ comes from the natural map $R(X) \otimes R(Y) \longleftarrow R(X \times Y)$ dual to~~

$$\cancel{\prod_{k\geq 1}{}' H^*_{\Sigma_k}((X \times Y)^k, A) \longleftarrow \prod_{k\geq 1}{}' H^*_{\Sigma_k}(X^k, A) \times \prod_{k \geq 1}{}' H^*_{\Sigma_k}(Y^k, A)}$$

*Question:* Is the map

$$\Phi : \quad H_*(X) \longrightarrow \mathrm{Hom}_{rgs}(R, H_*(X))$$

YES
see p. 8.

a ring homomorphism? This maps takes $x \in H_*(X)$ and $\sigma \in H_*(\Sigma_k)$ and forms the element $\sigma \int x \in H_*(B\Sigma_k, H_*(X)^{\otimes k}) \cong H_*(E\Sigma_k \times_{\Sigma_k} X^k)$ which then maps by $\mu_k$ into $H_*(X)$. The element $\sigma \int x$ is the image of $\sigma$ under

$$H_*(B\Sigma_k) \longrightarrow H_*(B\Sigma_k, H_*(X)^{\otimes k})$$

induced by the equivariant map

$$\mathbb{Z}_2 \longrightarrow H_*(X)^{\otimes k}$$
$$1 \longmapsto x^{\otimes k}.$$

~~I think~~ that if it is true that this corresponds to the map $(*)$ on page 4, then this above map is additives. Here is a direct proof: Let $x, y \in H_*(X)$. Recall how one adds the elements $\Phi(x)$ and $\Phi(y)$. One has

$$[\Phi(x) + \Phi(y)](\sigma) = \sum \Phi(x)(\sigma_j') \cdot \Phi(y)(\sigma_j'')$$

where

$$\Delta\sigma = \sum \sigma_j' \otimes \sigma_j''$$

$$H_*(B\Sigma_k) \longrightarrow \bigoplus_{i=0}^{k} H_*(B\Sigma_i) \otimes H_*(B\Sigma_{k-i})$$

On the other hand

$$[\Phi(x+y)](\sigma) = (\mu_k)_*(\sigma \int (x+y))$$

so it is necessary to understand $\sigma \int (x+y)$.

$$\sigma \longmapsto \sigma S(x+y)$$

$$H_*(B\Sigma_k, \mathbb{Z}_2) \longrightarrow H_*(B\Sigma_k, (\mathbb{Z}_2 a \oplus \mathbb{Z}_2 b)^{\otimes k}) \longrightarrow H_*(B\Sigma_k, H_*(X)^{\otimes k})$$

$$\cong$$

$$\sum_{i=0}^{k} H_*(B\Sigma_k, \operatorname{ind}_{\Sigma_i}^{\Sigma_i \times \Sigma_{k-i}} \mathbb{Z}_2 a^{\otimes i} \otimes \mathbb{Z}_2 b^{\otimes k-i})$$

$$\cong$$

$$\sigma^{\epsilon} \longmapsto \sum_j \sigma_j' \otimes \sigma_j'' \in \sum_{i=0}^{k} H_*(B\Sigma_i) \otimes H_*(B\Sigma_{k-i}) \longrightarrow \oplus_i H_*(B\Sigma_i, H_*(X)^{\otimes i}) \otimes \cdots$$

$$\longmapsto \sum_i \sigma_j' S x \otimes \sigma_j'' S y.$$

Thus

$$\sigma S(x+y) = \sum \text{im } \sigma_j' S x \otimes \sigma_j'' S y$$

On the other hand it is pretty clear that

$$\begin{array}{ccc}
H_*(B\Sigma_i, H_*(X)^{\otimes i}) \otimes H_*(B\Sigma_{k-i}, H_*(X)^{\otimes k-i}) & \longrightarrow & H_*(B\Sigma_k, H_*(X)^{\otimes k}) \\
\downarrow \cong & & \downarrow \cong \\
H_*(E\Sigma_i \times_{\Sigma_i} X^i) \otimes H_*(E\Sigma_{k-i} \times_{\Sigma_{k-i}} X^{k-i}) & \longrightarrow & H_*(E\Sigma_k \times_{\Sigma_k} X^k) \\
\downarrow \mu_i \otimes \mu_{k-i} & & \downarrow \mu_k \\
H_*(X) \otimes H_*(X) & \xrightarrow{\mu} & H_*(X)
\end{array}$$

is commutative, the last square being the fact that $\oplus \mu_k$ is a ring homomorphism.

I ~~[illegible]~~ now wish to check carefully whether $\Phi$ is a ring homomorphism. ~~[illegible]~~ Recall that the multiplicative coproduct of $R$ comes from The maps

$$\Delta_m : H_*(B\Sigma_k) \longrightarrow H_*(B\Sigma_k \times B\Sigma_k)$$

induced by the ordinary diagonal homomorphism $\Sigma_k \to \Sigma_k \times \Sigma_k$. Let

$$\Delta_m(\sigma) = \sum{}' \sigma_j' \otimes \sigma_j''$$

and let $x, y \in H_*(X)$, so that

$$[\Phi(x)\cdot\Phi(y)](\sigma) \overset{\text{defn}}{=} \sum \mu_{k*}(\sigma_j' \textstyle\int x)\cdot \mu_{k*}(\sigma_j'' \textstyle\int y).$$

Now I want to show this is the same as $\Phi(x\cdot y)(\sigma)$. Now

$$x\cdot y = \mu_*(x\otimes y) \qquad x\otimes y \in H_*(X\times X),$$

and we have a diagram

$$\begin{tikzcd}
\overset{\sigma}{H_*(B\Sigma_k)} \arrow[r] \arrow[dddd, "\Delta_m"] & \overset{\sigma\int x\otimes y}{H_*^{\Sigma_k}((X\times X)^k)} \arrow[rr, "(\mu^k)_*"] \arrow[d, equal] & & H_*^{\Sigma_k}(X^k) \arrow[ddddd, "\mu_{k*}"] \\
& H_*^{\Sigma_k}(X^k\times X^k) \arrow[ddd, "\text{res wrt } \Delta:\Sigma_k\to\Sigma_k\times\Sigma_k"] & & \\
& & & \\
& & & \\
\underset{\Sigma\sigma_j'\otimes\sigma_j''}{H_*(B\Sigma_k\times B\Sigma_k)} \arrow[r] & \underset{\Sigma \sigma_j'\int x\,\otimes\,\sigma_j''\int y}{H_*^{\Sigma_k\times\Sigma_k}(X^k\times X^k)} \arrow[d, "\mu_{k*}\otimes\mu_{k*}"] & & \\
& H_*(X)\otimes H_*(X) \arrow[rr, "\mu_*"] & & H_*(X).
\end{tikzcd}$$

The first square is commutative by properties of the operation $S$ which we haven't completely checked but which should offer no difficulty, and the second square is commutative since it expresses a homotopy commutativity property of the $\mu_k$. Thus we have checked the following:

<u>Proposition</u>: Is $X$ is an infinite loop space, then the Dyer–Lashof–Kudo–Araki–Browder operation (KADL-operations)

$$\Phi : H_*(X) \longrightarrow \mathrm{Hom}_{rgs}(R, H_*(X))$$

is a ring homomorphism.

(Remarks: In writing this up you should use $\sigma S x$ to denote the element of $H_*^{\Sigma_k}(X^k)$ ~~rather than the elem~~ obtained as the image of $\sigma$ under the map

$$H_*(B\Sigma_k) \xrightarrow{\cdot x^{\otimes k}} H_*(B\Sigma_k, H_*(X)^{\otimes k}) \overset{\text{canon.}}{\simeq} H_*^{\Sigma_k}(X^k).$$

and establish as preliminary lemmas the properties of $S$ that you need.)

Actually we know by our earlier work that it should be possible to describe the functor $\mathrm{Hom}_{rgs}(R, A)$ from rings to rings in terms of the power operations associated to the regular permutation representation of elementary abelian 2-groups. We shall now carry this out in detail. Now the basic idea is that $\mathrm{Hom}_{rgs}(R, ?)$ is an analogue of the Witt ring functor which one first begins to describe using the ring homomorphism $w_k : W(A) \to A$. Though this hasn't been checked carefully, $w_k$ should come from the element of $R(\Sigma_n)_*$ corresponding to the class of an n-cycle and geometrically furnishing the operation $\psi^n$. We begin by discussing the (analogous) ring operations on

Fix a positive integer a and let $\mathbb{Z}_2^a$ act on itself by translations; one thus get an embedding $i_a : \mathbb{Z}_2^a \hookrightarrow \Sigma_{2^a}$. Let $N$ be the normalizer of $\mathbb{Z}_2^a$ in $\Sigma_{2^a}$; it is the semidirect product of $\mathbb{Z}_2^a$ and $Gl(a, \mathbb{Z}_2)$. The restriction homomorphism

$$\begin{array}{ccc} H^*(B\Sigma_{2^a}) & \longrightarrow & H^*(B\mathbb{Z}_2^a)^N \\ & \searrow & \cong \;\leftarrow \text{Dicksons theorem} \\ & & \mathbb{Z}_2[w_{2^a-2^{a-1}}, \ldots, w_{2^a-1}] \end{array}$$

is surjective, where $w_{2^a-2^i}$ denotes the corresponding Whitney class of the regular representation of $\mathbb{Z}_2^a$, since $reg(\mathbb{Z}_2^a)$ is the restriction of the standard repn. $\Delta_{2^a}$ of $\Sigma_{2^a}$. We let $\psi_a$ be the composition

$$\mathrm{Hom}_{rgs}(R, A) = \prod_{k \geq 1}' H^*(B\Sigma_k, A) \xrightarrow{pr_{2^a}} H^*(B\Sigma_{2^a}, A) \longrightarrow H^*(B\mathbb{Z}_2^a, A)^N \cong A[[w_{2^a-2^{a-1}}, \ldots, w_{2^a-1}]]$$

Claim $\psi_a$ is a ring homomorphism. Indeed $w_{2^a-1}(\Delta_{2^a})$ kills the image of the induction map from $\Sigma_i \times \Sigma_j$ to $\Sigma_{2^a}$, $i+j = 2^a$, $i, j > 0$, and yet multiplication by $w_{2^a-1}$ is injective in

$H^*(B\mathbb{Z}_2^a, A)^N$, hence $\psi_a$ is additive; multiplicativity is clear.

Set $T_i = w_{2^a-2^i}$, $0 \le i < a$. If $z \in \mathrm{Hom}_{rgs}(R, A)$ let $z_{\beta_0 \cdots \beta_{a-1}}$ be defined by

$$\psi_a(z) = \sum{}' \, z_{\beta_0 \cdots \beta_{a-1}} T_0^{\beta_0} \cdots T_{a-1}^{\beta_{a-1}}$$

Then the natural transf. $z \longmapsto z_{\beta_0 \cdots \beta_{a-1}}$ from $\mathrm{Hom}_{rgs}(R, A)$ to $A$ is represented by an element

$$\delta_{\beta_0 \cdots \beta_{a-1}} \in H_*(B\Sigma_{2^a}) \subset R$$

$$\deg(\delta_{\beta_0 \cdots \beta_{a-1}}) = \sum_{i=0}^{a-1} \beta_i (2^a - 2^i).$$

Now I want to explore the relation between $\psi_a$ and $\psi_{a-1}$. Note that by our previous calculations we know that $\{\delta_{\beta_0 \cdots \beta_{a-1}}\}_{\beta_0 > 0,\ a \ge 1}$ is a minimal system of generators for $R$, in fact a polynomial system of generators. Now we know that

$$\begin{tikzcd}
\mathbb{Z}_2^{a-1} \arrow[r, "(i_{a-1}, i_{a-1})"] \arrow[d, "j"] & \Sigma_{2^{a-1}} \times \Sigma_{2^{a-1}} \arrow[d] \\
\mathbb{Z}_2^a \arrow[r, "i_a"] & \Sigma_{2^a}
\end{tikzcd}$$

commutes and that $j^*(\mathrm{reg}(\mathbb{Z}_2^a)) = 2(\mathrm{reg}\, \mathbb{Z}_2^{a-1})$, hence

$$j^*(w_{2^a-2^i}) = \begin{cases} 0 & i = 0 \\ w^2_{2^{a-1}-2^{i-1}} & i > 0 \end{cases}$$

On the other hand if $z_{2^a}$ is the component of $z$ in $H^*(B\Sigma_{2^a}, A)$ we know that

$$\mathrm{res}^{\Sigma_{2^a}}_{\Sigma_{2^{a-1}} \times \Sigma_{2^{a-1}}} z_{2^a} = z_{2^{a-1}} \otimes z_{2^{a-1}}$$

whence

$$* \qquad \boxed{j^* \psi_a = \psi_{a-1}^2} \qquad \text{(recall } \psi_a(z) = \iota_a^* z_{2^a}\text{)}$$

or written out

$$\sum z_{0\beta_1 \cdots \beta_{a-1}} w_{2^{a-1}-1}^{2\beta_1} \cdots w_{2^{a-1}-2^{a-2}}^{2\beta_{a-1}} = \sum z^2_{\gamma_0 \cdots \gamma_{a-2}} w_{2^{a-1}-1}^{2\gamma_0} \cdots w_{2^{a-1}-2^{a-2}}^{2\gamma_{a-2}}$$

so we conclude that

$$z_{0\beta_1 \cdots \beta_{a-1}} = z^2_{\beta_1 \cdots \beta_{a-1}} \qquad \text{or}$$

$$\delta_{0\beta_1 \cdots \beta_{a-1}} = \delta^2_{\beta_1 \cdots \beta_{a-1}}$$

(In terms of cohomology operations the above box ~~this~~ checks because

$$\psi_a : H^*(X) \longrightarrow H^*(X)[[w_{2^a-1}, \ldots, w_{2^a-2^{a-1}}]] \qquad \text{sends}$$

$$\psi_a(e(L)) = \sum_{i=0}^{a} w_{2^a-2^i} e(L)^{2^i} \quad )$$

Let $V$ be the following functor from rings $/\mathbb{Z}_2$ to rings $/\mathbb{Z}_2$: $V(A) =$ set of functions $\beta \longmapsto z_\beta$ where $\beta = (\beta_0, \ldots, \beta_a)$ is a finite sequence of non-negative integers for all $a \geq 0$ ~~and~~ such that

$$Z_{(0,\beta_1,\dots,\beta_{a-1})} = Z^2_{(\beta_1,\dots,\beta_{a-1})}$$

The addition on $V(A)$ is component-wise, hence

$$V(A) \cong A^I$$

as abelian groups, where $I$ runs of the set of such sequences $\beta$ with $\beta_0 \geq 1$ and the empty sequence (this last corresponds to the generator of $H_0(B\Sigma_1)$). Multiplication is given by the rule

$$(Z' \cdot Z'')_\beta = \sum_{\beta'+\beta''=\beta} Z'_{\beta'} \cdot Z''_{\beta''}$$

Thus $V$ is represented by the polynomial ring $\mathbb{Z}_2[\delta_\beta]_{\beta \in I}$ with

$$\Delta_{add}(\delta_\beta) = \delta_\beta \otimes 1 + 1 \otimes \delta_\beta$$

$$\Delta_{mult}(\delta_\beta) = \sum_{\beta'+\beta''=\beta} \delta_{\beta'} \otimes \delta_{\beta''}$$

and we have a natural ring homomorphism

$$\operatorname{Hom}_{rgs}(R, A) \longrightarrow V(A),$$

which we know is an isomorphism if we use the fact that $R$ is a polynomial ring. However this ~~is one~~ assumption is not really needed at this point, since we know that $R$ is primitively generated and that Frobenius is injective on the primitive elements since

$$\delta^2_{(\beta_0,\dots,\beta_{a-1})} = \delta_{(0,\beta_0,\dots,\beta_{a-1})}.$$

Thus from Hopf algebra theory, $R$ is a polynomial ring.

<u>Remark:</u> Nakaoka's Hopf algebra

$$\varinjlim_k H_*(B\Sigma_k) = R/(\sigma-1)$$

gives rise to the functor from $\mathbb{Z}_2$-algs. to $\mathcal{Ab}$ which associates to $A$ the subgroup of $\mathcal{B}_m(V(A))$ consisting of those $z$ with $z_{\underbrace{(0,\dots,0)}_{a}} = z_\phi^{2^a} = 1$ for all $a \geq 0$.

January 8, 1970 (still groggy)

Here's how to think of KADL operations. Recall how you learned from Grothendieck ~~[illegible]~~ to think of $H_*(B)$ where $B$ is an H-space. Given a ring $A$ over $\mathbb{Z}_p$ (not nec. comm.)

$$\mathrm{Hom}_{rgs}(H_*(B), A) = \mathrm{Hom}_{gp\ functors}\left([?, B], \cancel{H(\ldots)}\ h_A(?)^{\times}\right)$$

where $\times$ denotes group of units and $h_A = H^* \otimes A$.

Now let us suppose that $B$ is everyway-commutative H-space and let $b(?) = [?, B]$ be the represented functor. Then we have natural transformation

$$\mu_k^*: \ b(X) \longrightarrow b(E\Sigma_k \times_{\Sigma_k} X^k)$$

which is additive, for $k \geq 1$. ~~[illegible]~~ Moreover

$$\mathrm{res}^{\Sigma_{k+l}}_{\Sigma_k \times \Sigma_l} \mu_{k+l}^*(x) = \mu_k^*(x) \boxplus \mu_l^*(x) \quad \text{in } b(E\Sigma_k \times_{\Sigma_k} X^k \times E\Sigma_l \times_{\Sigma_l} X^l)$$

Suppose given an additive transf. $\theta: b \longrightarrow h_A^*$ ($A$ now comm.), or equivalently a ring hom. $H_*(B) \longrightarrow A$. Then $x \longmapsto (\theta(\mu_k^* x))$ is a new such transformation

$$b(X) \longrightarrow \prod_{k \geq 1}{}' H^*(E\Sigma_k \times_{\Sigma_k} X^k) \otimes A$$

$$\wr\wr$$

$$\mathrm{Hom}_{rgs}(R(X), A)$$

Now if we use the isomorphism which one gets from external Steenrod operations

$$\prod_{k\geq 1}' H^*(E\Sigma_k \times_{\Sigma_k} X^k)\otimes A \cong \mathrm{Hom}_{rgs}(R,A)\otimes H^*(X)$$

we have a new map

$$b(X) \longrightarrow \left(\mathrm{Hom}_{rgs}(R,A)\otimes H^*(X)\right)^*$$

which must come from a homomorphism $H_*(B) \longrightarrow \mathrm{Hom}_{rgs}(R,A)$. This morphisms of functors of $A$ is represented by the KADL map

$$H_*(B) \longrightarrow \mathrm{Hom}_{rgs}(R, H_*(B)).$$

In general we know it is better to work with the $\psi_a$ than with $\mathrm{Hom}_{rgs}(R, H_*(B))$. So let $a$ be an integer $\geq 1$. Then we have a commutative diagram

$$\begin{tikzcd}
b(X) \arrow[r, "x \mapsto \theta(\mu_k^*(x))"] \arrow[dr] & \prod_{k\geq 1}' H^*_{\Sigma_k}(X^k)\otimes A \arrow[d, "i_a^*\Delta_X^*"] & \mathrm{Hom}_{rgs}(R,A)\otimes H^*(X) \arrow[l, "\cong"'] \arrow[d, "\wr"] \\
 & & \prod_{k\geq 1}' H^*(B\Sigma_k)\otimes A\otimes H^*(X) \arrow[d, "\psi_a"] \\
\theta(rg\otimes x) & H^*(B\mathbb{Z}_{2^a})^N\otimes H^*(X)\otimes A & H^*(B\mathbb{Z}_{2^a})^N\otimes A\otimes H^*(X) \arrow[l, "\Gamma"']
\end{tikzcd}$$

where $\Gamma$ is a $H^*(B\mathbb{Z}_{2^a})^N\otimes A = A[[w_{2^a-1},\ldots, w_{2^a-2^{a-1}}]]$ algebra hom. and is the Steenrod operation

$$\Gamma : e(L) \longmapsto \sum_{i=0}^{a} w_{2^a-2^i}\, e(L)^{2^i}$$

Here $reg \otimes x$ denotes the element of $b(B\mathbb{Z}_{2^a} \times X)$ which is $i_a^* \Delta_X^* \mu_k^*(x)$, or intuitively the $2^a$-fold sum of $x$ with itself regarded equivariantly ~~for~~ under $\mathbb{Z}_{2^a}$. An alternative description is the trace or norm of $f^*x$ for the equivariant map $f: \mathbb{Z}_{2^a} \times X \longrightarrow X$.

Now the above is all very complicated and involved with the use of $R$. Ultimately we are claiming the following which has essentially been proved.

Proposition: Let $\Gamma: A[w_{2^a-1}, \ldots, w_{2^a-2^{a-1}}] \otimes H^*(X) \circlearrowleft$ be the Steenrod endomorphism with

$$\Gamma(e(L)) = \sum_{i=0}^{a} w_{2^a-2^i}\, e(L)^{2^i}.$$

Let $\theta: b \longrightarrow h_A^\times$ be a mult. char. class. Then there is a unique mult. char class

$$\theta^\#: b \longrightarrow h^\times_{A[w_{2^a-1}, \ldots, w_{2^a-2^{a-1}}]}$$

such that

$$\Gamma\, \theta^\#(x) = \theta(reg \otimes x).$$

Remark: Note that $\Gamma$ itself is essentially map $Norm_f\, f^*$ ~~the norm~~ for cohomology classes for the equivariant map $\mathbb{Z}_{2^a} \times X \longrightarrow X$. This raises the question of whether for an arbitrary finite group $G$ acting on a finite set $S$, there is an operation $\theta^\#: b \longrightarrow h_R^\times$ such that for any $f: P \times_G S \longrightarrow X$ and $y \in b(P \times_G S)$

$$Norm_f\, \theta^\#(y) = \theta(Norm_f\, y).$$

$A \subset X$ $\qquad U \cap V \cap A = \phi$

compact + Hausdorff.

Bredon's book

Serre Top paper

Swan – Venkov

X Swan, R.G., The non-triviality of the restriction map in the cohomology of groups, Proc. Amer. Math. Soc 11 (1960) 885-887

Serre, J.-P., Sur la dimension cohomologique des groupes profinis, Topology 3 (1965) 413-420

Serre, J. P. Cohomologie des groupes discrets, C.R. Acad. Sc. Paris 268 (1969) 268-271.

X Swan R.G. Groups of coh. dim. one, J. of Algebra 12 (1969) 585-610

$$G \overset{\text{closed}}{\subset} GL_n(\mathbb{C}) = K$$

coefficients $\mathbb{Z}_p$

S.S. I. $\quad K/G \longrightarrow BG \longrightarrow BK$

$$H^*(BK) \otimes H^*(K/G) \Rightarrow H^*(BG)$$

wants to know $H^*(K/G)$ f.g.

S.S. II

$$\begin{array}{c} K \\ \downarrow \\ K \longrightarrow K/G \longrightarrow B_G \end{array}$$

$$H^*(BG, H^*K) \Longrightarrow H^*(K/G)$$

~~so~~ Venkov assume $H^*(BG; \mathbb{Z}_p)$ f.g. for $0 \leq * \leq 4n^2$

~~actually~~ [illegible]

point is that $G$ acts trivially on $H^*(K)$ because it acts through the translation action of K on itself + K is connected

$n^2$

~~must supp~~ $\quad H^*(BG) \otimes \underset{\leq n^2}{\underline{H^*(K)}} \Longrightarrow \underset{\leq 2n^2}{\underline{H^*(K/G)}}$

Venkov, B.B., ~~[illegible]~~. Cohomology of groups of units in algebras with division, Dokl. Akad. Nauk SSSR 137 (1961) 1019-1021

$\mathfrak{A}$ division alg over $\mathbb{Q}$ rank $n$
$G$: group of units norm 1 in maximal order

$$H^k(G,M) \xrightarrow{\sim} H^{n+k}(G,M) \qquad k \geq \frac{n(n+1)}{2}$$

make $G$ acts on spaces quad. forms $n$-vbls discriminant 1.

Venkov, B.B. Cohomology algebras for some classifying spaces. Dokl. Akad. Nauk SSSR 127 (1959), 943-944