Mumford's Bowdoin Lectures on Abelian Varieties

Prerequisites:

Theory of group schemes (infinitesimal gp. schemes char. p)
Descent
Picard functor representability of
Intersection theory
General th. of analytic spaces
Cartier operator on differential forms in char. p.
Nakai-Moish criterion for ampleness

§1. First properties

/k algebraically closed.

Def: X abelian var. if X a complete ~~connected~~ group variety.
O origin

i) X non-singular

$\left.\begin{matrix}\Omega^1_X \\ T_X\end{matrix}\right\}$ globally free $\qquad \omega = \Omega^1_{X,0} \otimes k(0)$
$\qquad t = T_{X,0}$

$\Rightarrow \Omega^1_X \simeq \omega \otimes \mathcal{O}_X$

a vector field (1-form) is regular $\Leftrightarrow$ left inv. $\Leftrightarrow$ rt. inv.

Thm. X, Y abelian varieties
$f: X \to Y$ morph.
$\Rightarrow f(x) = h(x) + a \qquad h$ homom.

subvar of X $\quad (W\cdot D^r) > 0 \quad r = \dim W$.

Take $U \subset X$ affine open. Then $X - U = \bigcup_{i=1}^{n} D_i$, D_i irreducible codim 1. Let $D = \sum_{i=1}^{n} D_i$. Given W let $a \in U$, $b \in W$ and ~~let~~ T_{b-a} be $b-a$ translate

$$T_{b-a}^{-1} W \cap U \neq 0.$$

Thus $\forall i$ $\quad D_i \cap T_{b-a}^{-1} W$ is a proper intersection. Inductively suppose $\mathbb{R}$ $\sum a_{ij} Z_j$. Calculate

$$(W \cdot D^r) = \sum a_{ij}(Z_{ij} \cdot D^{r-1}) > 0 \quad \text{if}$$

need at least one $D_i \cap T_{b-a}^{-1}(W) \neq \emptyset$ — if not $\underbrace{T_{b-a}^{-1} W}_{\text{comp}} \subset \underbrace{U}_{\text{affine}}$ impos.

Special ground fields:

$k = \mathbb{C}$

Thm: X abelian variety, then $X \simeq \mathbb{C}^n/\Lambda$, Λ lattice holomorphic isom.

Pf: $\exp : t \longrightarrow X$ a homomorphism, $\exp$ is open and image is a subgp, thus onto

Dual to $(\ )^{(p)}$ on 1-form is Cartier operator

$$\mathcal{C} : \omega \longrightarrow \omega \qquad p^{-1} \text{ linear}$$

Generally if X any n.s. variety. $\exists$ a p^{-1} linear map
$C : \Omega^1_{X, \text{closed}} \longrightarrow \Omega^1_X$

$$0 \longrightarrow \mathcal{O}_X / \mathcal{O}_X^p \xrightarrow{d} \Omega^1_{X, cl} \xrightarrow{C} \Omega^1_X \longrightarrow 0$$

$$\left. \begin{cases} C\omega = 0 \iff \omega = df. \\ C(\alpha^p \omega) = \alpha\, C\omega \\ C\left(\frac{dx}{x}\right) = dx/x \end{cases} \right\} \text{ characterize } C.$$

~~defining~~ characterize C.

Cartier's lemma: $\forall\, \omega \in \Omega^1_{K/k, cl}$ choose separating transcendence basis $x_1, \ldots, x_n$ of K/k. Then $\exists$ unique fns. $\theta, a_i \in K$ $\ni$

$$\omega = d\theta + \sum a_i^p \frac{dx_i}{x_i}$$

+ complement that if ω regular at a simple pt. of a model θ + a_i are then regular.

FMLA: $\langle C\omega, D\rangle^p = \langle \omega, D^p\rangle - D^{p-1}\langle \omega, D\rangle$

$$\omega \in \Omega^1_{K/k} \qquad D \in \mathrm{Der}(K/k)$$

so if X abelian + if ω, D invariant $\langle \omega, D\rangle \in k$

Mumford

General theory of Picard scheme.

X non-singular projective variety
$\forall$ parameter schemes S

Def: "family of invertible sheaves on X param. by S" to be ~~invertible~~ $\mathrm{Pic}(X\times S)/\mathrm{Pic}\, S$

Thm: ~~[illegible]~~ $S \rightsquigarrow \mathrm{Pic}(X\times S)/_{\mathrm{Pic}\, S}$ is representable

i.e. $\exists$ scheme ~~[illegible]~~ $\underline{\mathrm{Pic}}_X$

equivalently $\exists$ Poincare sheaf L on

$$X \times \underline{\mathrm{Pic}}_X$$

such that all M on $X\times S$

$$\exists !\ f : S \longrightarrow \underline{\mathrm{Pic}}_X$$

for which $(1\times f)^* L = M \otimes p_2^* K$

If $x_0 \in X$ basepoint let $\mathrm{Pic}(X\times S, x_0 \times S)$ = iso classes of L on $X \times S \ni L|_{x_0\times S}$ trivial
~~[illegible]~~

classical topology. $H^1(X, \mathcal{O}_X)$

$$0 \longrightarrow \frac{H^1(X, \mathcal{O}_X^h)}{H^1(X, \mathbb{Z})} \longrightarrow H^1(X^h, \mathcal{O}_X^{*h}) \longrightarrow \underbrace{\mathrm{Ker}\,\{H^2(X,\mathbb{Z}) \to H^2(X, \mathcal{O}_X^h)\}}_{\text{finite gen abelian gp.}}$$

lattice in $H^1(X, \mathcal{O}_X)$

$H^1(X^h, \mathcal{O}_X^{*h}) \overset{\text{GAGA}}{\cong} H^1(X, \mathcal{O}_X^*) = \mathrm{Pic}\, X.$

complex torus $\cong \mathrm{Pic}^\circ X.$

finite gen abelian gp. $\cong \mathrm{Pic}/\mathrm{Pic}^\circ$

Using Pic we prove theorem cube: X, Y, Z (n.s.) proj.

L on $X \times Y \times Z$

L trivial on $\{x\} \times Y \times Z$

$X \times \{y\} \times Z$

$X \times Y \times \{z\}$

Pf: View L as defining ~~or~~ $Y \times Z \xrightarrow{\lambda} \mathrm{Pic}\, X$

element of $\mathrm{Pic}(X \times Y \times Z, \{x\} \times Y \times Z)$

where $\lambda(y_0 \times Z) = \lambda(Y \times z_0) = 0.$

Lemma: X, Y, Z three vars. X complete

$f: X \times Y \to Z \quad \ni f(X \times y_0) = f(x_0 \times Y) = z_0$

$\Longrightarrow \mathrm{Im}\, f = z_0.$

X abelian $\quad X \times X \times X \longrightarrow X$

$$p_{123}(x, y, z) = x + y + z$$

$$p_{12}(x, y, z) = x + y$$

7 non-trivial maps.

<u>Corollary:</u> L invertible on $X \times X \times X$

$$p_{123}^* L \simeq p_{12}^* L \otimes p_{13}^* L \otimes p_{23}^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1} \otimes p_3^* L^{-1}$$

<u>Corollary:</u> $f, g, h : S \longrightarrow X^{L}$

$$(f+g+h)^* L \simeq (f+g)^* L \otimes (f+h)^* L \otimes (g+h)^* L \otimes f^* L^{-1} \otimes g^* L^{-1} \otimes h^* L^{-1}.$$

($f^* L$ quadratic in f).

<u>Example:</u> $S = X$

$f = l\delta$

$g = m\delta$

$h = n\delta$

<u>Cor:</u> $L' = (-\delta)^* L$

$$(n\delta)^* L \simeq L^{\frac{n^2+n}{2}} \otimes (L')^{\frac{n^2-n}{2}}$$

$$(n\delta)^{-1}(D^g) = ((n\delta)^{-1}D)^g = (n^2 D)^g = n^{2g} D^g$$

$$(n\delta)^{-1}(D^g) = (\deg n\delta)(D^g)$$

$(D^g) > 0 \quad \therefore \quad \deg n\delta = n^{2g}.$

§2. Albanese varieties

S variety $s_0 \in S$ basepoint. Then $f: S \longrightarrow \mathrm{Alb}\text{-}S$ is universal map into an abelian variety $\ni f(s_0) = 0$.

$\mathrm{Alb}.S =$ "coeff. free H_1 of S."

$k = \mathbb{C}$ can $\int$ & get maps into complex tori which turn out to be ~~all~~ abelian vars. Take $\omega_1, \ldots, \omega_g$ basis for regular 1-form

$$s \longmapsto \left(\int_{s_0}^{s} \omega_1, \ldots, \int_{s_0}^{s} \omega_g \right) \in \mathbb{C}^g$$

well-defined up to $\Lambda \subset \mathbb{C}^g$ where

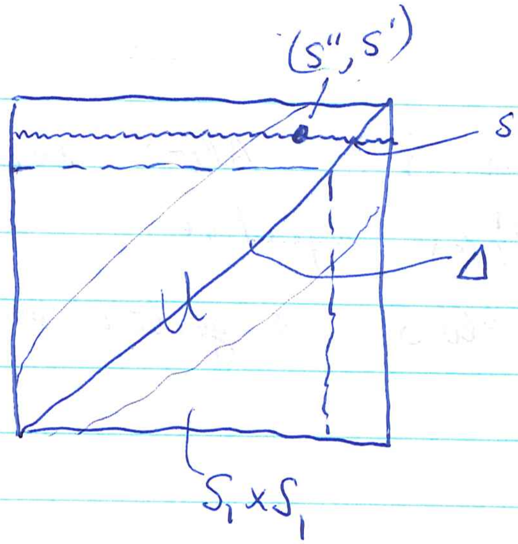
$$\Lambda = \int_{\sigma} (\omega_1, \ldots, \omega_g) \qquad \sigma \in H_1$$

so get holom. map

$$S \longrightarrow \mathbb{C}^g/\Lambda$$

Use Hodge decomp. to show Λ a lattice. More instrinsically

$$S \longrightarrow H^0(\Omega^1)^* / \mathrm{Im}\, H_1(S, \mathbb{Z}).$$



choose $X^* \subset X$ affine containing O
$\forall\, a \in R$ (" Spec R) $g_1^* a$ is a rational fn on $S_1 \times S_1$
is a regular fn on $g_1^{-1} X^*$ hence on $\Delta \cap (S_1 \times S_1)$
$\therefore$ also regular on all of Δ. ~~since~~ $\therefore$ regular on $U_a \supset \Delta$.
Hence R fin. generated $\Rightarrow \exists\, U \supset \Delta \ni g^*(a)$ regular on U
all $a \in R$. $\Rightarrow g_1$ extends $U \to X^*$. Take
$s' \in S - S_1$. $\{(s, s')\}$ = line $\exists\, s'' \ni (s'', s') \in U$ &
$s'' \in S_1$. Look at $g_1(s, s'') + f_1(s'')$ defined in some
neighborhood of s'. Also an extension of f_1 to a
nbd. of S' and so f_1 defined everywhere.

K/k fn. field, Recall $\omega \in \Omega^1_K$ is called regular
if for all n.s. varieties X with fn. field K ω has
no pole over X. Easy to see that ω regular on
one complete non-singular variety $\Rightarrow$ regular on all.
$\Omega^1_{K, reg.}$ (finite dimensional since if S complete normal then
$\Omega^1_{K, reg} \subset (\Omega^1_{S/k})^{**}$.)

Cor 2: $\forall\, f: S \to X$ $\omega \in H^0(\Omega^1_X) \Rightarrow f^* \omega \in \Omega^1_{k(S), reg.}$

You get these Y's as follows.

$$\forall\ L \subset \operatorname{Lie}(X)$$

let $Y = \begin{cases} X & \text{basespace} \\ \mathcal{O}_Y = \{ f \overset{1/p}{\in} \mathcal{O}_X \mid Df = 0 \quad D \in L \}. \end{cases}$

Now let $V = \{ \omega \in H^0(\Omega^1_X) \mid f^*\omega = 0 \}$ and let

$$L = V^0 \subset \operatorname{Lie} X.$$

(Lie alg since $C(f^*\omega) = f^* C\omega$ $\Downarrow$ so L closed ~~under~~ under $(\ )^{(p)}$.

If $a \in \underline{\mathcal{O}}_X$ $Da = 0$ $D \in L \Rightarrow da \in \sum_{\omega \in V} k(X)\,\omega$

$\Longrightarrow f^* da = 0 \Longrightarrow f^* a \in k(S)^p \Longrightarrow f^*(a)^{1/p} \in k(S).$

$$\Longrightarrow \begin{tikzcd} S \arrow[r, "f"] \arrow[dr, dotted, "\exists"'] & X \\ & Y \arrow[u] \end{tikzcd}$$

QED.

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b) + lemma + a.c.c. on subextensions of a fin. gen. extension will imply Alb S $\exists$.

K/k fn. field regular diff one which has no pole on any non-singular model. example $f: S \longrightarrow X$

$\omega \in H^0(\Omega^1_X) \Longrightarrow f^*\omega \in \Omega^1_{k(S), reg}$.

$$\begin{tikzcd} S_0 \arrow[rd, "\exists"] & \\ \supset S' \arrow[r] & X \\ S \supset \arrow[ru] & \end{tikzcd}$$

$K = k(S)$ S complete non-singular

$\omega \in H^0(\Omega^1_S) \Longrightarrow \omega \in \Omega^1_{k(S), reg}$

S S_0

$\mathrm{Im}(\pi - \pi') \subset A$

$(\pi - \pi')a = a - a = 0$

Tate-residues

A DVR quotient field K containing k_0 $\ni$ $[k : k_0] < \infty$. Then $\forall\, x \in K$ $[xA + A : A] < \infty$ Choose a projection

$\pi : K \longrightarrow A$

then $x \mapsto (x\pi - \pi x) \bmod [K, \mathrm{End}_0(K, A)]$ is a derivation so

$$\mathrm{res}(b\,da) = \mathrm{tr}\; b(a\pi - \pi a)$$

$$
\begin{array}{c}
\chi \in H^n(f_1, \dots, f_n; A) \\
\wr\!\wr \\
\mathrm{Ext}^n\!\left(A/(f_1, \dots, f_n), A\right) \\
\big\downarrow \scriptstyle \text{using } \omega \\
H^n_{\mathfrak{m}}\!\left(\Omega^n_{A/k}\right) \xrightarrow{\ \mathrm{res}\ } k.
\end{array}
$$

Proof of b) Step I. Given $f: S \to X$ set

$$Z_n = \overline{f(S^n)} \quad \text{irred increasing}$$

(both chain conditions on primes)

$$Z = \overline{f(S^n)} \quad n \text{ large}$$

$$\begin{cases} Z + Z \subset Z \\ -Z = Z \end{cases} \quad \text{since} \quad Z + a \subset Z \text{ , } = \text{ by dim}$$

$\therefore$ Z subabelian var. so can factor

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[dr] & X \\ & Z \arrow[u, phantom, "\cup"] \end{tikzcd}$$

Thus

$\ast$ $\therefore$ If
$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d] & X \\ Y \arrow[ur, "\pi \text{ hom. same dimension}"'] & \end{tikzcd}$$
then $\overline{f(S^n)} = X \Rightarrow \overline{g(S^n)} = Y$.

(exercise show $\overline{f(S^n)} = X \Rightarrow f(S^n) = X$.)

Lemma: $\forall f: S \to X$ generating map

char 0: $f^*: H^0(\Omega^1_X) \to \Omega^1_S$ injective

char p: $\exists$
$$\begin{tikzcd} S \arrow[r] \arrow[dr, "g"'] & X \\ & Y \arrow[u, "\pi"'] \end{tikzcd}$$

$$g^*: H^0(\Omega^1_Y) \to \Omega^1_S \text{ inj.}$$

Pf: In char p fix $n \ni \overline{f(S^n)} = X$ get $k(S^n) \supset k(Y) \supset k(X)$

hence (by *) $\exists$ single $N \ni \overline{f(S^N)} = X_n$ all $n \geq n_0$
$\Longrightarrow k(X_n) \subset k(S^N)$ so use a.c.c. on subfields.

Cor. 1: If S rational, then Alb $S = 0$
(no everywhere regular differentials)

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general fact: X any variety such that $\forall f : S \longrightarrow X$
S rational are constant $\longrightarrow \forall S_0 \subset S$ non-sing $\exists$
extension. (Artin proof of thm. last time)

Cor. 2: X, Y n.s. varieties $\longrightarrow$ $\mathrm{Alb}(X \times Y) = \mathrm{Alb}\, X \times \mathrm{Alb}\, Y$

"Alb is a linear functor"

Pf:

$$\begin{tikzcd}
X \times Y \arrow[r, "p_1"] \arrow[dr, "p_2"] & X \arrow[l, bend right, "s_1"'] \\
& Y \arrow[ul, bend left, "s_2"]
\end{tikzcd}$$

s_i basepoints

yields by functorality
$$\mathrm{Alb}(X \times Y) \rightleftarrows \mathrm{Alb}\, X \times \mathrm{Alb}\, Y$$
$$\therefore \quad '' \qquad\qquad '' \quad \times B \quad '' \ \mathrm{Ker}(p_1, p_2)$$

Every reg. diff on $X \times Y$ is a sum of one on X and on Y.
so differential of B, which pull back to diff on $X \times Y$
trivial on sections, are 0.

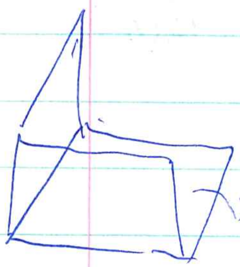
__Seesaw principle:__ L on $X \times Y$, $L \simeq \mathcal{O}_{X\times Y}$ on $X \times \{y_0\}$ and on $\{x\} \times Y$ all $x \implies L \simeq \mathcal{O}_{X\times Y}$ everywhere (see Mumford's book).

Use this to reduce to X projective. Let $L_{y,z} = L|_{X\times\{y\}\times\{z\}}$

(I) $\exists M_X$ ample inv. sheaf on $X \ni$

$$H^1(X, M_X \otimes L_{y,z}) = 0 \quad \text{all } y, z$$

$$\underbrace{H^0(\quad " \quad) \neq 0}$$



$X \times Z$

defines a family $P_{y,z}$ of divisors on X.

Know: $L_{y,z_0} \simeq L_{y_0,z} \simeq \mathcal{O}$ all y, z.

$$P_{y_0,z} = P_{y,z_0} = P_0$$

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general facts:

$$L \quad X \xrightarrow{\text{flat proper}} Y$$

$\forall y \in Y \quad H^1(X_y, L_y) = 0$

$\Rightarrow$ (USC) $\Rightarrow$ $\pi P = P_0 \times Z \Rightarrow P_{y,z} \subset P_0$ all y,z

$\Rightarrow$ $M_x \otimes L_{yz} \simeq M_x$ all y,z done by seesaw.