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Operations in the theory ΩT.

Let h be a generalized cohomology theory with products defined on the category (FCW) of finite CW complexes. By this we mean that h is a contravariant functor from (FCW) to the category of graded anti-commutative rings which satisfies the homotopy axiom, is half-exact and is endowed with a suspension isomorphism. (half-exact means that if Y is a subcomplex of X, then

$$h(X,Y) \longrightarrow h(X) \longrightarrow h(Y)$$

is exact where

$$h(X,Y) = \mathrm{Ker}(h(X/Y) \longrightarrow h(pt)).$$

A suspension isomorphism is an isomorphism

$$\delta : h^q(X) \xrightarrow{\sim} h^{q+1}(\Sigma X)$$

which is natural and since we have products is given by cupping with a canonical element in $h^1(\Sigma)$.) This data is subject to axioms to be found in Dold's tract.

To determine the algebra (under composition) of operations of h, that is, all natural transformations $\theta : h(X) \longrightarrow h(X)$ which are <u>stable</u> (commute with δ). A stable operation is additive (<u>Proof</u>: From half exactness axiom (stated correctly) one has that if X, Y are pointed, then there is a diagram

$$\begin{tikzcd}
 & h(Y) \arrow[d, "p_2^*"] \arrow[dr, "id"] & \\
h(X) \arrow[r, "p_1^*"] \arrow[dr, "id"'] & h(X \vee Y) \arrow[r, "in_2^*"] \arrow[d, "in_1^*"] & h(Y) \\
 & h(X) &
\end{tikzcd}$$

with exact row + column and hence a direct sum decomposition

$$h(X \vee Y) \cong h(X) \oplus h(Y)$$

$$p_1^* u + p_2^* v \longleftarrow u \oplus v$$

$$z \longmapsto in_1^* z \oplus in_2^* z.$$

Any natural op θ clearly preserves this direct sum decomposition. Next note that the $\boxed{\text{geometric map } \Sigma X \xrightarrow{\nu} \Sigma X \vee \Sigma X}$ has $p_1 \nu = id$, $p_2 \nu = id$ and therefore

$$\begin{tikzcd}
h(\Sigma X \vee \Sigma X) \arrow[dr, "\nu^*"] \arrow[d, "\cong"'] & \\
h(\Sigma X) \oplus h(\Sigma X) \arrow[r, "sum"'] & h(\Sigma X)
\end{tikzcd}$$

commutes. Thus if $\theta : h \to h$ is natural it commutes with ν^*, hence

$$\theta : h(\Sigma X) \longrightarrow h(\Sigma X)$$

is always additive. So if θ is stable, θ is additive.)

It is convenient to extend h to a functor on the stable homotopy category of finite complexes whose objects are pairs (X, n) (think of as $\Sigma^{n} X$) X is a pointed finite complex and where the morphisms are

$$\{\Sigma^n X, \Sigma^m Y\} = \varinjlim_N \left[\Sigma^{N+n} X, \Sigma^{N+m} Y\right].$$

$$h^q(\Sigma^n X) = h^{q-n}(X) \qquad \text{defn.}$$

Before we had products

$$h^p(X) \otimes h^q(Y) \longrightarrow h^{p+q}(X \wedge Y)$$

which appear the same now for "stable finite complexes" where $\wedge$ is the reduced join:

$$(\Sigma^n X) \wedge (\Sigma^m Y) = \Sigma^{n+m}(X \wedge Y).$$

From now on let $\mathcal{H}$ be the category of "stable (finite) complexes" and consider h as a functor on this category.

<u>Proposition:</u> Let $X \in \mathrm{Ob}\,\mathcal{H}$. TFAE

(i) $h(X)$ is a flat $h(pt)$-module $\qquad (pt = S^0 \text{ in } \mathcal{H})$

(ii) $h(X)$ is a finitely gen. projective $h(pt)$-module

(ii) $\forall Y$

$$h(X) \otimes_{h(pt)} h(Y) \xrightarrow{\sim} h(X \wedge Y)$$

(ii)′ If DX is the Spanier-Whitehead dual of X, then

$$h(X) \otimes h(DX) \longrightarrow h(X \wedge DX)$$

is surjective.

Proof: (i) ⇒ (ii) build Y up from spheres
(ii) ⇒ (ii)′, (i)′ ⇒ (i) clear.
(ii) ⇒ (i)′: By S-W duality there are maps

$$\Phi : X \wedge DX \longrightarrow pt$$

$$\Psi : pt \longrightarrow DX \wedge X$$

such that

$$X \cong X \wedge pt \xrightarrow{id \wedge \Psi} X \wedge DX \wedge X \xrightarrow{\Phi \wedge id} pt \wedge X \simeq X$$

is the ~~identity~~ identity. (In fact one defines a duality to be a map Φ inducing an isomorphism

$$\# : \{Z, DX \wedge Y\} \longrightarrow \{X \wedge Z, Y\}$$

where $\#(u)$ is comp.

$$X \wedge Z \xrightarrow{id \wedge u} X \wedge DX \wedge Y \xrightarrow{\Phi \wedge id} pt \wedge Y \simeq Y$$

first when $Z = S^n$ any n & $Y = pt$ and then $\#$ is an isom in general. Ψ is defined to be $\#^{-1}$(can. isom $X \wedge pt \simeq X$).)

Now $\Phi^* 1 \in h(X \wedge DX)$ can by assumption be written

$$\Phi^* 1 = \sum_{i=1}^{n} u_i \boxtimes v_i \qquad \text{where}$$

$$u_i \in h(X) \qquad v_i \in h(DX)$$

Then given $x \in h(X)$ we have

$$x \boxtimes 1 = \sum_i u_i \boxtimes \Psi^*(v_i \boxtimes x)$$

or setting $\langle v_i, x \rangle = \Psi^*(v_i \boxtimes x) \in h(pt)$ we have

$$\boxed{x = \sum u_i \langle v_i, x \rangle}, \quad \text{for all } x \in h(X).$$

This shows that $h(X)$ is a direct summand of $h(pt)^n$ proving (i)'. **QED.**

<u>Definition</u>: X is of <u>Künneth</u> type (wrt h) if the equivalent conditions of the preceding proposition hold.

<u>Remark</u>: The pairing $\langle\ ,\ \rangle$ of $h(DX) \otimes h(X) \longrightarrow h(pt)$ is a perfect duality so

$$\boxed{h(DX) \simeq \mathrm{Hom}_{h(pt)}(h(X), h(pt)).}$$

Indeed the boxed formula above shows that

$$\lambda = \sum \lambda(u_i) \langle v_i, ? \rangle$$

so the map is onto. To get injectivity interchange X and DX to get $y = \sum v_i' \langle u_i', y \rangle$, whence if $\langle y, x \rangle \equiv 0 \Rightarrow y = 0$.

From now on we make the following assumption.

<u>Hypothesis:</u> h is generated by stable complexes of Künneth type, i.e. for any $x \in h(X)$ $\exists$ map $u: X \longrightarrow Y$ in $\mathcal{H}$ and $y \in h(Y)$ such that $u^*(y) = x$ and Y is of Künneth type.

<u>Proposition:</u> Let M be an $h(pt)$ module. Then the stable operations $h(X) \longrightarrow M \otimes_{h(pt)} h(X) \ (\overset{\text{defn}}{=} h_M(X))$ $X \in \mathcal{H}$ coincide with the stable operations $h \longrightarrow h_M$ as X runs over the (full subcategory of stable) complexes of Künneth type.

<u>Proof:</u> To prove $\mathrm{Homst}_{\mathcal{H}}(h, h_M) \longrightarrow \mathrm{Homst}_{/\mathcal{H}K}(h, h_M)$ is bijective. Clearly injective, so suppose given $\theta: h \longrightarrow h_M$ defined for X of Künneth type. Given Y and $y \in h(Y)$ choose $u: Y \longrightarrow X$ $x \in h(X)$ where X is of Künneth type such that $u^*(y) = x$ and set $\theta y = u^* \theta x$. To show independent suppose $y_i \in h(Y_i)$, $u_i: X \longrightarrow Y_i$ $i = 1, 2$ are two choices and let Z be the cone of $u_1 - u_2$

$$X \xrightarrow{u_1 - u_2} Y_1 + Y_2 \xrightarrow{j} Z.$$

$$h(X) \ni 0 \longleftarrow\!\shortmid\ y_1 \oplus y_2 \in h(Y_1 + Y_2)$$

so by half-exactness $\exists$ $z \in h(Z)$ with $j^* z = y_1 \oplus y_2$. By hypothesis may assume Z of Kunneth type. Then

$$u_1^* \theta y_1 - u_2^* \theta y_2 = (u_1 - u_2)^* \theta (y_1 \oplus y_2) = (u_1 - u_2)^* \theta j^* z$$

$= (u_1 - u_2)^* j^* \theta z = 0$. Thus θx is well-defined. It's now easy to show θ is natural + stable. QED

Remark: We have really shown that the category $\mathcal{I}$ of pairs (X, y) X of Künneth type, $y \in h(X)$ is filtering and that for any $X \in \mathrm{Ob}\, \mathcal{H}$

$$(*) \qquad \boxed{h(X) = \varinjlim_{(Y,y) \in \mathcal{I}} \{X, Y\}}.$$

Let End h be the ring of stable operations of h. $h(pt)$ is a subring of End h and so we get left and right $h(pt)$ module structures on End h. For a stable complex X of Künneth type and elements $x \in h(X)$, $\lambda \in \mathrm{Hom}_{h(pt)}(h(X), h(pt)) = h(DX)$ we obtain a linear functional

$$[\lambda, x] : \theta \longmapsto \lambda(\theta x)$$

for the left $h(pt)$-module structure on End h, i.e.

$$a\theta \longmapsto \lambda((a\theta)x) = \lambda(a\theta x) = a\lambda(\theta x)$$

Let $A \subset \mathrm{Hom}_{h(pt)}(\mathrm{End}\, h, h(pt))$ be the set of $\mathbb{Z}$-linear combinations of linear functionals of the form $[\lambda, x]$. Note that A is a sub-$h(pt)$-module since

$$(a\{\lambda, x\})\theta = a\lambda(\theta x) = (a\lambda)(\theta x) = \{a\lambda, x\}.$$

Let M be an $h(pt)$-module. Then by $\star$ and Yoneda's lemma we have

$$\mathrm{Homst}(h, h_M) \cong \varprojlim_{(Y,y)\in \mathcal{J}} h_M(Y) \cong \mathrm{Hom}_{h(pt)}(h(Y)', h(pt))$$

where $h(Y)' = \mathrm{Hom}_{h(pt)}(h(Y), h(pt))$. Thus

$$\boxed{\mathrm{Homst}(h, h_M) \cong \mathrm{Hom}_{h(pt)}(A, M)}$$

where

$$A = \varinjlim_{(Y,y)\in \mathcal{J}} h(Y)'$$

If $(Y, y) \in \mathcal{J}$ and $\lambda \in h(Y)'$ let $\{\lambda, y\}$ be the element of A represented by the inductive limits. Then the isomorphism in the box is given by

$$\theta \longmapsto \left(\{\lambda, y\} \longmapsto \lambda(\theta y)\right).$$

Define a ring ~~structure~~ on A by the formula

$$\{\lambda_1, y_1\} \cdot \{\lambda_2, y_2\} = \{\lambda_1 \boxtimes \lambda_2, y_1 \boxtimes y_2\}$$

where the external products are defined by means of the isom

$$\begin{array}{ccc} h(Y_1 \wedge Y_2) & \simeq & h(Y_1) \otimes_{h(pt)} h(Y_2) \\ y_1 \boxtimes y_2 & \longleftarrow & y_1 \otimes y_2 \end{array}$$

Proposition: Let R be an (anti-commutative) $h(pt)$-algebra. Then an operation $\theta: h \longrightarrow h_R$ preserves products iff the corresponding map of $h(pt)$-modules $A \longrightarrow R$ is a ring homomorphism. In other words

$$\boxed{\mathrm{Homst}^{\otimes}(h, h_R) \cong \mathrm{Hom}_{h(pt)\text{-alg}}(A, R)}$$

~~Proof:~~ ~~(... that there is a ring homomorphism ... $h(pt)$ ...).~~ ~~To say that $A \longrightarrow R$ is a hom. means that for all λ_1, λ_2~~

$$(\lambda_1 \boxtimes \lambda_2)(\theta(y_1 \boxtimes y_2)) = \lambda_1(\theta y_1)\, \lambda_2(\theta y_2)$$
$$= (\lambda_1 \boxtimes \lambda_2)(\theta y_1 \boxtimes \theta y_2)$$

~~or equivalently that $\theta(y_1 \boxtimes y_2) = \theta y_1 \boxtimes \theta y_2$, that is, that θ preserves products. QED.~~

Proof: Let $\theta: h \longrightarrow h_R$ be a stable operation and let $\varphi: A \longrightarrow R$ be the corresponding $h(pt)$-module map so that

$$\varphi\{\lambda, y\} = \lambda(\theta(y)).$$

To say that φ is a ring homomorphism is to have

$$\begin{array}{ccc}
\varphi\{\lambda_1, y_1\}\{\lambda_2, y_2\} & \overset{?}{=} & \varphi\{\lambda_1, y_1\} \cdot \varphi\{\lambda_2, y_2\} \\
\| & & \| \\
\lambda_1 \boxtimes \lambda_2 \left(\Theta(y_1 \boxtimes y_2)\right) & & \lambda_1(\Theta y_1) \cdot \lambda_2(\Theta y_2) \\
 & & \| \\
 & & \lambda_1 \boxtimes \lambda_2 \left(\Theta y_1 \boxtimes \Theta y_2\right)
\end{array}$$

This is so for all $\lambda_i \in h(Y_i)'$ iff

$$\Theta(y_1 \boxtimes y_2) = \Theta y_1 \boxtimes \Theta y_2$$

i.e. iff Θ preserves products. QED.

(Observe that A is an $h(pt)$-module by

$$\alpha\{\lambda, y\} = \{\alpha\lambda, y\} \qquad \alpha \in h(pt)$$

which is of course the structure coming from the fact that A is an inductive limit of $h(pt)$-modules. This makes A into an $h(pt)$-algebra.)

Having now described ring operations $h \longrightarrow h_R$ ~~onto~~ in terms of A, we can carry over the category (scheme) structure to A as follows:

Define a functor C from (anti-commutative) rings to categories: Given R set

$$\mathrm{Ob}\, C(R) = \mathrm{Hom}_{\text{rings}}(h(pt), R)$$

$$\mathrm{Hom}_{\mathcal{C}(R)}(\varphi_1, \varphi_2) = \mathrm{Homst}^{\otimes}_{/R}(h_{\varphi_1}, h_{\varphi_2})$$

Claim this functor is representable with

object ring — $h(pt)$

morph. ring — A

and with ~~[illegible] other~~ maps to be determined.
By prop. above

$$\mathrm{Homst}^{\otimes}_{/R}(h_{\varphi_1}, h_{\varphi_2}) \cong \mathrm{Hom}_{h(pt)\text{-alg}}(A, R_{\varphi_2})$$

which proves representability. ~~and shows that the natural $h(pt)$-alg structure~~

~~$h(pt) \longrightarrow A \qquad \alpha \longmapsto \{\alpha, 1\} \quad 1 \in h(pt)$~~

~~is in fact the "target" map. To determine the "source" map: Given~~

~~$\Theta : R_{\varphi_1} \otimes_{h(pt)} h \longrightarrow R_{\varphi_2} \otimes_{h(pt)} h$~~

~~Let~~

~~$\hat{\Theta} : A \longrightarrow R_{\varphi_2}$~~

~~be the corresponding map so that~~

~~$\hat{\Theta}\{\lambda, y\} = \lambda\, \Theta(y)$~~ $\qquad \Theta(y) \in R_{\varphi_2} \otimes_{h(pt)} h(Y)$

$\lambda \in h(Y)^{\wedge}$

~~(Check $\hat{\Theta}\{\alpha, 1\} = \alpha\, \Theta(1) = \alpha = \varphi_2(\alpha) \otimes 1$ in $R_{\varphi_2} \otimes_{h(pt)} h(pt)$)~~

We must be more explicit: Given

$$\Theta : R_{\varphi_1} \otimes_{h(pt)} h \longrightarrow R_{\varphi_2} \otimes_{h(pt)} h$$

R-alg. homomorphism define

$$\boxed{\begin{aligned} &\hat{\theta} : A \longrightarrow R \\ &\hat{\Theta}\{\lambda, y\} = m\langle 1 \otimes \lambda, \Theta(1 \otimes y)\rangle \end{aligned}}$$

where

$$m : R_{\varphi_2} \otimes_{h(pt)} h(pt) \longrightarrow R$$

is given by

$$r \otimes u \longmapsto r\varphi_2(u).$$

How to recover φ_2 from $\hat{\Theta}$: Take $Y = pt$, $y = 1 \in h(pt)$.

$$\hat{\Theta}\{u\hat{1}, 1\} = m\langle 1 \otimes u\hat{1}, 1 \otimes 1\rangle$$
$$= m(1 \otimes u) = \varphi_2(u)$$

~~Here~~ we let $\hat{1} \in h(pt)'$ be the dual base to $1 \in h(pt)$. How to recover φ_1 from $\hat{\Theta}$:

$$\hat{\Theta}\{\hat{1}, u\} = m\langle 1 \otimes \hat{1}, \Theta(1 \otimes u)\rangle = m\,\Theta(1 \otimes u)$$
$$= m\,\Theta(\varphi_1(u) \otimes 1)$$
$$= m(\varphi_1(u) \otimes 1) = \varphi_1(u)$$

Conclusion:

$$\begin{array}{ccc} u & \longmapsto & \{u\hat{1}, 1\} \\ h(pt) & \overset{\text{target}}{\underset{\text{source}}{\rightrightarrows}} & A \\ u & \longmapsto & \{\hat{1}, u\} \end{array}$$

Suppose now that $\varphi_1 = \varphi_2 = \varphi$ and $\Theta : h_\varphi \to h_\varphi$ is the identity. Then

$$\hat{\Theta}\{\lambda, y\} = m\langle 1\otimes\lambda, \Theta(1\otimes y)\rangle$$
$$= m(1\otimes\langle\lambda, y\rangle)$$
$$= \varphi(\langle\lambda, y\rangle).$$

Conclusion:

$$A \xrightarrow{\text{identity section}} h(pt)$$
$$\{\lambda, y\} \longmapsto \langle\lambda, y\rangle$$

Suppose given

$$h_{\varphi_0} \xrightarrow{\Theta_0} h_{\varphi_1} \xrightarrow{\Theta_1} h_{\varphi_2}$$

$$\widehat{\Theta_1\Theta_0}\,\{\lambda, y\} = m_2\langle 1\otimes\lambda, \Theta_1\Theta_0(1\otimes y)\rangle$$

Now
$$\theta_0(1\otimes y) \in R_{\varphi_1} \otimes_{h(pt)} h(Y)$$

write
$$\theta_0(1\otimes y) = \sum r_i \otimes y_i = \sum m_1\langle 1\otimes\lambda_i, \theta_0(1\otimes y)\rangle \otimes y_i$$

$$\theta_1\theta_0(1\otimes y) = \sum (r_i\otimes 1)\,\theta_1(1\otimes y_i)$$

$$m_2\langle 1\otimes\lambda, \theta_1\theta_0(1\otimes y)\rangle = m_2\sum (r_i\otimes 1)\langle 1\otimes\lambda, \theta_1(1\otimes y_i)\rangle$$

$$= \sum r_i \; \hat{\theta}_1\{\lambda, y_i\}.$$

$$= \sum_i \hat{\theta}_0\{\lambda_i, y\} \cdot \hat{\theta}_1\{\lambda, y_i\}$$

$\therefore$

$$\boxed{\widehat{\theta_1\theta_0}\{\lambda, y\} = \sum_i \hat{\theta}_1\{\lambda, y_i\}\cdot\hat{\theta}_0\{\lambda_i, y\}}$$

where $y_i \in h(Y)$ $\lambda_i \in h(Y^*)'$ are such that for all $z \in h(Y)$
$$z = \sum y_i\langle\lambda_i, z\rangle$$

<u>Conclusion:</u>

$$\begin{array}{ccc} A & \xrightarrow{\text{comp.}} & A \,{}_s\!\otimes_{h(pt)}{}_t A \\ \{\lambda, y\} & \longmapsto & \sum_i \{\lambda, y_i\}\otimes\{\lambda_i, y\} \end{array}$$

where $y_i \in h(Y)$ $\quad \lambda_i \in h(Y)'$ are $\ni$

$$z = \sum y_i \langle \lambda_i, z \rangle \quad \text{all } z \in h(Y).$$

(**Check:** Suppose that $z = \sum \bar{y}_j \langle \bar{\lambda}_j, z \rangle$ all $z \in h(Y)$. Then

$$\sum_i \{\lambda, \bar{y}_j\} \otimes \{\bar{\lambda}_j, y\} = \sum_{i,j} \{\lambda, y_i \langle \lambda_i, \bar{y}_j \rangle\} \otimes \{\bar{\lambda}_j, y\}$$

$$= \sum_{i,j} \{\lambda, y_i\}\{\hat{1}, \langle \lambda_i, \bar{y}_j \rangle\} \otimes \{\bar{\lambda}_j, y\}$$

$$= \sum_{i,j} \{\lambda, y_i\} \otimes \{\langle \lambda_i, \bar{y}_j \rangle, 1\}\{\bar{\lambda}_j, y\}$$

$$\cancel{= \sum_i \{\lambda, y_i\} \otimes \{\textstyle\sum \langle \lambda_i, \bar{y}_j \rangle \bar{\lambda}_j, y\}}$$

$$\cancel{= \sum_i \{\lambda, y_i\} \otimes \{\lambda_i, y\}} \;)$$

$$= \sum_i \{\lambda, y_i\} \otimes \{\sum_j \langle \lambda_i, \bar{y}_j \rangle \bar{\lambda}_j, y\}$$

$$= \sum_i \{\lambda, y_i\} \otimes \{\lambda_i, y\}.$$

This shows that the formula for Δ is independent of the ~~decomposition~~ choice of the y_i and λ_i.)

<u>Example</u>: $h = \Omega$. If $\varphi_i : \Omega(pt) \longrightarrow R$, $i=1,2$ are homomorphisms, then a ring homomorphism

$$\theta : h_{\varphi_1} \longrightarrow h_{\varphi_2}$$

is given by a power series $\varphi(X) = \sum a_i X^{i+1}$ $\quad a_0 = 1$ with coefficients in $h_{\varphi_2}(pt) = R \otimes_{\varphi_2\, h(pt)} h(pt) \xrightarrow[\cong]{m_2} R$ by the rules

$$\begin{cases} \theta(r \otimes z) = (r \otimes 1) \cdot \hat{\varphi}(z) \\ \hat{\varphi}\left(c_1^{\Omega}(L)\right) = \varphi\left(1 \otimes c_1^{\Omega}(L)\right) \end{cases}$$

Then φ_1 is the composition

$$h(pt) \xrightarrow{in_2} R \otimes_{\varphi_1\, h(pt)} h(pt) \xrightarrow{\theta} R \otimes_{\varphi_2\, h(pt)} h(pt) \xrightarrow[\cong]{m_2} R$$

which is given by what the group law goes to. But

$in_2' \, c_1^{\Omega}(L)$, where $in_2' : h(pt) \to R \otimes_{\varphi_2} h(pt)$

$$\theta\left(1 \otimes c_1^{\Omega}(L)\right) = \varphi\left(1 \otimes c_1^{\Omega}(L)\right).$$

so $$\theta(in_2 F^{\Omega})\left(\theta\, in_2\, c_1^{\Omega} L_1, \theta\, in_2\, c_1^{\Omega} L_2\right) = \varphi\left((in_2' F^{\Omega})(in_2' c_1^{\Omega} L_1, in_2' c_1^{\Omega} L_2)\right)$$

$$\left(\varphi\, in_2' c_1^{\Omega} L_1, \varphi\, in_2' c_1^{\Omega} L_2\right)$$

$\therefore$ $\cancel{m_2 \theta(in_2 F^{\Omega}) = m_2(\varphi \otimes in_2 F^{\Omega})}$ $\cancel{\varphi_1 F^{\Omega}}$

$$\therefore \left(\theta\, in_2 F^{\Omega}\right)(\varphi X, \varphi Y) = \varphi\left((in_2' F^{\Omega})(X, Y)\right)$$

$$\therefore \quad m_2\,\Theta\, in_2 F^{\Omega} = (m_2\varphi) * \varphi_2 F^{\Omega}. \qquad \text{since } m_2 \circ in_2' = \varphi_2$$

In other words a map

$$\Theta : h_{\varphi_1} \longrightarrow h_{\varphi_2}$$

is given by a power series $\varphi(X) = \sum a_i X^{i+1} \in R[[X]]$, $a_0 = 1$ with

$$\varphi_1(F^{\Omega}) = \varphi * (\underline{\varphi_2} F^{\Omega}).$$

Therefore

$$\begin{array}{ccc} h(pt) & \xrightarrow{\text{target}} & A \\ & & \wr\!\wr \\ & & h(pt)[a_1, a_2, \ldots]. \end{array}$$

and

$$h(pt) \xrightarrow{\text{source}} A$$

is given by the group law

$$\varphi_{\underline{a}} * F^{\Omega} \qquad\qquad \varphi_{\underline{a}} = \sum a_i X^{i+1}$$

The identity map is given by

$$\begin{array}{ccc} A & \xrightarrow{\text{ident.}} & h(pt) \\ a_i & \longmapsto & 0 \end{array} \qquad i \geq 1.$$

Finally we compute the composition:

$$h_{\varphi_0} \xrightarrow{\Theta_\phi} h_{\varphi_1} \xrightarrow{\Theta_1} h_{\varphi_2}$$

Suppose

$$\Theta_\phi\big(\overset{in_2^0}{\downarrow}c_1^{\Omega}(L)\big) = \psi_\phi\big(\overset{in_2^1}{\downarrow}c_1^{\Omega}(L)\big) \qquad m_1\psi_\phi(X) \in R[[X]]$$

$$\Theta_2\big(\overset{in_2^1}{\downarrow}c_1^{\Omega}(L)\big) = \psi_2\big(\overset{in_2^2}{\downarrow}c_1^{\Omega}(L)\big) \qquad m_2\psi_2(X) \in R[[X]]$$

Then

$$\Theta_2\Theta_1\left(in_2^0\, c_1{}^\Omega L\right) = \Theta_2\left(\psi_1\left(in_2^1\, c_1{}^\Omega L\right)\right)$$
$$= \left(\Theta_2\psi_1\right)\left(\psi_2\, in_2^2\, c_1{}^\Omega L\right).$$

so $\Theta_2\Theta_1$ is given by the power series

$$(\Theta_2\psi_1)\circ\psi_2\,.$$

To calculate $\Theta_2\psi_1$: Recall $\psi_1(X)$ ~~[illegible]~~ has coefficients in $R_{\varphi_1}\otimes_{h(pt)} h(pt) \overset{m_2}{\cong} R$ and that Θ_2 is R-linear. Thus $\Theta_2\Theta_1$ is given by the power series

$$\psi_1\circ\psi_2$$

and the ~~[illegible]~~ source group law is

$$\varphi_0 F^\Omega = (\psi_1\circ\psi_2) * \varphi_2 F^\Omega = \psi_1 * \underbrace{\left(\psi_2 * \varphi_2 F^\Omega\right)}_{\varphi_1 F^\Omega}$$

<u>Conclusion:</u> If $\Theta_1 : h_{\varphi_0} \longrightarrow h_{\varphi_1}$ is given by ψ_1 and $\Theta_2 : h_{\varphi_1} \longrightarrow h_{\varphi_2}$ is given by ψ_2, then $\Theta_2\Theta_1 : h_{\varphi_0} \longrightarrow h_{\varphi_1}$ is given by $\psi_1\circ\psi_2$. Thus

$$\begin{array}{ccc} A & \xrightarrow{\ \Delta\ } & \underset{s}{A} \otimes_{h(pt)} \underset{t}{A} \\ \wr\!\| & & \wr\!\| \\ h(pt)[\underline{a}] & \longrightarrow & h(pt)[\underline{a}', \underline{a}''] \end{array}$$

$$\Delta\left(\sum a_i X^{i+1}\right) = \left(\sum a_i'' X^{i+1}\right)\circ\left(\sum a_i' X^{i+1}\right)$$

<u>The ring structure of End h:</u>

Given $\quad h \xrightarrow{\theta_1} h \xrightarrow{\theta_2} h \quad$ stable operations

let

$$\hat{\theta}_i : A \longrightarrow h(pt)$$

be the corresponding $h(pt)$-~~algebra~~ module maps, where A is regarded as an $h(pt)$-~~algebra~~ module by means of the target map. Thus

$$\hat{\theta}_i \,\{\lambda, y\} = \lambda(\theta_i y).$$

I claim that $\widehat{\theta_2\theta_1}$ may be calculated in terms of the composition map as follows

$$\begin{tikzcd}
A \arrow[r, "comp"] \arrow[ddrr, bend right, "\widehat{\theta_2\theta_1}"'] & A_s \otimes_{h(pt)} {}_tA \arrow[d, "id \otimes \hat{\theta}_1"] & \\
& A_s \otimes_{h(pt)} h(pt) \arrow[r, "m"] & A \arrow[d, "\hat{\theta}_2"] \\
& & h(pt)
\end{tikzcd}$$

<u>Proof</u>:

$$\left(\hat{\theta}_2 \circ m \circ id \otimes \hat{\theta}_1 \circ \Delta\right)\{\lambda, y\} = \hat{\theta}_2\, m \sum_i \{\lambda, y_i\} \otimes \lambda_i(\theta_1 y)$$

$$= \hat{\theta}_2 \sum_i m\left(\{\lambda, y_i\}\{1, \lambda_i(\theta_1 y)\} \otimes 1\right) = \hat{\theta}_2 \sum_i \{\lambda, y_i \lambda_i(\theta_1 y)\}$$

$$= \hat{\theta}_2 \{\lambda, \theta_1 y\} = \{\lambda, \theta_2\theta_1 y\} = \widehat{\theta_2\theta_1} \{\lambda, y\},$$

where we have used that $z = \sum y_i \lambda_i(z)$ for all $z \in h(Y)$.

Note that the monoid of stable ring homs. $h \to h$ is not directly related to the category associated to $h(pt)$ since the latter is a groupoid and the former has idempotent elts. Thus $\mathrm{End}^{s}\Omega$ = the set of ways of associating to a formal group law a power series, ~~[illegible]~~, i.e. sections of the target map $Fl \xrightarrow{t} Ob$. Given sections s_1 and s_2 their composition is roughly the rule which associates to a group law F the ~~[illegible]~~ power series which is the composite of $s_1(F)$ and $s_2(s_1(F) * F)$. This is not the same as the composition of $s_1(F)$ and $s_2(F)$.

~~[illegible]~~

Inverse and antipode: Recall that we have identified $h(DY)$ and $h(Y)'$ by means of the pairing

$$\langle \lambda, y \rangle = \Phi^*(\lambda \boxtimes y) \qquad \lambda \in h(DY),\ y \in h(Y)$$

where ~~[illegible]~~ $\Phi : pt \longrightarrow DY \wedge Y$ is the trivial duality map. Define

$$I : A \longrightarrow A$$

by

$$I\{\lambda, y\} = (-1)^{d^\circ y \cdot d^\circ \lambda}\{y, \lambda\}$$

where $y \in h(Y) = h(DDY)$, and given $\theta : h_{\varphi_1} \longrightarrow h_{\varphi_2}$ define $\bar{\theta} : h_{\varphi_2} \longrightarrow h_{\varphi_1}$

$$\hat{\bar{\theta}}\{\lambda, y\} = \hat{\theta} I\{\lambda, y\}.$$

Calculate:

$$\widehat{\bar{\theta}\theta}\{\lambda, y\} = \sum_i \hat{\bar{\theta}}\{\lambda, y_i\}\, \hat{\theta}\{y_i, y\}$$

$$\cancel{\sum_i I\{\lambda, y_i\} \{\lambda_i, y\}}$$

$$\cancel{\sum_i (-1)^{d^\circ\lambda \cdot d^\circ y_i} \{y_i, \lambda\}\{\lambda_i, y\}}$$

$$= \sum_i (-1)^{d^\circ\lambda \cdot d^\circ y_i} \langle y_i, \theta\lambda\rangle \langle \lambda_i, \theta y\rangle$$

(? signs θ not homog?)

$$= \sum_i \langle \theta\lambda, y_i\rangle \langle \lambda_i, \theta y\rangle$$

$$= \langle \theta\lambda, \theta y\rangle = \Phi^*(\theta\lambda \boxtimes \theta y) = \theta\langle \lambda, y\rangle = \langle \lambda, y\rangle$$

since θ acts as identity on R. Thus

$$\bar{\theta}\theta = id$$

and similarly $\theta\bar{\theta} = id$ as $\bar{\bar{\theta}} = \theta$. Thus $\bar{\theta} = \theta^{-1}$ and we find that (putting back in $\otimes$)

$$\boxed{\widehat{\theta^{-1}}\{\lambda, y\} = m\langle \theta(1 \otimes \lambda), 1 \otimes y\rangle}$$

Conclude the inverse is given by

$$A \xrightarrow{\text{inverse}} A$$
$$\{\lambda, y\} \longmapsto (-1)^{d^\circ\lambda \cdot d^\circ y}\{y, \lambda\}$$

~~The above inverse map gives rise to a transpose map End h → End h as follows. Given Θ: h → h define ${}^t\Theta$ by~~

Remarks: In rewriting the above you should ~~identify h_φ(pt) with R and define Θ̂: A → h_φ(pt) by~~

have maps

$$j_\varphi : h(Y) \longrightarrow h_\varphi(Y)$$
$$m_\varphi : h_\varphi(pt) \xrightarrow{\cong} R$$

Also use a different letter than φ for the homs $h(pt) \to R$

such that

$$m_\varphi j_\varphi = \varphi$$

and define $\hat{\Theta} : A \to R$ by

$$\hat{\Theta}\{\lambda, y\} = m_{\varphi_2}\langle j_{\varphi_2}\lambda, \Theta j_{\varphi_1} y\rangle$$

(Notation?)

<u>Question</u>: Given an operation $\theta : h \longrightarrow h$ can one always define a transpose operation ${}^t\theta : h \longrightarrow h$?

The operations in ΩT:

Let $\Omega_{(p)} = \Omega \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ and let

$$\xi(X) = c(F^{\Omega}) \in \Omega_{(p)}(pt)[[X]]$$

be the power series of Cartier $\ni$

$$c(F^{\Omega}) * F^{\Omega}$$

is a typical group law. Define ΩT as the base extension

$$\Omega T(X) = LT \otimes_L \Omega(X)$$

where $L \longrightarrow LT$ sends the universal law F_u into the universal typical law F_t, and $L \longrightarrow \Omega$ sends F_u to F^{Ω}. It is clear that an (multiplicative) operation $\theta : \Omega T \longrightarrow (\Omega T)_R$ where R is an LT-algebra is the same as an operation $\Omega \longrightarrow (\Omega T)_R$ carrying F^{Ω} to a typical law. Therefore we have

$$\mathrm{Homst}^{\otimes}(\Omega T, (\Omega T)_{\varphi}) \cong \left\{\begin{array}{l} \text{power series } \overbrace{\psi(X) = \textstyle\sum a_i X^{i+1}} \in R[[X]] \\ a_0 = 1 \text{ such that } \psi * \underbrace{(\varphi F^{\Omega T})}_{F^{\Omega T_\varphi}} \text{ is} \\ \text{typical.} \end{array}\right.$$

(Here we use for the 1000th time, the

<u>Lemma</u>: If $\psi(X) \in Q(pt)[[X]]$ has leading term aX with $a \in Q(pt)^*$ and if $\hat{\psi} : \Omega \longrightarrow Q$ is the ~~unique~~ multiplicative operation with

$$\hat{\psi}(c_1^\Omega(L)) = \psi(c_1^Q(L))$$

then $\hat{\psi} : \Omega(pt) \longrightarrow Q(pt)$ is given by

$$\boxed{\hat{\psi} F^\Omega = \psi * F^Q}$$

<u>Proof</u>:

$$\hat{\psi}\, F^\Omega(c_1^\Omega L_1, c_1^\Omega L_2) = \hat{\psi}\, c_1^\Omega(L_1 \otimes L_2) = \psi(c_1^Q(L_1 \otimes L_2))$$

$$\|\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\|$$

$$(\hat{\psi} F^\Omega)(\hat{\psi}\, c_1^\Omega L_1, \hat{\psi}\, c_1^\Omega L_2) \qquad\qquad \psi\, F^Q(c_1^Q L_1, c_1^Q L_2)$$

$$\|$$

$$\hat{\psi} F^\Omega(\psi(c_1^Q L_1), \psi(c_1^Q(L_2)))$$

Therefore

$$\hat{\psi} F^\Omega(\psi X, \psi Y) = \psi F^Q(X, Y)$$

$$\text{or } (\hat{\psi} F^\Omega)(X, Y) = \psi F^Q(\psi^{-1} X, \psi^{-1} Y) = (\psi * F^Q)(X, Y).$$

So to use this we need to know the $\psi \ni \psi * (\varphi F^{\Omega T})$ are typical. But given a typical law F over R, to say that $\psi * F$ is typical is the same as saying that ψ^{-1} is a typical curve for F. Indeed if q is a prime $\neq p$

$$(F_q\, \psi^{-1})(X^q) = \sum_{\zeta^q = 1}^{F} \psi^{-1}(\zeta X) = \psi^{-1}\left(\sum_{\zeta^q = 1}^{\psi * F} \zeta X\right).$$

But any typical curve is uniquely expressible in the form

$$\psi^{-1} = \sum_{n\geq 0}^{F} V^n[r_n]\,\gamma_0$$

ie

$$\psi^{-1}(X) = \sum_{n\geq 0}^{F} r_n X^{p^n}.$$

Conclude that

$$\mathrm{Homst}^{\otimes}(\Omega T, (\Omega T)_\varphi) \cong \{\text{sequences } (r_1, r_2, \cdots) \in R\}$$

$$\hat{\varphi}_{\underline{r}} \longleftarrow\!\!\shortmid\ \underline{r}$$

where

$$\varphi_{\underline{r}}^{-1}(X) = \sum_{n\geq 0}^{F_\varphi} r_n X^{p^n} \qquad r_0 = 1$$

and $F_\varphi = \varphi F^{\Omega T}$. Thus

$$\mathrm{Hom}^{\otimes}_{\Omega T(pt)}(A, R_\varphi) \cong \{\text{sequences } \underline{r}\}$$

$$\Longrightarrow \qquad A \cong \Omega T(pt)[a_1, a_2, \cdots],$$ where the target $\Omega T(pt)$-algebra structure is indicated. To calculate the source map: Given $\underline{r}$ and $\varphi : \Omega T(pt) \longrightarrow R$, we then have

$$\Omega T \xrightarrow{\hat{\varphi}_{\underline{r}}} (\Omega T)_\varphi$$

and the source map is the map $\Omega T(pt) \xrightarrow{\hat{\varphi}_r} (\Omega T)_\varphi$, or equivalently the group law $\hat{\varphi}_{\underline{r}}(F^{\Omega T}) = \varphi_{\underline{r}}^{-1} * F_{\varphi}$. Therefore

$$\begin{tikzcd} \Omega T(pt) \arrow[r, "\text{source}"] \arrow[dr, bend right] & A \arrow[d, "\cong"] \\ & \Omega T(pt)[a_1, a_2, \dots] \end{tikzcd}$$

$$F^{\Omega T} \longmapsto \varphi_{\underline{a}}^{-1} * F^{\Omega T}.$$

<u>Proposition</u>: Let $R = \Omega T(pt)[a_1, a_2, \cdots]$ and let

$$r_{\underline{a}} : \Omega T \longrightarrow (\Omega T)_R = \Omega T[a_1, a_2, \cdots]$$

be the ring homomorphism given by the power series $\varphi_{\underline{a}}$ where

$$\varphi_{\underline{a}}^{-1}(X) = \sum_{n \geq 0}^{F^{\Omega T}} a_n X^{p^n} \qquad a_0 = 1.$$

Write

$$r_{\underline{a}}(x) = \sum_{\alpha} r_{\alpha}(x)\, a^{\alpha}$$

where α runs over multi-indices. Then every stable operation $\theta : \Omega T \longrightarrow \Omega T$ is uniquely expressible as an infinite sum

$$\theta x = \sum u_{\alpha} r_{\alpha}(x)$$

where $u_{\alpha} \in \Omega T(pt)$.

<u>Proof</u>: We know that ΩT is a direct summand of $\Omega_{(p)}$ and that the latter is generated by complexes of Künneth type, namely the skeletons of MU. Hence ΩT is also generated by Kunneth complexes and (so by preceding work) we know that any (stable) operation $\Theta : \Omega T \to \Omega T$ is the composition

$$\begin{tikzcd}
x \arrow[r, mapsto] & \sum r_\alpha(x) \otimes a^\alpha \\
\Omega T(X) \arrow[r, "\Delta"] \arrow[dr, bend right, "\Theta"'] & \Omega T(X) \otimes_{\Omega T(pt)} A \arrow[d, "id \otimes \hat{\Theta}"] \\
& \Omega T(X)
\end{tikzcd}$$

where $\hat{\Theta} \in \mathrm{Hom}_{\Omega T(pt)}(A, \Omega T(pt))$. Now $A = \Omega T(pt)[a_1, a_2, \cdots]$ so $\hat{\Theta}$ is given by its values on a^α. If $\hat{\Theta}(a^\alpha) = u_\alpha \in \Omega T(pt)$, then clearly $\bar{\Theta}(x) = \sum u_\alpha \cdot r_\alpha(x)$.

Problem: After prolonged absence due to illness & take up the old stuff.

Operations in ΩT:

$$\Omega T = LT \otimes_L \Omega$$

universal for standard theories with typical law

$$c(F^\Omega) = \xi \in \Omega(pt)$$

$$\hat{\xi} \cdot c_1^\Omega(L) = \xi(c_1^\Omega(L))$$

$$\therefore \quad \hat{\xi} \cdot F^\Omega = \xi * F^\Omega$$

$$\therefore \quad i' F^{\Omega T} = \xi * F^\Omega = c(F^\Omega) * F^\Omega \text{ is typical}$$

Now ~~ ~~ an operation $\Omega \longrightarrow \Omega T$ same as

$$\varphi(X) = X + \cdots \in \Omega T(pt)[[X]]$$

$$\hat{\varphi}\, c_1^\Omega(L) = \varphi(c_1^{\Omega T}(L))$$

$$\hat{\varphi} F^\Omega = \varphi * F^{\Omega T}$$

and therefore $\hat{\varphi}$ induces an operation from $\Omega T \longrightarrow \Omega T \otimes_{Z_{\varphi}} R$ iff $\varphi * F^{\Omega T}$ is typical.

~~[crossed out]~~

$\varphi * F^{\Omega T}$ ~~typical~~ $\Longleftrightarrow$ $\varphi^{-1}(X)$ is a typical curve for $F^{\Omega T}$

γ typ. for F

$\Longrightarrow$ $\varphi\circ\gamma$ typical for $\varphi * F$.

---

so write

$$\varphi_a^{-1}(X) = \sum^{F^{\Omega T}} a_n X^{p^n} \qquad a_0 = 1$$

let $\hat{\varphi}_a = r_{\underline{a}} : \Omega T \longrightarrow \Omega T[a_1, a_2, \cdots]$

Then

$$r_{\underline{a}} F^{\Omega T} = \varphi_* F^{\Omega T} \qquad \text{typical ?}$$

$$\overset{\text{over } \mathbb{Q}}{=} \underbrace{\varphi \circ l^{\Omega T^{-1}}}\circ F_+ \circ l^{\Omega T} \circ \varphi^{-1}$$

new log $= l^{\Omega T} \circ \varphi^{-1} = l \circ \sum_{n\geq 0}^{F^{\Omega T}} a_n X^{p^n}$

$$= \sum_{n\geq 0} l(a_n X^{p^n}) = \sum_{n\geq 0} \sum_{m\geq 0} P_{p^m-1} \frac{(a_n X^{p^n})^{p^m}}{p^m} \quad : \text{typical}$$

$$\therefore \quad \boxed{r_{\underline{a}}(P_{p^\nu - 1}) = \sum_{m+n=\nu}' p^n P_{p^m - 1}\, a_n^{p^m}}$$

Next want to calculate the power series assoc to

$$r_{\underline{a}} \circ r_{\underline{b}}$$

$$r_{\underline{a}}(r_{\underline{b}}\, c_1^{\Omega T}(L)) = r_{\underline{a}}\left(\varphi_{\underline{b}}^{\#}(c_1^{\Omega T}(L))\right)$$

$$= r_{\underline{a}}(\varphi_{\underline{b}}^{\#})\left(\varphi_{\underline{a}}^{\#}(c_1^{\Omega T}(L))\right).$$

$$\cancel{= (\varphi_{\underline{a}} \circ r_{\underline{a}}\varphi_{\underline{b}})(c_1^{\Omega T}(L))}$$

$$= \left[\underline{r_{\underline{a}}(\varphi_{\underline{b}})} \circ \varphi_{\underline{a}}\right](c_1^{\Omega T}(L)).$$

$$\ell^{r_{\underline{a}} F^{\Omega T}} = \ell^{F^{\Omega T}} \circ \varphi_{\underline{a}}^{-1}$$

$$\varphi_{\underline{a}}^{-1} r_{\underline{a}}(\varphi_{\underline{b}}^{-1}) = \varphi_{\underline{a}}^{-1} r_{\underline{a}} \sum^{F^{\Omega T}} (b_n X^{p^n})$$

$$= \varphi_{\underline{a}}^{-1} \sum^{r_{\underline{a}} F^{\Omega T}} (b_n X^{p^n})$$

$$= \ell^{\Omega T^{-1}} \sum \ell^{\Omega T}\, \varphi_{\underline{a}}^{-1}(b_n X^{p^n})$$

~~Λ~~ $\therefore$ $\quad \Lambda_{\underline{a}}(\varphi_{\underline{b}}) \circ \varphi_{\underline{a}} = \varphi_{\underline{c}}$

where

$$\varphi_{\underline{c}}^{-1} = \sum^{F^{\Omega T}} \varphi_a^{-1}(b_n X^{p^n}) = \sum^{F^{\Omega T}} \sum^{F^{\Omega T}} a_q b_n^{p^q} X^{p^{n+q}}.$$

$$\sum^{F^{\Omega T}} c_n X^{p^n}$$

or

$\Lambda_{\underline{a}} \circ \Lambda_{\underline{b}} = \Lambda_{\underline{c}}$ where

$$\therefore \quad \sum P_{p^m-1} c_n^{p^m} \frac{X^{p^{m+n}}}{p^m}$$

$$= \sum \frac{P_{p^m-1}}{p^m} \left(a_k b_n^{p^k} X^{p^{n+k}}\right)^{p^m}$$

$$\boxed{\sum_{h=0}^{N} P_{p^h-1} p^{N-h} c_{N-h}^{p^h} = \sum_{k+m+n=N} P_{p^m-1} p^{k+n} a_k^{p^m} b_n^{p^{k+m}}}$$

April 17, 1969:

Notes on category schemes from late fall of 68.

Unsolved problems:

1. Milnor–Moore thm in char p where p is odd
2. Fiber functors + the reduction of a groupoid to a group.
3. Formal groupoids of dimension 1.

~~The Milnor-Moore thm. in char 2:~~
~~Suppose given a formal category scheme in characteristic 2 such that the augmentation ideal is killed by Frobenius.~~

The situation in characteristic 2:

Given a category scheme in characteristic 2

$$\begin{tikzcd} R \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & A \arrow[l, bend right, "\varepsilon"'] \arrow[r, "\Delta"] & A \otimes_R A \end{tikzcd}$$

consider

$$\delta : A \longrightarrow A \otimes_R A$$

$$\delta(x) = \delta_0 x - \delta_1 x + \delta_2 x$$

$$= 1 \otimes x - \Delta x + x \otimes 1$$

remember
$s = \delta_1$
$t = \delta_0$

Then

$$\delta(xy) = (\delta x)(\Delta y) + (1 \otimes x + x \otimes 1)\,\delta y + x \otimes y + y \otimes x$$

so if we consider the map

$$I \xrightarrow{\ \delta\ } I \otimes_R I \longrightarrow \Omega \otimes \Omega \longrightarrow S_2\,\Omega$$

we see that I^2 goes to 0 and hence δ induces a map

$$\begin{tikzcd} \psi: \Omega \arrow[r] \arrow[dr, bend right, "d"'] & S^2\Omega \arrow[d] \\ & \Lambda^2\Omega \end{tikzcd}$$

ψ is something like the Cartier operator on forms. In effect I believe that in the case where $A = R\otimes_k R$ R smooth $/k$ ψ is given by

$$\boxed{\psi(\omega)(X) = \omega(X^2) - X\omega(X)}$$

If $\psi(\omega)$ is defined this way then one checks easily that

$$\begin{cases} \psi_\omega(X+Y) - \psi_\omega(X) - \psi_\omega(Y) = d\omega(X,Y) \\ \psi_\omega(fX) = f^2\psi_\omega(X) \\ \psi_{f\omega}(X) - f\psi_\omega(X) = Xf\cdot\omega(X). \end{cases}$$

Thus

$$\psi(f\omega) = f\psi(\omega) + df\cdot\omega$$

and so we can extend ψ to a derivation of $S^{\cdot}\Omega$ which we shall denote d. Thus there is a de Rham complex

$$R \xrightarrow{d} \Omega \xrightarrow{d} S^2\Omega \xrightarrow{d} S^3\Omega \longrightarrow \cdots\cdots$$

which I conjecture can be used to prove the same theorems in char. 0 but for formal cat. scheme with I killed by Frobenius.

(Conjectured corollary $\Rightarrow$ Steenrod operations are trivial $Sq^i u = 0$, $0 < i \le n = \dim u$)

In characteristic p, p odd, I lack the analogue of ψ. The Cartier operator

$$\psi_\omega(X) = \omega(X^p) - X^{p-1}\omega(X)$$

thus $\psi_\omega(X+\varepsilon Y) - \psi_\omega(X)$ not R-linear in Y, $\varepsilon^2 = 0$.

is not an R-polynomial function of X. ~~is a differential form again only if $d\omega = 0$.~~ Thus I calculated the deviation from linearity in char 3

$$\omega((X+Y)^3) - (X+Y)^2\omega(X+Y) = [\omega(X^3) - X^2\omega(X)] + [\omega(Y^3) - Y^2\omega(Y)]$$

$$+ \left(-Y d\omega(Y \wedge X) - d\omega(Y \wedge [Y,X])\right)$$

$$+ \left(-X d\omega(X \wedge Y) - d\omega(X \wedge [X,Y])\right)$$

which is not an R-polynomial function of X. However

$$\omega((fX)^3) - (fX)^2\omega(X) = f^3(\omega(X^3) - X^2\omega(X)).$$

We used the following

$$\star \quad \begin{cases} (X+Y)^3 = X^3 + [X[XY]] + [Y[YX]] + Y^3 \\ (fX)^3 = f^3X^3 + \left(f(Xf)^2 + f^2(X^2f)\right)X \end{cases}$$

My feeling was that if I assumed Ω proj. finite type over R, then I could effectively describe in terms of R and $L = \mathrm{Hom}(\Omega, R)$ Lie ring (endowed with pth power $X^{(p)}$) acting on R. with the identities $\star$.