The next ~~problem is~~ thing ~~to note~~ is to find candidates for classification. The method up to now has been to choose a partition and use this to map into a product space in which case one usually wants to know that that the map is injective. This means that the ~~intersection~~ sup partition $\bigvee_{n \in \mathbb{Z}} T^n a =$ ~~trivial~~ point partition I.

~~And that~~ ~~By ergodicity $\bigwedge T^n a =$ the 0 partition~~

$$\cancel{X \supset T^n A_n} \qquad \bigwedge T^n a = B$$

B partition invariant under T

Since a partition maps us into a product space if one assigns values to the different elements of the partition one has prediction theory.

Suppose the partition is given by a function $f : X \to \mathbb{R}$. The ~~entropy~~ entropy will be somewhat analogous to the arc length — independent of the actual parameterization.

<u>Guess</u>: f defines a prediction theory problem and the entropy is the associated distance function!!

The point is that the entropy is calculated using only f and its translates $T^k f$. Suppose f is a simple function $f = \sum_{i=1}^{n} a_i \chi_{E_i}$ $\quad a_i \neq a_j$ if $i \neq j$.

then $\quad E(\pi f) = -\sum_i \mu E_i \ln \mu E_i$

minimize the distance from f to $\sum_{i=1}^{n} \alpha_i T^{*i} f$

i.e. $\| f - \sum_{i=1}^{n} \alpha_i T^i f \|^2$ minimum

ie $\langle f - \sum_{i=1}^{n} \alpha_i T^i f , T^i f \rangle = 0$

or $\langle f, T^i f \rangle = \sum_{j=1}^{n} \alpha_j \langle T^j f, T^i f \rangle = \sum_{j=1}^{n} \alpha_j \langle T^{j-i} f, f \rangle$

Suppose that

$$\langle T^i f, f \rangle = \int z^i \, d\mu = \int z^i h \frac{d\theta}{2\pi}.$$

want

$$\int z^{-i} h \frac{d\theta}{2\pi} = \sum_{j=1}^{n} \alpha_j \int z^{j-i} h \frac{d\theta}{2\pi}$$

i.e. $\int z^{i} \left(1 - \sum_{j=1}^{\infty} \alpha_j z^j\right) h \frac{d\theta}{2\pi} = 0$ $\quad i > 0$

$\therefore$ $\int f(z) \overline{z^i} \, h \frac{d\theta}{2\pi} = 0$ $\quad i > 0.$

$\|$ $\quad f(z) = 1 - \sum_{j=1}^{\infty} \alpha_j z^j$

$\int |f(z)|^2 z^i h \frac{d\theta}{2\pi} = 0$ $\quad$ all $i \neq 0$

$$\Longrightarrow \quad |f(z)|^2 h = k.$$

distance $= e^{\frac{1}{2}\int \log h \frac{d\theta}{2\pi}}$

where $\quad h = 2\pi \frac{d\mu}{d\theta}$ $\qquad$ $\langle T^i f, f \rangle = \int z^i h \frac{d\theta}{2\pi}$.

suppose $\quad f = \sum_{i=1}^{n} a_i \chi_{E_i}$ $\qquad$ ~~a_i real~~

then

$$\langle T^n f, f \rangle = \sum_{i\,j} a_i \bar{a}_j \, \mu(E_i \cap T^{-n} E_j).$$

variational problem. Minimize

$$d = \exp \tfrac{1}{2} \int \log h \frac{d\theta}{2\pi}$$

where

$$\langle T^n f, f \rangle = \int z^n h \frac{d\theta}{2\pi} \qquad + \qquad \|f\|^2 = \int h \frac{d\theta}{2\pi} = 1.$$

$$f \chi_{E_i} = f.$$

$f = \sum a_i \chi_{E_i}$ $\qquad$ $\sum |a_i|^2 \mu(E_i) = 1.$

$f + \sum b_i \chi_{E_i}$ $\qquad$ ~~$\mathrm{Re}\, a_i \bar{b}_i$~~

$$\mathrm{Re}\left\{ \sum a_i \bar{b}_i \, \mu(E_i) = 0 \right\}.$$

I understand what a source is (H, R, T, v)
if ergodic then v may be ignored. unique $\ni$ $Tv = v$ & $\|v\| = 1$.

Channel

Start with (H_i, R_i, T_i, v_i)

and look in

$$(H_1 \otimes H_2, R_1 \otimes R_2, T_1 \otimes T_2, \underline{v_1 \otimes v_2})$$
$$Z$$

for a measure in product $\ni$
such that

$$f(x)g(y)$$

ω in $A^I \times B^I$ by

$$\int_X d\mu(x) \int f(x,y)\, d\nu_x(y) = \int f(x,y)\, d\omega(x,y)$$

$$d\omega(x,y) = d\nu_x(y)\, d\mu(x)$$

Any ω may be put in this form ~~provided~~. by R–N

$$\int_Y d\nu_x(y) = 1 \quad \forall x$$

$$\Updownarrow$$

$$pr_{1*}\omega = \mu$$

defines η by $(pr_2)_* \omega$.

On the classification of measure preserving transformations

Given H, R, T — H sep. Hilbert space, R max comm $*$ closed alg, T unitary $\ni$ $TRT^{-1} \subseteq R$.

Assume T ergodic, i.e. only $I \in R$ left fixed by R — equivalent to H being irreducible under T, R.

Method of attacking problem: generate invariants

example: <u>entropy</u>.

Choose a ~~[illegible]~~ vector $v \in H$ — under standard model corresponds to $L^2(X,\mu) = H$, v is a fn. $f \in L^2(X,\mu)$ ~~[illegible]~~ Given f we can set up a prediction theory problem, namely try to estimate

$$d(f, \langle T^{-n}f, n>0\rangle).$$

and maximize this over f. This can also be done for $L^p(X,\mu)$ since unit ball is <u>unif. convex</u> ~~[illegible]~~

given a convex set Q closed it has a point closest to 0; choose $q_n \ni |q_n| \searrow \inf\{|q| \mid q \in Q\} = K$. Claim q_n converge. Indeed uniform convexity ~~[illegible]~~ of unit ball means that $\forall \varepsilon\, \exists \delta \ni$ if $|u|, |v| \le 1$ and $\left|\frac{u+v}{2}\right| \ge 1-\delta$ then $|u-v| \le \varepsilon$.

by scalars this means that given δ ~~as $|q_n|$ $|q_m|$ and~~ $\exists n_0 \ni n \ge m \ge n_0$ $\Rightarrow |q_n|, |q_m| \le K+\varepsilon$. Then $\left|\frac{q_n+q_m}{2}\right| \ge K$

$\Rightarrow |q_n - q_m| \le \varepsilon$. QED.

$\boxed{\text{applications of information theory.}}$

Source $\quad X, \mu, T, f$

where f has values in a finite alphabet A

~~[illegible]~~ thus everything determined by (A, μ)
where μ is a measure on $A^{\mathbb{Z}}$, T shift.

Channel: $\quad (A, \nu, B)$.

$$\underbrace{A^{\mathbb{Z}} \times B^{\mathbb{Z}}}_{\nu} \xrightarrow{pr_2} A^{\mathbb{Z}}$$

ν is a measure on $A^{\mathbb{Z}} \times B^{\mathbb{Z}}$ and if $x \in A^{\mathbb{Z}}$ we let ~~$B^{\mathbb{Z}}$~~ ν_x be the measure induced on $x \times B^{\mathbb{Z}}$. Claim that $(pr_2)_* \nu = \mu$. A measure on $A^{\mathbb{Z}}$ since

---

~~[illegible]~~ channel $\quad (A, \nu, B)$

Thus joining (A,μ) (B,η) is a measure ω on $A^I \times B^I$

$\ni \begin{cases} pr_{1*}\,\omega = \mu \\ pr_{2*}\,\omega = \eta. \end{cases}$

~~Definition of Bockstein operations on $H^*(F, \underline{\Phi}_F)$.~~

Furstenberg's paper in Systems theory

problem is to write up correctly the constant coeff. thm. in the ~~[illegible]~~ constant coeff. Gaussian case.

Ω μ prob. space $\mathcal{F}$ subfield of meas. sets of μ.
X r.v. on Ω. Then conditional expectation of X wrt $\mathcal{F}$ is
a Y measurable wrt $\mathcal{F}$ $\ni$

$$\int_E Y\,d\mu = \int_E X\,d\mu \qquad \text{all } E \in \mathcal{F}$$

Idea: given $H = L^2(\Omega,\mu)$ $A =$ alg. $L^\infty(\Omega,\mu)$ of bdd. ops on H.
~~Then~~ suppose B is a weakly closed s.a. subalgebra of A. Then
Want to define a map

$$E : A \longrightarrow B$$

conditional expectation.

$$\mathrm{tr}_B(EX \cdot Y) = \mathrm{tr}_A(X \cdot Y).$$

ie if $Y = \chi_E$ $E \in \mathcal{F}$ this says that

$$\int_E EX = \int X.$$

which is what we want!!!!

$L^2(X,\mu)$ necessarily separable if T ergodic.

$\therefore$ Assume T ergodic and let $\mathcal{J}$ be any finite partition into two non-~~empty~~ measure 0 pieces. Then if E is the projection operator χ then the subspace generated by $\{T^n \chi\}$ is stable under

start with a

(X, μ, T, f) stochastic cx. valued process stationary.

If $\|f\|^2 = \int_X |f|^2 d\mu < \infty$, then get a Hillbert space problem

(H, R, T, v) where
| R max * closed comm. alg. of bdd ops on H
| v cyclic vector for T, R.
| T unitary op on H.

In my case ~~I~~ forget about v which is the initial time $t=0$ distribution of the numerical quantity I am interested in! and I try to classify only the triple (H, R, T). Assume ergodic and let 1 be the unique invariant vector.

Examples and methods: To each f

partitions and generating fns.

Suppose ~~B~~ S is a subalgebra * closed of R.

Assume ergodic!!!!!

Rohlin's thm. (X, μ, T) T ergodic aperiodic
$\Rightarrow$ $\exists$ a generator.

~~Proof Assume $(X, \mu$~~ X, μ is a lebesgue measure space so $L^2(X, \mu)$ is separable and T. One finds this is irred.
$\therefore$ any vector generates.

problem: make trace calculation work.

~~Theorem~~:

Stochastic process – discrete, stationary.

(X, μ, T, f) $\qquad f: X \to R$.

$$X \longrightarrow \prod_{\mathbb{Z}} R$$

induced measure, shift auto,

not much in the way of classification

~~Suppose we forget about~~

translation:

(H, A, T, v)

H. Hilbert space

A max comm. * closed algebra of bdd ops.

v cyclic vector $\qquad \|v\| = 1$.

T automorphism of H, A, v.

can reconstruct X as the maximal ideal space of A.

~~$\mu(X)$~~ $\quad \mu(A) = (Av, v)$.

$$\mu(TAT^{-1}) = (TAT^{-1}v, T^{-1}v) = (Av, v) = \mu(A).$$

hence one can recover ~~X, μ, v~~ $\quad X, \mu$

---

we drop the f and study only (X, μ, T).

~~Drop X~~

Further thoughts on entropy

Suppose we are given X, μ, T
X, μ ~~measure~~ prob space, T measure preserving transformation.
Better we are given a separable Hilbert space H a vector v, a weakly* closed commutative algebra A of operators on H such that $\overline{Av} = H$ and a unitary transf of (H, A, v). Can you classify such triples with autos. T.

H, A, v — H separable Hilbert space
A ~~weak~~ *closed commutative weak closed subalg of $\mathcal{O}p\, H$
v vector $\ni$ Av dense in H.

Then if Boolean algebra of projections in A has no atoms we know $\exists$ isom $(H, A, v) \simeq (L^2([0,1], \text{Leb}, 1)$.

In addition we give T auto. of (H, A, v). Can I classify such pairs.

first of all I can try to classify (H, T, σ) but in interesting cases ~~(H,T)~~ one knows that $C\sigma^{\perp}$ is countably infinite copies of $\left(L(S^1, \frac{d\theta}{2\pi}), z\right)$. In other words what is interesting is the way T and A act upon each other. Now Mackey's imprimitivity theorem says that when you do find an A and T it is of this type.

The Russians have tried to prove the analogous result I want in the case of Bernoulli transformations, and Adler claims the result for the 2-torus.

In the ergodic case the action of T on A is without eigenfns except 1.

Classification of measure-preserving transformations.

<u>Stage 1:</u> <u>**Decomposition into irreducible constituents**</u> (analogous to von Neumann decomposition of a ~~measurable~~ weakly closed *closed algebra into factors. The point is to use the Boolean subalgebra of projection operators ~~which are closed under stable under the action of T.~~ which commute with the action of T.

<u>Stage 2:</u> <u>Analysis of ~~the~~ an ergodic piece.</u> This is the important part. The first obvious thing to note is that the weakly closed algebra generated by A and T is all operators on Hilbert space since anything commuting with A is in A and since anything in A commuting with T is necessarily a constant!

$$\tilde{M} \xrightarrow{\tilde{\varphi}} \tilde{M}$$

$$f \mapsto \tilde{\varphi}^{-1} \cdot f\, \tilde{\varphi}$$

$$\bigcup_n \tilde{\varphi}^{-n} \pi \tilde{\varphi}^{n} \qquad \text{subgp of } \operatorname{Aut}(\tilde{M})$$

Take closure + call it G

If G is a lie gp., then $\tilde{\varphi}^{-1} G \tilde{\varphi} = G$ φ is an expansive automorphism so

G $\qquad$ g is nilpotent $\qquad\qquad$ $\tilde{\varphi}^{-1} G \tilde{\varphi}$

and $\tilde{\varphi}$ induces an auto. of G/G°.

finite?

$$\begin{tikzcd}
0 \arrow[r] & G^\circ \arrow[r] & G \arrow[r] & G/G^\circ \arrow[r] & 0 \\
& \pi^\circ \arrow[r] \arrow[d, "\varphi"] \arrow[dr, phantom, "\text{cart}"] & \pi \arrow[u, phantom, "\cup"] \arrow[r] \arrow[d, "\varphi"] & \pi/\pi^\circ \arrow[d] & \\
& \pi^\circ \arrow[r] & \pi \arrow[r] \arrow[d] & \pi/\pi^\circ & \\
& G^\circ & G & &
\end{tikzcd}$$

Therefore obvious thing to do is to look for a normal subgp. π_0 of finite index such that $\varphi \pi_0 \subset \pi_0$ and φ induces isom on π/π_0.

~~Definition: nilmanifold = manifold on which a nilpotent Lie group acts transitively. May assume the group G is simply-connected.~~ ~~$M = G/H$~~

~~Note that if we assume G acts faithfully that~~

~~$\bigcap_{x\in G} H^x = \{1\}$.~~

NO.

infinitesimally

compact nilmanifold = nilpotent Lie group modulo discrete uniform subgroup. $M = G/D$

each element of $\mathfrak{g}$ yields a vector field on M.

e^{tX}

Thus get a trivialization of the tangent bundle

$$G \longrightarrow G/D$$

Hence tangent bundle is trivialized in a canonical way so there is a connection which is in particular integrable and hence has curvature zero. Hence can describe.

Now ~~can~~ why one pass to the inverse limit?

$x_s \in a_n + m^s$ $\forall n$, but not to m^s ~~[illegible]~~.

$x_t \in a_n + m^t$ all n.

$x_s \equiv x_t$ m^s.

lim. comp.

Suppose have x_t

$x_s \in a_n + m^s$

$x_t = u_n +$

$\phi \neq a_n \cap (-x_t + m^t)$ A/m^{t+1}

all n

$\Longrightarrow \exists$ $z \in a_n \cap (-x_t + m^t)$

$x_t + z \in a_n + m^{t+1}$ all n

<u>Lemma:</u> Any subgp of finite index contains a normal one.
Any subgp. of a nilpotent group is nilpotent.

$$0 \longrightarrow \pi' \longrightarrow \pi \longrightarrow \pi'' \longrightarrow 0$$

$\boxed{\text{Hochschild–Serre}}$

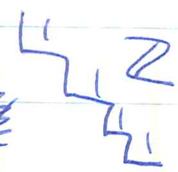
$$E_2^{pq} = H^p(\pi'', H^q(\pi', M)) \Longrightarrow H^{p+q}(\pi, M).$$

<u>Example:</u> $M = \mathbb{R}$

$$\underbrace{H^q(\pi', M)^{\pi''}}_{\mathbb{R}} = \underbrace{H^q(\pi, M)}_{\mathbb{R}}$$

$$H^1(\pi, \mathbb{R})$$

If Γ nilpotent and grs F torsion-free $\Longleftrightarrow$ $\Gamma \subset \cancel{SL}$



Thus basic fact is that if $\pi' \hookrightarrow \pi$ finite index

then

$$H^q(\pi', M) \longleftrightarrow H^q(\pi, M)$$

all M char 0.

M compact manifold smooth

$\varphi: M \longrightarrow M$ expansive mapping. This means that if $|\ |$ is a Riemannian metric on $T(M)$, then

$$|d\varphi^n(v)| \geq c\lambda^n |v| \qquad c>0 \quad \lambda>1.$$

This condition is independent of the choice of $|\ |$, for if $\|\ \|$ is another then

$$a\|v\| \leq |v| \leq b\|v\| \qquad \text{all } v$$

so

$$\|d\varphi^n(v)\| \geq \frac{1}{b} c\lambda^n a\|v\| = c'\|v\| \qquad \text{all } v.$$

~~Suppose we set consider~~

$$\cancel{\liminf_{n\to\infty} \frac{1}{n} \log \frac{|d\varphi^n(v)|}{|v|} = B(v)}$$

~~Then $B(\lambda v)$~~

$$\cancel{\log|d\varphi^n(v)| \geq \log c + n\log\lambda + \log|v|}$$

$$\cancel{\therefore \liminf \frac{\log|d\varphi^n(v)|}{n} \geq \log\lambda.}$$

~~Define~~ $\cancel{\lambda(v) = \liminf \frac{1}{n}\log|d\varphi^n(v)|}$

Example $H^*(\mathbb{R}^n/\mathbb{Z}^n, \mathbb{R}) = \Lambda(\mathbb{R}^n) = H^*_{L.A.}(\mathbb{R}^n)$

G

---

h) Euler characteristic of a covering multiplies

$$\chi(\pi) \cdot [\pi : \pi'] = \chi(\pi')$$

Thus taking $\varphi\pi = \pi'$ one sees that

$$\boxed{\chi(M) = 0.}$$

Γ finitely generated torsion free nilpotent group

$G = \widehat{\mathbb{R}\Gamma}$ $\quad$ G simply-connected nilpotent Lie gp.

why is $\underline{H^*(G/\Gamma, \mathbb{R})} \simeq H^*(\mathfrak{g}, \mathbb{R})$

answer topologically G is contractible.

$$H^*_{gp}(\Gamma, \mathbb{R}) \simeq H^*_{gp}(G, \mathbb{R}) \simeq H^*(\mathfrak{g}, \mathbb{R}).$$

~~Theorem~~

$\pi' \longrightarrow \pi$

I know that there is a finite-type π resolution

$$0 \to \underbrace{\boxed{P_n \to \quad \cdots \quad \to P_1 \to P_0}}_{\text{acyclic.}} \to \mathbb{Z} \to 0$$

with luck can assume 1 cell of top-dimension?
Poincare duality then

Correct statement of P-D says: that for any twisted constant sheaf F

$$H^i(X, F) \times \cancel{H^{n-i}(X, \mathrm{Hom}(F, \omega))}$$
$$\mathrm{Ext}^{n-i}(X; F, \omega) \to H^n(X, \omega) \simeq \mathbb{Z}$$

doesnt look very good!!

?

Bert claimed that if ~~F is a finitely generated~~ ~~subgp~~ Γ is a discrete uniform subgp of a real Lie ~~algebra of~~ group G, then

$$\boxed{H^*(G/\Gamma, \mathbb{R}) \simeq H^*(\mathfrak{g}).}$$

G compact, true by ~~[illegible]~~. Chevalley-Eilenberg.

Consider $\tilde{\varphi} : \tilde{M} \longrightarrow \tilde{M}$ universal covering.

Note that $\varphi : M \longrightarrow M$ has $d\varphi$ injective $\therefore$ φ etale hence φ is a covering map with finite fibers.

deg φ. Let G (π_1) be the ~~universal~~ group covering of M

$$\begin{tikzcd} \tilde{M} \arrow[r, "\tilde{\varphi}"] \arrow[d] & \tilde{M} \arrow[d] \\ M \arrow[r, "\varphi"] & M \end{tikzcd}$$

$\tilde{\varphi}$ must be a covering map and hence a diffeomorphism since both are connected.

$\tilde{\varphi}$ commutes with action of π_1 on $\tilde{M}$.

a) $\varphi_* : \pi_1 \longrightarrow \pi_1$ injective since its a covering map

b) π_1 finitely generated since M is compact.

c) $\varphi_*\pi_1$ is of finite index in π_1 since φ is a finite covering.

~~$\tilde{M} \simeq \mathbb{R}^n$~~

d) $\tilde{M} \simeq \mathbb{R}^n$ because: $\tilde{\varphi}$ is a diffeo of $\tilde{M}$ which is expanding $\Rightarrow$ $(\tilde{\varphi})^{-1}$ is a diffeo which is contracting $\Rightarrow$ ~~$(\tilde{\varphi})^{-1}$ has a unique fixed point~~ $\tilde{M}, \tilde{\varphi}$ top. conjugate to V, A where A is linear + contracting.

e) $\bigcap_n (\varphi_*)^n \pi_1 = \{e\}$

Proof: Suppose this intersection is K. Then $\varphi(K) \subset K$. and if $x \in K$ then ~~[illegible]~~ $x = \varphi y$ + $x = \varphi^n z \Rightarrow y = \varphi^{n-1} z$ φ inj. $\Rightarrow y \in K \Rightarrow \varphi K = K$.
Thus if N is the covering space of M ~~belonging to~~ $\tilde{M}/K$ we have that $\tilde{\varphi}$ induces an expanding diffeo. of N so $N \simeq \mathbb{R}^n \Rightarrow N$ s.c. $\Rightarrow K = \{e\}$.

---

d) and c) use that φ is a covering map.
f) π_1 has no elements of finite order. (finite gp π cannot act freely on $\mathbb{R}^n$ since that $K(\pi, 1)$ not inf. dim)

Shub's conjecture: $\pi_\#$ is an extension of a finitely generated torsion-free nilpotent group by a finite group.

$$1 \longrightarrow N \longrightarrow \pi_\# \longrightarrow F \longrightarrow 1$$

cohomological observations:

$$\tilde{M} = K(\pi_\#, 1)$$

hence making $\tilde{M}$ orientable if necessary get "Poincare duality" for $H^*(\pi_\#, \mathbb{Z})$.

---

g) If π' is a normal subgp. of π of finite index. Then Cartan–Leray s.s. vanishes so $H^8(\pi, \mathbb{Q}) \simeq H^8(\pi', \mathbb{Q})^{\pi/\pi'}$

$$H^8(\pi, \mathbb{Q})$$

Consider $\tilde{\varphi} : \tilde{M} \longrightarrow \tilde{M}$ universal covering.

Note that $\varphi : M \longrightarrow M$ has $d\varphi$ injective $\therefore$ φ etale hence φ is a covering map with finite fibers.

deg φ. Let $\underset{\pi_1}{G}$ be the ~~universal~~ covering group of M

$$\begin{tikzcd} \tilde{M} \arrow[r, "\tilde{\varphi}"] \arrow[d] & \tilde{M} \arrow[d] \\ M \arrow[r, "\varphi"] & M \end{tikzcd}$$

$\tilde{\varphi}$ must be a covering map and hence a diffeomorphism since both are connected.

$\tilde{\varphi}$ commutes with action of π_1 on $\tilde{M}$.

a) $\varphi_* : \pi_1 \longrightarrow \pi_1$ injective since its a covering map

b) π_1 finitely generated since M is compact.

c) $\varphi_* \pi_1$ is of finite index in π_1 since ~~[illegible]~~ φ is a finite covering.

~~d) $\tilde{M} \simeq R^n$.~~

d) $\tilde{M} \simeq R^n$ because: ~~[illegible]~~ $\tilde{\varphi}$ is a diffeo of $\tilde{M}$ which is expanding $\Rightarrow$ $(\tilde{\varphi})^{-1}$ is a diffeo which is contracting $\Rightarrow$ ~~$(\tilde{\varphi})^{-1}$ has a unique fixed point~~ $\tilde{M}, \tilde{\varphi}$ top. conjugate to V, A where A is linear + contracting.

Example: If φ endom of G expanding and $\varphi\Gamma \subset \Gamma$. $\boxed{\Gamma \text{ nilpotent}}$ finitely-gen. torsion-free

Let $d = \text{degree } \varphi = [\Gamma : \varphi\Gamma]$.

$$\begin{tikzcd} \Gamma \arrow[r, "\varphi"] \arrow[d] & \Gamma \arrow[d] \\ \Gamma_{ab} \arrow[r, "\varphi_{ab}"] & \Gamma_{ab} \end{tikzcd}$$

what is degree of φ_{ab}.

$$\Gamma_{ab} / \varphi\Gamma_{ab} = \Gamma / \varphi\Gamma \cdot [\Gamma, \Gamma]$$

$$[\Gamma : \varphi\Gamma \cdot [\Gamma,\Gamma]] \mid [\Gamma : \varphi\Gamma].$$

~~Therefore one sees that~~

if φ_{ab} is an isom.
then φ is.
hence letting $m = \deg \varphi_{ab}$

Lefschetz formula says that

$$\zeta(z) = \text{char poly } \varphi_* \text{ on } H^*(G/\Gamma, \mathbb{R})$$
$$= \text{char poly } \varphi_* \text{ on } H^*(\mathfrak{g}, \mathbb{R})$$
$$= \text{char poly } \varphi_* \text{ on } \Lambda(\mathfrak{g}^*) = \text{easy formula.}$$

~~char poly of φ on $\mathfrak{g}^*$~~

A/m^δ

$0 \neq a \in \bar{m}_k + m^\delta = m_{k+1} + m^\delta$

Chevalley thm.

m_n decreasing sequence of ideals $\ni \bigcap m_n = 0$.

in a complete local ring $\Rightarrow \forall_\delta \exists k \ni m_k \subset m^\delta$.

if not $0 \to m_n \to A \to A/m_n \to 0$

~~Prove that if G simply connected~~

$$\boxed{H^*(G/\Gamma, \mathbb{R}) \simeq H^*_{LA}(\mathfrak{g}).}$$

<u>explicit isomorphism if possible.</u>

present method

$$\boxed{H^*(G/\Gamma) = H^*_{gp.}(\Gamma) = H^*_{gp}(\hat{\Gamma}) = H^*_{LA}(\mathfrak{g}).}$$

nilpotent gp is unimodular. det adjoint action is 1 since adjoint action is unipotent. Thus $\exists$ nice G invariant measure on G/Γ.

~~Let $\Gamma' \subset \Gamma$ be a subgp of finite index.~~
~~Then is~~

Γ finitely generated + nilpotent.

<u>Method:</u> Start with forms on G — Γ acts on G to right

$$0 \longrightarrow A^0(G) \longrightarrow A^1(G) \longrightarrow A^2(G) \longrightarrow \cdots\cdots$$

This is an acyclic left Γ module resolution of $\mathbb{R}$.

Suppose $\underbrace{\Gamma \subset \Gamma'}_{\text{bigger}}$ then

Real problem is to show that the G invariant and Γ invariant stuff yield same cohomology. So if $\underbrace{\Gamma \subset \Gamma'}_{\text{finite index}}$ the restriction map $A^*(G)^{\Gamma'} \hookrightarrow A^*(G)^{\Gamma}$ is a cohomology ~~injection~~ because of the trace.

Problem: Prove an estimate of the form

$$\dim H^*(\pi, R) \leq \dim \Lambda^* \mathbb{R}^n \qquad n = \dim.$$

First case: $M = G/\pi$ $\quad$ G s.c. nilpotent group + π discrete uniform subgroup. Claim there is a canonical isom.

$$\boxed{H^*(M) \simeq H^*_{LA}(\mathfrak{g})}$$

Stable manifold thm. gives a definite homeomorphism

$$V \longrightarrow \overset{0}{\tilde{M}}.$$

such that $\tilde{\varphi}$ appears linear + expanding.

hence have notion of lines issuing from 0

?

Next stage

$$Q = \tilde{\varphi}^{-n} \Gamma$$

$$\pi \xrightarrow{\varphi} \pi \rightarrow$$

$$Q = \varinjlim$$

it is a non-finitely generated group.

But it's dense in $\tilde{M}$ supposedly!

~~Show that~~

observe that if π is nilpotent then for any $\pi' \subset \pi$ of finite index we have

$$H^*(\pi, \mathbb{R}) \xrightarrow{\simeq} H^*(\pi', \mathbb{R}).$$

This is because both π and π' have the same Malcev completion. Thus it is reasonable to ask whether there is an estimate of the form

$$\dim H^q(\pi, \mathbb{R}) \le \dim \Lambda^q \mathbb{R}^n \qquad n = \dim M.$$

On the other hand suppose that π' is of finite index in π and $H^*(\pi, \mathbb{R}) \simeq H^*(\pi', \mathbb{R})$. If $\pi' \triangleleft \pi$ this means that π/π' acts trivially on $H^*(\pi'; \mathbb{R})$.

Problem: Can there exist a normal subgp N with

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{\varphi}} & \tilde{M} \\ \downarrow & & \downarrow \\ \tilde{M}/N & & \tilde{M}/N \\ \downarrow & & \downarrow \\ M & & M \end{array} \qquad \pi \text{ equivariant}$$

$\boxed{\zeta(z) = \text{char. poly. of } \varphi_* \text{ on } H^*(G/D, \mathbb{R}).}$

Nomizu $H^*(G/D) \simeq H^*(\mathfrak{g})$.
used induction ugh!

Start with C^∞ forms on G
and want D invariants.
Look at algebraic functions on D. $\boxed{\text{poly functions on } \mathfrak{g}}$
why are these D acyclic.
Because poly functions $= \left(\hat{\mathbb{R}}(D)\right)'$. should be D acyclic
since $\mathbb{R}(D)$ is.

The first problem is to show that the ~~xxxx~~ de Rham cohomology

First problem is to bound cohomology.

~~xxxx~~ Question: Suppose π is a ~~finitely generated~~ polycyclic group such that for every subgroup π' of finite index we have

$$H^*(\pi) \xrightarrow{\sim} H^*(\pi')$$

Then is π ~~is~~ nilpotent?

$\underbrace{\Gamma \subset \Gamma'}_{\text{normal}}$

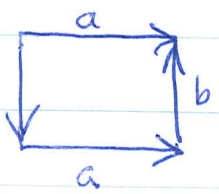
Somehow want π small enough so that the cohomology stabilizes out.

$\Gamma = \pi O$

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{\varphi}} & \tilde{M} \\ \cup & & \cup \\ \Gamma & \hookrightarrow & \Gamma \end{array}$$

So have structure of a group on $\bigcup_n \varphi^{-n}\Gamma$

Klein bottle



generators a, b

relation $aba^{-1}b = e.$

$$aba^{-1} = b^{-1}$$

$$a^2ba^{-2} = (aba^{-1})^{-1} = b.$$

$\therefore$ $a^2 \in$ center $\qquad \mathbb{Z} \to \mathbb{Z} \times_{\sigma} \mathbb{Z} \to \mathbb{Z}$

~~Autosates~~

$(a^2, b) \longrightarrow G \longrightarrow \mathbb{Z}/2\mathbb{Z}$

gives rise to the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Solvable + not nilpotent

$\cancel{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}}$ no center

$\cancel{a^2 \quad b}$

no center. $\quad \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

$\qquad b \quad a$

$aba = b^{-1}$

M manifold, f diffeomorphism of M.
to classify the pair M, f up to isomorphism i.e. conjugacy.
What is the analogous cobordism question.

Say that (M,f) cob. to (M',f') if $\exists$ manifold with boundary (W,g) etc.

Get an interesting problem.
Thom classification theorem; does this $\exists$?

In the homotopy theory the invariant one expects is self cobordism group of self cobordisms of M i.e. manifolds W together with $\theta: M \cup (-M) \simeq \partial W$.

$M \cup (-M) \simeq \partial W.$

Can you classify such (M,f) cobordism classes by homotopy information.
For example if $f_t: M \to M$ is a 1-parameter family of automorphisms of M then f_0 and f_1 are necessarily equivalent.

First problem: Can you embed (M,f) into a nice sphere?

Take M embed in sphere + dual SW so that one get

$M * M'$

$$M * M' \simeq S^m$$

Now I think M' should be a manifold In any case we have an action on the sphere. $\mathbb{Z}$ acts on tubular nbd so tangent bundle has an equivariant action!

serious problem of reducing a homotopy ~~ep~~ action to

If G acts on X, If G acts freely, then there is an equivariant map

$$X \longrightarrow E_G$$

unique up to equivariant homotopy. ~~[illegible]~~

Last but not least.

If G acts on a manifold M, there is associated a unitary representation on the ~~[illegible]~~ intrinsic hilbert space. If $G = \mathbf{Z}$, then get a unitary transformation which may be put in the form

$$U = \oint_{0 \le \theta \le 2\pi} e^{i\theta}\, dE_\theta$$

where E_θ is a projection operator!

Eigenfunction = ~~[illegible]~~ function $f\sqrt{\mu}$ such that

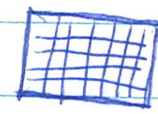
$\dfrac{\mu}{T^*\mu}$ smooth non-zero fn.

$$T(f\sqrt{\mu}) = (f \circ T)\sqrt{\mu \circ T} = e^{i\theta} f\sqrt{\mu}$$

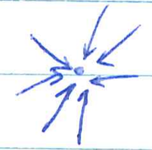
$$T^*(f) = f \cdot e^{i\theta} \cdot \sqrt{\frac{\mu}{T^*\mu}} \qquad \frac{f \circ T}{f} = e^{i\theta}\sqrt{\frac{\mu}{\mu \circ T}}$$

when does T leave invariant a measure. Then must solve the equation

$$T(f\mu) = f\mu$$



or $T(f)\,T(\mu) = f\mu.$



or $$\frac{T(f)}{f} = \frac{\mu}{T(\mu)}.$$

has to do with

$$H^1(\mathbb{Z}, C^\infty(X)^*)$$

$$H^1(\mathbb{Z}, C^\infty(X))$$

using log

If $\exists$ an eigenfunction for $\theta = 0$, i.e. $\dfrac{T(f)}{f} = \sqrt{\dfrac{\mu}{T\mu}}$

$$\Rightarrow \frac{T(f^2)}{f^2} = \frac{\mu}{T\mu}.$$

So the assertion that there is an eigenfunction for $t = 1$ is same as there existing an invariant measure at least if eigenfunction always positive. Suppose $\exists$ an invariant ~~[illegible]~~ measure μ. Then there is an ~~[illegible]~~ eigenfunction $T1 = 1$.

$$\boxed{Tf = e^{i\theta} f.}$$

Suppose

$$\{f(x) = c\}.$$

$$f(Tx) = e^{i\theta} c.$$

~~Suppose θ rational~~

~~ie~~ The functions $\ni$ $Tf = f$ are those fns. which are constant on the ~~[illegible]~~ orbits, in fact closed orbits.

Can use Lefchetz for <u>information</u> on fixpts.

$$L(f^n) = \operatorname{tr} H_*(f)^n$$

$$\zeta(f) = \sum \frac{1}{n}\{\overset{alg.}{\text{card}}\ \mathrm{Fix}(f^n)\} \cdot z^n$$

$$= \frac{1}{\det(1 - H_*(f) z)}.$$

Clearly rational

So the problem is to determine how the algebraic card $\mathrm{Fix}(f^n)$ relates to the <u>actual cardinality</u>!

In the case of an Anosov-diffeomorphism one has the hyperbolic structure on the tangent bundle to fool with!!

$$T = \underbrace{T^u}_{n-k} \oplus \underbrace{T^s}_{k}$$

T ergodic if f measurable $Tf = f$ a.e. $\Rightarrow$ f a.e. constant.

or if $T(A) = A \Rightarrow \mu(A) = 0$ or $\mu(A) = 1$.

metrically transitive

<u>topologically transitive</u>.

$$T(f\sqrt{\mu}) = f\sqrt{\mu}$$

$$T(f^2\mu) = f^2\mu.$$

Relate the following:

a) metrically transitive if T preserves a volume element

b) topologically transitive

c) $\Omega = M$.

d) periodic points are dense.

$\Downarrow$ If $\overline{\mathbb{Z}\cdot x} = M$ then $x \in \Omega$ + Ω closed + inv. dense.

$\Uparrow$

Anosov $\Rightarrow$ $\Omega = \overline{\{\text{periodic pts}\}}$.

$$\boxed{H^n(X, X-x)}$$

$\emptyset$

Example: x fixed point of f^n.

Then want Lefchetz no. which is $\dfrac{\det(1-df(x))}{|\det(1-df(x))|}$

$= \text{sign of } \det(1-df(x))$.

$= \pm 1$.

In the {Anosov case} no of the eigenvalues are on the unit circle and hence fixed points are simple and I is easy to determine.

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Summary sheet on ~~S~~hub's problem.

I spent Oct. 4 trying to make sense out of the holonomy group idea: Let $f: M \to M$ be an expanding map with fixedpoint 0 and let $f: M' \to M'$ be the universal covering map, and call 0 in M' the basepoint. ~~As 0 is an attracting~~ One can then ask whether f on M' is isomorphic to df on T_0M'. Sternberg says yes provided the eigenvalues of df_0 satisfy the nondegenracy condition $\lambda_i \neq \lambda^\alpha$ if $|\alpha| \geq 1$. One also sees that if our situation is isomorphic to a nilmanifold one then this non-degeneracy condition implies that the Lie group is abelian. Under the non-degeneracy condition it also follows that the isomorphism of T_0M' and M' which we call exp is unique. Thus it becomes possible to define a holonomy transformation $\theta_x: T_0M' \to T_0M'$ by the formula

$$\theta_x(v) = \exp^{-1}(x\exp(v)) - \exp^{-1}(x0) \quad \forall\, v \text{ in } T_0M'=V,\ x \text{ in } \pi$$

Basic formula: $\theta_{fxf^{-1}} = df \cdot \theta_x \circ df^{-1}$.

From the basic formula one has $\theta_{f^m x f^{-m}} = Y^m \theta_x Y^{-m}$. where Y=df. The hope was to let m go to infinity and by proving a bound on the RHS conclude that θ_x was linear, at least in the special case wher Y is multiplication by a scalar. From this it would follow that relative to exp each x in π is affine from which the desired result follows.

The only trouble with the above approach is that it will not work because ~~the nature of~~ df_0 may be perturbed drastically without affecting the way f acts on π.

~~A side result. Let f be and expanding endomorphism of a finitely generated nilpotent group π and let $X = G/\pi$. If θ is an automorphism of X,f Then $\theta(x) =$~~

Let f' be the map on M' lifting f. The problem posed by Hirsch is to show that the closure in the group of homeomorphisms of $U(f')^{-n}\pi(f')^n$ is a Lie group whose connected component acts transitively and freely on M'. It is very likely than that f8 induces an expanding map on g, and consequently that G is nilpotent. Letting π_0 be the subgroup of π which lies in G_0, it follows that $U f^{-m}\pi_0 f^m$ is dense in G_0 and hence that G_0 is the real Malcev completion of π_0. Consequently M'/π_0 is compact and so π/π_0 is finite and everything is clear.

Nedd to know (i) closure of $Uf^{-n}\pi f^n = G$ is a Lie group on which f induces an expansion (ii) the connected component of G acts transitively on M'.

Problem: Show that $\{u: f^m u f^{-m} \longrightarrow id\}$ is a normal subgroup of π of finite index.

Clearly a subgroup since in a topological group if x_n and y_n are tending to 1 so is $x_n y_n^{-1}$. It should be normal because for any element v the sequence $f^m v f^{-m}$ should be bounded?, i. e. it should lie in a compact subset of Aut M. This subgroup is stable under f. ~~Finally it~~ Finally ~~it show~~ how can we show it's of finite index.

Conjecture; Show that if K is a compact subset of M', then the set of elements of $U f^{-n}\pi f^n$ which send 0 into K is relatively ~~compact~~ compact in the group of homeomorphisms of M. From this it will follow that G is a locally compact group. Moreover it will follow that the isotropy group H at 0 is compact. On the other hand it is clear that G acts transitively, Hence $G/H = M'$ is Eucildean. But H acts on M' faithfully and

Question: Is there any hope of constructing a metric?

Suppose I can find a locally determined metric on S^1 such that

$$\rho(3x, 3y) = \alpha\, \rho(x,y)$$

Claim that $\alpha = 3$.

Proof: Define a measure μ on S^1 by

$$\mu([x,y]) = \rho(x,y). \qquad \text{for } x \text{ and } y \text{ close together.}$$

Then $\mu(3I) = \alpha\, \mu(I)$. for any interval I.

Now choose the three ~~quadrants~~ arcs. Then for each arc

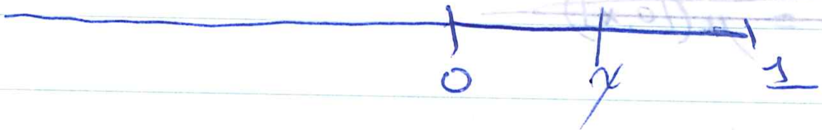
$$\mu(S^1) = \mu(3I) = \alpha\, \mu(I)$$

so $$\mu(I_1) + \mu(I_2) + \mu(I_3) = \mu(S^1)$$

$$\frac{3}{\alpha} = 1 \Rightarrow \alpha = 3$$

Suppose $f: S \to S$ is expanding.

does there exist a *positive* measure μ on S^1 such that
μ has no atoms, $\mu[x,y]=0 \Longrightarrow x=y$.
$\mu([3x,3y]) = 2\mu([x,y])$.



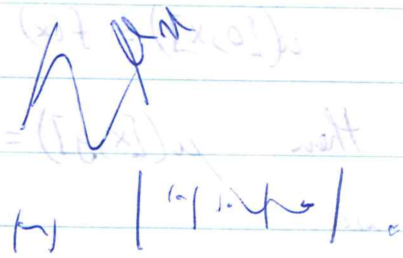
$$\int_0^x \mu = f$$

2

3

$f(2)=$

$$x = \sum{}' a_j 3^j$$



$$f(x) = \sum{}' a_j 2^j$$

Then

$$f\left(\tfrac{x}{3}\right) = \tfrac{1}{2} f(x).$$

so $f(3x) = 2f(x)$ ← all x.

$$\mu([3x,3y]) = f(3x) - f(3y) = 2f(x) - 2f(y) = 2\mu([x,y]).$$

$f(3x) = 2f(x)$?

$$f(3x) = (3x)^{\frac{\log 2}{\log 3}} = 3^{\frac{\log 2}{\log 3}} \cdot x^{\frac{\log 2}{\log 3}} = 2f(x).$$

want all monotone solutions of this equation.
Note: $f(0)=0$.

under what conditions is the measure going to pass to S^1

ie $f(x+1) - f(x)$ constant fn. of x.

$$\begin{array}{ccc} C(M,M) & \longleftarrow & \Gamma(TM) \\ \exp X & \longleftarrow\!\shortmid & X \end{array} \qquad \text{homeo near } 0.$$

so given f Anosov define

$$\begin{array}{ccc} C(M,M) & \longrightarrow & C(M,M) \\ \varphi & \longmapsto & f\circ\varphi\circ f^{-1} \end{array}$$

has differential $\varphi \longmapsto Df\circ s\circ f^{-1}$, at identity
which is hyperbolic if f is

<u>Thm</u>. (Generalized Stable Manifold) f Anosov. $\forall x$ $\exists^{1-1}$ immersed submanifold $W^s(x) \subset M$

a) $y \in W^s(x) \Leftrightarrow d(f^n x, f^n y) \to 0$ as $n \to \infty$

b) $f\, W^s(x) = W^s(f(x))$.

c) $\bigcup W^s(x) = M$

d) $\forall\, x, y \in M$ $\quad W^s(x) \cap W^s(y) \neq \phi \Longrightarrow W^s(x) = W^s(y)$.

e) $W^s(x)$ tangent to T^s_x

f) $W^s(x)$ and $W^s(y)$ are C^1 close for x, y close.

$$1 + \sum_{n=1}^{\infty} n^{-2z} (\gamma^n + \bar{\gamma}^n) \qquad |\gamma| = 1.$$

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

χ a character of $(\mathbb{Z}/q\mathbb{Z})^*$

Assume $\chi(mn) = \chi(m) \cdot \chi(n)$

$\chi(m$

Suppose $\gamma = e^{\frac{2\pi i}{p}} = \zeta$

define

$$\chi(j) = \begin{cases} \zeta^j & j \equiv 0 \quad (p) \\ 0 & j \equiv 0 \quad (p). \end{cases}$$

$$\sum_{n=1}^{\infty} n^{-s} \gamma^n = L(s, \chi) + \sum_{m=1}^{\infty} (pm)^{-s}$$

$$= L(s, \chi) + p^{-s} \zeta(s).$$

Thus want to calculate

$$1 + \sum_{n=1}^{\infty} n^{-2z} (\gamma^n + \bar{\gamma}^n) = \begin{array}{l} 1 + \\ L(2z, \chi) + p^{-2z} \zeta(2z) \\ L(2z, \bar{\chi}) + p^{-2z} \zeta(2z) \end{array}$$

$\zeta(0) = -\frac{1}{2}$

now as $z \to 0$ this

clearly $\zeta(s)$ has a zero at $s=0$

functional eqn.

$$\Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \underbrace{\zeta(s)}_{\substack{\text{simple pole} \\ s=0}} = -\Gamma\left(\frac{1-s}{2}\right) \pi^{\frac{1-s}{2}} \underbrace{\zeta(1-s)}_{\text{pole } s=0}.$$

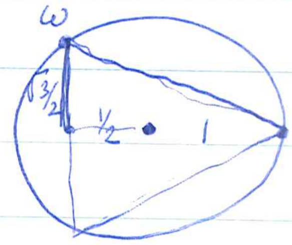
$\zeta(0) \neq 0.$

$$\sum n^{\delta} z^n$$

$\frac{1}{2} - 1$

$$1+\sum_{n=1}^{\infty}(1+n^2)^{-z}(\lambda^n+\lambda^{-n})$$

$$(1+in)^{-z}(1-in)^{-z}(\lambda^n+\lambda^{-n})$$



$\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$1-\omega = \frac{3}{2} + i\frac{\sqrt{3}}{2}$

calculate

tr T_n on $\Gamma(1)_n$

$\Gamma(1)_n = \{\alpha z^n + \beta \bar{z}^n\}$.

$z \mapsto \gamma z$

$\bar{z} \mapsto \bar{\gamma}\bar{z}$

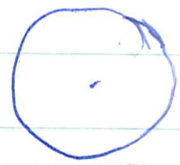
$$1 + 2\frac{\mathrm{Re}\,\gamma - 1}{|1-\gamma|^2} \checkmark$$

$\gamma = i$ $\quad 1+2\frac{-1}{2} = 0$

$\gamma = -1$ $\quad 1+2\frac{-2}{4} = 0$

$$\sum \frac{\gamma^n}{(1+n^2)^z}$$

$\gamma = i$



$$1+\sum_{n=1}^{\infty}(1+n^2)^{-z}(\gamma^n+\bar{\gamma}^n)$$

$$\boxed{\sum_{n=0}^{\infty}\frac{\gamma^n}{1+n^2}}$$

suppose that ~~[illegible]~~ z is real and $> \frac{1}{2}$

then this series converges

$$\sum_n \frac{1}{(1+n^2)^z}$$

$f(z) = e^{2\pi i z} - 1 \qquad \frac{1}{n^z}$

$f'(n) = 2\pi i$

$$\sum_{n=-\infty}^{\infty}(1+n^2)^{-z}\gamma^n = \frac{1}{2\pi i}\oint \frac{dw}{e^{2\pi i w}-1}\,\frac{\gamma^w}{(1+w^2)^{+z}}$$

Therefore go back ~~[illegible]~~ M compact manifold $T: M \to M$ Anosov. Show T has a ~~fixed~~ periodic pt. If not we conclude that $\frac{1}{\det(1-zT_*)} = 1$ by Lefschetz. But in fact if we consider the flat trace of T on $\Gamma(\Lambda^\delta T_*)$ we conclude it must be zero.

$$\sum_{n=0}^{\infty} \binom{n}{k} z^n = \frac{1}{k!}\left(\frac{d}{dz}\right)^k \sum_{n=0}^{\infty} z^n = \frac{1}{k!}\left(\frac{d}{dz}\right)^k \frac{1}{1-z}$$

$$= \frac{1}{(1-z)^{k+1}}$$

$$\sum_{n=1}^{\infty} \frac{1}{(1+n^2)^z} - \sum_{n=1}^{\infty} \frac{1}{(n^2)^z} = \underline{\sum_{n=1}\left\{\frac{1}{(1+n^2)^z} - \frac{1}{n^{2z}}\right\}}$$

$\operatorname{Re} z > \frac{1}{2}$ $\qquad$ $\operatorname{Re} z > \frac{1}{2}$

$$\frac{\dfrac{1}{(1+n^2)^z} - \dfrac{1}{n^{2z}}}{\dfrac{1}{n^{2z}}} = -1 + \frac{1}{\left(1+\frac{1}{n^2}\right)^z}$$

$$= -1 + \left\{1 + \frac{(-z)}{n^2} + \frac{(-z)(-z+1)}{2}\frac{1}{n^4}\right\}.$$

$$= -\frac{z}{n^2} + o($$

$$\therefore \quad \frac{1}{(1+n^2)^z} - \frac{1}{n^{2z}} = \left\{-\frac{z}{n^2} + o\left(\frac{1}{n^3}\right)\right\}\frac{1}{n^{2z}}$$

$$\frac{1}{n^{2z+2}}$$

converges for

$2z+2 > 1$

or $2z > -1$

or $z > -\frac{1}{2}$.

therefore can replace $(1+n^2)$ by n^2.

26 George, street after blinker right

~~notes on~~ Atiyah–Bott.

elliptic complex E^i $d: E^i \to E^{i+1}$

introduce metric and Laplacian's $dd^* + d^*d$.

eigenspaces

$E^\bullet_\lambda$ stable under d because

Hodge thm. $d\Delta = dd^*d = \Delta d$.

so $\Delta x = \lambda x \Longrightarrow$ ~~$d\lambda x$~~ $\Delta dx = d\lambda x = \lambda dx$

$$H^*(E^\cdot_\lambda) = \begin{cases} 0 & \lambda > 0 \\ H^*(E) & \lambda = 0. \end{cases}$$

Now given T form $T_\lambda = \pi_\lambda \circ T \circ L_\lambda$ so that

where $E^\cdot_\lambda \underset{\pi_\lambda}{\overset{L_\lambda}{\rightleftarrows}} E^\cdot$

Then
$$\begin{cases} \operatorname{tr} T^\cdot_0 = \operatorname{tr} H^*T \text{ on } H^*(E^\cdot) \\ \operatorname{tr} T^\cdot_\lambda = 0. & \lambda > 0 \end{cases}$$

Set $\operatorname{tr}^z T^i = \sum_\lambda (1+\lambda)^{-z} \operatorname{tr} T^i_\lambda$ analytic fn. of z.

$\operatorname{tr}((1+\square)^{-s} T^i)$

~~Claim $\operatorname{tr}^z T = \ldots$?~~

Prop: $\operatorname{tr}^0 T^i = \operatorname{tr}^b T^i$ (local sum).

$\sum_i (-1)^i \operatorname{tr}^z T^i = L(T, E^\cdot)$ Lefschetz no.

what about $L(0,\chi) + L(0,\overline{\chi})$.

$$L(s,\chi) = \frac{1}{m}\sum_{k=0}^{m-1} \Big({\sum_{(x,m)=1}}' \chi(x)\,\zeta^{xk} \Big) \sum_{n=1}^{\infty} \frac{\zeta^{-nk}}{n^s}$$

$$\sum_{i=1}^{p-1} \zeta^{i} = 0.$$

to $\gamma \in \pi$ we associate ~~[illegible]~~ the holonomy tranformation $\theta_\gamma : V \to V$ such that $\theta_\gamma(0) = 0$ and

$$\theta_\gamma(\lambda x) = \lambda\, \theta_{f^{-1}\gamma f}(x)$$

now θ_γ is smooth

$$\lambda^{-m}\, \theta_\gamma(\lambda^m x) = \theta_{f^{-m}\gamma f^m}(x)$$

The idea somehow is to show that

$$\{u_m = f^{-m}\gamma f^m\}$$

is a "bounded" family of diffeomorphisms in the sense that

$\|u_m - id\| \leq K$ some constant independent of m.

But

$$\cancel{\|f^{-m}\gamma f^m x - x\| \leq \frac{1}{\mu^m}\|\gamma f^m x - f^m x\|}$$

$$\cancel{\lambda^m \mu 0}$$

$$\lambda^{-m}\theta_\gamma \lambda^m \log \mu(0) = \lambda^{-m}\theta_\gamma \log f^m \mu(0)$$

f^m

$$\lim_{m\to\infty} \theta_{f^m \gamma f^{-m}}(x)$$

$$\dot{\Theta}^{(0)}_{\gamma\gamma}(x) = \dot{\Theta}_{\bar{\gamma}}(\log \gamma 0)\, \dot{\Theta}_\gamma x.$$

$$\frac{1}{\lambda}\Theta_\gamma(\lambda x) = \sum_{n=0}^{\infty} \frac{\lambda^{-n}}{n!}\Theta_\gamma^{(n)}(x)$$

let $f \to +\infty$

$$\| $$

$$\Theta_{f^{-1}\gamma f}(x)$$

let $m \to +\infty$.

$$0 \to \underbrace{W/\Pi_0}_{\text{torus}} \to \underbrace{V/\Pi}_{\text{manifold compact}} \to V/W \,/\, \Pi/\Pi_0 \to 0$$

$\therefore x$

$$\underbrace{\Theta_{f^{-m}\gamma f^m}}_{\text{bdd.}}(x) = \frac{1}{\lambda^m}\Theta_\gamma(\lambda^m x)$$

I seem to have a problem: namely the operators

$$\{ f^{-m}\gamma f^m \qquad m \geq 0 \}$$

Should be bounded because f expands

Some estimates needed. If can be done, then it follows that $\frac{1}{\lambda^m}\Theta_\gamma(\lambda^m x)$ is bounded and hence(?) that Θ_γ is linear.

Assume I can show Θ_γ linear. It follows that Π is a set of affine transformations on V which acts discontinuously so that the quotient is compact. To show Π_0 = translations in Π. Then Π/Π_0 is finite. Let W be subspace generated by $\Pi_0 0$. Π Π preserves W and $\therefore$ acts on V/W. $\therefore$ Π/Π_0 acts on V/W so that quotient is compact.

$\therefore$ Π/Π_0 finite + $W = V$. impossible

$0 \in \tilde{M}$

define a linear map θ_γ from $V = T_0\tilde{M}$ to itself as follows:

θ_γ = difference.

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\gamma} & \tilde{M} \\ \uparrow \exp_0 & & \uparrow \exp_0 \\ T_0\tilde{M} & \xrightarrow{\log_0^{-1}\gamma 0} & T_0\tilde{M} \end{array}$$

This pen is hell

Yes

$$\theta_\gamma x = \exp_0^{-1} \gamma(\exp_0 x) - \exp_0^{-1}(\gamma 0)$$

Certainly maps V into itselfs and leaves 0 fixed.

$$\theta_\gamma(\lambda x) = \exp_0^{-1}(\gamma f \exp_0 x) - \exp_0^{-1}(\gamma 0)$$

$$= \exp_0^{-1}(f \cdot f^{-1}\gamma f(\exp_0 x)) - \text{''}$$

$$= \lambda\{\exp_0^{-1}(f^{-1}\gamma f(\exp_0 x)) - \exp_0^{-1}(f^{-1}\gamma f(0))\}.$$

$$\boxed{\theta_\gamma(\lambda x) = \lambda\, \theta_{f^{-1}\gamma f}(x).}$$

$f^{-1}\gamma f$

"small displacement"

This is the wrong map. The one I had in mind is definitely linear.

$$\begin{array}{ccc} \tilde{M} & & T_{\gamma 0}(\\ \uparrow & & \uparrow \\ T_0\tilde{M} & \xrightarrow{d\gamma} & T_{\gamma 0}\tilde{M} \end{array}$$

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$$\theta_{\bar\gamma}\theta_\gamma x = e_0^{-1}(\gamma e_0(\theta_\gamma x)$$

$$\boxed{\gamma(\exp x) = \exp(\theta_\gamma x + \log \gamma 0)}$$

$$\gamma(\exp \lambda x) = \gamma(f \exp x) = f(f^{-1}\gamma f)(\exp x)$$

$$= f \exp(\theta_{f^{-1}\gamma f} x + \log f^{-1}\gamma 0)$$

$$\exp(\theta_\gamma \lambda x + \log \gamma 0) = \exp(\lambda \theta_{f^{-1}\gamma f} x + \log \gamma 0)$$

$$\theta_\gamma(\lambda x) = \lambda \theta_{f^{-1}\gamma f} x$$

$$\bar\gamma(\gamma \exp x) = \bar\gamma \exp(\theta_\gamma x + \log \gamma 0)$$

$$\exp(\theta_{\bar\gamma\gamma} x + \log \bar\gamma\gamma 0) = \exp(\theta_{\bar\gamma}(\theta_\gamma x + \log \gamma 0) + \log \bar\gamma 0)$$

$$\therefore \theta_{\bar\gamma\gamma} x + \log(\bar\gamma\gamma 0) = \theta_{\bar\gamma}(\theta_\gamma x + \log \gamma 0) + \log \bar\gamma 0.$$

set $x=0$ $\quad \log(\bar\gamma\gamma 0) = \theta_{\bar\gamma}(\log \gamma 0) + \log \bar\gamma 0$

$$\therefore \theta_{\bar\gamma\gamma} x = \theta_{\bar\gamma}(\theta_\gamma x + \log \gamma 0) - \theta_{\bar\gamma}(\log \gamma 0)$$

$$\theta_{\bar\gamma\gamma}(x + \varepsilon y) = \theta_{\bar\gamma\gamma}(x) + \varepsilon \dot\theta_{\bar\gamma\gamma}(x, y)$$

$$\parallel$$

$$\theta_{\bar\gamma}(\theta_\gamma x + \varepsilon \dot\theta_\gamma(x,y) + \log \gamma 0) - \theta_{\bar\gamma}(\log \gamma 0) = \theta_{\bar\gamma}(\theta_\gamma x + \log \gamma 0) + \dot\theta_{\bar\gamma}(\theta_\gamma x + \log \gamma 0, \dot\theta_\gamma \ldots)$$

~~classify two dimensional expanding maps.~~

suppose that $\phi: M \to M$ has the property that

$d\phi_p : T_pM \to T_pM$ is a scalar λ. $\lambda > 1$

Then consider $\tilde{M} \simeq V$ with $T_pM. \simeq V$

and we have transformations $\gamma \in \pi$ γ acts on V

π acts on V without fixed points
+ discontinuously

and $\lambda \pi \lambda^{-1} \subset \pi$

$$\lambda(\gamma x) \subset \pi(\lambda x).$$

now define a map

given γ consider the map

$$V = T_0(V) \xrightarrow{d\gamma} T_{\gamma(0)}(V)$$

$$(d\gamma)(v) = \gamma(0) + \theta_\gamma(v)$$

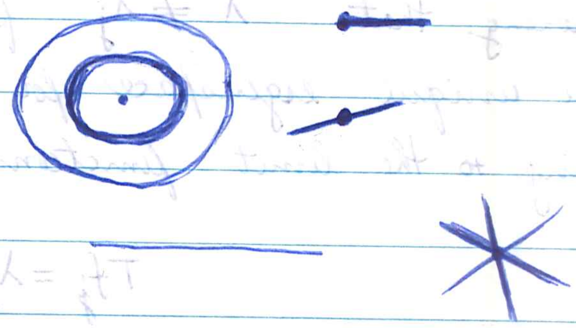
Therefore there exist contractions which are not-linear.

φ $\tilde{M}$ $\boxed{\text{assume generic eigenvalues}}$

Then $\tilde{M}$ has a unique linear structure. with a frame which is invariant under φ.

~~eigendistribution~~

$\tilde{M}$ has φ acting on it



take eigen-space thru origin

this gives an invariant immersed submanifold of M

consisting of points which converge to M λ-quickly.

So it should be possible to prove that these eigenspaces must be preserved.

$$\begin{array}{ccccc} \pi_0 & \longrightarrow & \pi & \longrightarrow & \pi/\pi_0 \\ \downarrow & & & & \downarrow \varphi \end{array}$$

$a \in \pi$ $\quad$ $b \in \pi_0$

$$\begin{aligned} \varphi(aba^{-1}) &= \varphi a\, \varphi b \cdot (\varphi a)^{-1} \\ &= \varphi a\, \lambda b \cdot (\varphi a)^{-1} \\ &= \lambda\, \varphi a \cdot b \cdot (\varphi a)^{-1} \end{aligned}$$

$$\begin{aligned} \varphi^n(aba^{-1}) &= a \cdot \lambda^n b \cdot a^{-1} \\ &= \lambda^n (aba^{-1}) \end{aligned}$$

$\Longrightarrow$ $aba^{-1} = \mu b,$

$\therefore$ ~~φ~~ But $\varphi^n a \in a\pi_0$ for $n =$ order of φ on π/π_0 ~~large~~

so

$\mu =$ root of 1

$\Longrightarrow \mu = \pm 1.$

$(\exp tX)$

Construction: Write

$$\frac{d}{dt}\left\{ f\circ(\exp tX)^{-1} \right\} =$$

$$W = W^u \oplus W^s$$

and construct W^u_{loc} as image of a map

$$W^u \longrightarrow W^s$$

$$W^u_{loc} = \{\text{graph } g \quad g : W^u \to W^s\}.$$

Then graph g $\exists$ and is !

Next

$$f\circ(\exp tX)\circ f^{-1} = \exp tY$$

$Y = df(X)$. clear.

$$\Gamma T(M) \longrightarrow \mathrm{Hom}(M,M) \quad \text{iso at } 0 \longmapsto I.$$

$$X \longmapsto \exp X.$$

$$X$$

$$\theta \longmapsto f\circ\theta\circ f^{-1}$$

identity is a fixed point even hyperbolic

Hence $\exists$

---

want global choice of stable submanifolds ie. an ~~iso~~ map

$$\begin{array}{ccc} T^s & \longrightarrow & M\times M \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle pr_1} \\ M & \xrightarrow{id} & M \end{array} \qquad \Gamma(T^s) \longrightarrow \mathrm{Hom}(M,M).$$

Conclusion: M smoothly Anosov. Then the distributions T^s and T^u are integrable and there are maps

$$\begin{tikzcd} T^s \arrow[rr, "\Theta^s"] \arrow[dr, "\pi"'] & & M \times M \arrow[dl, "pr_1"] \\ & M & \end{tikzcd}$$

also for u

f equivariant such that $\Theta^s(T^s_p) = W^s_p$ all $P \in M$.

we have a map

$$\Gamma(T^s) \longrightarrow \mathrm{Hom}(M, M)$$

sections of π $\longrightarrow$ sections of pr_1

Question: Does Θ^s have the property that

$$\boxed{\Gamma\Theta^s = \exp \ ?}$$ probably not.

Question: M manifold with a contraction f. Let P be the unique fixpt. Is (M, f) isomorphic to $(T_p(M), df)$?
The answer is NO — see Sternberg

Anosov diffeomorphisms

$$T = T^s \oplus T^u$$

$$J_1(T) \simeq J_1(T^s) \oplus J_1(T^u)$$

$$0 \longrightarrow T^s \otimes T^* \longrightarrow J_1(T^s) \longrightarrow T^s \longrightarrow 0.$$

$$0 \longrightarrow T^{10} \otimes T^* \longrightarrow J_1(T^{10}) \longrightarrow \underset{\text{Con.}}{\underline{T^{10}}} \longrightarrow 0$$

first problem

An Anosov manifold has an exponential map

$$\boxed{\exp : T(M) \longrightarrow M \times M}$$

defined by stable and unstable manifolds.

Don't understand! understand algebra involved

Review Shub

stable manifold theorem. f hyperbolic local diffeomorphism around the origin of a Banach space W, (i.e. $f: U \longrightarrow W$ C^1 U nbd of 0 $f(0)=0$ and $Df(0): W \xrightarrow{\sim} W$.) Then there exists unique W^s_{loc} germ of submanifold at W stable under f tangent at 0 to E^s.

V

Suppose $T = T^s \oplus T^u$

let $X \in \Gamma(M, T^s)$. Then if $P \in M$ is a fixpt. of f, can we conclude that the curve

$$t \longmapsto e^{tX} P$$

lies in $W^s(P)$?

Keep t fixed.

$$f^m(e^{tX}P) = \left[f^m \circ e^{tX} \circ (f^{-1})^m\right] P$$

$$= e^{t(f^m X)} P$$

~~but $f^m X$ not proportional to X.~~

~~So~~ Let $m \to \infty$

as $X \in \Gamma(M, T^s)$

$$\|f^m X\| \longrightarrow 0$$

So $f^m(e^{tX}P) \longrightarrow P$

and therefore $e^{tX}P \in W^s(P)$.

Therefore if we choose

~~Note also that if $x, y \in$ same~~

Recall that x, y belong to same $W^s \Longleftrightarrow d(f^m x, f^m y) \to 0$ as $m \to \infty$.

so $d(f^m P, f^m e^{tX} P) = d(f^m P, \underline{e^{tf^m X}} f^m P) \longrightarrow 0$ tending to 0 uniformly.

$$\exp_0 : V \xrightarrow{\sim} \tilde{M}$$

$$\begin{tikzcd}
T_0 V \arrow[r, "d(\exp_0)_0", "\sim"'] \arrow[d, "t\exp_0^{-1}\gamma 0"] & T_0 \tilde{M} \arrow[d, "d\gamma"] \\
T_{\exp_0^{-1}\gamma 0}(V) \arrow[r, "\sim", "d(\exp_0)_{\exp_0^{-1}\gamma 0}"'] & T_{\gamma 0}\tilde{M}
\end{tikzcd}$$

$$\begin{tikzcd}
\tilde{M} \arrow[r, "r"] & {} \\
V \arrow[u, "s"] &
\end{tikzcd}$$

$r = \exp_0^{-1}(\gamma 0)$

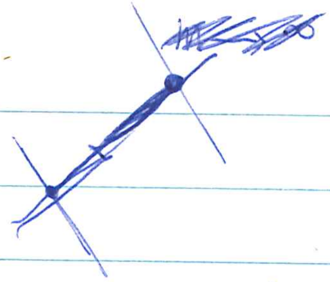
~~On the other~~ ~~So the map I w~~

I claim this need map is only the derivative of the first at 0.

Now form $d\Theta$.

$$\dot{\Theta}_\gamma(\lambda x) = \lambda \dot{\Theta}_{f^{-1}\gamma f}(x)$$

$$\therefore \boxed{\dot{\Theta}_\gamma = \dot{\Theta}_{f^{-1}\gamma f}.}$$



$\boxed{\Theta_{f^{-m}\gamma f^m}(x) \longrightarrow 0 \text{ if } x \longrightarrow 0 \quad \text{unif. in } m.}$

constant

$$\Theta_\gamma(\lambda^m x) = \dot{\Theta}_\gamma(x) + \underline{\lambda^m x^2 a} + \lambda^{2m} x^3$$

$\lambda^m x^2 \to \infty$

$\lambda^{2m} x^3 \longrightarrow 0$

$f^m \gamma f^m \to \gamma$

$\cancel{\lambda^m x_m^2 \cdot \lambda^{2m} x_m^3 \longrightarrow 1}$

$\cancel{\lambda^{3m} x_m^5 \longrightarrow 1}$

$\cancel{x_m \sim \lambda^{3/5 m}}$

$\lambda^m x^2 \to 1 \qquad x \to 0$

$\lambda^{2m} x^4 \to 1$

$\lambda^{2m} x^3 \to \infty.$

$$\lim_{m\to\infty} \lambda^{-m}\Theta_\gamma(\lambda^m x) = A(x)$$

---

can't seem to let $m \to \infty$. so try $m \to -\infty$.

$$\underline{\lambda^{-m}\Theta_\gamma(\lambda^m x) = \Theta_{f^{-m}\gamma f^m}(x).}$$

~~$m \to \infty$~~ If Θ_γ linear then

$$\Theta_{f^{-m}\gamma f^m} = \Theta_\gamma$$

In other words f induces the identity on the rotational part.

Prop: $X = G/\Gamma$ compact nilmanifold with expanding map f.
Suppose ~~(finite) group~~ η acts freely on X so that f passes down to $X/\langle\eta\rangle = M$. Then η comes from ~~a linear~~ an auto ~~[illegible]~~ of (G,f) followed by a right translation ~~in~~ from the normalizer of Γ.

□

f Anosov diffeo of M. Assume $T = T^s \oplus T^u$ is smooth. ~~Th~~ Then these distributions are smooth and define a two transversal foliations of M and hence of $\tilde{M}$. By Pugh + structural stability Ω = closure of periodic points. Hence M has ~~a~~ a periodic point, + may assume replacing f by f^n, that ~~f~~ f has a fixed point P which we use as basepoint for $\tilde{M}$. Assertions

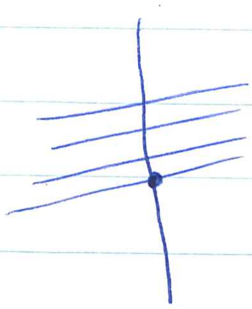
(i) no homoclinic points in $\tilde{M}$ except P.

(ii) P only fixed point of f

(iii) A global coordinate system ~~of~~ on $\tilde{M}$ is obtained by the transversal foliations.

If our picture is correct, every Anosov diffeo. has a fixed point.

Proof: By Anosov-Pugh, f has a periodic point O. Let $(\tilde{M}, O)$ be universal covering. Our picture asserts that the unstable manifold through O cuts each stable manifold in exactly one point.



~~But on $W^s(O)$ then f is a contraction, hence there is a unique periodic fixed point~~

~~the~~ Now consider the mapping g induced on $W^u(O)$ by taking f on $\tilde{M}$ modulo the stable equivalence relation. Clearly $g^n = f / W^u(O)$ so g is ~~contract~~ expanding and so g has a unique periodic point which must be O. Thus f fixes the leaf $W^s(O)$ and is a contraction so O is a fixed point of f and so (M, f) has a fixed point.

measure preserving transformations

H, R, T, σ ergodic   Transitive Z set.

and a map of transitive G sets is always surjective.

$$f : (H, R, T, \sigma) \longrightarrow (H_1, R_1, T_1, \sigma_1)$$

$$X \longrightarrow Y$$

$$H_2 \hookleftarrow H_1 \qquad \text{isometry}$$

$$R_2 \hookleftarrow R_1$$

commute with T, T_1

$$\sigma \longleftarrow \sigma_1$$

A factor automorphism. No notion of conjugacy $\boxed{\text{since abelians.}}$ To such an f there is an f_* map called conditional expectation.

Such an (H, R, T, σ) is an ergodic source.

Fdl Thm of Information theory theory says given ~~#~~ a ~~source~~ channel can transmit from one source to another.

A channel is a different kind of morphism.