Witt rings + Bockstein

Formula for the Bockstein $\beta : H^{g}(A) \to H^{g+1}(A)$ where A is a cosimplicical ring of characteristic p.

Idea is to take

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$$

and replace by

$$0 \to A \longrightarrow W_2(A) \longrightarrow A \to 0$$

This seems to work because if X is a simplicial set and

$$A = (\mathbb{Z}/p\mathbb{Z})^X$$

then

$$W_2(A) = (\mathbb{Z}/p^2\mathbb{Z})^X$$

Thus the procedure I am going to use ~~[illegible]~~ works for any cosimplicial ring and yields the old answers.

need a formula for $\boxed{W_2(A)}$

Suppose k perfect. Then get maps

$$k \times k \longrightarrow W_2(k)$$
$$(a, b) \longmapsto s(F^{-1}a) + pb$$

in other words

$$s(a^{1/p}) + p(b).$$

now try to calculate the product and ring structure

$$s(a^{1/p}) + p(b) + s(\bar{a}^{1/p}) + p(\bar{b}) = s((a+\bar{a})^{1/p}$$

Check your classification in char 2.

Suppose given A $\ni$ if $x \in I$ then $x^2 = 0$

$\Rightarrow$ i

$$\Lambda(I/I^2) \longrightarrow gr^I(A) \to 0$$

$$\begin{tikzcd}
 & & 0 \arrow[d] & & & \\
0 \arrow[r] \arrow[d] & \Lambda^2\Omega \arrow[r, "\delta"] \arrow[d] & \Omega\otimes\Omega^0 \arrow[r] \arrow[d] & S_2\Omega \arrow[r] \arrow[d] & 0 & \text{exact} \\
0 \arrow[r] & R \arrow[r] \arrow[d, equal] & A_2 \arrow[r, "D"] \arrow[d] & \Omega\otimes A_1^0 \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R \arrow[r] \arrow[d] & A_1^0 \arrow[r, "D"] \arrow[d] & \Omega \arrow[r] \arrow[d] & 0 & \text{exact} \\
 & 0 & 0 & 0 & &
\end{tikzcd}$$

exact exact exact

follows that $S_2\Omega \simeq Q$

that $\Lambda^2\Omega \simeq I/I^2$

& that $R \to A_2 \to \Omega\otimes A_1$ exact

then

Actually one may just defines $D: S\Omega\otimes A_k \longrightarrow S\Omega\otimes A_{k-1}$

$\boxed{\text{want } D^2 = 0}$ ~~which is I believe clear~~ !

$D(f$

define

$$= -\{\gamma(a+a') - \gamma(a) - \gamma(a')\}$$ not

$$W_2(k) = k \times k$$

and ~~set~~ if k is perfect then we may unwind ~~write~~ the first factor and get that

$$W_2(k) \xrightarrow{\pi} k$$

$$\pi(a,b) = a^{1/p}$$

$$s(a) = (a^p, 0).$$

$$W_2(k) = k \times k$$

$$= \cancel{k}$$

Then $$s\pi(a,b) = (a,0)$$

and $$s[\pi(a,0)]^p = s(a) = (a^p, 0).$$

The product

$$(a,b) \longmapsto b - \gamma(a)$$

$s(a)$.

$$(s(a) + i(b))(s(\bar{a}) + i(\bar{b}))$$

$$= s(a\bar{a}) + \underset{\substack{\| \\ i(a\bar{b}}}{s(a)\,i(\bar{b})} + s(\bar{a})\,i(b)$$

$$\therefore \boxed{(a,b)(\bar{a},\bar{b}) = (a\bar{a},\ a\bar{b} + \bar{a}b)}$$

Steenrod operations

~~$K(-n$~~

$K(V, -n) \longrightarrow A$

$\underbrace{S_2 K(V, -n)}_{L} \longrightarrow A$

$\boxed{(L \otimes L)_{\Sigma}}$

$\mathrm{Tor}_*^{\Sigma}(L \otimes L, \mathbb{Z}/_2)$

$0 \to \Lambda_2 L \to L \otimes L \to S_2 L \to 0$

$0 \to \Gamma_2 L \to L \otimes L \to \Lambda_2 L \to 0$

$0 \to \Lambda_2 L \to \Gamma_2 L \to L^{(2)} \to 0$

$0 \to L^{(2)} \to S_2 L \to \Lambda_2 L \to 0$

$0 \to \Lambda_2 L \to \Gamma_2 L \to L^{(2)} \to 0$

in fact this is easy to calculate, i.e.

periodic of period 1 (in char. 2)

and the cohomology is $\Gamma_2 L / \Lambda_2 L \simeq L^{(2)}$

hence may define Steenrod operations easily

two spectral sequence

$H^*\{(L \otimes L)_{\Sigma}\}$

$H^*\{\mathrm{Tor}_*^{\Sigma}(L \otimes L, \mathbb{Z}/_2)\} \Longrightarrow$

$$f : X_1 \longrightarrow k$$

$$\underbrace{f(x,z)}_{\alpha} = \underbrace{f(x,y)}_{\beta} + \underbrace{f(y,z)}_{\gamma}$$

$$\boxed{(\alpha,0) - (\beta,0) - (\gamma,0)}$$

zero $= (0,0)$

$-(a,b) = (-a,\ a^2 - b)$

Check $(a,b) + (-a,\ a^2-b) = (0,\ \cancel{b} + \cancel{a^2} \cancel{-b} + \cancel{-a^2})$

$$\begin{aligned}
(\alpha,0) - (\beta,0) - (\gamma,0) &= (\alpha,0) + (-\beta,\ \beta^2) + (-\gamma,\ \gamma^2) \\
&= (\alpha-\beta,\ \beta^2 - \alpha\beta) + (-\gamma,\ \gamma^2) \\
&= (\alpha-\beta-\gamma,\ \beta^2 - \alpha\beta + \gamma^2 + (-\gamma)(\alpha-\beta)) \\
&= (0,\ \beta^2 - (\beta+\gamma)\beta + \gamma^2 + (-\gamma)(\gamma)) \\
&= (0,\ -\gamma\beta)
\end{aligned}$$

$$\boxed{(\beta f)(x,y,z) = f(x,y)\, f(y,z)}$$

Let R, A, etc. be a c-ring. We intro

<u>Steenrod operations</u>

$$\begin{tikzcd}
C^*(X^n) \arrow[d] & & C^*(X)^{\otimes n} \otimes W \arrow[ll] \arrow[d] & \\
C^*(X^n)/\Sigma(n) \arrow[d] & & C^*(X)^{\otimes n} \otimes W/\Sigma(n) \arrow[ll] \arrow[r, leftrightarrow] & \left[C^*(X)^{\otimes n}\right]^{inv} \otimes W/\Sigma(n) \\
C^*(X) & & f \otimes \cdots \otimes f \otimes \underline{a} \arrow[ll, mapsto] & a \in W/\Sigma(n)
\end{tikzcd}$$

<u>Theorem (Dold)</u>: generalized Steenrod ops generate all ops

<u>Problem</u>: Calculate $\beta : H^1 \to H^2$

<u>Conjecture</u>: We know already of

$$H^1(X, \mathcal{O}_X^*[[\varepsilon]]^*)$$

of Witt vectors and of the Bockstein operations of Serre

Thus $$\mathcal{O}_X[[\varepsilon]]^* \simeq \mathcal{O}_X^* \times \underbrace{\left(1 + \varepsilon\mathcal{O}_X + \varepsilon^2\mathcal{O}_X + \cdots\right)}_{G(\mathcal{O}_X)}$$

and

$$G(\mathcal{O}_X) \simeq \text{Bergman-Witt scheme}$$

so in particular we find an extension

$$0 \to \mathbb{G}_a \to W_p \to \mathbb{G}_a \to 0$$

W_p = Witt vectors of length p.

and so get an operator $H^1(X, \mathcal{O}_X) \to H^2(X, \mathcal{O}_X)$ which is first Bockstein.

$$f: X_1 \longrightarrow \mathbb{Z}_2$$

$\cancel{f(x+g)}$

$$f(x,z) = f(x,y) + f(y,z)$$

$$\begin{array}{c} k \\ \uparrow \\ W_2(k) \\ \uparrow \\ k \\ \uparrow \\ 0 \end{array}$$

exact sequence of abelian groups.

$$W_2(k) = \{1 + at + bt^2 \quad \text{mod } t^3\}.$$

$$(1+at+bt^2)(1+a't+b't^2)$$
$$= (1+(a+a')t + (aa'+b+b')t^2).$$

$W_2(k) = (k \times k)$ with group law

$$\boxed{(a,b)+(a',b') = (a+a', b+b'+aa')}$$

~~good section~~ In fact $W_2(k)$ is a ring.

$\cancel{a^2+bp}$

$\cancel{a^2+bp}$

$$(a,b) \longmapsto a^2+2b$$

$$(\mathbb{Z}_2 \times \mathbb{Z}_2) \longrightarrow \mathbb{Z}/4$$

$$(a,b)+(a',b') \longmapsto a^2+2b+(a')^2+2b'$$

$$(a+a', b+b'+aa') \longmapsto (a+a')^2 + 2(b+b'+aa')$$

$$(a^2+2b)(\bar{a}^2+2\bar{b}) = (a\bar{a})^2 + 2(a^2\bar{b} + b\bar{a}^2)$$

$$\boxed{(a,b)(\bar{a},\bar{b}) = (a\bar{a}, a^2\bar{b} + \bar{a}^2 b)}$$

ring law up to sign

mult section is $a \longmapsto (a,0)$

$\mathbb{Z}/p\mathbb{Z} \xrightarrow{s} \mathbb{Z}/p^2\mathbb{Z}$

~~It is more difficult for p^3~~

I am now beginning to understand Witt vectors a bit

Thus if k is perfect and A has residue field p there is a multiplicative section $s: k \to A$ defined by ~~these~~ choosing any

suppose k perfect (char p), A complete local ring residue field k. maximal ideal $m \ni p$. Observe that if

$$(x-y) \in m^k \qquad x = y + a \qquad a \in m^k$$

then

$$x^p = (y+a)^p = y^p + \underline{p}y^{p-1}\underline{a} + \underline{\binom{p}{2}}y^{p-2}a^2 + \cdots + a^p$$

$$\underline{1+k} \qquad 1+2k \qquad pk$$

$$\therefore x^p - y^p \in m^{k+1}$$

and so if we choose a section $t: k \to A$

we may define

$$s(x) = \lim_{n\to\infty} t\left(x^{p^{-n}}\right)^{p^n}$$

$- (a^p + bp)$

$\|$

$(-a)^p + (-b)p$

p odd.

$$\boxed{\begin{array}{l} (p \overset{=2}{\text{even}} \text{ then} \\ -(a^2+2b) = -a^2+2b \\ \qquad = (a)^2 + 2(b+a^2) \\ -(a,b) = (a, b+a^2) \end{array}}$$

$$f(y,z) - f(x,z) + f(x,y) = 0$$

$$\beta \quad - \quad \alpha \quad + \quad \gamma \quad = 0$$

$$(\beta, 0)$$

$$(\beta,0) + (-\alpha,0) + (\gamma,0) = \ ?$$

$$(-\alpha, 0) + \left(\beta+\gamma, \sum_{\substack{i+j=p \\ 1\le i\le p-1}} \frac{\beta^i \gamma^j}{i!\,j!}\right)$$

$$= \Big(0, \ \underbrace{\frac{(-\beta-\gamma)^i}{i!} \frac{(\beta+\gamma)^j}{j!}} \quad + \sum \frac{\beta^i}{i!}\frac{\gamma^j}{j!}$$

$$(\beta+\gamma)^p \left\{ \sum_{\substack{i+j=p \\ 1\le i\le p-1}} (-1)^i \frac{1}{i!}\frac{1}{j!} \right\}$$

$$\cancel{\sum_{i=1}^{p-1} \frac{1}{i!\,(p-i)!} (-X)^i}$$

$$1 + \varphi(X)$$

$$\left(t(x^{p^{-n}})\right)^{p^n} - \left(\bar{t}(x^{p^{-n}})\right)^{p^n} \in m^{p^n} \quad \text{by above.}$$

so can take $\cancel{t} = s$. $\boxed{s(x^p) = s(x)^p}$

$$s(x^{p^n})s(y^{p^n}) - s(x^{p^n}y^{p^n}) \in m^{p^n}$$

Thus the **unique** multiplicative section $\mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{Z}_p$

is $a \longmapsto \lim_{n\to\infty} (a^{p^n})$

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next try to calculate the **Bockstein**

$$\begin{array}{c} f(x,y) \in \mathbb{Z}/p\mathbb{Z} \\ \Big\downarrow \text{lift to } \mathbb{Z} \text{ and } \theta \text{ take } p\text{th power} \\ \mathbb{Z}/p^2\mathbb{Z} \end{array}$$

and then calculate

$$g(x,y,z) = \underline{\theta f(y,z) - \theta f(x,z) + \theta f(x,y)}$$

$$f(x,z) = f(x,y) + f(y,z)$$

$f(x,z)$

$$\begin{tikzcd} \mathbb{Z}/p^2\mathbb{Z} \arrow[d, shift right, "\pi"'] \\ \mathbb{Z}/p\mathbb{Z} \arrow[u, shift right, "j"'] \arrow[d] \\ 0 \end{tikzcd}$$

$$D_a = i^{-1}(a - j\pi a).$$

somehow is very similar to jet theory

$$A = \mathbb{Z}_p \qquad R = \mathbb{Z}/p\mathbb{Z}$$

have $A \xrightarrow{\varepsilon} R$ and $j : R \to A$ multiplicative

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$$W_p(k) = k \times k = \{(a,b)\} \qquad a^p + bp$$

addition is:

$$a^p + bp + \bar{a}^p + \bar{b}p = (a+\bar{a})^p + \left[-\frac{(a+\bar{a})^p - a^p - \bar{a}^p}{p} + b + \bar{b} \right] p$$

$$\frac{(a+\bar{a})^p - a^p - \bar{a}^p}{p} = \sum_{i=1}^{p-1} -\frac{(p-1)!}{i!\,j!} a^i \bar{a}^{p-i}$$

recall that $(p-1)! \equiv -1 \ (p)$

so we get

$$\boxed{(a,b) + (\bar{a},\bar{b}) = \left(a+\bar{a},\ \sum_{\substack{i=1 \\ i+j=p}}^{p-1} + \frac{a^i}{i!}\frac{\bar{a}^j}{j!} + b + \bar{b}\right)}$$

~~note that we can form this whenever we~~

$$\cancel{\Gamma_p(k) \longrightarrow W_p(k)}$$

given $f : X_1 \to \mathbb{Z}/p\mathbb{Z}$ $\to$ $\delta f = 0$

$\| \quad fd_0 - fd_1 - fd_2 = 0.$

lift f to $\mathbb{Z}$.

choose a section of $\mathbb{Z} \underset{s}{\leftrightarrows} \mathbb{Z}/p\mathbb{Z}$

i.e. label $0, 1, \ldots, p-1$

and consider

$$\delta(sf) \in p\mathbb{Z}.$$

reduce mod $p^2\mathbb{Z}$

Actually the point is that in $\mathbb{Z}_p$ there is a canonical multiplicative system of coset representatives.

Thus let u be a primitive root mod p $\quad u^{p-1} = 1$

~~There are lifts to a primitive root~~

Probably not that difficult.

$$2 + ap \qquad 0 \le a < p$$

$$\cancel{(2+ap)^{p-1} = 2^{p-1} + (p-1)\,2^{p}} \qquad 2^p \equiv 2 \quad (p)$$

$$(2+ap)^p = 2^p + \cancel{p\,2^{p-1}ap} \equiv 2 + ap$$

$$\cancel{\text{want } 2+ap} \qquad \therefore\ a = \frac{2^p - 2}{p}$$

$$2 + ap = 2 + \frac{2^p-2}{p}\,p = 2^p.$$

Thus

$$\cancel{\gamma(-\beta-\gamma+\beta+\gamma)} - \cancel{\gamma(-\beta-\gamma)} - \cancel{\gamma(\beta+\gamma)}$$

$$+ \quad \gamma(\beta+\gamma) - \gamma(\beta) - \gamma(\gamma)$$

$$\varphi(X) = \sum_{i=1}^{p-1} \frac{(-1)^i}{i!(p-i)!} X^i$$

$$1 + p!\,\varphi(X) - X^p = \sum_{i=0}^{p} \binom{p}{i} (-X)^p = (1-X)^p$$

Backstein

β in dim 1

$*$ in dim n

$$\therefore \; p!\,\varphi(X) = (1-X)^p - (1-X^p)$$

$$-p!\,\varphi(X) = (X-1)^p - (X^p - 1)$$

$$\therefore -p!\,\varphi(1) = 0$$

$$\therefore \; \varphi(1) = 0.$$

$$0 \to A \to W_p(A) \rightleftarrows A \to 0$$

These are after all abelian groups

Γ

Conclude that

$$\big(\mathrm{Bock}(f)\big)(x,y,z) = \sum_{i=1}^{p-1} \frac{f(x,y)^i}{i!} \frac{f(y,z)^{p-i}}{(p-i)!}$$

i.e. intuitively $\gamma_p\{f(x,z)\} - \gamma_p\{f(x,y)\} - \gamma_p\{f(y,z)\}$.

$$A_1 \; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \; A_2$$

$$\begin{array}{c} A_2^{(p)} \\ \uparrow \\ S_p^0(A_2) \longrightarrow \Gamma_p(A_2) \end{array}$$

$a, b \in W_2(k)$

$$S(\pi a^p) = a^p$$

$$S(\pi a^p + \pi b^p) = (a+b)^p$$
$$= a^p + \underbrace{\frac{(a+b)^p - a^p - b^p}{p}}\, p + b^p$$

if $\quad x = \pi a^p \qquad$ ~~$\varphi(a,b)$~~
$\qquad y = \pi b^p$

$(\pi a)^p = x \qquad x^{1/p} = \pi a$

we have

$$S(x+y) = S(x) + S(y) + \frac{(x^{1/p} + y^{1/p})^p - x - y}{p} \cdot p$$

therefore let us ~~$\varphi(\pi a)$~~ instead try to form

~~$S(x+y$~~

$$k \longrightarrow W_2(k) \searrow k$$

$$W_2(k) = k \times k$$

$$(a,b) + (a', b') = \left(a+a',\ b+b' + \frac{(a+a')^p - a^p - (a')^p}{p}\right)$$

$$(a+a')^p = \sum \frac{p!}{i!\,j!} a^i (a')^j \qquad i+j=p$$

$$\frac{(a+a')^p - a^p - (a')^p}{p} = -\sum_{i=1}^{p-1} \gamma_i(a)\, \gamma_{p-i}(a')$$

$$A \xrightarrow{D} \Omega \otimes A$$

so I consider the cosimplicial ring

$$\begin{tikzcd} A \arrow[r, shift left, "\Delta"] \arrow[r, shift right, "1\otimes id"'] & A \otimes_R A \arrow[l, bend right, "\varepsilon \otimes id"'] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & A \otimes_R A \otimes_R A \end{tikzcd}$$

which comes by shifting 1 dimensions and forgetting the rest; corresponds to taking maps to be the objects and a morphisms ~~to be~~ ~~with~~ from $\longrightarrow$ to $\searrow$ to be

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so the arguments should then be the same!!

ie Ω in this case is

$$\mathrm{Ker}\left\{ A \otimes_R A \xrightarrow{\varepsilon \otimes id} A \right\}$$

which is

$$\frac{I \otimes_R A}{I^2 \otimes_R A} = \Omega \otimes_R A$$

and $D = \Delta - 1 \otimes id$

$$\eta_l - \eta_r$$

This means that I ought to be able to carry out the arguments in characteristic 2.