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If $p \nmid n$

$$\mathrm{Ker}\, n\delta \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}$$

$$g = \dim X$$

~~Pf~~ Another proof by Russell ~~[illegible]~~

$d(n\delta) = n$ is injective on differentials so OKAY if $n \nmid p$.

to show $p\delta : X \to X$ surj. If not w.m.a $p\delta \equiv 0$.

$g = \dim X$. Need

$\dim H^1(\mathcal{O}_Y) = \dim Y$ all ab. var.

If $pX = 0$ take $g+1$ independent points of order p

$$H = (\mathbb{Z}/p\mathbb{Z})^{g+1} \subset X$$

Set $Y = X/H$

Use $\forall$ n.s. proj. Z p-cyclic coverings classified by

$$\{\alpha \in H^1(Z, O_Z) \ni F\alpha = \alpha\}.$$

Thus get $g+1$ independent elts in $H^1(Z, O_Z)$ contradicts.

§ 4. Λ, H and $\mathcal{G}$

~~Thm~~ $x \in X$ $\quad T_x : X \to X$ translation by x

<u>Theorem of square</u>: $\forall L$ on X

$$T_{x+y}^* L = T_x^* L \otimes T_y^* L \otimes L^{-1}$$

$$T_{x+y}^{-1} D \underset{\text{lin equiv.}}{\simeq} T_x^{-1} D + T_y^{-1} D - D$$

<u>Proof</u>: Restrict $p_{123}^* L = p_{12}^* L \otimes \cdots$
to $\{x\} \times \{y\} \times X$.

<u>Defn</u>: $\forall L$ on X

$$M = p_{12}^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1} \quad \text{on } X \times X$$

Universal property of Pic scheme

$\Rightarrow \exists\ \Lambda(L) : X \longrightarrow \underline{\mathrm{Pic}}\, X$

$\ni \boxed{(1 \times \Lambda(L))^* P \simeq M}$ $\quad P =$ Poincare inv. sh. on $X \times \mathrm{Pic}\, X$

note $\Lambda(L)(0) = 0$, X conn. $\Rightarrow \Lambda(L)(X) \subset \underline{\mathrm{Pic}}^0 X$.

$\Lambda(L)(x) =$ pt. of $\underline{\mathrm{Pic}}^0 X$ corresponding to $T_x^* L \otimes L^{-1}$.

Thm of square says $\Lambda(L)$ is a homomorphism.

$\Lambda(L\otimes M) = \Lambda(L)+\Lambda(M)$

<u>Defn</u>: $H(L) = \mathrm{Ker}$ of $\Lambda(L)$.

$x \in H(L) \iff T_x^* L \simeq L$.

true also if x is an R valued point.

$$T_x^* L \simeq L \otimes p_2^* K \qquad K \text{ on } \mathrm{Spec} R$$

on $X \times \mathrm{Spec} R$.

If L inv. sheaf on X, $\mathbb{L}$ be principal bundle

$$0 \to \mathbb{G}_m \to \mathbb{L} \to X \to 0$$

<u>Defn</u>: $\mathcal{G}(L) = \mathbb{L}\,|\,H(L)$

$*$ { algebraic group scheme $/k$
acts on $\mathbb{L}$.

<u>Pf</u>: Define $\sigma: \mathcal{G}(L) \times \mathbb{L} \to \mathbb{L}$ by restricting

$\not{\!\!4}$ to $X \times H(L)$. Get M trivial $X \times H(L)$

or equivalently

$$p_{12}^* L \simeq p_1^* L \otimes p_2^* L \quad \text{on } X \times H(L).$$

(this isom. unique if we choose $l_0 \in \mathbb{L}_0$ or equivalently $L_0 \otimes k(0) \xleftarrow{\simeq} k$ and ask that above isom. because id on $0 \times H(L)$.

This induces

$$p_{12}^*\mathbb{L} \xrightarrow[\sim]{\Phi} p_1^*\mathbb{L}\otimes p_2^*\mathbb{L}$$

So now to define σ

$$\begin{aligned}
\sigma:\ \mathbb{L}\times \mathcal{G}(L) &= p_1^*L \underset{X\times H(L)}{\times} p_2^*\mathbb{L} \\
&\longrightarrow p_1^*\mathbb{L}\otimes p_2^*\mathbb{L} \\
&\xrightarrow[\Phi]{\sim} p_{12}^*\mathbb{L} \\
&\underset{\text{defn.}}{=\!=} \mathbb{L}\times_X (X\times H(L), p_{12}) \\
&\longrightarrow \mathbb{L}
\end{aligned}$$

Remark: $\mathcal{G}(L) \simeq \underbrace{\mathrm{Aut}}_{\text{functor}}(\mathbb{L}/X)$ — autos of arrow $\mathbb{L}\to X$ which are translations on X

Exact sequence of group schemes

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathcal{G}(L) \longrightarrow H(L) \longrightarrow 1$$

Example: $A(L)\equiv 0$

$H(L) = X.$

$\mathcal{G}(L) = \mathbb{L}$ so $\mathbb{L}$ has a group structure

Check $\mathbb{L}$ commutative

(III.) Application to descent

$$X \xrightarrow{\ f\ \ \text{surjective}\ } Y = X/H \qquad H \subset X \text{ finite subgp. scheme.}$$

Given L on X, question how many M on $Y \ni f^*M = L$.

~~[illegible]~~ If so make H act on $\mathbb{L} \simeq X \times_Y M$ by

$$\theta(x,m) = (\theta x, m) \qquad \theta \in H.$$

So get action of H on $L \ni$

a) $\mathbb{L} \to X$ is H-equivariant

b)
$$\begin{array}{ccc} \mathbb{L}/H & \longrightarrow & X/H \\ \parallel & & \parallel \\ M & \longrightarrow & Y \end{array}$$

Conclusion: 1-1 correspondence ~~between M's and~~ actions of H on $\mathbb{L}$ covering translation action of H on X and M's on Y + ~~[illegible]~~ isos. $f^*M = L$.

Necessary condition to have lifting of H on X to H on $\mathbb{L}$ is that $H \subset H(L)$. Then have to lift in

$$\begin{tikzcd} & \mathcal{G}(L) \arrow[d] \\ H \arrow[ur, dotted, "\tau"] \arrow[r, phantom, "\subset"] & H(L) \end{tikzcd}$$

Cor: If $\Lambda(L)=0$ then $\forall f: X \to Y$ isogeny $\exists M$ on $Y \ni f^*M \simeq L$.

Proof: All $H \subset X$ finite have to show

$$\begin{tikzcd} & L \arrow[d] \\ H \arrow[ur, dashed, "\tau"] \arrow[r, phantom, "\subset"] & X \end{tikzcd}$$

So enough to know $\mathrm{Ext}^1(H, \mathbb{G}_m) = 0$ in category of commutative alg. gp. schemes. (Use composition series to reduce to simple cases.)

Tate

k $|\ |$ complete, C complete valued $\supset k$, $C = \bar{C}$.

<u>Examples of rigid analytic spaces</u>

1. global holomorphic function on C

$$\varphi(v) = \sum_{\nu=0}^{\infty} a_\nu v^\nu \qquad a_\nu \in k$$

entire if $|a_\nu| \rho^\nu \to 0 \quad \forall \rho \in \mathbb{R}$.

2. global holom. function on $C^* = C - 0$

$$\varphi(v) = \sum_{m \in \mathbb{Z}} a_m v^m \qquad \text{everywhere cgt. } v \in C^*$$

These form a ring $\mathcal{L}$ holom. fns. on C^* defd $/k$.

Let $\mathcal{K}$ = q.f. of $\mathcal{L}$ = meromorphic fns. on C^*.

If $b \in k^*$ $\qquad \varphi_b(v) = \varphi(b^{-1}v)$.

Let $q \in k^*$ $\quad 0 < |q| < 1$.

$$q^{\mathbb{Z}} = \{q^n / n \in \mathbb{Z}\}.$$

<u>Proposition:</u> $C^*/q^{\mathbb{Z}}$ is an abelian variety of dim $1/k$.

Let K_q = merom. fns. on $\mathbb{C}^*/q^{\mathbb{Z}}$

= " " " $\mathbb{C}^*$ invariant by q.

preceding result precisely

<u>Theorem</u>: ① $\exists !\ A_q$, ab var of dim 1 / k with $k(A_q) = K_q$
$\ni\ A_q(\mathbb{C}) \simeq \mathbb{C}^*/q^{\mathbb{Z}}$.

In fact $A_q(k_1) \simeq k_1^*/q^{\mathbb{Z}}$ $\forall$ complete k_1 $k \subset k_1 \subset \mathbb{C}$

② $K_q = k(x,y)$

$$x(v) = \sum_{m\in\mathbb{Z}} \frac{q^m v}{(1-q^m v)^2} - 2\sum_{m=1}^{\infty} \frac{q^m}{1-q^m}$$

$$y(v) = \sum_{m\in\mathbb{Z}}' \frac{(q^m v)^2}{(1-q^m v)^3} + \sum_{m=1}^{\infty}{}' \frac{q^m}{1-q^m}$$

Weierstrass form which works in all characteristics

$$\begin{cases} y^2 + xy = x^3 - b_2 x - b_3 \\ b_2 = 5\displaystyle\sum_{n=1}^{\infty}{}' \frac{n^3 q^n}{1-q^n} \\ b_3 = \displaystyle\sum_{n=1}^{\infty} \left(\frac{7n^5+5n^3}{12}\right)\frac{q^n}{1-q^n} \end{cases}$$

Classically

$$\wp(u) = x + \frac{1}{12} \qquad v = e^u$$

$$\wp'(u) = x + 2y$$

$$j_q = \frac{(1+48b_2)^3}{\Delta} \qquad \Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24}$$

$$j_q = \frac{1}{q} + 744 + 196884\, q + \cdots$$

(3) $k = \mathbb{C}$ get all elliptic curves

<u>k non-archimedean:</u> get for $k = \mathbb{C}$ all curves with $|j| > 1$.

if $|k|$ is discrete get those A's whose Neron model has $\mathbb{G}_m$ as connected component of special fibre.

Only units in $\mathcal{L}$ of form $a\sigma^m$.

S theory for groups.

defns: ~~S-equiv is a map~~
category = reduced s. gps.
S-equiv. = map $\ni$ $\Updownarrow$ $\pi_* f \otimes S^{-1}\mathbb{Z}$ iso
$H_*(f, S^{-1}\mathbb{Z})$ ".

S-fibration = red-fibration $\ni$ $\begin{cases} \pi_q \operatorname{Ker} f & S\text{! div} \\ \operatorname{Coker} \pi_1 f & S \text{ torsion free.} \end{cases}$

cofibration = ordinary cofibration.

Remarks: S-equiv + S-fib $\Longleftrightarrow$ t.f. $\Longrightarrow$ M1 for c. + t.f.
t.f. characterized by RLP with $\langle d\sigma^q \rangle \hookrightarrow \langle \sigma^q, d\sigma \rangle$ $q \geq 2$
$\langle 0 \rangle \hookrightarrow \langle \sigma^1, d\sigma = 0 \rangle$ $q = 1$
hence get M2 fact. for c + t.f.

M0, M5 clear as well as retract.

~~Obstruction theory~~

~~Hom X^K~~ $\operatorname{Hom}_{red}(\Delta(q)/\Delta(q)^{(0)}, X^K)$

$= \operatorname{Hom}\left(\Delta(q)/\Delta(q)^{(0)} \times K, X\right)$

$r(X^K) \subset X^K$

consist of $\Delta(q) \times K \longrightarrow X$

should be $E_0(X^K)$ $\ni$ $\Delta(q)^0 \times K \hookrightarrow \{x_0\}$.

Thus $\boxed{X \wedge K = \frac{X \times K}{\{x_0\} \times K}}$

$$X^K = E_0(X^K)$$

$$Y \longrightarrow X^K$$

$$\ni \quad Y \times K \longrightarrow X$$

$$\{y_0\} \times K \longrightarrow \{x_0\}.$$

this approach seems to lead to <u>confusion</u>.

Don't care about simplicial ~~structure~~

~~Key~~ lemma 1. S-fib = fib wrt RLP wrt $M(s,q) \to CM(s,q)$ $q \geq 1$ $s \in \mathbb{Z}$ and $S^q \to MC(s,q)$ $q \geq 1$. $s \in \mathbb{Z}$.

lemma 2: Cobase extension by an S-equiv + cof. is same. — need basic lemma on long exact sequence for homology

$H_q(\ \ ) \longrightarrow \quad \longrightarrow \quad \longrightarrow$

Kleiman on Pic

Thm. X proper over k = alg cl. field, L inv. sheaf on X, eq. cond

a) L τ equiv. to O

b) $\chi(F \otimes L) = \chi(F)$ all coh. F

c) L num. equiv. O.

Reduction to X projective. Choose $f: X' \to X$ Chow
f birational proper, X' projective.

$$L \ \tau \text{ equiv } O \iff f^*L \underset{\tau}{\sim} O$$

$$L \underset{num}{\sim} O \iff f^*L \underset{num}{\sim} O$$

$b_{X'} \Rightarrow b_X$. Consider $0 \to K \to F \to f_*f^*F \to G \to 0$

isom on open dense set.

By induction on $\dim X$ may assume b_X for K, G, $R^+f_*f^*F$

Leray

$$\chi(f^*F \otimes f^*L) = \sum(-1)^q \chi(R^q_{\ } f_* f^*F \otimes L)$$

$$\|$$

$$\chi(f^*F) = \sum(-1)^q \chi(R^q f_* f^*F)$$

$\Rightarrow b_X$ for $f_*f^*F \Rightarrow b_X$ for F.

Thm. X proj. H ample L as above, eq. cond:

a), b), c)

d) $L^{\otimes m} \otimes H$ ample for all m.

If further X is irreducible

e) $\chi(L^{\otimes m} \otimes H^{\otimes n}) = \chi(H^{\otimes n})$

f. $\begin{cases} (L \cdot H^{(n-1)}) = 0 \\ (L^2 \cdot H^{(n-2)}) = 0 \end{cases}$ $n = \dim X$ (≥ 2) (Hodge index thm.)

Reductions: c. $\Rightarrow$ f.

$$\chi(L^{\otimes n_1} \otimes H^{\otimes n_2} \otimes \cdots \otimes H^{\otimes n_{r-1}}) = \chi(H$$

$$= \cdot \; n_1 n_2 \cdots n_{r-1} + \cdots$$

using $(L_1 \cdots L_n) =$ coeff c ~~of~~ in $\chi(L_1^{\otimes n_1} \cdots L_n^{\otimes n_n}) = c\, n_1 \cdots n_n + \cdots$

f. $\Rightarrow$ c. Reduce to $\dim X = 2$ by Bertini + use Hodge I th.

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§5. Dual

§6. Coh. of inv. sheaves

§7. Curves

§8. Hom(X,Y).

Cor $\mathrm{Pic}^\tau = \mathrm{Pic}^0$.

<u>Theorem</u>: $\Lambda(L) = 0 \iff L$ is num. equiv. to 0. $\iff$ L alg. equiv 0

i.e. $L \leftrightarrow \lambda \in \mathrm{Pic}^\tau_X$.

<u>Proof</u>: $(\Leftarrow)$ L^n corres to $n\lambda \in (\underline{\mathrm{Pic}}^0_X)_{red}$ abelian var.

Generalize Λ: given R^S valued pt. of $\underline{\mathrm{Pic}}^\tau_X$ get family $\Lambda : X \times_S \cancel{\mathrm{Spec}}\, R \longrightarrow \underline{\mathrm{Pic}}^0_X \times_S \cancel{\mathrm{Spec}}\, R$ over $\cancel{\mathrm{Spec}}\, R.\ S$

Take universal case $S = \mathrm{Pic}^\tau_X$ get

$$\begin{tikzcd}
X \times \underline{\mathrm{Pic}}_X \arrow[rr] \arrow[dr] & & \underline{\mathrm{Pic}}^0_X \times \underline{\mathrm{Pic}}_X \arrow[dl] \\
& \underline{\mathrm{Pic}}_X &
\end{tikzcd}$$

Look at $X \times \overset{P}{\overset{\shortparallel}{(\underline{\mathrm{Pic}}^0_X)_{red}}}$ get a hom.

$$\begin{tikzcd}
X \times P \arrow[r] \arrow[rr, bend left] \arrow[dr, "p_1"'] & P \times P \arrow[r, "p_1"] \arrow[d, "p_2"] & P \\
& P &
\end{tikzcd}$$

trivial on $X \times \{0\}$ $\quad 0 \times P$ $\quad \therefore$ map 0.

Conclude $\Lambda(L) = 0$ if $L \in \underline{\mathrm{Pic}}^0 X$. Thus

$L^n \in \mathrm{Pic}^\tau \Rightarrow L^n \in \mathrm{Pic}^0 \Rightarrow n\Lambda(L) = 0 \Rightarrow \mathrm{Im}(\Lambda(L)) < P_n$ finite set

$\Rightarrow \Lambda(L) = 0$.

($\Longrightarrow$) Basic step $\Lambda(L)=0 \Rightarrow \forall n \quad L \simeq M^n$ some M.

this $\Rightarrow$ $Pic^\tau = Pic^0$.

a) $\Lambda(L)=0 \Longrightarrow (n\delta)^*L = L^{\otimes n}$.

Pf:

$$X \longrightarrow X \times X$$
$$x \qquad (x, -x)$$

pull back $\quad p_{12}^*L \simeq p_1^*L \otimes p_2^*L$

get $\quad \mathcal{O}_X \cong L \otimes (-\delta)^*L$.

So use

$$(n\delta)^*M = M^{\frac{n^2+n}{2}} \otimes (-\delta)^*M^{\frac{n^2-n}{2}}$$

$$= L^n.$$

$L \simeq (n\delta)^*M_n$

$$(n\delta)^*\{T_{nx}^*M_n \otimes M_n^{-1}\} \simeq \{T_{nx}^*M_n \otimes M_n^{-1}\}^n$$
$$\wr$$
$$T_x^*L \otimes L^{-1} \simeq 0$$

since $\Lambda(T_y^*M_n \otimes M_n^{-1}) = 0$ (thm 8)

$y = nx$ some x

So $T_y^*M_n \otimes M_n^{-1}$ n torsion $\forall n$.

so $\operatorname{Im} \Lambda(M_n) \subset P_n \to \Lambda(M_n) = 0$

proves ~~the~~ basic step $\qquad$ QED.

Cor: $(n\delta)^*L$ alg. eq. L^{n^2}, esp $(-\delta)^*L$ alg. eq. L.

Deep result.
Y, Z ~~[illegible]~~ cycles on X complementary dim on X.

$$(Y \cdot Z) = (-\delta^{-1} Y \cdot Z)$$

Corollary of etale cohomology theory.

Cor: L ample $\Rightarrow H(L)$ ~~[illegible]~~ finite.

§5. Duals

Theorem (Cartier): $\forall$ isogenies $f: X \to Y$

$\operatorname{Ker} f$ Cartier dual of $\operatorname{Ker}(f^*: \underline{\mathrm{Pic}}^0_Y \to \underline{\mathrm{Pic}}^0_X)$

Pf: Show first that k valued pts. of $\operatorname{Ker} f^*$ corres. bij. to characters of $\operatorname{Ker} f$.

k-valued pt of $\operatorname{Ker} f^*$
is an M on $Y \ni f^*M \simeq \mathcal{O}_Y$

$\longleftrightarrow$ liftings of $\operatorname{Ker} f$ actions on X
to action on $\operatorname{Ker} f$ on $\mathbb{1} = X \times \mathbb{G}_m$

liftings

$$\begin{tikzcd} & X \times \mathbb{G}_m \arrow[d] \\ \operatorname{Ker} f \arrow[ur, dashed] & \subset X \end{tikzcd}$$

same as Char. of $\operatorname{Ker} f$.

In fact theorem is just generalization to S valued points.

Cor 1: $\mathrm{Pic}^\circ X$ is reduced, same dimension as X.

Proof: $P = (\mathrm{Pic}^\circ X)_{red}$ dim h., $\dim X = g$

$$\underset{\substack{\| \\ n^{2g}}}{\operatorname{rank} \operatorname{Ker}(n\delta_X)} = \operatorname{rank}(\operatorname{Ker} n\delta \text{ on } P) = n^{2h} \cdot \underset{\substack{\| \\ p^a}}{(\operatorname{Ker} n\delta \text{ on nilp.})}$$

taking $(n,p) = 1 \Rightarrow g = h$

$n = p^\nu \Rightarrow$ no nilpotent.

Defn: $\hat{X} = \mathrm{Pic}^\circ X$ dual of X.

Cor 2: L ample $\Rightarrow \Lambda(L) : X \longrightarrow \hat{X}$ isogeny.

(Ker finite $\Rightarrow$ surj.)

esp. $\left(L \text{ ample}, M \underset{alg}{\approx} \mathcal{O} \Rightarrow \exists x \in X \ni M \simeq T_x^* L \otimes L.\right.$

Defn: $\hat{f} : \hat{Y} \to \hat{X}$

__Defn__: (X, x_0) (Y, y_0) 2 ptd. vars.
divisorial correspondence in an inv. sheaf on $X \times Y$ trivial on $X \times y_0$, $x_0 \times Y$. Call set of these $B(X, Y)$.

__Prop 1__: X, Y complete n.s.

$$B(X,Y) = \mathrm{Hom}_{ptd}(X, \underline{Pic}^\circ_Y) = \mathrm{Hom}(Y, \underline{Pic}^\circ_X)$$

X abelian, Poincare sheaf P on $X \times \hat{X}$.

~~Defn~~:

$$B(X, \hat{X}) = \mathrm{Hom}(X, \hat{\hat{X}})$$

$$P \in \quad \cong \quad \mathrm{Hom}(\hat{X}, \hat{X})$$

$$P \longleftrightarrow id_{\hat{X}}$$

Let p define $K_X : X \longrightarrow \hat{\hat{X}}$.

__lemma__: K_X isom.

__Defn__: X, Y two ab var, a non-degenerate pairing is a divisorial corr. P on $X \times Y$ defining

$$X \xrightarrow{\sim} \hat{Y} \qquad Y \xrightarrow{\sim} \hat{X}$$

__Lemma__:

$$\begin{tikzcd} \mathrm{Ker}(n\delta_X) \arrow[r, "\sim"] \arrow[d, phantom, "\cap"] \arrow[dr, phantom, "-1"] & (\mathrm{Ker}\, n\delta_{\hat{X}}) \arrow[d, phantom, "\cap"] \\ X \arrow[r, "K_X", "\sim"'] & \hat{X} \end{tikzcd}$$

both are c-duals of $\mathrm{Ker}\, n\delta_X$

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(I.) k alg. cl. complete wrt real abs. value $|\;|$
A/k abelian variety

$R =$ integers, $\bar{k}$ = res.

(A) To show $\exists\ \mathcal{A}$ group scheme over Spec R.

$$\begin{array}{ccccc} A & \subset & \mathcal{A} & \supset & \bar{A} \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} k & \subset & \operatorname{Spec} R & \supset & \operatorname{Spec} \bar{k} \end{array}$$

$\bar{A}$ extension $\quad 0 \longrightarrow \mathbb{G}_m^{h} \longrightarrow \bar{A} \longrightarrow \underset{\text{abelian}}{\bar{B}} \longrightarrow 0$

(B) In case $\bar{A} = \mathbb{G}_m^{g}$ "totally degenerate reduction"
show there is a k-analytic uniformization of A.

Mumford answers these if char $\bar{k} \neq 2$.

Direct proof

$$X \xrightarrow[(id,0)]{s_1} X \times X \xrightarrow{p_{12}} X$$

$$H^i(X, L) \xrightarrow{p_{12}^*} H^i(X \times X, p_{12}^* L) \xrightarrow{s_1^*} H^i(X, L)$$

$$\wr\wr\wr$$

$$\bigoplus H^j(X, L) \otimes H^{i-j}(X, L)$$

Use induction to show $H^0(X, L) = 0 \Rightarrow H^*(X, L) = 0$.

Case 3: $P^{\pm 1}$ on $X \times \hat{X}$ Poincare divisor.

$$\begin{array}{c} X \times \hat{X} \\ \downarrow p_2 \\ a \in \hat{X} \end{array} \qquad H^i(\hat{X}, R^j p_{2*} P^{\pm 1}) \Rightarrow H^{i+j}(X \times \hat{X}, P^{\pm 1})$$

P on $X \times \{a\}$ is alg. equiv. to 0 and $\neq 0_X$ if $a \neq 0$

$\Longrightarrow$ $R^j p_{2*} P^{\pm 1}$ torsion sheaf concentrated at 0.

$\Rightarrow$ $H^i(\quad) = 0$ $i > 0$

So degenerates.

$$H^i(X \times \hat{X}, P^{\pm 1}) = \Gamma(R^i p_{2*} P^{\pm 1})$$

$$= 0 \qquad i > g$$

$$= 0 \qquad i < g \quad \text{Serre duality.}$$

$$R^i p_{2*} P = 0 \qquad i \neq g.$$

lemma: $\dim H^g(X\times\hat{X}, P) = 1$

equivalently

$$R^g p_{2*}(P) \simeq k(0).$$

Pf: $$R^k p_{2*}(P) \otimes k(0) \xrightarrow[\sim]{} H^k(X\times\{0\}, P|_{X\times a})$$

if $k = \dim$ fibers.

$\therefore$ ~~[illegible]~~.

$$R^g p_{2*}(P) \otimes k(0) \simeq H^g(X\times 0, \mathcal{O}_X) = 1 \text{ dim}$$

$$R^g p_{2*}(P) \simeq A/I \qquad \sqrt{I} = m$$

$$I \overset{?}{=} m$$

Enough to check $\quad R^g p_{2*}(P)\otimes \mathcal{O}_Y \simeq k$

all tangent vectors Y at 0

$$R^g p_{2*}(P) \otimes \mathcal{O}_Y \simeq H^g(X\times Y, P|_{X\times Y})$$

$$0 \to m/n \to \mathcal{O}_{X\times Y} \to \mathcal{O}_{X\times(0)} \to 0$$

$$\Rightarrow \quad 0 \to \tilde{P}\otimes\mathcal{O}_X\otimes m/n \to \tilde{P} \to P|_{X\times(0)} \to 0$$

$$\begin{array}{ccc} \tilde{P}\otimes\mathcal{O}_X\otimes m/n & & P|_{X\times(0)} \\ \wr\| & & \wr\| \\ \mathcal{O}_X & & \mathcal{O}_X \end{array}$$

$$\Rightarrow \quad 0 \to \mathcal{O}_X \to \tilde{P} \to \mathcal{O}_X \to 0$$

$$H^{g-1}(\mathcal{O}_X) \xrightarrow{\text{onto?}} H^g(\mathcal{O}_X) \to H^g(\tilde{P}) \to H^g(\mathcal{O}_X) \to 0$$

$$H^0(\mathcal{O}_X) \xrightarrow{\delta} H^1(\mathcal{O}_X) \qquad \delta 1 = \alpha$$

So ~~cob. $x \to \alpha \cup x$~~

$$\delta x = \alpha \cup x$$

But $H^* = \Lambda H^1$.

thus must know $\alpha \neq 0$.

But if $\alpha = 0$ sequence splits

contradicts universal mapping property!.

Case 4: L non-degenerate i.e. $H(L)$ finite or $\Lambda(L)$ isogeny.

Let $M = p_{12}^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}$

By defn $M = (1_X \times \Lambda(L))^* P$.

$$\begin{array}{ccc} X \times X & \xrightarrow{1 \times \Lambda} & X \times \hat{X} \\ M \downarrow p_2 & & P \downarrow p_2 \\ X & \xrightarrow{\Lambda} & \hat{X} \end{array}$$

Λ is flat

$$\Lambda^* \left[R^i p_{2*} P \right] \simeq R^i p_{2,*} \left((1_X \times \Lambda)^* P \right)$$

$$\simeq R^i p_{2,*} (M)$$

$$\simeq R^i p_{2,*} \left(p_{12}^* L \otimes p_1^* L^{-1} \right) \otimes L^{-1}$$

$$\| \\ \begin{cases} 0 & i \neq g \\ A & i = g \end{cases} \qquad \Lambda = \Gamma H(L).$$

So $\deg \Lambda = \dim A = d$

$$R^i p_{2,*}(p_{12}^* L \otimes p_1^* L^{-1}) \simeq \begin{cases} 0 & i \neq g \\ A & i = g \end{cases}$$

$$\therefore \quad H^i(X\times X, p_{12}^* L \otimes p_1^* L^{-1}) = \begin{cases} 0 & i \neq g \\ d \text{ dim} & i = g. \end{cases}$$

autom
$$\begin{array}{ccc} X\times X & \xrightarrow{\tau} & X\times X \\ (x, y) & \longmapsto & (x, y-x) \end{array}$$

$$\begin{cases} p_{12}\circ\tau = p_2 \\ p_1 \circ \tau = p_1 \end{cases}$$

So
$$H^*(X\times X, p_{12}^* L \otimes p_1^* L^{-1}) \simeq H^*(X\times X, p_2^* L \otimes p_1^* L^{-1})$$
$$\cong$$
$$H^*(X, L) \otimes H^*(X, L^{-1})$$
$$\wr\!\wr$$
$$H^{g-*}(X, L)$$

<u>Conclusion</u>: At ~~most~~ exactly one non-vanishing cohomology gp.

say $H^{i(L)}(X, L) \neq 0$.

<u>Frobenius Theorem</u>: L non-degenerate $\exists\, i(L) \ni H^k(X,L) = \begin{cases} 0 & k \neq i(L) \\ \sqrt{d} \text{ diml} & k = i(L) \end{cases}$

$d = \deg \Lambda(L)$.

Cor. ~~[illegible]~~ $\chi(L^n) = n^g \chi(L) = \frac{(D^g)}{g!} n^g$ if $L = \mathcal{O}_X(D)$.

Proof: $\chi(L)^2 = \deg \Lambda(L)$.

$$\begin{aligned}\chi(L^n)^2 &= \deg \Lambda(L^n) \\ &= \deg (n\delta \circ \Lambda(L)) \\ &= \deg n\delta \cdot \deg \Lambda(L) \\ &= n^{2g} \chi(L)^2.\end{aligned}$$

Conjectures: L non-degenerate, H ample

$$P(n_1, n_2) = \chi(L^{n_1} \otimes H^{n_2})$$

homog. poly of degree g.

$P(0,n) \neq 0$

$P(n,0) \neq 0$.

a). $P(1,\alpha)$ has only real roots

b) $i(L) =$ no. of positive roots.

Cor: $\mathbb{G}_m =$ center of $\mathcal{G}(L)$ if L non-deg.

Cor: X abelian, L ample $\Longleftrightarrow \begin{cases} H^0(L) \neq 0 \\ H(L) \text{ finite} \end{cases}$