March 12, 1969.

Decomposition of the Lazard scheme over $\mathbb{F}_p$

Let $L_p \simeq \Omega(pt) \otimes \mathbb{F}_p$ be the Lazard ring over $\mathbb{F}_p$, let F_u denote the universal group law. Let $X = \operatorname{Spec} L$ and let

$$X = Z_1 \supset Z_2 \supset Z_3 \supset \cdots\cdots \supset Z_\infty$$

be the closed subschemes given by

$$Z_i(T) = \{\text{formal group laws over } T \text{ of height } \geq i\}$$

Let

$$X_i = Z_i - Z_{i+1}$$

so that

$$X_i(T) = \{\text{formal group laws over } T \text{ of height} = i\}$$

for any $\mathbb{F}_p$-scheme T.

Fix an integer $h \geq 1$ and let F_0 be a group law of height h over $\mathbb{F}_p$. It is clear that the group scheme N of power series $a_1X + \cdots$ a_1 a unit acts on X_h and hence there is a map

$$N \longrightarrow X_h$$
$$\varphi \longmapsto \varphi * F_0$$

<u>Theorem:</u> The stabilizer H_{F_0} of the law F_0 is an ~~[illegible]~~ etale profinite group scheme over $\mathbb{F}_p$ and the above map induces an isomorphism of schemes

$$N/H_{F_0} \xrightarrow{\sim} X_h$$

Thus X_h is a "homogeneous space" scheme under N.

<u>Proof</u>: Given an ~~ring~~ $\mathbb{F}_p$-algebra R and a formal group F over R consider the algebra $R' = R[a_1, a_1^{-1}, a_2, \ldots\ldots]$ obtained by adjoining elements a_i, $i \geq 1$ a_1 invertible subject to the relations

$$(*) \qquad \varphi * F_0 = F$$

where $\varphi(x) = \sum a_i X^i$.

Following Lazard we are going to analyze the equations (*) and show that R' is obtained from R by successively adjoining roots of etale equations. Thus $R \to R'$ will be free and ind-etale. Applying this ~~when~~ R = coordinate ring of X_h we obtain a map $X_h' \longrightarrow X_h$ which is flat integral and pro-etale. $X_h'(T) = \{(F, \varphi)$ where F is a formal group law of height h and φ is an isomorphism of F with $F_0\} = \{\varphi \text{ over } T\} = N(T)$. Thus $X_h' \simeq N$ over X_h, so $N \longrightarrow X_h$ is covering for the ffqc topology and so $N/H_{F_0} \simeq X_h$. Also as $N \to X_h$ is pro-etale, H_{F_0} will be a pro-etale group scheme over $\mathbb{F}_p$.

A. If q is not a power of p, then a_q is a polynomial in $a_1, \ldots, a_{q-1}$ in R:

In effect let $\varphi_{q-1}(X) = \sum_{i=1}^{q-1} a_i X^i$ so that

$$\varphi_{q-1} * F_0 \equiv F + c\, B_q \qquad \text{mod degree } q+1$$

where $c \in R[a_1, \ldots, a_{q-1}]$. One then has that $a_q = -c$.

B. If $q = 1$, then a_1 is a root of the equation $\boxed{a_1^{p^h - 1} \lambda_0 = \lambda}$ where

$$[p]_{F_0}(X) = \lambda_0 X^{p^h} + \cdots$$
$$[p]_F(X) = \lambda X^{p^h} + \cdots$$

($\lambda_0, \lambda \in R^*$ since the law has height h).

C. If $q = p^k$ $k \geq 1$, then a_q is a root of an equation

$$\boxed{\lambda (a_q)^{p^h} - \lambda^{p^k} a_q = \mu_k}$$

where $\mu_k \in R[a_1, \ldots, a_{q-1}]$.

In effect one has $\varphi_{q-1} * F_0 \equiv F \mod \deg q+1$ * hence

$$\varphi_{q-1} * [p]_{F_0} \equiv [p]_F + \mu_k X^{p^{k+h}} \mod \deg p^h(p^k+1)$$

and also that $\varphi_q(X) = \varphi_{q-1}(X) + a_q X^q$ so

$$\varphi_q * [p]_{F_0} \equiv \varphi_{q-1} * [p]_{F_0} + (\lambda^q a_q - \lambda a_q^{p^h}) X^{q p^h} \mod \deg p^h(q+1).$$

so a_q satisfies the boxed equation.

*(since

$$\varphi_{q-1} * [p]_{F_0} \equiv [p]_F \mod \deg q+1$$

one has that $\varphi_{q-1} * F_0 \equiv F \mod q+1$, lemme 6).

Therefore one has

$$R \underset{\substack{\uparrow \\ \text{etale} \\ \text{of group} \\ \mu_{p^h-1}}}{\longrightarrow} R[a_1] \xrightarrow{\sim} R[a_1, \dots, a_{p-1}] \underset{\substack{\uparrow \\ \text{etale} \\ \text{of degree} \\ p^h}}{\longrightarrow} R[a_1, \dots, a_p] \simeq R[a_1, \dots, a_{p^2-1}] \underset{\substack{\uparrow \\ \text{etale} \\ \text{degree} \\ p^h}}{\longrightarrow} R[a_1, \dots, a_{p^2}]$$

<u>Questions:</u> Over $\mathbb{F}_p$ suppose $[p]_F(X) = Q_p X^p + \cdots + Q_{p^2} X^{p^2} + \cdots$, are the ~~[illegible]~~ coefficients of degree not a power of p zero when the law is typical? Can the above proof of Lazard be simplified ~~in case~~ using ~~[illegible]~~ a reduction to a typical law?

Problem sheet, March 12, 1969

1. Let F be a group law over $\mathbb{F}_p$ with $[p]_F(X) = X^{p^h}$. Let A be the scheme of autos of F, and let A_1 be the subgroup scheme of autos $\equiv$ id mod deg 2. Calculate the cohomology with coefficients $\mathbb{F}_p$ of A and A_1. Over $\mathbb{F}_{p^h}$ A is the group scheme associated to the profinite group E^* where E is the maximal order in the central division algebra D over $\mathbb{Q}_p$ with invariant $\frac{1}{h}$. E^* is a semi-direct product $(\mathbb{F}_{p^h})^* \times E_1$ where E_1 is ~~endos.~~ group of autos $\equiv$ id mod deg 2. (In fact Lubin claimed that one could assume F had only terms of degree divisible by p^h so that $(\mathbb{F}_{p^h})^*$ acts as endos. in the obvious way e.g. αX. Then any endo is uniquely expressible as an infinite sum $\sum_{n\geq 0} \alpha_n \Phi^n$, where $\alpha_n \in (\mathbb{F}_p)^*$, Φ is the Frobenius (endo since F defined over $\mathbb{F}_p$) and where the sum is taking in the sense of endos. E_1 consists of those autos with $\alpha_0 = 1$). E_1 is a p-adic Lie group of dimension h^2 and as it is torsion free it's cohomology should have Poincaré duality of dimension h^2. It is unlikely that the cohomology algebra be an exterior algebra except if $h < p-1$. In effect the logarithm $\log : E^* \longrightarrow D$ doesn't have a matching exponential except for valuation $\geq \frac{1}{p-1}$ and one has the element ψ in E_1 with valution $\frac{1}{h}$ since ψ^h = "the series X^{p^h}" = $[p]_F$.

To what extent does the cohomology of the scheme of autos. of a group law of height h over $\mathbb{F}_p$ approach the cohomology of the Steenrod algebra as $h \to \infty$

2. Can one find a cohomology theory given by a MU-algebra spectrum (convergent) having a group law of a given height such that the endos. of the formal group law give rise to **Adams operations** in the cohomology theory.

3. Operations in BP theory corresponding to the operators $1 + a_1 V_p + a_2 V_p^2 + \cdots\cdots$ on typical coordinates.

4. Lifting laws from char p to char 0. Using Lazard we have the canonical group law on X_1 becomes canonically isom. to $\hat{G}_m$ over N

$$\begin{tikzcd} N \arrow[d] & M \arrow[l, dashed] \arrow[d, dashed] \\ X_1 & Y_1 \arrow[l] \end{tikzcd}$$

integral ind etale of group $\mathbb{Z}_p^*$

should exist by thm. on equivalence of etale topology of Y_1 and X_1

$X_1 = \operatorname{Spec}(L[P_{p-1}^{-1}] \otimes \mathbb{F}_p)$ represents laws of ht 1 in char. p

$Y_1 = \operatorname{Spec} L[P_{p-1}^{-1}] \otimes_{\mathbb{Z}} \mathbb{Z}_p$ laws of ht 1 over $\mathbb{Z}_p$

and using Lubin-Tate it should be true that the canon. law on Y_1 becomes canonically isomorphic to $\hat{G}_m$ over M

5.) Let ~~A~~ E be the ~~group of units~~ endo. ring of the group law of height h over $\mathbb{F}_p$ ~~[illegible]~~, ~~let~~ $D =$ quotient field of E, $\mathfrak{p}$ = maximal ideal of E, so that $E/\mathfrak{p} \simeq \mathbb{F}_{p^h}$. We want to calculate the cohomology of $A_1 = 1 + \mathfrak{p}$ with coefficients in $\mathbb{Z}/p\mathbb{Z}$. As a first approximation (?) we calculate the cohomology

$$\varinjlim H^*(U, \mathbb{Q}_p)$$

where U runs over the open subgroups of A_1. By a theorem of Lazard this is the same as the Lie algebra cohomology $H^*(\mathfrak{g}, \mathbb{Q}_p)$ where $\mathfrak{g}$ is the Lie algebra of A_1. Here one has an exponential-logarithm correspondence between $A_a = 1 + E \cdot \Phi^a$ and $E \cdot \Phi^a$ where ~~a~~ $a\,v(\Phi) > \frac{1}{p-1}$ (v = normalized valuation, Φ = Frobenius so that $\Phi^h = p$ and $v(\Phi) = \frac{1}{h}$); hence ~~[illegible]~~ a must be an integer $> \frac{h}{p-1}$. In any case the Lie algebra of A_1 as a p-adic analytic group is D under bracket. After base extension (from $\mathbb{Q}_p$ to K a splitting field) D becomes isomorphic to $\operatorname{Hom}_K(V, V)$ where $\dim_K V = h$ and one knows that

$$H^*(\mathfrak{gl}(V), K) = \text{an exterior algebra with generators of degrees } 1, 3, \dots, 2h-1.$$

Hence the same formula will hold before base extension as D being reductive will have its cohomology an exterior algebra on the primitive elements (Koszul). Thus

$$H^*(A_a, \mathbb{Z}_p) \simeq \text{exterior algebra on generators of degrees } 1, 3, \dots, 2h-1 \text{ mod } p\text{-torsion}.$$

~~[illegible]~~ for a large. Actually this formula should hold for

$a = 1$ as the action of A_n/A_{n+1} on $H^*(A_{n+1}, \mathbb{Z}_p)$ should be trivial as it coincides with an inner auto. of the Lie algebra.

Note that in Fröhlich, prop. 3, page 80, E/pE is isomorphic to the ring of endos. $\sum_{i=0}^{h-1} a_i X^{p^i}$ of the additive group over $\mathbb{F}_{p^h}$, hence $A/A_h = E^*/1+pE$ is isomorphic to the group of autos. of the additive group truncated at order h over $\mathbb{F}_{p^h}$. This suggests as $h \to \infty$ that A/A_h tends to the group of points (over $\overline{\mathbb{F}}_p$) of the group scheme given by the dual of the Steenrod algebra mod Bockstein. Hopefully this means that the cohomology of A converges to that of the Steenrod algebra as $h \to \infty$.

<u>Lemma</u>: Let $X = Z_0 \supset Z_1 \supset Z_2 \supset \cdots$ be a topological space filtered by closed subsets and let F be a sheaf on X. Then there is a $\underbrace{\text{spectral sequence}}_{}$ (not necessarily convergent)

$$E_1^{pq} = H^{p+q}_{Z^p/Z^{p+1}}(X; F) \Longrightarrow H^{p+q}(X, F)$$

<u>Proof</u>: Recall that if Y, Z are closed subsets of X with $Y \subset Z$, then we have a long exact sequence,

$$\xrightarrow{\delta} H^q_Y(X, F) \longrightarrow H^q_Z(X, F) \longrightarrow H^q_{Z/Y}(X, F) \xrightarrow{\delta}$$

It is the long exact sequence of Ext resulting from applying $\mathrm{Ext}^*(?; X; F)$ to

$$0 \longrightarrow \mathbb{Z}_{Z,Y} \longrightarrow \mathbb{Z}_Z \longrightarrow \mathbb{Z}_Y \longrightarrow 0.$$

So one gets an exact couple

$$\begin{tikzcd}
H^{p+q}_{Z^{p+1}} \arrow[d] & & & \\
H^{p+q}_{Z^p} \arrow[r] & H^{p+q}_{Z^p/Z^{p+1}} \arrow[r, "\delta"] & H^{p+q+1}_{Z^{p+1}} \arrow[r, dashed] \arrow[d] & H^{p+q+1}_{Z^{p+1}/Z^{p+2}} \\
& & {} &
\end{tikzcd}$$

Thus if $E_1^{pq} = H^{p+q}_{Z^p/Z^{p+1}}$, we have

$$d_1 : E_1^{pq} \longrightarrow E_1^{p+1,q}$$

which is correct.

<u>QED</u>

<u>Lemma</u>: If Y, Z are closed subsets of X, with $Z \subset Y$, then

$$H^*_{Z/Y}(X, F) \simeq H^*_{Z-Y}(X-Y, F)$$

<u>Proof</u>: If I is injective, we have

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & \Gamma_{Z-Y}(X-Y, I) \arrow[d] & \\
0 \arrow[r] & \Gamma_Y(X, I) \arrow[r] \arrow[d] & \Gamma(X, I) \arrow[r] \arrow[d, "\wr"] & \Gamma(X-Y, I) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma_Z(X, I) \arrow[r] & \Gamma(X, I) \arrow[r] & \Gamma(X-Z, I) \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}$$

yielding by serpent an exact sequence

$$0 \longrightarrow \Gamma_Y(X, I) \longrightarrow \Gamma_Z(X, I) \longrightarrow \Gamma_{Z-Y}(X-Y, I) \longrightarrow 0$$

and hence an isomorphism

$$\Gamma_{Z/Y}(X, I^\cdot) \simeq \Gamma_{Z-Y}(X-Y, I^\cdot)$$

for any injective <u>complex</u>. QED.

Suppose now that we want to calculate the equivariant cohomology of the Lazard scheme $X = \operatorname{Spec}(\Omega(pt) \otimes \mathbb{F}_p)$ with values in $\mathcal{O}_X$ where the group is N_1 using the decomposition of X into orbits under N. The problem is to relate the cohomology of X, U, Z where Z is defined by $f = 0$, f a non-zero divisor in $A(X)$ which we can even assume invariant under f. All I can say is that there is a long exact sequence

$$\longrightarrow H^*(A(N_1), A(X)) \longrightarrow H^*(A(N_1), A(U)) \longrightarrow \varinjlim_n H^*(A(N_1), A(X)/f^n A(X))$$

$$(= H^*(X)) \qquad\qquad (= H^*(U)) \qquad\qquad (= H^{*-1}_Z(X))$$

and a Bockstein spectral sequence for calculating

$$H^*(A(N_1), A(X)/f^n A(X))$$

in terms of

$$\underset{\substack{\| \\ A(Z)}}{H^*(A(N_1), A(X)/f A(X))} \qquad (= H^*(Z)).$$

Not much help. Possibly one can get to a position where ~~the~~ other cohomology theories than $H^*(X, \mathcal{O}_X)$ are used.

March 17, 1969

Review of local cohomology

X manifold, A closed subset of X.

(i) $H^*_A(X) = H^*(X, X-A)$ elements are represented by ~~[illegible]~~ $\begin{cases} Z \xrightarrow{f} X,\ g: W \to X-A & \text{prop-or} \\ \alpha: Z|X-A \simeq \partial W. \end{cases}$

(ii) $H^*(A) = \varinjlim_{U \supset A} H^*(U)$ elts. rep. by $\begin{cases} W \longrightarrow X & \text{pr-or.} \\ \partial W | A = \phi. \end{cases}$

(iii) $H^*(X, A) = H^*_{pr/X}(X-A)$

exact sequences

$$H^*_A(X) \longrightarrow H^*(X) \longrightarrow H^*(X-A) \longrightarrow \cdots$$

$$H^*_{pr/X}(X-A) \longrightarrow H^*(X) \longrightarrow H^*(A) \longrightarrow \cdots$$

excision

$$H^*_A(X) \simeq H^*_A(U) \quad \text{where } U \text{ open} \supset A.$$

Gysin isom.

$$H^*(A) \simeq H^{*+d}_A(X) \quad \text{if } A \hookrightarrow X \text{ oriented submanifold of codim } d.$$

(iv) $H_*(X, X-A)$ elts rep. by $\begin{cases} W \longrightarrow X & W \text{ comp-or} \\ \partial W | A = \phi \end{cases}$

(v) $H_*(X, A)$ elts. rep. by $\begin{cases} Z \xrightarrow{f} X-A & Z \text{ oriented};\ \partial Z = \phi \\ f^{-1}F \text{ compact for all } F \text{ closed in } X \\ F \cap A = \phi. \end{cases}$

(vi) $H_*(A)$ $\begin{cases} Z \longrightarrow X & Z \text{ comp-or} \\ Z|X-A = \partial W & f: W \to X-A \ni f^{-1}F \text{ comp} \quad W \text{ orientable} \\ \text{all } F \text{ closed in } X \ni F \cap A = \phi. \end{cases}$

exact sequences:

$$H_*(A) \longrightarrow H_*(X) \longrightarrow H_*(X, A) \longrightarrow \cdots$$

$$H_*(X-A) \longrightarrow H_*(X) \longrightarrow H_*(X, X-A) \longrightarrow \cdots$$

excision:

$$H_*(X, X-A) = H_*(U, U-A) \qquad \text{if } A \subset U \text{ open}$$

$$H_*(X, A) = H_*(X-F, A-F) \qquad \text{if } F \text{ closed and } F \subset \operatorname{Int} A.$$

Gysin isomorphism:

$$H_{*+d}(X, X-A) \simeq H_*(A) \qquad A \text{ oriented submanifold}$$

continuity:

$$\begin{cases} H_*(A) = \varinjlim_{K \subset A} H_*(K) & K \text{ compact} \\ H_*(X, A) = \varinjlim_{K \subset A} H^*(X, K) & \text{"} \\ H_*(X, X-A) = \varinjlim_{K \subset X-A} H_*(X, K) & \text{"} \end{cases}$$

(It seems reasonable therefore if L is locally closed in X to set

$$H_*(X, L) = \varinjlim_{K \subset L} H_*(X, K)$$

$$H_*(L) = \varinjlim_{K \subset L} H_*(K)$$

whence one has a single exact sequence

$$H_*(L) \longrightarrow H_*(X) \longrightarrow H_*(X, L) \longrightarrow \cdots$$

and a single excision

$$H_*(X, L) = H_*(X-F, L-F)$$

if F closed $\subset \operatorname{Int} L$.)

<u>Duality results:</u>

If X is oriented, then

$$\ast \quad \begin{cases} H^*_K(X) \cong H_{n-*}(K) \\ H^*(K) \cong H_{n-*}(X, X-K) \end{cases} \qquad K \text{ compact.}$$

If X is ~~compact~~ oriented and ~~A closed~~, then we also have

$$\ast \quad \begin{cases} H^*(X-A) \cong H_{n-*}(X, A) \\ H^*(X, A) \cong H_{n-*}(X-A) \end{cases}$$

~~More generally the last two hold~~ [if $X-A$ is relatively compact in X.] The basic duality results are

$$\begin{cases} H^*_K(X) \cong H_{n-*}(K) & K \text{ compact} \\ H^*(X-A) \cong H_{n-*}(X, A) & X-A \text{ rel. compact in } X \end{cases}$$

for they imply the others by passage to the limit

Must work supports into the axioms

basic object is $H^*_A(X)$ A closed.

because then can define

$$H^*_\Phi(X) = \varinjlim_{A\in\Phi} H^*_A(X).$$

basic properties:

excision $H^*_A(X) = H^*_A(U)$ U open $\supset A$.

exact sequence

$$\cdots \xrightarrow{\delta} H^*_A(X) \longrightarrow H^*(X) \longrightarrow H^*(X-A) \xrightarrow{\delta} \cdots$$

functorality: if $f: X^A \longrightarrow Y^B$

then get $f^*: H^*_B(Y) \longrightarrow H^*_{f^{-1}B}(X) \left(\longrightarrow H^*_A(X)\right.$ need $\left.A \supset f^{-1}B\right)$

and $f_*: \underline{H^*_A(X) \longrightarrow H^*_{fA}(Y)} \left(\longrightarrow H^*_B(Y)\right.$ need $\left.fA\subset B\right)$

closed since f proper + A closed

Note that the map

$$H^*_A(X) \longrightarrow H^*_{A'}(X) \quad \text{if } A\subset A'$$

is a special case of an f_*.

variances

I $\quad f : X \longrightarrow Y^{B}$

$$H^{\bullet}_{B}(Y) \longrightarrow H^{\bullet}_{f^{-1}B}(X)$$

II $\quad f : X^{A} \longrightarrow Y^{A'}$ proper-or $\quad fA \subset A'$

$$f_* : H^{\bullet}_{A}(X) \longrightarrow H^{\bullet}_{A'}(Y).$$

$$f^* : H^{\bullet}_{B}(Y) \longrightarrow H^{\bullet}_{f^{-1}B}(X)$$

$$f_* : H^{\bullet}_{A}(X) \longrightarrow H^{\bullet}_{fA}(Y)$$

(X, A) objects

two kinds of morphisms, ordinary map $\quad f^{-1}B \subset A$

$f : (X, A) \longrightarrow (Y, B)$ $\quad$ ~~$f^{-1}B \subset A$~~ $\quad$ ~~$A \subset f^{-1}B$~~

want f^* : ~~$H(X, A) \longrightarrow H(Y, B)$~~ $H_B(Y) \longrightarrow H_A(X)$.

~~$H(X, A) \longrightarrow H(Y, B)$~~

proper oriented map

$f : (X, A) \longrightarrow (Y, B) \quad \ni \quad fA \subset B$

Then one wants

$$f_* : H_A(X) \longrightarrow H_B(Y)$$

(b) compatibility axiom

$$(X, A) \qquad\qquad (X, A')$$

suppose that $\quad A \subset A'$. Then you have two ~~[illegible]~~ maps

$$\Omega_A(X) \underset{id^*}{\overset{id_*}{\rightrightarrows}} \Omega_{A'}(X)$$

$\iota^{-1} A \subset A'$
$\iota A \subset A'$.

this follows from b)

$$\begin{array}{ccc} {}^{A}X & \xrightarrow{id} & X^{A} \\ \big\downarrow{\scriptstyle id} & & \big\downarrow{\scriptstyle id} \\ X_{A'} & \xrightarrow{id} & X^{A} \end{array}$$

(c) excision axiom

Point is to prove Thm isom

$$\mathrm{Hom}(K, Q^+) \longleftrightarrow \mathrm{Hom}^{\otimes}\{\Omega, Q\}$$

ι:

given a theory you can twist it by a char. class

$\beta : \Omega \longrightarrow Q$ when does it come from an α.

Begin by defining $\quad \bar{\beta}(E) = \iota_*^{-1} \beta\, \iota_* 1$

~~where ι is an immersion with normal bundle E~~

to know $\boxed{A}$ independent of choice of ι