Problem

K theory – important papers

- Borel – Serre
- Atiyah's power operations in K theory
- Cartier – Bergman on W, ~~Similarity~~ Structure of $H^*(BU)$ as Hopf algebra.
- Bass – Milnor – Serre | algebraic K_1
- traces + characteristic polynomials on the derived cat.
- Motives + similarity of Groth. Galois theory + usual polynomial equations in K theory.
- Atiyah character + cohomology spectral sequence

Jacobs, Some new results of Ergodic theory, Jahres. der Deutschen Math. - 67 (1965) 144-182

Abramov

Gurevic

Kushnirenko DAN 161 (1965) 37-38

Analytical problem 1: Assume f expanding and show that similar estimates hold for ?

$f: M \longrightarrow M$ $\quad$ f does not act on $\Gamma(T_M)$

$f^*: C_0(T_M^*) \longleftarrow C_0(T_M^*)$ $\quad$ but only on $\Gamma(T_M^*)$

where f^* has what kind of analytical properties?

ω form.

$\|\omega\|_p = \sup \frac{|\omega(X)|}{\|X\|}$. $\quad$ X runs over $T_p(M)$.

$$\begin{array}{c} M \\ \downarrow f \\ M \end{array}$$

$$\|f^*\omega\|_p = \sup \frac{|f^*\omega(X)|}{\|X\|} \qquad X \text{ runs over } T_p$$

$$= \sup \frac{|\omega(f_*X)|}{\|X\|} \qquad f_*: T_pX \longrightarrow T_{fp}X$$

isom.

$$= \sup \frac{|\omega(f_*X)|}{\|f_*X\|} \frac{\|f_*X\|}{\|X\|}$$

assume Mather

$\|f_*X\| \geq \lambda\|X\|$ all X

$$\geq \|\omega\|_{fp} \cdot \lambda$$

Therefore $\quad \|f^*\omega\| = \sup_p \|f^*\omega\|_p \geq \sup_p \|\omega\|_{fp} \cdot \lambda \geq \lambda \cdot \|\omega\|.$

Thus we find that f^* expands on $C_0(T_M^*)$. $\quad$ How about L^2 estimates

$$\|f^*\omega\|_{L^8} = \int \|f^*\omega\|_p^8 V \geq \lambda^8 \int \|\omega\|_{fp}^8 V = \lambda^8 \sum_{U_i} \int_{U_i} \|\omega\|_{fp}^8 V$$

$\cancel{\int f(g(x))\,dx}$

$\cancel{\int f(y)\,d(g^{-1}(y))}$

~~To get the variance correct.~~

$$\begin{array}{ccc} f^*\omega & & \omega \\ Y & \xrightarrow{f} & X \end{array}$$

$$\int_X \omega = \langle [X], \omega \rangle$$

$$\langle [Y], f^*\omega \rangle = \langle f_*[Y], \omega \rangle.$$

$$\cancel{\int g^* f^* \omega = \int (f \circ g)^* \omega}$$

$$\int_X \omega = \int_Y f^*\omega$$

if degree is 1.

because $\Vert$

$$\sum_i{}' \int_{\mathbb{R}^n} \varphi_i^*(\rho_i \omega)$$

$\mathbb{R}^n \xrightarrow{\varphi_i} U_i$ $\qquad \sum \rho_i = 1.$

$$\sum_i{}' \int_{U_i} \|\omega\|^g_{fp} (\rho_i V) \qquad U_i \longrightarrow f U_i$$

$$\underbrace{f^* f_* \rho_i V}$$

$$\int \|\omega\|^g_{fp}\, V = \int \|\omega\|^g_{p} \cdot \underline{f_* V}$$

defined because f is a covering.

Then there is a function g non-zero such that

$$\frac{f_* V}{V} = g$$

M compact ~~smooth~~ manifold.

T tangent bundle.

f diffeomorphism of M.

assume f expanding, i.e. there is a Riemannian metric $\| \|$ on T such that $\forall v \in T$

$$\|df^n(v)\| \geq c\lambda^n \|v\| \qquad \text{where } c>0,\ \lambda>1 \text{ ind. of } n.$$

Question: Does there exist an absolute bound on the expansion, i.e. an estimate of the form

$$\|df^n(v)\| \leq C\mu^n \|v\| \qquad \mu > 1. \quad \text{all } n.$$

Yes this is clear because take $\mu = \|df\|$.

So now can look at the spectrum of ~~#~~ f or df acting on $C^0(TM)$. Question: Is this spectrum ~~indep~~ the same for $C^k(TM)$?

Th: If A is a bdd linear transformation on a Banach space V, then $\|A\| \geq \sup\{|\lambda| : \lambda \in \operatorname{spec} A\} = \cancel{\lim_{n\to\infty} \|A^n\|^{1/n}}$ $\limsup_{n\to\infty} \sqrt[n]{\|A^n\|}$

Proof: $\operatorname{spec} A = \{\lambda \mid (A-\lambda)^{-1} \nexists\}$.

if $|\lambda| > \|A\|$, then $\dfrac{1}{A-\lambda} = -\dfrac{1}{\lambda}\left(\dfrac{1}{1-\frac{A}{\lambda}}\right) = -\dfrac{1}{\lambda}\left(1 + \dfrac{A}{\lambda} + \dfrac{A^2}{\lambda^2} + \cdots\right)$

Converges so $\lambda \notin \operatorname{spec} A \Longrightarrow$ ~~spec A ⊆ {λ|~~ $\operatorname{spec} A \subseteq \{\lambda \mid |\lambda| \leq \|A\|\}$.

X compact topological space

E vector bundle over X

f acts on E and on X.

Say f expanding if for any norm $\| \ \|$ on E we have

$$\|f^n v\| \geq c\lambda^n \|v\| \quad \text{all } n \quad (c > 0,\ \lambda > 1)$$

independent of choice of $\| \ \|$.

① may modify $\| \ \|$ and λ so that $c = 1$.

~~let~~ ~~$1 < \mu < \lambda$ and set~~

$$\cancel{|v| = \inf \frac{\|f^n v\|}{\lambda^n}}$$

~~are two norms on~~ ~~Suppose~~ $\cancel{c\lambda^n}$

Choose $1 < \mu < \lambda$ and $m \ni$

$$\|f^n v\| \geq \mu^n \|v\| \quad \text{for } n \geq m.$$

$$\frac{\|v\| + \|fv\| + \cdots + \|f^n v\|}{n+1} = N(v)$$

then $N(fv) - N(v) = \dfrac{\|f^{n+1} v\| - \|v\|}{n+1}$

$$\frac{N(fv)}{N(v)} - 1 = \frac{\|f^{n+1} v\| - \|v\|}{n+1} \frac{1}{N(v)} \geq 0.$$

f acts on E over f on X.
Then f acts on $\Gamma(E^*)$.

$\omega \in \Gamma(E^*)$.

$$\langle v, f^*\omega \rangle = \langle f(v), \omega \rangle .$$

Assume $f(X) = X$. Then

(3) f expanding $\Longleftrightarrow$ f^* expands on $C^0(E^*)$.

Proof: f expanding $\Longrightarrow$ $\dfrac{\|fv\|}{\|v\|} \geq \mu > 1$. so

$$\|f^*\omega\|_p = \sup_{v \in E_p} \frac{\langle v, f^*\omega\rangle}{\|v\|} = \sup_{v \in E_p} \frac{\langle fv, \omega\rangle}{\|fv\|} \cdot \frac{\|fv\|}{\|v\|}$$

$$\geq \mu \cdot \|\omega\|_{fp} .$$

$$\Longrightarrow f^* \text{ expanding.}$$

Conversely suppose f^* expands on $C^0(E^*)$, i.e.

$$\|f^{*m}\omega\| \geq c\lambda^m \|\omega\| .$$

NO

$$\|f^m v\|_{f^m p} = \sup_{\omega \in C^0(E^*)} \frac{\langle f^m v, \omega\rangle_{f^m p}}{\|\omega\|} = \sup \frac{\langle v, (f^*)^m \omega\rangle_p}{\|\omega\|}$$

$$\|f^m v\|_{f^m p} = \langle v, (f^*)^m \omega \rangle \qquad \text{where } \|\omega\|_{f^m p} = \|\omega\| = 1$$

$$f^n(e'+e'') = (f')^n e' + (f')^{n-1}(ge'') + (f')^{n-2}(gf''e'') + (f')^{n-3}(g(f'')^2e'') + \cdots g(f'')^{n-1}e'' + (f'')^n e''.$$

assume contracting $\quad C = \|g\|.$

$$\|f^n(e'+e'')\| \le \|(f')^n e'\| + \|(f'')^n e''\| + C\left(\|f'\|^{n-1} + \|f'\|^{n-2}\|f''\| + \cdots\right)\|e''\|$$

$$\le \lambda^n\|e'\| + \lambda^n\|e''\| + Cn\lambda^{n-1}\|e''\|$$

$$\le \lambda^n\|e'\| + \underbrace{\left(\lambda^n + Cn\lambda^{n-1}\right)}\|e''\|.$$

But if $\lambda < \mu < 1$

then $\dfrac{\lambda^n + Cn\lambda^{n-1}}{\mu^n} = \left(\dfrac{\lambda}{\mu}\right)^n + \dfrac{Cn}{\mu}\left(\dfrac{\lambda}{\mu}\right)^{n-1} \to 0.$

$n \to 0$

This works for contracting and so for expanding in case that f^{-1} is exists.

$E = E' \oplus E''$

$$E_x \longrightarrow E_{fx} \longrightarrow E_{f^2x} \longrightarrow E_{f^3x} \longrightarrow \cdots\cdots$$

$\|\ \ \|\qquad \|\ \ \|\qquad \|\ \ \|\qquad \|\ \ \|$

Suppose that $\dfrac{\|f^n(e'+e'')\|}{\|e'+e''\|}$

Proposition: X compact, $f: X \to X$ map (surjective) ~~(surjective)~~
$0 \to E' \to E \to E'' \to 0$ bundles over X on which f acts.
Then if f expands on E' and E'' it expands on E.

Proof. Let $\tilde{X} =$ lim inverse of

$$\begin{tikzcd} & E \arrow[r, "f"] \arrow[d] & E \arrow[d] & \\ \arrow[r] & X \arrow[r, "f"] & X \arrow[r, "f"] & X \end{tikzcd}$$

And let $\tilde{E} = \varprojlim E$. Then $\tilde{E}/\tilde{X}$ is a vector bundle. Now f yields an isomorphism $\tilde{f}$ of $\tilde{X}$ by

$$\tilde{f}\{x_n\} = \{f x_n\} = \{x_{n+1}\}.$$

$\{x_n \mid f x_n = x_{n-1}\}$

given $\{y_n\} \mid f y_n = y_{n-1}$

want ~~to~~ $\tilde{f}\{x_n\} = \{y_n\}$?

let $\begin{cases} x_{n+1} = y_n \\ x_0 = f(x_1) = \cancel{x} \end{cases}$ ~~[illegible] $\tilde{f}$ onto.~~

$\tilde{f}$ injective $\quad$ & $\quad \tilde{f}\{x_n\} = \tilde{f}\{x_n'\}$.

$x_{n+1} = x'_{n+1} \qquad n \geq 0.$

As diagram

$$\begin{tikzcd} \tilde{E} \arrow[r, "p_n"] \arrow[d] & E \arrow[d] \\ \tilde{X} \arrow[r, "p_n"] & X \end{tikzcd}$$

commutes and p_n is surjective, it suffices for f to be expanding on E to show $\tilde{f}$ expands on $\tilde{E}$. But then we are reduced to contractions since $\tilde{f}$ is an iso.

~~let $x_n = y_{n+1}$ $n \geq 0$.~~

~~Then if $n \geq 1$ $f(x_n) = f(y_{n+1}) = (y_n) = x_{n-1}$ so $(x_n) \in \tilde{X}$~~

~~$f(x_n) = (y_n)$?~~

~~So f is onto.~~

I have shown now that if f is an expanding diffeo of M, then it's also expanding on ΛT^*, $J_g(1)$ $J_g(T^*)$ etc. in which case f^* on $A^*(X)$ is expanding.

<u>Question:</u> $f^* : A^\circ \hookrightarrow A^\circ$ homotopy equivalence. Under what conditions can we conclude that

$$\bigcap_n (f^*)^n A^\circ \subset A^\circ$$

is a homotopy equivalence

$\omega \in A^g$ $\quad d\omega = 0$.

$\cancel{\omega = \frac{1}{d} f^* f_* \omega}$

$$\boxed{\omega_1 = \frac{1}{d} f_* \omega.}$$

$$\omega_n = \left(\frac{1}{d} f_*\right)^n \omega$$

how much do we know about g?
It doesn't matter since if we take q large f^* will be ~~contract~~ expanding!

One can ask that $f_* V = V$. Invariant measure.
example $S^1 \xrightarrow{2} S^1$

$$2_* dz = dz.$$

$$\sum_i \underbrace{\gamma_i}\,(\varphi^* dz)$$

coset representatives for $[\pi / \varphi\pi]$.

In the group case what can we say about the tangent bundle of M. If there is no finite group part, then it is trivial because one may take the vector fields on M and left translate them, commutes with the right γ translation, but not with φ. So have finite group acting on M_0 with trivial tangent bundle. What can we do? ~~What is that if acts of~~ By general non-sense we get an element of $H^1(\sigma, Gl(C^\infty(M_0))^*)$. Note that $C^\infty(M_0)$ is a ring on which π acts acyclically.

Suppose a finite gp σ acts nicely on a

Suppose $M_0 = G/\pi_0$ nil-manifold with expansion f and π/π_0 finite gp acting freely thereon. What forms on M can be written in the form $(f^*)^n \omega_n$ all n. Do there exist any?

Case 1: $\pi = \pi_0$.

example $2 : S^1 \longrightarrow S^1$.

$$\omega = a(z)\,dz$$

$$2^* \omega = \boxed{2a(2z)\,dz}$$

Thus

$$a(z)\,dz \in \operatorname{Im}(2^*)^n \iff \cancel{a(z) = b(2z)}$$

$a(z/2^n)$ defined.

$$a(z) = a\left(z + \tfrac{1}{2^n}\right) \quad \text{all } n$$

$\iff$ a is constant.

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f function on G/π $\quad \pi = \pi_0$

$$f = \varphi^* g$$

$$f(x) = g(\varphi x). \qquad \cancel{x \in (\varphi \pi \varphi^{-1})}$$

Thus

$$\cancel{f(x \cdot \gamma) = g(\varphi(x) \cdot (\varphi^{-1} \mu \varphi)) = g(\varphi^{-1}) \cdot R_\mu \varphi}$$

① may modify $\| \ \|$ and λ so that $c = 1$.

Proof. Choose ~~...~~ n such that $c\lambda^n > \cancel{\ } 1$

~~Then let ... $N(\sigma) \geq \|\sigma\|$~~ Then $\|f^n \sigma\| > \cancel{\ } \|\sigma\|$. if $\|\sigma\| \neq 0$ Let

$N(\sigma) = \|\sigma\|^2 + \dots + \|f^{n-1}\sigma\|^2$ (a new norm). ~~and~~ Then

$$\frac{N(f\sigma)}{N(\sigma)} - 1 = \frac{N(f\sigma) - N(\sigma)}{N(\sigma)} = \frac{\|f^n\sigma\|^2 - \|\sigma\|^2}{N(\sigma)} > 0 \quad \text{unless } \|\sigma\| = 0.$$

As SE is compact conclude that this is bounded away from 0 so

$$\frac{N(f\sigma)}{N(\sigma)} - 1 \geq \varepsilon > 0$$

$$N(f\sigma) \geq (1+\varepsilon) N(\sigma).$$

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② if $0 \to E' \to E \to E'' \to 0$ is an exact sequence on which f acts, then

~~①~~ f exp. on $E \Leftrightarrow f$ exp. of E' and f exp on E''.

($\Longrightarrow$ clear).

Proof: Write $E = E' \oplus E''$ so

$$f(e') = f'e'$$

$$f(e'') = f''e'' + ge'' \quad \text{where}$$

$$\begin{tikzcd} E'' \arrow[r, "g"] \arrow[d] & E' \arrow[d] \\ X \arrow[r, "f"] & X \end{tikzcd}$$

assume

$$\|f'e'\| \geq \lambda \|e'\|$$
$$\|f''e''\| \geq \lambda \|e''\|$$

take quotient norm on E''

so then

$$\|f''e''\| \leq \|fe''\| \leq \lambda \|e''\|.$$

contracting case

$$\cancel{\|f^n(e'+e'')\|} = (f''e'')^n + \cancel{(f'')^n (ge)}$$

~~note that where the series~~ $\cancel{-\frac{1}{\lambda}\left(1+\frac{A}{\lambda}+\frac{A^2}{\lambda^2}+\cdots\right)}$

~~Converges~~ Suppose that $(A-\lambda)^{-1}$ exists for $|\lambda| > c$.
Let f be a linear functional on $\mathcal{L}(V,V)$. Then

$$f\left(\frac{1}{A-\lambda}\right) = -\frac{1}{\lambda}\left(1+\frac{f(A)}{\lambda}+\frac{f(A^2)}{\lambda^2}+\cdots\right)$$

holds for $|\lambda| \geq \|A\|$ and is an equality of analytic function so must have

$$\frac{|f(A^n)|}{\lambda^n} \leq M \qquad \text{all} \quad |\lambda| > c$$

$$\text{or} \qquad f\left(\frac{A^n}{\lambda^n}\right) \leq M \qquad |\lambda| > c.$$

thus by Banach-Steinhaus $\left\|\frac{A^n}{\lambda^n}\right\|$ bounded $\forall$

$$\frac{\|A^n\|}{|\lambda|^n} \leq C(\lambda) \qquad |\lambda| > c.$$

$$\overline{\lim} \; \frac{\|A^n\|^{1/n}}{|\lambda|} \leq 1$$

$$\therefore \limsup \sqrt[n]{\|A^n\|} \leq c.$$

$$|\lambda| > \limsup \sqrt[n]{\|A^n\|}$$

$$\Longrightarrow \quad |\lambda| > \sqrt[n]{\|A^n\|} \qquad \text{all } n > n_0$$

$$\Longrightarrow \quad 1 > \frac{\|A^n\|}{|\lambda|^n} \qquad \Longrightarrow \text{ conv.}$$

V n dimensional representation of G

G acts on $SV \sim S^{2n-1}$ and $c_n(V)$ is transgression class

$c_n(V) \in H^{2n}(G, \mathbb{Z})$

You have a cochain formula for $H^{2n}(G,\mathbb{Z})$ can you give a simple formula for $c_n(V)$ by making suitable choices?

An element of $H^{2n}(G,\mathbb{Z})$ is ~~an element~~ function

$f : G^{2n} \longrightarrow \mathbb{Z} \quad \ni \quad \delta f = 0.$

We are trying to construct a section of SV over the $2n$ skeleton. It is possible over the $2n-1$ skeleton. This means that for each simplex of B_G of dim $< 2n$ we have a covering simplex in SV.

$$\begin{tikzcd} & E_G \times_G V \arrow[d] \\ \Delta(q) \arrow[ur] \arrow[r] & B_G \end{tikzcd}$$

residue ~~of a~~ or trace of a transformation

nth Chern class

G acts on S^{2n-1} then it acts on the singular chains

$$\begin{tikzcd} & \mathbb{Z} & & & & \\ \arrow[r] & C_{2n-1}(S^{2n-1}) \arrow[u] \arrow[r] & \cdots \arrow[r] & C_1(S^{2n-1}) \arrow[r] & C_0(S^{2n-1}) \arrow[r] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

which gives an element of $Ext^{2n}_G(\mathbb{Z},\mathbb{Z}) \simeq H^{2n}_G(\mathbb{Z})$ $\boxed{QED}$

which is c_n.

Similarly for the other Chern classes.

Theorem (Shub): ~~Category of pointed expanding maps equivalently to the~~

The functor π_1 from expanding maps with fixpt. to groups is fully faithful.

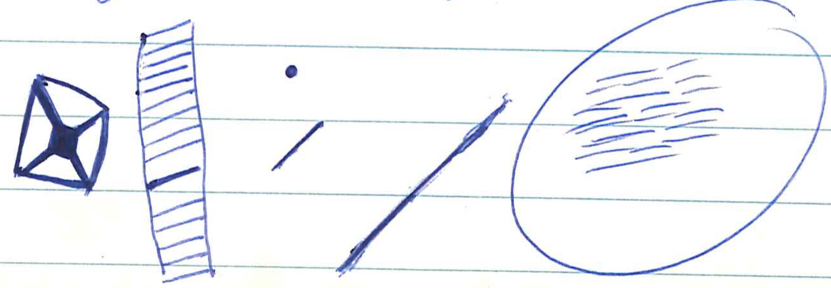
Theorem (Quillen): M compact smooth manifold with an expanding map f then ~~for any continuous function φ on M~~

~~converges uniformly to~~ on $C^0(M)$

$$\lim_{n\to\infty} \frac{I + f^* + \cdots + (f^*)^{n-1}}{n}$$

exists uniformly and converges to the unique invariant measure μ.
This also works for C^α α slightly better than 0.

Problem: Define ~~an invariant~~ a metric on M with the property that $d(f^{-m}\gamma f^m x, f^{-m}\gamma f^m y) = d(x,y)$ on the universal covering of M. Perhaps you need the notion of a geodesic joining two points of M.



stable

Fundamental problem: Define and solve generic partial differential operators.

Requirements of the definition:

1. Stability for lower order perturbations
2. Stability for variable coefficients.
3. (Topological conjugacy)

Lots of questions which should be answered.

ⓐ Suppose we give a symbol ~~on the~~ sheaf on $\mathbb{P}(T^*)$ flat over Ω. Then what are the integrability conditions, e.g. integrability of the characteristic equations.

ⓑ Supposing the integrability conditions are satisfied what about arbitrary variation of lower terms.

Hormander's fundamental insight into the problem is the local character of the estimates and what to expect as one goes to the boundary.

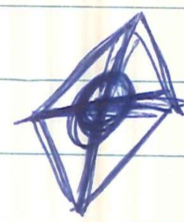
I wish to prove the exactness of the transpose sequence of compactly supported distributions. Therefore I want a homotopy operator

Example Fairly critical:

Suppose that we consider dist.

$$\mathcal{D}(E^0)^\sim \longleftarrow \mathcal{D}(E^1)^\sim \longleftarrow \cdots$$

with compact support. We suppose this sequence to be exact in



Problem:

$T = \mathbb{C}^n$ coordinates $z_1, \ldots, z_n$.

~~Let M be a finitely generated homogeneous A = ST module. There a~~

~~Let M be~~ Let F be a coherent sheaf on $P(T^*)$. ~~and let M~~

Suppose F comes from applying the associated sheaf functor to M and there is an exact sequence

$$A \otimes V_2 \longrightarrow A \otimes V_1 \longrightarrow A \otimes V_2 \longrightarrow M$$

of homogeneous maps. Then what?

Problem: Prove that to F there is a ~~functor~~ number associated which ~~measures of the problem~~ measures loss of smoothness. Show that this number is stable under change of metric. and ~~that it is~~ small variations!

Dunford - Miller

$$\begin{cases} \varphi : \Omega \longrightarrow \Omega \\ p(A) = 0 \Longrightarrow \mu(\varphi^{-1}A) = 0 \end{cases}$$

$A(n) f = \frac{1}{n}(1 + \varphi^* + \cdots + (\varphi^*)^{n-1}) f$ conv. a.e. all $f \in L^1$

$$\Longleftrightarrow \underbrace{\frac{1}{n} \sum_{\ell=0}^{n-1} p(\varphi^{-\ell}(A)) \le K \cdot p(A)}_{\cancel{\forall A}\ \exists K \ni \forall A}$$

$$\mu = h \frac{d\theta}{2\pi} + \mu_s$$

Szegö

$$\boxed{\exp \int \log h \frac{d\theta}{2\pi}}$$

$$\shortparallel$$

$$d(X_0, \bigvee_{n<0} X^n)$$

$$h(a, T) = d(a, \bigvee_{n>0} T^{-n} a)$$

Rabinowitz trick

Given $\quad P_1, \dots, P_n = 0 \Rightarrow Q = 0$

Look at polynomials

$$P_1, P_2, \dots, P_n, \ 1 - QT$$

Conclude that

$$A_i(T) P_i + B(T)(1 - QT) = 1$$

Set $\quad T = 1/Q$

$$\sum_i A_i\left(\tfrac{1}{Q}\right) P_i = 1 \qquad\qquad \frac{Q^m A_i(T)}{Q^k T^k}$$

or $\boxed{\sum{}' \tilde{A}_i P_i = Q^n}$

$P_1, \dots, P_n$ homogeneous of degree m

$P_i = 0 \quad \forall i \Rightarrow Q = 0$, $\quad Q$ of degree k

$$P_i \qquad\qquad S^{k+1} - QT$$

$$\sum_i{}' A_i(T,S) P_i + B(T,S)(S^{k+1} - QT) = S^N$$

$$Q^{d} \sum_{0 \le j \le k}{}' A_{ij}(T) S^j \ P_i \qquad\qquad \text{set } S = 1.$$

Returning to our manifold, can we ~~[illegible]~~ prove ~~[illegible]~~ the formula for toral automorphisms?

Map $M \longrightarrow$ simplex in Hilbert space from the partition.

[What I want is to use scattering theory to factor my automorphism so as to get eigenvalues outside of the unit circle.

Prediction theory:

H Hilbert space, U unitary operator on H; v cyclic vector

$$D^+ = \langle U^n v : n \geq 0 \rangle.$$

Assume $\bigcap_{n\geq 0} U^n D^+ = 0$. $\bigcup_{n\in\mathbb{Z}} U^n D^+ = H$.

and therefore if can represent

$$\langle H, U, v \rangle = \langle L^2(S^1, \mu), z, 1 \rangle$$

D^+ closure of holom. fns.

on the other hand I can also make

$$\langle H, D^+, U \rangle = \langle L^2(S^1, \text{Lebesgue}), \text{holom}, z \rangle$$

Everything clearly works and again we may define the entropy in the obvious way. All this works for a compact Hausdorff space and a measure ω on it.

<u>Theorem</u>: Entropy thus defined coincides with usual entropy.

<u>Proof</u>: Clear from continuity of $h(\mathcal{a},T)$ in $\mathcal{a}$. ‖

The point is that $h_n(\mathcal{a},T) = H(\mathcal{a}/\bigvee_{i=1}^{n} T^{-i}\mathcal{a})$ is monotone decreasing and continuous in $\mathcal{a}$. Therefore? NO The point is that

$$d(\mathcal{a}\vee\mathcal{a}', \mathcal{B}\vee\mathcal{B}') \leq d(\mathcal{a},\mathcal{B}) + d(\mathcal{a}',\mathcal{B}').$$

hence

$$d(\mathcal{a}\vee T^{-1}\mathcal{a}\vee\cdots T^{-n}\mathcal{a},\ \mathcal{B}\vee T^{-1}\mathcal{B}\vee\cdots\vee T^{-n}\mathcal{B}) \leq (n+1)\, d(\mathcal{a},\mathcal{B})$$

so

$$\left|\frac{1}{n+1} H(\mathcal{a}\vee\cdots\vee T^{-n}\mathcal{a}) - \frac{1}{n+1} H(\mathcal{B}\vee\cdots\vee T^{-n}\mathcal{B})\right| \leq d(\mathcal{a},\mathcal{B})$$

$$\therefore\quad |h(\mathcal{a},T) - h(\mathcal{B},T)| \leq d(\mathcal{a},\mathcal{B}).$$

Basic problem: Let M be a compact C^∞ manifold and let f be an ~~diff~~ endomorphism of M which leaves a smooth volume element ω invariant. Find a formula for the entropy of f in terms of topological data of f assuming that f is structurally stable.

Can you replace measure partitions by partitions of unity?

Suppose $\sum \rho_i = 1$ is a partition of unity on M. Define **its entropy** with respect to ω to be the entropy of the simplicial complex ass. to $\sum \rho_i$, the induced measure and the simplex decomposition.

$\cancel{\rho_i \ln \rho_i}$

Better $\displaystyle\sum_i \int \rho_i\,\omega = 1$

so define the entropy to be

$$\sum - \int \rho_i\,\omega \ln\left(\int \rho_i\,\omega\right) = H(\mathcal{P})$$

How about conditional entropy?

Given $\mathcal{P}$: $\sum \rho_i = 1$ and $\mathcal{Q}$: $\sum q_j = 1$

$$H(\mathcal{P}/\mathcal{Q}) = \sum_{i,j} -\ln\left(\frac{\omega(\rho_i q_j)}{\omega(q_j)}\right)\cdot \omega(\rho_i q_j)$$

<u>Old definition:</u>

$$h(a,T) = \lim_{n\to\infty} \frac{1}{n} H(a \vee T^{-1}a \vee \cdots\cdots T^{-n+1}a).$$

$$H(a \vee T^{-1}a) = \cancel{H(a) + H(T^{-1}a/a)}$$

$$= H(T^{-1}a) + H(a/T^{-1}a)$$

$$H(a \vee T^{-1}a \vee T^{-2}a) = H(T^{-1}a \vee T^{-2}a) + H(a/T^{-1}a \vee T^{-2}a)$$

$$= H(T^{-2}a) + H(T^{-1}a/T^{-2}a) + H(a/T^{-1}a \vee T^{-2}a).$$

Now that $$H(T^{-1}\xi/T^{-1}\eta) = \sum_{i,j} -\ln\left\{\frac{\mu(T^{-1}A_i \cap T^{-1}B_j)}{\mu(T^{-1}B_j)}\right\} \mu(T^{-1}A_i \cap T^{-1}B_j)$$

$$= H(\xi/\eta).$$ true for finite partitions

$$\therefore \quad H(a \vee T^{-1}a \vee \cdots \vee T^{-n+1}a) = \sum_{i=0}^{n-1} H(a/\bigcap_{0<i<n} T^{-i}a)$$

and as these are decreasing non-neg. quantities the limit exists.

$$\boxed{h(a,T) = \lim_{n\to\infty} H(a/T^{-1}a \vee \cdots \vee T^{-n}a).}$$

Question: Suppose $\sum_{i=1}^{\infty} p_i = 1$ $\quad p_i \geq 0$

Is $\sum -p_i \ln p_i < \infty$.

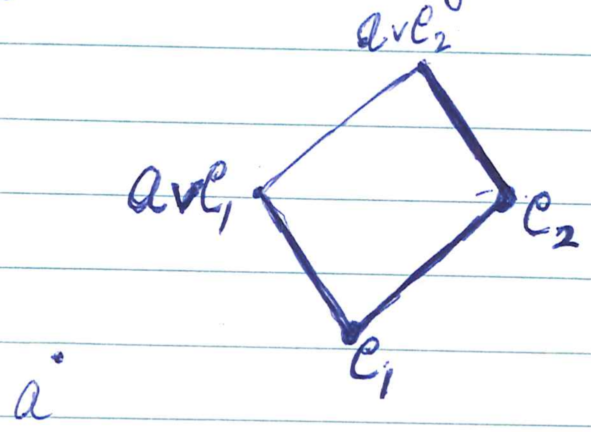
NO: First take $\sum 2^{-i} = 1$. Then divide each 2^{-i} up into 2^{2^i} pieces, whence

$$\sum_j -p_j \ln p_j = -\sum_i \frac{1}{2^i} \ln 2^{-2^i} = \sum_i \ln 2 = \infty.$$

If the ultimate partition has finite entropy all transformations have entropy 0

Generalizations:

Somehow entropy is a ~~[scribble]~~ means of measuring partitions which tends ~~to~~ monotonely toward infinity as the partition gets finer. Another fundamental property is its convexity, i.e.



$$H(a/\mathcal{C}_1) = H(a \vee \mathcal{C}_1) - H(\mathcal{C}_1)$$
$$\geq H(a \vee \mathcal{C}_2) - H(\mathcal{C}_2)$$

$$h(a, T) = H\left(a \Big/ \bigvee_{n>0} T^{-n}a\right) = H\left(\bigvee_{n\geq 0} T^{-n}a \Big/ \bigvee_{n>0} T^{-n}a\right)$$

$$= H(\mathcal{B} \mid T^{-1}\mathcal{B}) \longleftarrow$$ not rigorous because $H(\mathcal{B}) = \infty$ most likely.

and $T^{-1}\mathcal{B} \subset \mathcal{B}$.

$$\therefore \quad h(a, T) = H(\mathcal{B}) - H(T^{-1}\mathcal{B})$$

now why is this the same as the ~~preceding~~ old definition?

① 85

Brumer's talk:

$[K:\mathbb{Q}] < \infty$. E_K units of K. There are ~~[illegible]~~ various topologies on $E_K = \mathbb{G}_m(A)$ where $A =$ integers of K.

congruence topology – nbd. basis $\{u \in E_K \mid u \equiv 1 \mod \mathfrak{a}\}$.
profinite topology – ——— $n E_K$

Theorem of Chevalley: ~~[illegible]~~ Two topologies are the same.

Brumer's problem: To show that if p is a prime number, then the following two topologies are the same:

p-congruence topology – nbd. basis $\{u \in E_K \mid u \equiv 1 \mod p^n\}$
equivalently induced topology from $\mathbb{G}_m(A) \to \mathbb{G}_m(\hat{A})$
where $\hat{A} = \varprojlim A/p^n A = \prod_{\mathfrak{p}|p} A_{\mathfrak{p}}{}^{\wedge}$.
p-topology $p^n E_K$.

Brumer's theorem: If K is real and abelian, then the p-congruence is finer than the p-topology on E_K. ~~coincide~~

He proves this by taking a large p-adic field Ω containing K and considering the map

$$\mathcal{L}: E_K \otimes \Omega \longrightarrow \Omega[G]$$

$$\mathcal{L}(u \otimes \alpha) = \sum_{\sigma} (\alpha \cdot \log \sigma u)\, \sigma^{-1}$$

where $\log_p$ is the p-adic logarithm. By a result of Minkowski there is a unit u generating $E_K \otimes \Omega$ as an $\Omega[G]$ modules. One assume $\mathcal{L}$ not injective and get a relation

Weinstein tells that Avez has shown that the entropy of a diffeomorphism of a compact manifold ^{M} is $\leq \dim M$ (max. change in $n-1$ dimensional area), at least provided it leaves a smooth measure on M invariant.

Let X, μ be a probability space. — Boolean σ-algebra with a trace. — Commutative W^* algebra with a **trace**.

distinguished positive linear functional.

Given ~~[illegible]~~ a partition of 1

(3)

<u>Entropy</u>:

~~Let~~ (Ω, μ) probability space

If $\mathcal{a}: \Omega = A_1 \vee \cdots \vee A_r$ is a partition of Ω we set

$$H(\mathcal{a}) = -\sum{}' \mu(A_i) \ln \mu(A_i)$$

Entropy of the partition $\mathcal{a}$. Equivalently

$$H(\mathcal{a}) = \int_{\Omega/\mathcal{a}} -\ln(\text{measure of fibers})\ (\text{induced measure})$$

$$= \int_{y \in \Omega/\mathcal{a}} \left(-\ln \int d\lambda_y\right) d\nu(y)$$

Conditional entropy of a finite partition $\mathcal{a}$ wrt another $\mathcal{B}$

$$H(\mathcal{a}/\mathcal{B}) = \ ?$$

Weinstein ~~argues~~ says that $-\log \mu(A_i)$ ~~is~~ measures the amount of information about a point one gets from ~~having a~~ knowing the point is in A_i so that $H(\mathcal{a})$ is the ~~average amount of~~ expectation of the information offered by the partition.

Hence

$$H(\mathcal{a}/\mathcal{B}) = \sum_j \left\{ \sum_i \underbrace{-\ln\left(\frac{\mu(A_i \cap B_j)}{\mu(B_j)}\right)} \cdot \frac{\mu(A_i \cap B_j)}{\mu(B_j)} \right\} \mu(B_j).$$

amount of information obtained from having a point in $A_i \cap B_j$ when we already know its in B_j.

$$\therefore \boxed{H(\mathcal{A}/\mathcal{B}) = H(\mathcal{A}\vee\mathcal{B}) - H(\mathcal{B})}$$

$$\therefore \mathcal{A} \le \mathcal{B} \Rightarrow H(\mathcal{A}/\mathcal{B}) = 0.$$

Claim ~~[illegible]~~ that

$$d(\mathcal{A},\mathcal{B}) = H(\mathcal{A}/\mathcal{B}) + H(\mathcal{B}/\mathcal{A})$$

is a metric on the set of finite partitions.

$$d(\mathcal{A},\mathcal{B}) + d(\mathcal{B},\mathcal{C}) = 2H(\mathcal{A}\vee\mathcal{B}) - H(\mathcal{A}) - H(\mathcal{B}) + 2H(\mathcal{B}\vee\mathcal{C}) - H(\mathcal{B}) - H(\mathcal{C})$$

$$H(\mathcal{A}\vee\mathcal{B}\vee\mathcal{C}) = H(\mathcal{A}\vee\mathcal{B}/\mathcal{C}) + H(\mathcal{C})$$
$$= H(\mathcal{A}/\mathcal{B}\vee\mathcal{C}) + H(\mathcal{B}\vee\mathcal{C}).$$

$$\left.\begin{array}{c} d(\mathcal{A},\mathcal{B}) \\ + \\ d(\mathcal{B},\mathcal{C}) \end{array}\right\} = \left\{\begin{array}{l} \sum^{i} - \left\{ \ln\left(\frac{\mu(A_i\cap B_j)}{\mu(B_j)}\right) + \ln\left(\frac{\mu(A_i\cap B_j)}{\mu(A_i)}\right)\right\} \mu(A_i\cap B_j\cap C_k) \\ \sum - \left\{ \ln\left(\frac{\mu(B_j\cap C_k)}{\mu(B_j)}\right) + \ln\left(\frac{\mu(B_j\cap C_k)}{\mu(C_k)}\right)\right\} \mu(A_i \cap B_j\cap C_k) \end{array}\right.$$

?

8

If T is an automorphism of (Ω, μ), enough to have $T_* \mu = \mu$.
Then

$$H\Big(\bigvee_{n\geq 0} T^{-n}a \Big/ \bigvee_{n>0} T^{-n}a\Big) = d\Big(\bigvee_{n\geq 0} T^{-n}a, \bigvee_{n>0} T^{-n}a\Big)$$

is ~~...~~ calculated how?

$$H(\mathcal{B}/T^{-1}\mathcal{B}) \qquad T^{-1}\mathcal{B} \subset \mathcal{B}$$

$$\| $$

$$H(\mathcal{B}) - H(\underbrace{T^{-1}\mathcal{B}})$$

$$\|$$

$$H(\mathcal{B}) \;?$$

?

$H(a$

$$H(a \vee \mathcal{B}/\mathcal{C}) = H(a/\mathcal{C}) + H(a/\mathcal{B}\vee\mathcal{C})$$

$$\mathcal{B} \geq \mathcal{C} \text{ (finer)} \Rightarrow H(a/\mathcal{B}) \leq H(a/\mathcal{C})$$

$$H(a/\mathcal{B}) + H(\mathcal{B}/\mathcal{C}) = H(a\vee\mathcal{B}) - H(\mathcal{B}) + H(\mathcal{B}\vee\mathcal{C}) - H(\mathcal{C})$$

$$\neq H(a\vee\mathcal{B}/\mathcal{C}) + H(\mathcal{C}/\mathcal{B})$$

$$= H(a/\mathcal{C}) + H(a/\mathcal{B}\vee\mathcal{C}) + H(\mathcal{C}/\mathcal{B})$$

$$H(a/\mathcal{B}) + H(\mathcal{B}/\mathcal{C}) = H(a\vee\mathcal{B}) - H(\mathcal{B}) + \underline{H(\mathcal{B}\vee\mathcal{C})} - H(\mathcal{C})$$

$$= \underbrace{H(a\vee\mathcal{B}\vee\mathcal{C}) - H(a\vee\mathcal{B}/\mathcal{C})} + \underline{H(\mathcal{B}\vee\mathcal{C}) - H(\mathcal{C})} - \underline{H(\mathcal{B})}$$

$$=$$

(3.3) 90

$$H(a/c) = H(a \vee c) - H(c)$$

first assume ~~B~~ $a \geq B \geq c$.

then

$$H(a/B) + H(B/c) = [H(a) - H(B)] + [H(B) - H(c)] = H(a/c).$$

suppose we work with measurable sets and

$$d(A,B) = \mu(A-B) + \mu(B-A) = \mu(A \cup B) - \mu(A \cap B).$$

then triangle inequality $= \mu(A) + \mu(B) - 2\mu(A \cap B)$.

$$= 2\mu(A \cup B) - \mu(A) - \mu(B).$$

$$d(A,B) + d(B,C) = \mu(A \cup B) - \mu(A \cap B) + \mu(B \cup C) - \mu(B \cap C)$$

$$=$$

$$\underbrace{\mu(A \cup B) + \mu(B \cup C)}_{\shortparallel} - \underbrace{(\mu(A \cup C) + \mu(B))}_{} \geq 0$$

$$\mu(A \cup B \cup C) - \mu(\underbrace{B \cup (A \cap C)}) - \{\mu(A \cup B \cup C) - \mu(\underbrace{B \cap (A \cup C)})\}$$

3.4.

$$d(A,B) = 2\mu(A\cup B) - \mu(A) - \mu(B)$$
$$d(B,C) = 2\mu(B\cup C) - \mu(B) - \mu(C)$$
$$d(A,C) = 2\mu(A\cup C) - \mu(A) - \mu(C)$$

$$d(A,B) + d(B,C) - d(A,C) = 2\{\mu(A\cup B) + \mu(B\cup C) - \mu(A\cup C) - \mu(B)\}$$

$$= 2\{\mu(A\cup B\cup C) + \mu(B\cup(A\cap C)) - \mu(A\cup B\cup C) - \mu(B\cap(A\cup C))\}$$

$$\geq 0.$$

$$d(a,B) = 2H(a\vee B) - H(a) - H(B)$$
$$d(B,C) = 2H(B\vee C) - H(B) - H(C)$$
$$d(a,C) = 2H(a\vee C) - H(a) - H(C)$$

$$\cancel{H(C/B) + H(a/C\vee B) = H(a\vee C/B)}$$

$$d(a,B) + d(B,C) - d(a,C) = 2\{H(a\vee B) + H(B\vee C) - H(a\vee C) - H(B)\}.$$

$$\cancel{H(a\vee B) + H(B\vee C) = H(a\vee B\vee C) + H}$$

$$H(a\vee B) + H(B) = H(a\vee B\vee C) - H(B/a\vee C) + H(B)$$

$$\cancel{H(a\vee B) + H(B\vee C)}$$

$$H(B/a\vee C) + H(a/C) = H(a\vee B/C).$$

$$H(a\vee B) + H(B\vee C) = H(a\vee B\vee C) - H(C/A\vee B) + H(B) + H(C/B)$$

$$= H(A\vee B\vee C) + H(B) -$$

$$H(A\vee B) + H(B\vee C) = H(a\vee B\vee C) - H(C/a\vee B) + H(B) + H(C/B)$$

3.5

$$\begin{cases} H(a \vee B) + H(B \vee C) \geq H(a \vee B \vee C) + H(B) & ? \\ H(a \vee C) + H(B) \leq H(a \vee B \vee C) + H(B). \end{cases}$$

because

$$H(a \vee B) + H(B \vee C) = H(a \vee B \vee C) - \underline{H(C/a \vee B) + H(C/B)} + H(B)$$

$$\cancel{H(C/a \vee B) + H(B/a) = H(B \vee C/a)}$$

$$\cancel{H(C/B) \leq H(C/a \vee B) + H(a/B)}.$$

$$\cancel{H(C/a)}$$

$$H(C/C_1) \leq H(C/C_2)$$

if $C_1 \geq C_2$ by convexity of $-\log$ ~~entropy fn.~~

$-x \log x$

$$\cancel{H(C/C_1) = H(C)}$$

so this proves Δ inequality

Prediction theory:

H, U, a

Bochner $\Rightarrow \exists\, \mu$ on S^1 $\ni$ $\mu(z^n) = \langle U^n a, a \rangle$

whence
$$\begin{array}{ccc} H & \simeq & L^2(S^1, \mu) \\ U & & z \\ a & & 1 \end{array}$$

Let D^+ be closed subspace of H gen. by $U^n a$ $n \geq 0$.

$\therefore\ D^+ \simeq$ closure of polys in z in $L^2(S^1, \mu)$.

We assume that $D^+ < H$ in which case we look at

$$H = \bigcap_{n \geq 0} U^n D^+ \oplus H_1$$

Assume that $H_1 = 0$. In this case we may project 1 wrt $D^+ = UD^+ \oplus N$, getting $1 = (1-f) + f$. Then

$$H = \sum_{n \in \mathbb{Z}}' \mathbb{C}\, z^n f. \quad \text{and} \quad f \in \text{closure of polynomials.}$$

Can define map
$$L^2(S^1, \tfrac{1}{2\pi} d\theta) \longrightarrow L^2(S^1, \mu) \quad \text{by}$$
$$1 \longmapsto f$$

Thus $\langle z^k f, z^\ell f \rangle = \begin{cases} 0 & k \neq \ell \\ \|f\|^2. & k = \ell. \end{cases}$

~~In addition we have D~~

Assume v generates

Then

$$\bigcap_{n\geq 0} U^n D^+ = 0 \iff \left\{ \begin{array}{l} \mu \text{ absolutely cont. with} \\ \text{respect to Lebesgue} \\ \text{measure.} \end{array} \right.$$

($\Longrightarrow$) Choose $v = v_0 + v_1$ $\quad v_1 \in D^+ \quad v_0 \perp D^+$. Then

$$\begin{aligned} D^+ &= Cv_0 \oplus UD^+ \\ &= Cv_0 + CUv_0 + \cdots + CU^n v_0 + U^{n+1} D^+ \end{aligned}$$

$\therefore$ v_0 generates H

so measure from v_0 is absolutely cont. wrt $\frac{dz}{2\pi i}$

~~[illegible]~~ Converse trivial.

Thus $\mu = \mu_s + h \frac{dz}{2\pi i}$

corresponds to

$$\overset{v}{H} = \overset{v_\infty}{\bigcap_{n\geq 0} U^n D^+} \oplus \overset{v_0}{H_1}$$

where H_1 has $\bigcap$

and so therefore

$$|f|^2 d\mu = \frac{\|f\|^2}{2\pi} d\theta$$

i.e. $$d\mu = \frac{\|f\|^2}{2\pi} \frac{1}{|f|^2} d\theta$$

~~On the other~~ In particular ~~#~~ $f \neq 0$ a.e.

$$\frac{1}{|f|^2} \in L^1(S^1, d\theta)$$

since we get an isomorphism.

Therefore $\frac{1}{f} = g$ is a fn in $L^2(S^1, d\theta)$ ~~which bdry~~ which is the bdry. values of a holom. fn. in the interior. $\therefore g \in H^2(S^1)$.

So now we need the theorem which says that where $g^{\in H^2}$ ~~[crossed out]~~ $\Longleftrightarrow$ $\int |\log h| d\theta < \infty$?

$$\boxed{h = |g|^2 \in L^1}$$

Take $\log h = 2 \operatorname{Re} \log g$

Thus want

$$\log h = k + \bar{k}$$

Thm: Indeterminate case occurs when if we write $\mu = \mu_s + \underline{h} \frac{d\theta}{2\pi}$ we have $\int \log \underline{h} \, d\theta < \infty$.

Question: What is the distance from z^{-1} to closure of analytic functions?

~~[illegible]~~ Want $\|f\|^2 = \int |f|^2 \, d\mu$

$h = |g|^2$

$\alpha \, \frac{1}{f} \rightsquigarrow 1$

$\alpha_1 \; 1 \rightsquigarrow f$

$\alpha\left(\frac{1}{f} - 1\right) \rightsquigarrow 1 - f$

$\int \left(\frac{1}{f} - 1\right) \frac{d\theta}{2\pi} = 0$

I am after

$\|f\|^2 \int \frac{1}{|f|^2} \frac{d\theta}{2\pi} = \int d\mu = \|a\|^2$. assume $\|a\| = 1$

$1 - \|f\|^2$.

$\|f\|^2 \int |g-1|^2 \frac{d\theta}{2\pi} =$

$$\|f\|^2 \int h \frac{d\theta}{2\pi} = 1$$

where $d\mu = \|f\|^2 \, h \frac{\theta}{2\pi}$

∴ Up to a constant

$$\alpha^2 \int h \frac{d\theta}{2\pi} = 1$$

$\alpha = \|f\|$.

$$d\mu = \frac{\|f\|^2}{2\pi} h\, d\theta \implies \|f\|^2 \int h\, d\theta = 1$$

$$h = \frac{1}{|f|^2} = |g|^2.$$

$$\int \left(\frac{1}{f} - 1\right) \frac{d\theta}{2\pi} = 0. \qquad \int (g-1)\, d\theta = 0$$

$$g = 1 + a_1 z + a_2 z^2 + \cdots \qquad g = e^p$$

$$2\operatorname{Re} p = \log h.$$

$$\int |g|^2 \frac{d\theta}{2\pi} = \int \left(1 + a_1 z + a_2 z^2 + \cdots\right)^2 \frac{d\theta}{2\pi}$$

$$= 1 + a_1^2 + a_2^2 + \cdots\cdots$$

$$\frac{d\mu}{d\theta} = \|f\|^2 \cdot h \; \frac{1}{2\pi}$$

$$\int$$

$$d\mu = h \frac{d\theta}{2\pi} = |\underline{g}|^2 \frac{d\theta}{2\pi}$$

$\underline{g}$ unique up to constant of absolute value 1. normalize by requiring $\underline{g}(0)$ positive. Then

$$\underline{g} = \|f\| g$$

$$\int \underline{g} \frac{d\theta}{2\pi} = \|f\|.$$

$$\int \underline{g}\,\frac{d\theta}{2\pi} = \underline{g}(0).$$

$$\boxed{\log \underline{g}(0)} =$$

$$|\underline{g}|^2 = \underline{h}$$

$$k + \bar{k} = \log \underline{h}$$

$k(0)$ real.

$$\underline{g} = e^k$$

$$2\,k(0) = \int \log \underline{h}\,\frac{d\theta}{2\pi}$$

harmonic fns.

$$\therefore \quad \underline{g}(0)^2 = e^{2k(0)} = e^{\int \log \underline{h}\,\frac{d\theta}{2\pi}}$$

$$\therefore \quad \underline{g}(0) = e^{\frac{1}{2}\int \log \underline{h}\,\frac{d\theta}{2\pi}}$$

$$\|f\| = e^{\frac{1}{2}\int \log \underline{h}\,\frac{d\theta}{2\pi}}$$

$$\|f\|^2 = e^{\int \log \underline{h}\cdot\frac{d\theta}{2\pi}}$$

$$\underline{g} = e^k$$

$$2\,\mathrm{Re}\,k = \log h$$

$$2k(0) = 2\,\mathrm{Re}\,k(0) = \int \log h\,\frac{d\theta}{2\pi}$$

$$k(0) = \frac{1}{2}$$

a kind of continuous entropy

In other words given the probability distribution

$$\underline{h} \cdot \frac{d\theta}{2\pi}$$

from our stochastic process, the distance

$$d(a, UD^{+})^2 = e^{\int \log \underline{h} \cdot \frac{d\theta}{2\pi}}$$

$\|a\| = 1$

$$\underline{h} \in L^1 \Rightarrow (\log \underline{h})^{+} \in L^1$$

so

$$-\infty \leq \int \log \underline{h} \frac{d\theta}{2\pi} < \infty.$$

A Basic problems: Given a measure $\underline{h}\,\frac{d\theta}{2\pi}$ why is

$$\int \log \underline{h} \cdot \frac{d\theta}{2\pi}$$

a good animal?

Can you use this somehow to calculate entropy?

work in the space of partitions rather than in the Hilbert space $L^2(X,\mu)$

First point is that a partition is like ~~a~~ a measurable set, so we need to embed partitions into some linear or quasi-linear manifold of some sort so the entropy appears as a kind of linear functional. Thus two partitions a and B are said to be independent if

~~we want~~

$$\mu(A_i \cap B_j) = \mu(A_i) \cdot \mu(B_j)$$

in which case the entropies add.

$$\boxed{H(\mathfrak{a} \vee \mathcal{B}) = H(\mathfrak{a}) + H(\mathcal{B})}$$

~~Make an algebra out of the free abelian group generated by a poset by~~

$\cancel{\delta_x \cdot \delta_y =}$

$$\boxed{(f * g)(x, y) = \sum_{x \leq z \leq y} f(x, z) \cdot g(z, y)}$$

given a partially ordered set it is a category so one can form its group ring free abelian group generated by intervals ~~and~~ with obvious multiplication. It is a non-commutative ring.

Given a partition it defines an operator on measurable fns namely conditional expectation. A projection operator satisfying the Reynolds identity $E(fEg) = Ef \cdot Eg$. Consider the algebra generated by these conditional expectations. What does this mean for partitions of unity.

problems.

moduli space for Anosov diffeomorphisms + expanding maps.

Problem. Let $M \xrightarrow{f} M$ be an expanding map and let f_t be a 1-parameter deformation. According to Shub f_t is conjugate to f by a ! homeomorphism.

$$\underline{h_t f h_t^{-1}} = f_t$$

differentiate at $t=0$ so get

$$f_*(X) - X = Y$$

where $Y_m = \frac{d}{dt} f_t(m) \Big|_{t=0}.$

now the point is that $f_* - 1$ is invertible on the

$$\frac{d}{dt}\left(h_t f h_t^{-1}\right)\Big|_{t=0} = X \circ f - df \circ X = \frac{d}{dt} f_t \Big|_{t=0}.$$

$$f_t(m) = h_t\left(f\left(h_t^{-1} m\right)\right)$$

$$\boxed{Y_{fm} = X_{fm} - df_* X_m}$$

Suppose we work with autos. Then this can be written

$$Y = X - f_* X = (1 - f_*)X.$$

And we must be able to solve the equation.

Problem: In general you want to consider only very smooth variations Y and you want to solve for X. So the obvious thing todo is to see if the space Coker $1 - f_*$ is of finite codimension in the space of all Y. Therefore it seems desirable to have the image of $1 - f_*$ closed and in fact of finite codimension in the ~~space~~ set of C^∞ sections of T.

Let's consider expanding maps on S^1. Then given Y we want to solve $Y_{fm} = X_{fm} - df(X_m)$ for X. Suppose

$$\boxed{Y = a(z)\frac{d}{d\theta}} \qquad a \text{ periodic}$$

$$f(z) = z^2 \qquad df\left(\frac{d}{d\theta}\right) = 2\frac{d}{d\theta}$$

$$a(z^2)\frac{d}{d\theta} = b(z^2)\frac{d}{d\theta} - 2b(z)\frac{d}{d\theta}$$

therefore we have to solve the equation

$$\boxed{a(2z) = b(z^2) - 2b(z)}$$

in periodic functions.

103

Go over your calculations **carefully**

$M = S^1$ $\qquad f(z) = z^2$.

$$f_t(m) = h_t(f(h_t^{-1}m))$$

$\left.\frac{d}{dt}\right|_{t=0}$ $\qquad Y_m t + f(m) = f_t(m)$

$$\underline{Y_m} = X_{fm} - \underline{df(X_m)}.$$

$$\boxed{Y_m \in T_{fm}(M)}$$

$$Y \in \Gamma(f^*T). \qquad \underline{T \xrightarrow{df} f^*T}$$

$$\boxed{Y = f^*X - df(X)} \qquad M \longrightarrow M$$

$$a(z)\left(\frac{d}{d\theta}\right)_{z^2} = b(z^2)\left(\frac{d}{d\theta}\right)_{z^2} - 2b(z)\left(\frac{d}{d\theta}\right)_{z^2}$$

To solve

$$a(z) = b(z^2) - 2b(z)$$

$$\sum a_n z^n = \sum b_n z^{2n} - \sum 2b_n z^n$$

Write $\qquad a(z) = \sum_{n \in \mathbb{Z}} a_n z^n$

$$b(z) = \sum_{n \in \mathbb{Z}} b_n z^n$$

then

$$a(z^2)$$

$$\|$$

$$\sum' a_n z^{2n} = \sum' b_n z^{2n} - 2b_n z^n$$

$\therefore$

Therefore

$$n \text{ odd} \Rightarrow \quad b_n = 0.$$

$$a_n = b_n - 2b_{2n}$$

Thus given a_n can solve for b_n as follows:

$$a_1 = -2b_2 \qquad a_3 = -2b_6 \qquad \therefore \; b_2 = -\tfrac{1}{2} a_1$$

$$a_2 = b_2 - 2b_4 \qquad b_4 = -\tfrac{1}{2} a_2 + \tfrac{1}{2} b_2$$

$$a_4 = b_4 - 2b_8 \qquad = -\tfrac{1}{2} a_2 - \tfrac{1}{4} a_1$$

$$b_8 = -\tfrac{1}{2} a_4 + \tfrac{1}{2} b_4$$

$$b_8 = -\tfrac{1}{2} a_4 - \tfrac{1}{4} a_2 - \tfrac{1}{8} a_1$$

$$2^{-m\delta}\left\{\frac{(2^{\delta-1})^{m+1}-1}{2^{\delta-1}-1}\right\} = O(2^{-m}).$$

$\cancel{a_n = \ldots}$

$$\therefore \begin{cases} a_n = -2b_n & n \text{ odd} \\ a_n = b_{n/2} - 2b_n & n \text{ even} \end{cases}$$

$$b_k = \left(-\tfrac{1}{2}\right) a_k \qquad k \text{ odd.}$$

$$b_{2^m k} = -\frac{1}{2} \cdot a_{2^m k} + \frac{1}{2} b_{2^{m-1} k}$$

$$\boxed{b_{2^m k} = -\left[\left(+\frac{1}{2}\right) a_{2^m k} + \left(\frac{1}{2}\right)^2 a_{2^{m-1} k} + \cdots + \left(\frac{1}{2}\right)^{m+1} a_k\right] \qquad k \text{ odd}}$$

Therefore there is a 1-1 correspondence between a and b sequences

Can you estimate size of b sequence in terms of the a sequence?

Suppose $\quad |a_N| \leq C\,(N)^{-q}$

Seems hopeless

$$-2b_{2^m} = a_{2^m} + \frac{1}{2} a_{2^{m-1}} + \cdots + \frac{1}{2^m} a_1$$

$$2|b_{2^m}| \leq C\left\{ (2^m)^{-q} + \frac{1}{2}(2^{m-1})^{-q} + \cdots + \frac{1}{2^m} \right\}$$

$2|b_{2^m}| \; C \qquad 2^{-mq \;\; -(mq+q-1)} \qquad m$

$$q-1 + \cdots + (q-1)m$$

$$2^{-mq}\left\{ 1 + 2^{q-1} + \cdots\cdots 2^{m(q-1)} \right\}$$