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Isomorphism of Induced representations

Start with a map

$$F : \mathrm{Hom}_B(G, \rho_1) \longrightarrow \mathrm{Hom}_B(G, \rho_2)$$

and write F in the form

$$F(f)(g_1) = \int K(g_1, g_2)\, f(g_2)\, dg_2$$

If we were working with finite groups then have

$$\begin{tikzcd}
\mathrm{Hom}_B(G,\rho_1) \arrow[d, "\Phi"', "\wr"] \arrow[dr, "F"] & \\
G \times_B \rho_1 \arrow[r, "\varphi^{\#}"'] & \mathrm{Hom}_B(G,\rho_2)
\end{tikzcd}$$

where $\varphi^{\#} \in \mathrm{Hom}_G(G \times_B \rho_1, \mathrm{Hom}_B(G, \rho_2)) \cong \mathrm{Hom}_B(G \times_B \rho_1, \rho_2)$ wealate

$$\wr$$

$$\varphi \in \underbrace{\mathrm{Hom}_{B\times B}(G, \mathrm{Hom}(\rho_1, \rho_2))}$$

Now

$$\varphi^{\#}(g_1, z)(g_2) = \varphi_{g_2 g_1} z \qquad \varphi_{gb} = \varphi_g b, \quad \varphi_{bg} = b\varphi_g$$

(Check: $\varphi^{\#}(g g_1, z)(g_2) \overset{?}{=} \varphi^{\#}(g_1, z)(g_2 g)$

$$\varphi_{g_2 g g_1}(z) \qquad\qquad \varphi_{g_2 g g_1}(z) .\ )$$

and $\quad \Phi(f) = \sum_i (u_i, f(u_i^{-1})) \qquad$ where $G = \coprod u_i B$

$$= \int (g, f(g^{-1}))\, dg \qquad \boxed{\text{if } \int_B 1 = 1.}$$

$$\therefore \quad (\varphi^{\#}\Phi)(f) \overset{(g)}{=} \varphi^{\#} \sum_i (u_i, f(u_i^{-1}))^{(g)}$$

$$= \sum_i \varphi_{g u_i} f(u_i^{-1}) = \int_G \varphi_{g x^{-1}} f(x)\, dx$$

$$\therefore \quad F(f)(g) = \int \varphi_{g x^{-1}} f(x)\, dx$$

where $\quad \begin{cases} \varphi_{bg} = b \varphi_g \\ \varphi_{gb} = \varphi_g b \end{cases} \qquad \varphi : G \to \mathrm{Hom}(\Upsilon_1, \Upsilon_2)$

<u>Remarks:</u> The above calculation holds for finite groups, however ~~in reality~~ one can rework it to hold when B is of finite index in G. By extrapolation it should also hold when G/B is <u>compact</u> which is the ~~~~ situation at hand. Of course with suitable analytical modifications.

Conclusion is that the natural kind of operator to look for is one of the form

$$F(f)(g_1) = \int L(g_1 g_2) f(g_2^{-1})\, dg_2$$

where L is a "function" on G with values in $\mathrm{Hom}(\Upsilon_1, \Upsilon_2)$ such that

$$L(bg) = b L(g)$$
$$L(gb) = L(g) b$$

and where the integral is taken over G/B.

~~Actually L can be any distribution of some sort~~

~~dx~~

Actually rewrite as follows:

$$\int L(g_1 g_2)\, f(g_2^{-1})\, dg_2 = \int L(x) f(x^{-1} g_1)\, dx$$

in which case $\boxed{L(x)\,dx}$ may be replaced by any distribution on G with values in $\mathrm{Hom}(\mathfrak{f}_1, \mathfrak{f}_2)$ such that

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Clearly the above formula makes no sense because of infinite size of a function on G stable under B. So we must transfer the integral to the compact space G/B.

Consider the measure on G/B obtained from the isom

$$K/M \simeq G/B.$$

~~It is necessary to~~ If G/B carried a G invariant measure then we could replace the above integral by $\int_{G/B} \text{▬}\,\mu$, however as it doesn't we must work ~~hard~~.

So we need to know the jacobian $g^*\mu/\mu$ = some function on G/B.

$$\rho_g(xB)\, d\mu(xB) = g^* d\mu(xB) \stackrel{?}{=} d\mu(gxB).$$

$$\int f(xB)\, g^* d\mu(x, B) = \int f(gxB)\, d\mu(xB)$$

Work things out discretely first. Assume that

$$(Ff)(g) = \sum_i \varphi(gu_i)\, f(u_i^{-1})\, \alpha_i \qquad G = \coprod u_i B. \quad \mu(u_i B) = \alpha_i$$

I want

$$F(x\cdot f)(g) = \sum_i \varphi(gu_i)\, f(u_i^{-1}x)\, \alpha_i$$

$$= \sum_i \varphi(gu_i)\, f((x^{-1}u_i)^{-1})\, \mu(u_i B)$$

$$= \sum_i{}' \varphi(gx\cdot x^{-1}u_i)\, f((x^{-1}u_i)^{-1})\, \mu(u_i B)$$

Now $\quad x^{-1}u_i = u_{x^{-1}(i)}\, \underline{b(i,x^{-1})} \quad$ so

$$\varphi(gx\cdot x^{-1}u_i) = \varphi(gx\cdot u_{x^{-1}(i)})\, \tilde{\rho}_2 \underline{b(i,x^{-1})}\, \tilde{\rho}_2 b(i,x^{-1})^{-1}\, f(u_{x^{-1}(i)}^{-1})$$

$$\cdot\, \mu(u_i B)$$

Seems impossible to get both G invariance + indep. of coset.
?

$$F(f)(g) = \sum_i \varphi(g, u_i)\, f(u_i^{-1})\, \alpha_i$$

where

$$\varphi(g, x\,u_i) = \varphi(gx, u_i)\, \psi_x(i)$$

$$\varphi(bg, u_i) = \rho_1(b) \cdot \varphi(g, u_i)$$

$$\varphi(g, u_i b) =$$

$$\varphi(g_1, g_2)$$

$$\begin{cases} \varphi(bg_1, g_2) = b\,\varphi(g_1, g_2) \\ \varphi(g_1, g_2 b) = \varphi(g_1, g_2)\, b \\ \varphi(g_1, x g_2) = \varphi(g_1 x, g_2) \cdot \psi(x, g_2 B) \end{cases}$$

If so define

$$F(f)(g) = \sum_i \varphi(g, u_i)\, f(u_i^{-1})\, \overset{\mu(u_i B)}{\overset{\parallel}{\alpha_i}} \qquad \text{ind of } u_i$$

$$F(x \cdot f)(g) = \sum_i \varphi(g, u_i)\, f(\overset{(x^{-1}u_i)^{-1}}{u_i^{-1} x})\, \alpha_i$$

$$[x \cdot F(f)](g) = \sum_i \varphi(gx, u_i)\, f(u_i^{-1})\, \alpha_i$$

$$= \sum_i \varphi(g, x x^{-1} u_i)\, f((x^{-1}u_i)^{-1})\, \alpha_i$$

$$= \sum_i \varphi(gx, x^{-1}u_i)\, \psi(x, x^{-1}u_i B)\, f((x^{-1}u_i)^{-1})\, \alpha_i$$

$$= \sum_i \varphi(gx, u_{x^{-1}(i)})\, f(u_{x^{-1}(i)}^{-1}) \cdot \alpha_{x^{-1}(i)} \cdot \left[\frac{\alpha_i}{\alpha_{x^{-1}(i)}} \cdot \psi(x, x^{-1}(i)) \right]$$

Thus we want

$$\psi(x, x^{-1}(i)) = \frac{\mu(x^{-1} u_i B)}{\mu(u_i B)}$$

$$\cancel{\psi(x, x^{-1} u_i B)}$$

$$\psi(x, x^{-1} u B) = \frac{\mu(x^{-1} u B)}{\mu(u B)} \qquad x^{-1} u = y$$

$$\boxed{\psi(x, yB) = \frac{\mu(yB)}{\mu(xyB)}}$$

<u>Conclusion:</u> $$F(f)(g_1) = \int_{G/B} \varphi(g_1, g_2) f(g_2^{-1})\, d\mu(g_2 B)$$

is a mapping ~~[illegible]~~ from $I(\gamma_1)$ to $I(\gamma_2)$ if

$$\begin{cases} \varphi(bg_1, g_2) = \gamma_1(b)\varphi(g_1, g_2) \\ \varphi(g_1, g_2 b) = \varphi(g_1, g_2)\gamma_2(b) \\ \varphi(g_1, xg_2) = \varphi(g_1 x, g_2) \dfrac{\mu(g_2 B)}{\mu(x g_2 B)} \end{cases}$$

Actually you should be able to reduce to a ~~single~~ fn. of a single vbl, e.g. φ determined by $\varphi | K \times K$ and μ K stable so that $\varphi($

$$\varphi(g_1, g_2) = \varphi(g_1 g_2, e) \frac{\mu(\cancel{g_2} B)}{\cancel{\mu(g_1 g_2)}\ \mu(g_2 B)}$$

~~$\varphi(bg$~~ set ~~$\varphi(g, e)$~~

$$\psi(g) = \varphi(g, e) \chi(g).$$

Then

$$\psi(bg) = \varphi(bg, e) \chi(bg) = \xi_1(b) \varphi(g, e) \chi(g) \frac{\chi(bg)}{\chi(g)}$$

$$= \left[\frac{\chi(bg)}{\chi(g)} \cdot \xi_1(b)\right] \psi(g)$$

$$\psi(gb) = \varphi(gb, e) \chi(gb)$$

~~$\psi \quad \varphi(g, b) \quad \mu B$~~

$$\varphi(g, b) = \varphi(gb, e) \cdot \frac{\mu(B)}{\mu(bB)} = \varphi(gb, e)$$ ← where discrete + cont differ

$$\therefore \psi(gb) = \varphi(g, b) \chi(gb) = \varphi(g, e) \xi_2(b) \chi(gb)$$

$$= \psi(g) \left[\frac{\chi(gb)}{\chi(g)} \cdot \xi_2(b)\right]$$

<u>Conclusion:</u> $$F(f)(g_1) = \int_{G/B} \psi(g_1 g_2) \frac{\mu(B)}{\mu(g_2 B)} f(g_2^{-1}) \, d\mu(g_2 B).$$

defines a map from $I(\xi_1)$ to $I(\xi_2)$ if

$$\begin{cases} \psi(bg) = \xi_1(b)\psi(g) \\ \psi(gb) = \psi(g)\,\xi_2(b). \end{cases}$$

<u>Check:</u> $$F(x \cdot f)(g_1) = \int_{G/B} \psi(g_1 g_2) \frac{\mu(B)}{\mu(g_2 B)}$$

$$\cancel{\varphi(gb, e)} \qquad \varphi(g, b) = \varphi(gb, e) \frac{\mu(B)}{\mu(bB)}$$

$$\begin{aligned} \psi(gb) &= \varphi(gb, B)\, \chi(gb) \\ &= \varphi(g, B) \cdot \frac{\mu(bB)}{\mu(B)} \frac{\chi(gb)}{\chi(g)} \chi(g)\, \xi_2(b) \\ &= \psi(g) \cdot \left[\frac{\mu(bB)}{\mu(B)} \frac{\chi(gb)}{\chi(g)} \xi_2(b) \right] \end{aligned}$$

Suppose we fix $\mathfrak{I}$ and we define maps

$$\varphi_t^s \; \cancel{\varphi(s,t)} : \mathfrak{I}^s \longrightarrow \mathfrak{I}^t$$

such that $\cancel{\varphi(s,t)\circ\varphi t}$

$$\varphi_u^t \circ \varphi_t^s = \varphi_u^s$$

Thus you have a category whose objects are the elements of W and whose morphisms are the morph. of W.

<u>Proposition</u>: Let G/B be endowed with the measure $d\mu$ coming from the isomorphism $K/M \simeq G/B$ and the Haar measure of K. ~~Then~~ ~~For $x \in G$ let $\rho(x)$~~ Let $\rho(x,y)$ be the function on $G \times G/B$ such that

$$\cancel{\int_{G/B} f(y)\, d\mu(y) = \int_{G/B} f(xy)\, \rho(x,y)\, d\mu(y).}$$

$$\rho(x,\cdot) = \frac{(x^{-1})_*\, d\mu}{d\mu}$$

ie

$$\int_{G/B} f(y)\,\rho(x,y)\,d\mu(y) = \int_{G/B} f(y)\,(x_*\,d\mu)(y) = \int_{G/B} f(xy)\,d\mu(y)$$

Then

$$F(f)(g_1) = \int_{G/B} \psi(g_1 g_2)\, f(g_2^{-1})\, d\mu(g_2 B)$$

is a map from $I(\gamma_1)$ to $I(\gamma_2)$ provided

$$\psi(bg) = \xi_2(b)\psi(g)$$
$$\psi(gb) = \psi(g)\,\xi_1(b)\,\rho(b,e)$$

Here $\rho(b,e) = \frac{b_* d\mu}{d\mu}(e) =$ ~~[illegible]~~ det of Ad b on $\mathfrak{g}/\mathfrak{b}$ (or Ad b^{-1}?)

Thus $b \mapsto \rho(b,e)$ is a character on B which must vanish on N, M and so comes from something in A in fact $+$ sum of roots in Σ!!

NUTS.

Want to map $I(\chi_1)$ to $I(\chi_2)$ using

$$F(f)(g_1) = \int_{G/B} \varphi(g_1, g_2) f(g_2^{-1}) \, d\mu(g_2 B)$$

In order that this be an integral over G/B I need that

(1) $$\varphi(g_1, g_2 b) = \varphi(g_1, g_2)\, \chi_1(b)$$

In order that $F(f)(bg_1) = \chi_2(b)\, F(f)(g_1)$ I need that

(2) $$\varphi(bg_1, g_2) = \chi_2(b)\, \varphi(g_1, g_2)$$

Finally in order that $F(xf) = x F(f)$ I need that

$$F(xf)(g_1) = \int_{G/B} \varphi(g_1, g_2) f(g_2^{-1} x) \, d\mu(g_2 B)$$

$$\parallel$$

$$F(f)(g_1 x) = \int_{G/B} \varphi(g_1 x, g_2) f(g_2^{-1}) \, d\mu(g_2 B)$$

Recall $$\int_{G/B} f(y) \rho(x,y)\, d\mu(y) = \int_{G/B} f(xy)\, d\mu(y)$$ defn. of ρ

$$\parallel$$

so that $$\int_{G/B} f(y)\, d\mu(x^{-1}y)$$

i.e. $$\frac{d\mu(x^{-1}y)}{d\mu(y)} = \rho(x,y)$$

So $$F(xf)(g_1) = \int_{G/B} \varphi(g_1, x(x^{-1}g_2))\, f((x^{-1}g_2)^{-1})\, d\mu(g_2B)$$

$$= \int_{G/B} \varphi(g_1, xg_2)\, f(g_2^{-1})\, \rho(x^{-1}, g_2^B)\, d\mu(g_2B).$$

Thus I need

(3) $$\varphi(g_1 x, g_2) = \varphi(g_1, xg_2)\, \rho(x^{-1}, g_2^B)$$

$$= \varphi(g_1, xg_2)\, \frac{d\mu(xg_2B)}{d\mu(g_2B)}$$

All this agrees with preceding formulas.

Observations

Equation (3) tells me that

$$\varphi(g_1, g_2) = \varphi(g_1, g_2 \cdot e) = \varphi(g_1 g_2, e)\, \rho(g_2^{-1}, e)^{-1}$$

better

$$\varphi(g_1, g_2) = \varphi(g_1 \underbrace{g_2\, g_2^{-1}}_{x}, g_2) = \varphi(g_1 g_2, e)\, \rho(g_2, g_2B)$$

(3)′ $$\boxed{\varphi(g_1, g_2) = \varphi(g_1 g_2, e)\, \frac{d\mu(B)}{d\mu(g_2B)}}$$

Note that $(3)' \Longrightarrow (3)$ i.e.

$$\varphi(g_1 x, g_2) = \varphi(g_1 x g_2, e)\,\frac{d\mu(B)}{d\mu(g_2 B)}$$

$$\varphi(g_1, x g_2) = \varphi(g_1 x g_2, e)\,\frac{d\mu(B)}{d\mu(x g_2 B)} = \quad \cdot\ \frac{d\mu(g_2 B)}{d\mu(x g_2 B)}$$

$$\frac{d\mu(g_2 B)}{d\mu(x g_2 B)} = \rho(x^{-1}, g_2 B)$$

So now let $\psi(g) = \varphi(g, e)$ and find what (2)+(1) mean if φ is defined by

$$\varphi(g_1, g_2) = \psi(g_1 g_2)\,\frac{d\mu(B)}{d\mu(g_2 B)} = \psi(g_1 g_2)\,\frac{1}{\rho(g_2^{-1}, eB)}$$

Clearly

(2)' $\qquad \psi(bg) = \xi_2(b)\psi(g)$

is equivalent to (2). (1) implies

$$\cancel{\psi(gb) = \varphi(gb, e)}$$

$$\psi(g)\xi_1(b) = \varphi(g, e)\xi_1(b) = \varphi(g, b) = \psi(gb)\,\frac{\cancel{d\mu(B)}}{\rho(b^{-1}, eB)^{-1}}$$

(1)' $\qquad \psi(gb) = \psi(g)\,\xi_1(b)\,\rho(b) \qquad\qquad \rho(b) = \rho(b, B)$

But does (1)' imply (1)?

$$\varphi(g_1, g_2 b) = \psi(g_1 g_2 b) \quad \rho(g_2 b, g_2 B)$$

$$\varphi(g_1, g_2) = \psi(g_1 g_2) \quad \rho(g_2, g_2 B)$$

$$\varphi(g_1, g_2)\, \xi_1(b) = \psi(g_1 g_2)\, \xi_1(b)\, \rho(g_2, g_2 B).$$

Thus if (1)'

$$\varphi(g_1, g_2 b) = \psi(g_1 g_2)\, \xi_1(b)\, \rho(b) \quad \rho(g_2 b, g_2 B) \; ;$$

do we have

$$\rho(b)\, \rho(g_2 b, g_2 B) = \rho(g_2, g_2 B) \quad ? \qquad \text{for all } g_2,\ b \in B.$$

$$\rho(b) \; \frac{d\mu(b^{-1} g_2^{-1} g_2 B)}{d\mu(g_2 B)} = \frac{d\mu(g_2^{-1} g_2 B)}{d\mu(g_2 B)}$$

ρ

Thus let μ be given by a form ω so that

$$\frac{x^* \omega}{\omega}(y) = \bar{\rho}(x, y) \qquad x \in G, \; y \in G/B$$

then

$$\int (x^* f)(y)\, (x^* \omega)(y) = \int f(y)\, \omega(y)$$

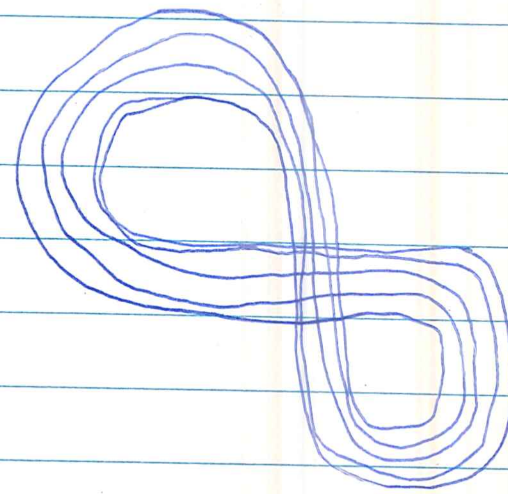
$$\int f(x \cdot y)\, \bar{\rho}(x, y)\, \omega(y) = \int f(y)\, \omega(y)$$

hence also

$$\int f(x^{-1}y)\,\omega(y) = \int f(y)\,\bar{\rho}(x,y)\,\omega(y).$$

$$\therefore\ \bar{\rho}(x^{-1},y) = \rho(x,y).$$

$$\therefore\ \rho(x,y) = \frac{(x^{-1})^*\omega}{\omega}(y).$$



$$\therefore\ \rho(b) = \frac{(b^{-1})^*\omega}{\omega}(B)$$

$$\rho(g_2 b, g_2 B) = \frac{(b^{-1})^*(g_2^{-1})^*\omega}{\omega}(g_2 B)$$

$$\rho(g_2, g_2 B) = \frac{(g_2^{-1})^*\omega}{\omega}(g_2, B).$$

$$\frac{(g_2^{-1})^*\omega}{(b^{-1})^*(g_2^{-1})^*\omega}(g_2 B) = \frac{(b^{-1})^*\omega}{\omega}(B)$$

$$(bg_2)^{-1*}\omega$$

$$\frac{x^* y^*\omega}{y^*\omega}\cdot\frac{y^*\omega}{\omega} = \frac{(yx)^*\omega}{\omega}(p)$$

$$\frac{b^*(g_2^{-1})^*\omega}{(g_2^{-1})^*\omega}(b^{-1}g_2 B) = \frac{(b^{-1})^*\omega}{\omega}(B)$$

Hence this fails.

$$\frac{b^{-1}d\mu}{d\mu}(B)\ \frac{(g_2 b)^{-1}d\mu}{d\mu}(g_2 B) = \frac{g_2^{-1}d\mu}{d\mu}(g_2 B) \qquad ?$$

$$\left(\frac{b^{-1}d\mu}{d\mu}\right)(B)\cdot \frac{b^{-1}g_2^{-1}d\mu}{g_2^{-1}d\mu}(g_2 B)\,\frac{g_2^{-1}d\mu}{d\mu}(g_2 B) = \frac{g_2^{-1}d\mu}{d\mu}(g_2 B)$$

We are after the result that if

$$(\mathfrak{J}_1 + \mathfrak{g})^S = \mathfrak{J}_2 + \mathfrak{g}$$

then there is a map. This seems to suggest that somehow should adjust φ both before + after to get ψ.

Understand Bruhat:

$$F(f) = \int_{G/B} \varphi(g_1, g_2)\, f(g_2^{-1})\, d\mu(g_2 B)$$

gives a map from $I\mathcal{J}_1$ to $I\mathcal{J}_2$ if

(1) $\varphi(b g_1, g_2) = \mathcal{J}_2(b)\, \varphi(g_1, g_2)$

(2) $\varphi(g_1, g_2 b) = \varphi(g_1, g_2)\, \mathcal{J}_1(b)$

(3) $\varphi(g_1 x, g_2) = \varphi(g_1, x g_2)\, \rho(x^{-1}, g_2 B)$

where
$$\rho(x, y) = \frac{(x^{-1})^* \omega}{\omega}(y) \underset{\text{imprecisely}}{=} \frac{\mu(x^{-1}y)}{\mu(y)}$$

where $\mu(Q) = \int_Q \omega$

The problem is to decide when such a φ $\exists$. Let $\bar{\omega} = \pi^* \omega$, $\pi : G \to G/B$ so that $(R_b)^* \bar{\omega} = \bar{\omega}$ $\bar{\omega}$ dies on the fibers and so that

$$\varphi(g_1 x, g_2) = \varphi(g_1, x g_2)\, \frac{x^* \bar{\omega}}{\bar{\omega}}(g_2)$$

(3) $\Longleftrightarrow$ $\cancel{\varphi(g_1 g_2)}$

$$\varphi(g_1 g_2, e) = \varphi(g_1, g_2)\, \frac{g_2^* \bar{\omega}}{\bar{\omega}}(e)$$

i.e.

$$\boxed{\varphi(g_1, g_2) = \varphi(g_1 g_2, e)\, \frac{\bar{\omega}}{g_2^* \bar{\omega}}(e)}$$

The problem is to compare this with

$$\varphi(g_1, g_2 b) = \varphi(g_1, g_2)\, \xi_1(b) = \varphi(g_1 g_2, e)\, \frac{\bar{\omega}}{g_2^* \bar{\omega}}(e)\, \xi_1(b)$$

$$\cancel{\varphi(g_1 g_2, b)\, \frac{\bar{\omega}}{(g_2)^* \bar{\omega}}(b) = \varphi(g_1 g_2, e)\, \xi_2(b)\, \frac{\bar{\omega}}{(g_2)^* \bar{\omega}}(b)}$$

$\therefore$ $\cancel{\frac{\bar{\omega}}{g_2^* \bar{\omega}}(e) = \frac{\bar{\omega}}{(g_2^{-1})^* \bar{\omega}}(b)}$ OKAY.

$\cancel{\varphi(g_1 g_2 b)}$

$\cancel{\varphi(g_1, b) = \varphi(g_1, e)\, \xi_1(b)}$

$\cancel{\varphi(g_1 b, e)}$

$$\varphi(g_1 g_2 b, e) \cdot \frac{\bar{\omega}}{(g_2 b)^* \bar{\omega}}\, e$$

Thus

$$\varphi(g_1 g_2 b, e)\cdot \frac{\bar\omega}{(g_2 b)^* \bar\omega}(e) = \varphi(g_1 g_2, e)\, \frac{\bar\omega}{g_2^* \bar\omega}(e)\cdot \xi_1(b)$$

Start over again:

You want to calculate maps from $I\xi_1$ to $I\xi_2$.

General fact: Let X be a ~~compact~~ manifold, E, F two bundles over X, and let $\varphi: \Gamma_c E \longrightarrow \Gamma F$. By the kernel thm. there is a distribution on $X \times X$ such that

$$\varphi(f)\cancel{(x)} = \int K(x,y)\, f(y)$$

in the following sense. If everything is smooth then, K is a section of

$$pr_2^* E^* \otimes pr_1^* F \otimes pr_2^* \omega_Y$$

Thus K is a linear function on $\Gamma(pr_2^* E \otimes pr_1 F^* \otimes pr_1^* \omega_X)$

$$\Gamma(X\times X, \mathrm{Hom}(pr_1^* F^*, pr_2^! E^*))$$

$$\varphi(f)(x) = \int K(x,y)\, f(y)\, dy$$

where $\quad K(x,y)\,dy \in \Gamma(X\times X,\ pr_2^* E^* \otimes pr_1^* F \otimes pr_2^* \omega)$

Consequently for φ to be G invariant means that K must be G invariant for some action.

Now $\quad \Gamma(X\times X,\ pr_2^*(E^*\otimes\omega)\otimes pr_1^* F)$

$$\varphi : G\times G \longrightarrow \rho_1^* \otimes \rho \otimes \rho_2$$

$$\varphi(b_1 g_1, b_2 g_2) = \rho_1(b_1)^* \otimes \rho(b_1) \otimes \rho_2(b_2)\ \varphi(g_1,g_2).$$

Let's assume G acts in the obvious way. Then

$$\varphi(g_1 x, g_2 x) = \varphi(g_1, g_2).$$

Now consider the homeom.

$$G\times G \longrightarrow G\times G$$
$$(g_1, g_2) \longmapsto (g_1 g_2^{-1}, g_2)$$

i.e. look at $\quad \tilde\varphi(g_1 g_2^{-1}, g_2) = \varphi(g_1, g_2)$

Then

$$\varphi(b_1 g_1, b_2 g_2) = \xi_2(b_2)\, \varphi(g_1, g_2)\, \xi_1(b_1)\, \rho(b_1)$$

$$\varphi(g_1 x, g_2 x) = \varphi(g_1, g_2).$$

Set

$$\psi(g_1 g_2^{-1}) = \varphi(g_1, g_2).$$

Then get

$$\psi(b_1 g_1 g_2^{-1} b_2^{-1}) = \xi_2(b_2)\, \psi(g_1 g_2^{-1})\, \xi_1(b_1)\, \rho(b_1).$$

ie. we have to study ψ $\ni$

$$\psi(b_1 x b_2^{-1}) = \xi_2(b_2)\, \psi(x)\, \xi_1(b_1)\, \rho(b_1).$$

~~Actually it is very reasonable to take dy to be the~~

$$pr_1^* F \otimes pr_2^* E^* \otimes pr_2^* \omega = (G \times G) \underset{B \times B}{\times} \left(\mathrm{Hom}(\mathfrak{I}_1, \rho) \otimes \mathfrak{I}_2 \right)$$

So K comes from a mapping $\psi(g_1, g_2) \in \mathrm{Hom}(\mathfrak{I}_1, \mathfrak{I}_2)$ such that

$$\psi(b_1 g_1, b_2 g_2) = \mathfrak{I}_2(b_1) \circ \psi(g_1, g_2) \circ \mathfrak{I}_1(b_2^{-1}) \rho(b_2)$$

The formula for F in terms of ψ is

$$(g_1, F(f)(g_1^{-1})) = \int_{g_2 B} \underline{K(g_1 B, g_2 B)} \cdot (g_2, f(g_2^{-1})) \cancel{d(g_2 B)}$$

$$= \int_{g_2 B} (g_1 \times g_2, \psi(g_1^{-1}, g_2^{-1}))(g_2, f(g_2^{-1})) \cancel{d(g_2 B)}$$

$$= \int_{g_2 B} (g_1 \times g_2, \psi(g_1^{-1}, g_2^{-1}) f(g_2^{-1})) \cancel{d(g_2 B)}$$

$$= \left(g_1, \int_{g_2 B} \psi(g_1^{-1}, g_2^{-1}) f(g_2^{-1}) \cancel{d(g_2 B)}\right)$$

$$F(f)(g_1) = \int_{g_2 B} \underline{\psi(g_1, g_2^{-1}) f(g_2^{-1})} \cancel{d(g_2 B)}$$

Try again: We are trying to construct "maps" from $I(\sigma_1)$ to $I(\sigma_2)$ and by following Bruhat we have shown that they are in 1-1 correspondence with ~~distributions~~ "functions" φ on G with values in $\mathrm{Hom}(\sigma_1, \sigma_2)$ satisfying

$$\varphi(b_1 x b_2^{-1}) = \sigma_2(b_2)\,\varphi(x).\,\sigma_1(b_1)\,\rho(b_1)$$

where $\rho(b)$ ~~measures the volume~~ is essentially the determinant of $\mathrm{Ad}\, b$ on $\mathfrak{g}/\mathfrak{b}$.

You must write this up carefully!

Let $X = G/B$ and let $F : I(\sigma_1) \longrightarrow I(\sigma_2)$ be a G mapping. Recall

$$\Gamma(G \times_B \sigma_i) = \cancel{f}\ I(\sigma_i)$$

$$\cancel{(g \mapsto f(g^{-1})) \longleftarrow f}$$

$$\bigcup_{g \in G} (g, f(g^{-1})) \longleftarrow f$$

By the kernel thm. F is given by

$$F(f)(x) = \int_{y \in X} K(x,y)\, f(y)\, dy$$

where $K(x,y)\,dy$ is a "section" over $X \times X$ of the bundle $pr_2^* E^* \otimes pr_2^* \omega \otimes pr_1^* F$

Bruhat 1

Proposition: Let $\varphi : G \longrightarrow \mathrm{Hom}(\mathcal{J}_1, \mathcal{J}_2)$ be a function such that

$$\varphi(b_1 g b_2) = \mathcal{J}_2(b_1)\cdot\varphi(g)\cdot\mathcal{J}_1(b_2)\cdot\rho(b_2^{-1})$$

where $\rho(b) =$ determinant of $\mathrm{Ad}\, b$ on $\mathfrak{n}$ (hence $\rho(\exp A) = \exp\big(\sum_{\alpha\in\Sigma''} \alpha(A)\big)$. Let $f \in I(\mathcal{J}_1)$ so that $f : G \to \mathcal{J}_1$ satisfies $f(bg) = \mathcal{J}_1(b)\, f(g)$. As the function

$$\theta_{g_1} : \; g_2 \longmapsto \varphi(g_1 g_2^{-1}) f(g_2)$$

satisfies

$$\theta_{g_1}(b g_2) = \rho(b)\, \theta_{g_1}(g_2),$$

it defines a section of the bundle $G \times_B \rho$ $= G \times_B \Lambda^r \mathfrak{n} \simeq G \times_B \Lambda^r(\mathfrak{g}/\mathfrak{b})^* = \Lambda^r T^*(G/B)$ where $r = \dim G/B$. This r form on G/B may be integrated over G/B since G/B is compact; denote the result by

$$F(f)(g_1) = \int_{g_2 B \in G/B} \varphi(g_1 g_2^{-1}) f(g_2).$$

Then $F : I(\mathcal{J}_1) \longrightarrow I(\mathcal{J}_2)$ is a map of G representations.

Proof. It is clear that F is well-defined and $F(f)(bg) = \mathcal{J}_2(b)\, F(f)(g)$. We must show it's a G-map. But

$$F(gf)(g_1) = \int_{g_2 B \in G/B} \varphi(g_1 g_2^{-1}) f(g_2 g)$$

$$= \int_{g_2 B \in G/B} \varphi(g_1 g (g_2 g)^{-1}) f(g_2 g)$$

Note that the r form on G/B represented by $g_2 \longmapsto \varphi(g_1 g (g_2 g)^{-1}) f(g_2 g)$ is the g-translate of the r form represented by $g_2 \longmapsto \varphi(g_1 g g_2^{-1}) f(g_2)$, hence has the same integral. Thus

$$F(gf)(g_1) = \int_{g_2 B \in G/B} \varphi(g_1 g g_2^{-1}) f(g_2)$$

$$= F(f)(g_1 g) = [g \cdot F(f)](g_1).$$

Recommend change of notation ~~from~~ to

$$F(f)(g_1) = \int_{g_2 B \in G/B} \underbrace{\varphi(g_1 g_2) f(g_2^{-1})}$$

As a fn. $\theta_{g_1}(g_2)$ satisfies

$$\theta_{g_1}(g_2 b) = \rho(b^{-1})\, \theta_{g_1}(g_2)$$

hence defines a r form on G/B.

$\cancel{\int_{g_1^{-1} z B} \varphi(z) f(z^{-1} g_1)}$ $\qquad$ $\cancel{z = g_1 g_2 \quad g_1^{-1} z = g_2}$

<u>Composition:</u>

Suppose $I(\zeta_1) \xrightarrow{F} I(\zeta_2) \xrightarrow{G} I(\zeta_3)$

given by

$$(Ff)(g_1) = \int_{g_2B \in G/B} \varphi(g_1g_2) f(g_2^{-1}) = \int_{g_2B \in G/B} \varphi(g_2) f(g_2^{-1}g_1)$$

$$(Gf)(g_1) = \int_{g_2B \in G/B} \psi(g_1g_2) f(g_2^{-1})$$

(shows that F defd if φ is a dist.)

Then

$$(GFf)(g_1) = \int_{g_2B} \psi(g_1g_2) \int_{g_3B} \varphi(g_2^{-1}g_3) f(g_3^{-1})$$

$$= \int_{g_3B} \left[\int_{g_2B} \psi(g_1g_2)\varphi(g_2^{-1}g_3)\right] f(g_3^{-1}).$$

Thus

$$\boxed{(\psi * \varphi)(g_1) = \int_{g_2B \in G/B} \psi(g_1g_2)\varphi(g_2^{-1})} = \int_{g_2B \in G/B} \psi(g_2)\varphi(g_2^{-1}g_1)$$

~~shows that operator well defd. for any dist. φ.~~

Next try to ~~classify~~ Construct these distributions φ.

<u>Main Remark:</u> Any invariant distributional ~~on G/H is~~ section of $G \times_H V$ over G/H is necessarily a smooth invariant section i.e. given by an element of V^H.

~~Proof~~

Suppose given double coset BxB, ~~so that~~ I want a "function" ? φ on BxB such that

$$\varphi(b_1 x b_2) = \rho_2(b_1)\,\varphi(x)\,\tilde{\rho}_1(b_2)$$

Fix x; ~~and~~ as usual this means that

$$\mathrm{Hom}_{B\times B}\big(BxB,\ \mathrm{Hom}(\tilde{\rho}_1, \rho_2)\big) = \mathrm{Hom}_{B\cap xBx^{-1}}(\tilde{\rho}_1^{\,x}, \rho_2)$$

where $\tilde{\rho}_1^{\,x}(b) = \tilde{\rho}_1(x^{-1}bx)$

One sees that if φ is an honest function on G, then φ lives only on the big cell.

$$b_1 x = x b_2$$

$$\Downarrow$$

$$\rho_2(b_1)\,\varphi(x) = \varphi(x)\,\tilde{\rho}_1(b_2)$$

$$\Downarrow$$

$$\varphi(x) \in \mathrm{Hom}_{B\cap xBx^{-1}}(\tilde{\rho}_1^{\,x}, \rho_2) = \mathrm{Hom}_{MA}(\tilde{\rho}_1^{\,x}, \rho_2).$$

Do the case of finite groups with a Tits system.

$$B, N \qquad B \cap N = \mathbb{T} \qquad N/\mathbb{T} \simeq W$$

here N is the normalizer of a torus.

$$G = \bigcup_{\sigma \in W} B\sigma B$$

Now proceed as follows: Let L, M be irred representations of T and show that there are no maps from

$$j_! L \text{ to } j_! M$$

unless M conjugate to L via some element of W. Then calculate the resulting category

$$\mathrm{Hom}_G(j_* L, j_* M) = \mathrm{Hom}_{B \times B}(G, \mathrm{Hom}(L, M))$$

$$\varphi^{\#} \longleftarrow\!\!| \ \varphi$$

where $$(\varphi^{\#} f)(g_1) = \sum_{g_2} \varphi(g_1 g_2)\, \varphi(g_2^{-1}) \ \frac{1}{|B|}$$

Thus $$\mathrm{Hom}_{B \times B}(G, \mathrm{Hom}(L, M)) = \mathrm{Hom}_B(G \times_B L, M)$$

$$\| $$

$$\prod_{B \times B} \mathrm{Hom}$$

$$\underbrace{\mathrm{Hom}_{B\times B}(G, \mathrm{Hom}(L,M))}= \prod_{BuB} \mathrm{Hom}_{B,B}(BuB, \mathrm{Hom}(L,M)).$$

$$= \prod_{u} \mathrm{Hom}_{B\cap uBu^{-1}}(L^{u^{-1}}, M)$$

$u \in G$ has stabilizer $\{(b_1,b_2) \mid b_1 u b_2^{-1} = u\} \xrightarrow{pr_1} B \cap uBu^{-1}$

where $L^{u^{-1}}$ is the rep of $B \cap uBu^{-1}$ given by
$b_1 \rightsquigarrow u^{-1}b_1u$ acting on L.

$$\cancel{= \prod_u \mathrm{Hom}_{B\times B}}$$

In our case we know that $B \cap uBu^{-1} \supset T$ and that the rest acts trivially, hence

$$\mathrm{Hom}_{B\cap uBu^{-1}}(L^{u^{-1}}, M) = \mathrm{Hom}_T(L^{u^{-1}}, M)$$

$$= \mathrm{Hom}_T(L, M^u)$$

where $\xi^u(t) = \xi(utu^{-1})$.

Conclusion

$$\mathrm{Hom}_G(j_*\xi_1, j_*\xi_2) = \prod_{u\in W} \cancel{\mathrm{Hom}_T(L^u, M)}\ \mathrm{Hom}_T(\xi_1^u, \xi_2)$$

now please calculate the composition.

Theorem: Let G be a finite group with a Bruhat decomposition B, T, W etc. Suppose that ρ_1 and ρ_2 are two representations of T, which are then extended to B so as to be trivial on N. Then there is an ~~canonical~~ isomorphism

$$a : \mathrm{Hom}_G(j_* \rho_1, j_* \rho_2) \cong \prod_{u \in W} \mathrm{Hom}_T(\rho_1^{\alpha_u}, \rho_2)$$

Definition of a: If $\varphi^\# \in$ have $\rho^{\alpha_u}(t) = \rho(\alpha_u^{-1} t \alpha_u)$

$$(\varphi^\# f)(g_1) = \frac{1}{|B|} \sum_{g_2} \varphi(g_1 g_2) f(g_2^{-1})$$

where $\varphi : G \longrightarrow \mathrm{Hom}(\rho_1, \rho_2)$ is $B \times B$ equiv.

Now choose an element $\alpha_u \in \hat{T}$ representing u.

$$a(\varphi) = \big(\varphi(\alpha_u)\big)$$

$$\varphi(\alpha_u)\, \rho_1^u(t)$$

$$\varphi(\alpha_u)\, \rho_1(\underbrace{\alpha_u^{-1} t \alpha_u}_{\in B}) = \varphi(t \alpha_u)$$

$$= \rho_2(t)\, \varphi(\alpha_u).$$

Thus a lands in the correct place.

The reason a is an isomorphism is because

$$G = \bigcup_u B\alpha_u B \qquad \text{Bruhat decomposition!}$$

So

$$\mathrm{Hom}_{B\times B}(G, \mathrm{Hom}(\rho_1, \rho_2)) = \prod_u \mathrm{Hom}_{B\times B}(B\alpha_u B, \mathrm{Hom}(\rho_1,\rho_2))$$

$$\wr\!\!\downarrow$$

$$\mathrm{Hom}_T(\rho_1^u, \rho_2)$$

Proof:

$$\text{Stabilizer of } \alpha_u = \{(b_1, b_2) \mid b_1\alpha_u = \alpha_u b_2\} \xrightarrow[\sim]{pr_1} B \cap \alpha_u B \alpha_u^{-1}$$

and $B \cap \alpha_u B \alpha_u^{-1}$ acts on $\mathrm{Hom}(\rho_1, \rho_2)$ by

$$(b, \alpha_u^{-1} b \alpha_u)\cdot \alpha_u = \cancel{b\alpha_u\alpha_u^{-1}b^{-1}}\alpha_u$$

$$b\cdot\varphi = \underline{(b, \alpha_u^{-1} b \alpha_u)\cdot\varphi} = \rho_2(b)\,\varphi\,\rho_1(\alpha_u^{-1} b^{-1} \alpha_u) = \varphi$$

ie
$$\rho_2(b)\varphi = \varphi\,\rho_1(\alpha_u^{-1} b\alpha_u)$$
$$= \varphi\,\rho_1^u(b).$$

Thus get
$$\mathrm{Hom}_{\underline{B\cap\alpha_u B\alpha_u^{-1}}}(\rho_1^u, \rho_2) = \mathrm{Hom}_T(\rho_1^u, \rho_2)$$

because $\rho_2(N) = 1$ and $\rho_1^u(N \cap \alpha_u N \alpha_u^{-1}) = 1.$

90
180
45
225

$$B = TN$$
$$\alpha_u B \alpha_u^{-1} = T\,\alpha_u N \alpha_u^{-1} \qquad \therefore\ B\cap\alpha_u B\alpha_u^{-1} = T\cdot(N\cap\alpha_u N\alpha_u^{-1})$$

How unique is α_u?

$$\bar{\alpha}_u = \alpha_u \cdot t_u$$

then $\varphi \mapsto \varphi(\bar{\alpha}_u) = \varphi(\alpha_u)\,\varphi(t_u) \in \mathrm{Hom}_T(\xi^u,$

Clearly changes what you get. ~~[illegible]~~

Inverse of $\mathcal{a}$:

Given φ_u define φ by

$$\varphi(b_1 \alpha_u b_2) = \xi_2(b_1)\,\varphi_u\,\xi_1(b_2)$$

Given $\psi(\alpha_u) = \psi_u$.

Now we want to calculate

$$\psi * \varphi(g_1) = \frac{1}{|B|}\sum{}' \psi(g_1 g_2)\,\varphi(g_2^{-1})$$

This is a bad approach. Instead ~~let~~

$$\Gamma(u,\rho)(\alpha_v) = \begin{cases} 0 & v \neq u \\ id_{\rho^u} & v = u \end{cases}$$

define $\Gamma(u,\rho) \in Hom_G(j_* \rho, j_* \rho^u)$

by

$$a\Gamma(u,\rho) = \delta_{uv} \quad \begin{matrix} 0 \text{ for } v \neq u \\ id \in Hom_T(\rho^u, \rho^u) \end{matrix}$$

So calculate

$$\Gamma(u,\rho)(g) = \begin{cases} 0 & \text{if } g \notin B\alpha_u B \\ \rho^u(b_1)\,\rho(b_2) & \text{if } g = b_1 \alpha_u b_2 \end{cases}$$

$$\Gamma(u,\rho)(g) = \begin{cases} 0 & \text{if } g \notin B\alpha_u B \\ \rho^u(b_1)\,\rho(b_2) & \text{if } g = b_1 \alpha_u b_2 \\ \rho(b) & \text{if } g = n\alpha_u b \end{cases}$$

Check well defined

$$g = b_1 \alpha_u b_2 = \bar{b}_1 \alpha_u \bar{b}_2 \quad \text{then}$$

$$b_1^{-1} \bar{b}_1 \alpha_u \bar{b}_2 b_2^{-1} = \alpha_u$$

$\underset{\sim\sim}{t \cdot n}$

$\underline{\alpha_u^{-1}(b_1^{-1}\bar{b}_1)\alpha_u} = \underline{b_2(\bar{b}_2)^{-1}}$

$\therefore \quad \varsigma^u(b_1^{-1})\,\varsigma^u(\bar{b}_1) = \varsigma(b_2)\,\varsigma(\bar{b}_2)^{-1}$

$b_1^{-1}\bar{b} = t \cdot n$

$$\varsigma(\alpha_u^{-1} b_1^{-1}\bar{b}\,\alpha_u) = \varsigma(\alpha_u^{-1} t\,\alpha_u)$$
$$= \varsigma^u(t) = \varsigma^u(tn) = \varsigma^u(b_1^{-1}\bar{b})$$

$G = \bigcup g_i B$

$$[\Gamma(v,\varsigma^u)\circ\Gamma(u,\varsigma)](\alpha_w) = \sum_i \Gamma(v,\varsigma^u)(\alpha_w g_i)\;\underbrace{\Gamma(u,\varsigma)(g_i^{-1})}$$

$$I(\varsigma) \xrightarrow{\Gamma(u,\varsigma)} I(\varsigma^u) \xrightarrow{\Gamma(v,\varsigma^u)} I((\varsigma^u)^v)$$

$\not\equiv 0 \Downarrow \qquad \not\equiv 0 \Downarrow$

$\alpha_w g_i \in B\alpha_v B \qquad g_i^{-1} \in B\alpha_u B$

$\Downarrow$

$\alpha_w \in B\alpha_v B\alpha_u B.$

$$(\varsigma^u)^v(t) = \varsigma^u(\alpha_v^{-1}t\alpha_v)$$
$$= \varsigma(\alpha_u^{-1}\alpha_v^{-1}t\alpha_v\alpha_u)$$
$$\simeq \varsigma(\alpha_{vu}^{-1}t\,\alpha_{vu})$$

It seems reasonable to conjecture that we get a constant times What is this constant?

$\Gamma(vu,\varsigma)$

Claim that

$$\boxed{\Gamma(v,\rho^u)\circ\Gamma(u,\rho) = c(v,u,\rho)\Gamma(vu,\rho)}$$

First problem: Show that if $w \neq uv$

$$\sum_i{}' \Gamma(v,\rho^u)(\alpha_w g_i)\Gamma(u,\rho)(g_i^{-1}) = 0.$$

Here $G = \bigcup g_i B$

But we can arrange the sum differently. Thus we group the g_i according to the double coset to which they belong. Thus write

$$n_i \cdot \alpha_{u(i)} = g_i \qquad \text{where } n_i \in N.$$

so that

$$\Gamma(u,\rho)(g_i^{-1}) = \Gamma(u,\rho)(\underline{\alpha_{u(i)}^{-1} n_i^{-1}})$$

$$= \begin{cases} 0 & \text{if } \underline{\alpha_{u(i)}^{-1}} n_i^{-1} \notin B\alpha_u B \\ & \end{cases}$$

Assume $(\alpha_u)^{-1} = \alpha_{u^{-1}}$?

Thus $u(i) = u^{-1}$ and we only have to sum over

$g_i = n_i\,\alpha_{u^{-1}}$ represent $U n_i \alpha_{u^{-1}} B = B\alpha_{u^{-1}} B$.

$\Gamma(u,\mathfrak{J})(g_i^{-1}) =$

$g_i = n_i(\alpha_u)^{-1}$ where $U n_i(\alpha_u)^{-1} B = B(\alpha_u)^{-1} B$.

and $\Gamma(u,\mathfrak{J})(g_i^{-1}) = \Gamma(u,\mathfrak{J})(\alpha_u n_i)$

$= \mathfrak{J}(\alpha_u)$

$$\left[\sum_i \underbrace{\Gamma(v,\mathfrak{J}^u)(\alpha_w\, n_i(\alpha_u)^{-1})}\right] \mathfrak{J}(\alpha_u).$$

0 if $\alpha_w\, n_i(\alpha_u)^{-1} \notin B\alpha_v B$

if $\alpha_w\, n_i(\alpha_u)^{-1} =$

Use fact that W generated by ~~refl~~ reflections !!!!

<u>Reflections:</u> It's up to you to calculate the **simple** case of a **reflection** and determine the formula for irreducibility.

First in the finite case. This is legitimate – think of $sl(n,k)$ where k is finite of characteristic p. Then representations are not completely reducible so that the intertwining no. criterion fails. Yet the induced representation may be **irreducible** by the same argument (here N is of order p^k and so we have Nakyama's lemma).

Assume that $s \in W$ is of order 2 i.e. reflection in the hyperplane $\alpha = 0$. Then I want to calculate

$$\Gamma(s, \xi^s)\Gamma(s, \xi).$$

Do the formulas become any easier? First problem is to determine cosets reps for BsB; this should be easier because s permutes all positive roots except α. (certainly OKAY for simple roots and they generate W). Thus $B \cap \alpha_s B \alpha_s^{-1}$ is of codim 1 in B. ~~missing just~~

$$B \cap \alpha_s B \alpha_s^{-1} = T \times \underbrace{N \cap \alpha_s N \alpha_s^{-1}}_{\text{missing a single root}}$$

$$N / N \cap \alpha_s N \alpha_s^{-1} \xrightarrow{\sim} B / B \cap \alpha_s B \alpha_s^{-1} \xrightarrow{\simeq} B \alpha_s B / B$$

Thus in fact ~~$N \cap \alpha_s N \alpha_s^{-1}$~~ there is an abelian subgroup $J \subset N$ such that

$$N = J \times (N \cap \alpha_s N \alpha_s^{-1}) \qquad \text{semi-direct since } N \cap \alpha_s N \alpha_s^{-1} \triangleleft N.$$

$$\Gamma(s, \xi^{s}) \Gamma(s, \xi)^{\alpha_u} = \sum_{j \in J} \Gamma(s, \xi^{\alpha_s})(\underline{\alpha_u j \alpha_s^{-1}}) \underbrace{\Gamma(s, \xi)(\alpha_s j^{-1})}_{\xi(\alpha_s)}$$

$$\Gamma(s, \xi^{\alpha_s})(\alpha_u j \alpha_s^{-1}) = \begin{cases} 0 & \text{if } \alpha_u j \alpha_s^{-1} \in B\alpha_s B \\ (\xi^{\alpha_s})^{\alpha_s}(b_1)\, \xi^{\alpha_s}(b_2) & \text{if } \alpha_u j \alpha_s^{-1} = b_1 \alpha_s b_2 \end{cases}$$

some nice helpful formulas

$$B\alpha_u B \cdot B\alpha_s B = \begin{cases} B\alpha_u\alpha_s B & \text{if } B\alpha_s B \not\subset B\alpha_u B \cdot B\alpha_s B \\ B\alpha_u B \cup B\alpha_u\alpha_s B & \quad '' \quad \subset \end{cases}$$

~~this get 0 unless~~

thus $\underline{\alpha_u j \alpha_s^{-1} \in \underbrace{B\alpha_u B \cup B\alpha_u\alpha_s B}}$

neither of these is $B\alpha_s B$

unless $u = s$ or $u = e$

therefore get 0 unless $u = s$.

now have to calculate

$$\alpha_s J \alpha_s^{-1} = \alpha_s \exp \mathbb{C} e_\alpha \alpha_s^{-1}$$

$$= \underline{\exp \mathbb{C} e_{-\alpha}} \quad \in B\alpha_s B$$

But $\quad B\alpha_s B/B \simeq N/N \cap \alpha_s N \alpha_s^{-1} = J$

Thus

$$\alpha_s J \alpha_s^{-1} = J\alpha_s \quad \text{mod } B.$$

should be true. Can you make this more explicit?

You are really reduced to $sl(2, \mathbb{R})$

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & \frac{1}{t} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}\begin{pmatrix} t & 1 \\ 0 & \frac{1}{t} \end{pmatrix}$$

$$\| $$

$$\begin{pmatrix} \frac{1}{t} & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} t & 1 \\ 0 & \frac{1}{t} \end{pmatrix}$$

Bruhat 3

Review: If $\psi \in \mathrm{Hom}_T((\xi_1 \otimes g^{-1})^{\alpha_s}, (\xi_2 \otimes g^{-1}))$, then

$$F(f)(g) = \int_J \psi f(\alpha_s^{-1} j g)\, dj$$

is a map $F: I(\xi_1) \longrightarrow I(\xi_2)$. $\qquad \xi_1 \otimes g^{-1} = \xi$

Now ~~calculate~~ suppose $\xi_2 \otimes g^{-1} = (\xi \otimes g)^{\alpha_s}$

take $\psi = id$

$$I(\xi \otimes g) \xrightarrow{F} I(\xi^{\alpha_s} \otimes g) \xrightarrow{G} I((\xi^{\alpha_s})^{\alpha_s} \otimes g)$$

$$f \longmapsto \left(g \mapsto \int_J f(\alpha_s^{-1} j g)\, dj\right) \longmapsto \left(g \mapsto \int_J dj_1 \int_J f(\alpha_s^{-1} j_2 \alpha_s^{-1} j_1 g)\, dj_2\right)$$

So calculate

$$[(GF)(f)](g) = \int_J dj_1 \int_J f(\alpha_s^{-1} j_2 \alpha_s^{-1} j_1 g)\, dj_2 \overset{?}{=}$$

finite case:

$$(GFf)(g) = \sum_{j_1} \sum_{j_2} f(\underbrace{\alpha_s^{-1} j_2 \alpha_s^{-1} j_1} g)$$

we know this is in $I((\xi^{\alpha_s})^{\alpha_s})$.

Recall $B\alpha_s B \cup B$ is a group.

hence $\underline{\alpha_s^{-1} j_2 \alpha_s^{-1} j_1} \in$ either $B\alpha_s B$ or B.

To calculate the operator

$$(Af)(g) = \int_J dj_1 \int_J dj_2 \; f(\alpha_s^{-1} j_2 \alpha_s j_1 g)$$

There are ~~[illegible]~~ really two integrals here when

$$\alpha_s^{-1} j_2 \alpha_s j_1 \quad \cancel{} \quad B\alpha_s B$$

and where

$$\alpha_s^{-1} j_2 \alpha_s j_1 \in B$$

$$\Updownarrow$$

ie

$$\alpha_s^{-1} j_2 \alpha_s \in B$$

$$\Updownarrow$$

$$j_2 \in \alpha_s B \alpha_s^{-1}$$

$$\Updownarrow$$

$$j_2 = e.$$

The point is that the first integral should be zero!!! ~~[illegible]~~ and this would be the case if f has compact support within the ~~[illegible]~~ set $B\alpha_s Bg$. What probably happens is that the integral over $B\alpha_s Bg$ may be transformed into an integral over the boundary Bg, leaving an integral kernel.

Problem: Transform

~~$(Af)(g)$~~

$$\iint_{J\times J} dj_1\, dj_2\; f(\alpha_s^{-1} j_2 \alpha_s j_1 g)$$

into

$$\int_J dj_1\; Q(j_1)\, f(\varphi(j_1) g)$$

Look carefully if $j_2 \neq 0$ then $\alpha_s^{-1} j_2 \alpha_s j_1 \notin B$

$$f(\alpha_s^{-1} j_2 \alpha_s j_1 g)$$

Write

$$\alpha_s^{-1} j_2 \alpha_s j_1 \in B\alpha_s B$$

in the form

$$b_1 \alpha_s b_2$$

Idea j_1 is OKAY.

$$\boxed{\alpha_s^{-1} j_2 \alpha_s = b_1(j_2)\alpha_s b_2(j_2)}$$

defined for $j_2 \neq 0$.

$$f(\alpha_s^{-1} j_2 \alpha_s j_1 g) = f(b_1(j_2)\alpha_s b_2(j_2) j_1 g)$$

$$=$$

Write

$$\alpha_s^{-1} j \alpha_2 = b(j)\alpha_s \varphi(j)$$

where $\varphi(j) \in J$ nice for $j \neq 0$

$b(j) \in B$.

Then

$$\int dj\, dj_1\, f(\alpha_s^{-1} j \alpha_s j_1 g)$$

$$\|$$

$$\int_{j\neq 0} dj\, dj_1\, f(\underline{b(j)}\alpha_s \underline{\varphi(j) j_1} g)$$

$$\|$$

$$\int_{j \neq 0} \underline{\xi_1(b(j))}\, f(\alpha_s j_2 g)\; dj\, dj_2$$

$j_2 = \varphi(j) j_1$

$$\|$$

$$\int \left[\int_{j\neq 0} \xi(b(j))\, dj\right] f(\alpha_s j_2 g)\, dj_2$$

clearly getting into singular ops.

This is clearly an $sl(2,\mathbb{R})$ calculation.

$$\int_{J\times J} dj\, dj_1\, f(\underline{\alpha_s^{-1} j \alpha_s j_1})$$

$\in \cancel{B\alpha_s B \cup B}$.

$J\alpha_s J \cup J$

Let $G_1 \subset G$ be generated by α_s and J.

Its Lie alg. is

$$e_\alpha,\ e_{-\alpha},\ H_\alpha$$

Then we are given a function f on $G_1 x$ and we want to calculate

$$\int_{J\times J} dj\, dj_1\, f(\alpha_s^{-1} j \alpha_s j_1 x)$$

The hope is that this is some multiple of $f(x)$ the multiple depending on J and its relation to α_s.

So we do for $sl(2)$. Except we know what the answer should be

formula for $\alpha_s =$

e_α

Choose α. This gives rise to $e_\alpha, e_{-\alpha}, H_\alpha$ related by

$$[H, e_\alpha] = \alpha(H) e_\alpha = \langle H, H_\alpha \rangle e_\alpha$$

$$[e_\alpha, e_{-\alpha}] = H_\alpha$$

$$\alpha(H_\alpha) = \langle \alpha, \alpha \rangle \quad \text{some number.}$$

have
$$s_\alpha(H) = H - 2 \frac{\langle H, \alpha \rangle}{\langle \alpha, \alpha \rangle} H_\alpha$$

want to show s_α is inner.

$$s_\alpha = \exp t \operatorname{ad} e_\alpha \operatorname{ad} e_{-\alpha}$$

$$(\operatorname{ad} e_\alpha \operatorname{ad} e_{-\alpha}) H = \operatorname{ad} e_\alpha (+\alpha(H) e_{-\alpha})$$
$$= \alpha(H) H_\alpha.$$

$$\exp t(\operatorname{ad} e_\alpha \operatorname{ad} e_{-\alpha}) H = \cancel{H + \frac{t\alpha(H)}{1!} H_\alpha + \frac{t^2 \alpha(H)^2}{2!} H_\alpha + \cdots}$$

$$\cancel{= H + H_\alpha - H_\alpha + e^{t\alpha(H)}}$$

$$= H + t\alpha(H) H_\alpha + \frac{t^2}{2!} \alpha(H) \alpha(H_\alpha) H_\alpha + \frac{t^3}{3!} \alpha(H) \alpha(H_\alpha)^2 H_\alpha$$

$$= H + \frac{\alpha(H)}{\alpha(H_\alpha)} \left[e^{t\alpha(H_\alpha)} H_\alpha - H_\alpha \right]$$

get $s_\alpha(H)$ if $e^{t\alpha(H_\alpha)} = -1$. ie $t = \frac{\pi i}{\langle \alpha, \alpha \rangle}$

Start with e_α and s. Then s gives you $e_{-\alpha}$ and get $\mathfrak{g}_1$ hence G_1 and now in G_1 you must choose α_s. G_1 is clearly $\simeq$ ~~...~~ some covering of $sl(2,\mathbb{R})$. So one considers in G_1 the normalizer of H_α and picks an element α_s in the compact part. At this stage we have almost everything and should be able to calculate things !!!

Return to $sl(2,\mathbb{R})$

$$\begin{cases} H\delta_\sigma = \sigma\delta_\sigma \\ X\delta_\sigma = \frac{1}{\sqrt{2}}(\lambda+\sigma)\,\delta_{\sigma+1} \\ Y\delta_\sigma = \frac{1}{\sqrt{2}}(\lambda-\sigma)\,\delta_{\sigma-1}. \end{cases}$$

$$\varphi\,\delta_\sigma = c_\sigma\,\delta'_\sigma.$$

$$\varphi\, X\delta_\sigma = X c_\sigma\,\delta'_\sigma$$

$$\|$$

$$\frac{1}{\sqrt{2}}(\lambda+\sigma)\,c_{\sigma+1}\,\delta'_{\sigma+1} = \frac{1}{\sqrt{2}}c_\sigma\,(1-\lambda+\sigma)\,\delta'_{\sigma+1}$$

$$\therefore\quad \frac{c_\sigma}{c_{\sigma+1}} = \frac{\lambda+\sigma}{1-\lambda+\sigma}$$

$$\varphi\,Y\delta_\sigma = Y c_\sigma\,\delta'_\sigma$$

$$\|$$

$$\frac{1}{\sqrt{2}}(\lambda-\sigma)\,c_{\sigma-1}\,\delta'_{\sigma-1} = \frac{1}{\sqrt{2}}c_\sigma\,(1-\lambda-\sigma)\,\delta'_{\sigma-1}$$

$$\frac{c_\sigma}{c_{\sigma-1}} = \frac{\lambda-\sigma}{1-\lambda-\sigma}$$

$$\frac{c_{\sigma-1}}{c_\sigma} = \frac{\lambda+\sigma-1}{1-\lambda+\sigma-1}$$

Calculate for $sl(2,\mathbb{R})$ and some choice of α_s the integral

$$\int_{J\times J} f(\alpha_s^{-1} j_1 \alpha_s j_2 x)\, dj_1\, dj_2 \qquad !$$

where f transform by ζ under b is

$$f(bx) = \zeta(b) f(x).$$

Case 1: $\alpha_s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\qquad J = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

$$\alpha_s^{-1}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\alpha_s = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & t \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}.$$

Write

$$\begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix} = \begin{pmatrix} \rho & \sigma \\ 0 & \tau \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\sigma & \rho \\ -\tau & 0 \end{pmatrix}\begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\sigma & -\sigma\phi+\rho \\ -\tau & -\tau\phi \end{pmatrix}$$

$\therefore \quad \sigma = -1 \qquad \tau = t \qquad \phi = -\frac{1}{t} \qquad \rho = \sigma\phi = \frac{1}{t}$

$$= \begin{pmatrix} 1 & 0 \\ \frac{1}{t} & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{t} & -1 \\ 0 & t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{t} \\ 0 & 1 \end{pmatrix}$$

$$f(\alpha_s^{-1} j_t \alpha_s j_\gamma x) = \cancel{\alpha_s} \quad \varphi\begin{pmatrix} \frac{1}{t} & -1 \\ 0 & t \end{pmatrix} f(\alpha_s j_{-\frac{1}{t}+\gamma} x)$$

$$\int_{t,\gamma} \underline{\varphi\begin{pmatrix} \frac{1}{t} & -1 \\ & t \end{pmatrix}} f\left(\alpha_s \begin{pmatrix} 1 & -\frac{1}{t}+\gamma \\ 0 & 1 \end{pmatrix} x\right) \quad dt\, d\gamma.$$

$$\int_{t,\gamma} t^\beta f\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{t}+\gamma \\ & 1 \end{pmatrix}\right) dt\, d\gamma$$

<u>Question</u>: What is

$$\int_{t,\gamma} \underbrace{t^\beta g(-\tfrac{1}{t}+\gamma)}\, dt\, d\gamma. \qquad g \text{ smooth}.$$

this is a smooth fn. of t & γ even at $t=0$.

g decays rapidly.

clearly if we int. wrt γ first get some constant

ie $\int_{\mathbb{R}} g(x)\,dx$.

Then have to integrate $c \cdot \int_t t^\beta dt$ — a most improper integral.

Important to note that β different for R_+ & R_- depending on t_s

Problem Calculate

$$\int f\left(\begin{pmatrix}1&0\\-t&1\end{pmatrix}\begin{pmatrix}1&\gamma\\0&1\end{pmatrix}\right)dt\,d\gamma$$

where f is a smooth function ~~of compact support~~ such that

$$f\left(\begin{pmatrix}u&a\\0&u^{-1}\end{pmatrix}z\right)=\chi(u)f(z)$$

where χ is a character on $\mathbb{R}^*$.

$$\begin{pmatrix}1&0\\-t&1\end{pmatrix}\begin{pmatrix}1&\gamma\\0&1\end{pmatrix}=\begin{pmatrix}1&\gamma\\-t&1-\gamma t\end{pmatrix}$$ has det 1.

~~and the integral is 0.~~

singular integral in t as $t\to\infty$.

$$\int t^\beta\, g\left(-\frac{1}{t}+\gamma\right)dt\,d\gamma.$$

The problem is that this integral I am trying to calculate doesn't make much sense.

NAK

So let $t \mapsto \frac{1}{t}$

$$\int t^{-\beta} g(-t+\gamma) \, -\frac{dt}{t^2} \, d\gamma$$

$$\parallel$$

$$= \int -t^{-(\beta+2)} g(\gamma - t)\, dt\, d\gamma.$$

this is well-defined because it's a sing. op.

Question:

Arrange that $g(\infty) = 0$.

$$\lim_{t\to 0} \chi\begin{pmatrix} \frac{1}{t} & -1 \\ 0 & t \end{pmatrix} \underbrace{f\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -\frac{1}{t}+\gamma \\ & 1 \end{pmatrix}\right)}_{g(-\frac{1}{t}+\gamma)} \Leftarrow \cancel{f(\gamma)} \; \underline{f\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

Review: f is a function on $sl(2,\mathbb{R})$ such that

$$f\left[\begin{pmatrix} t & a \\ 0 & t^{-1}\end{pmatrix} x\right] = \chi(t)\, f(x) .$$

To calculate

$$\int_{t,\gamma} f\left[\begin{pmatrix} 1 & 0 \\ -t & 1\end{pmatrix}\begin{pmatrix} 1 & \gamma \\ 0 & 1\end{pmatrix}\right] dt\, d\gamma$$

not clear that this integral is well-defined even for K-finite f.

This should be equal to $C_\chi\, f(id)$ hopefully.
Recall that χ is ~~under~~ a fn. of λ and ν.

$$\chi(t) = (\text{sign } t)^{\nu} \cdot |t|^{\lambda/2}$$

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1\end{pmatrix}\begin{pmatrix} t^{-1} & 0 \\ 0 & t\end{pmatrix} = \begin{pmatrix} t & ta \\ 0 & t^{-1}\end{pmatrix}\begin{pmatrix} t^{-1} & 0 \\ 0 & t\end{pmatrix}$$

$$= \begin{pmatrix} 1 & t^2 a \\ 0 & 1\end{pmatrix}$$

$$\int = \int f\left[\begin{pmatrix} \frac{1}{t} & -1 \\ 0 & t \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -\frac{1}{t}+\gamma \\ 0 & 1 \end{pmatrix}\right]$$

$$= \int \chi(t)^{-1} \underbrace{f\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -\frac{1}{t}+\gamma \\ 0 & 1 \end{pmatrix}\right)}_{g(-\frac{1}{t}+\gamma).} dt\, d\gamma$$

we know that

$$\lim_{t\to 0} \chi(t)^{-1} g\left(-\frac{1}{t}\right) = \lim_{t\to 0} f\begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix} = f(id).$$

$$\lim_{t\to 0} \chi(t)\, g\left(-\frac{1}{t}+\gamma\right) = f\begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} = f(id)$$

so at $t=0$ OKAY. except can't integ wrt. γ.

~~I will assume that $\mathrm{Re}\,\lambda >> 0$ ... then can integrate over t easily.~~

$\chi(t)^{-1} g(-\frac{1}{t}+\gamma)$ smooth in t and γ.

Let $t \to \infty$, then if g smooth $\mathrm{Re}\,\lambda >> 0$, goes to zero fast so can integrate in t and I get fn. of γ. ~~Then integrate~~ Can't integrate for $\mathrm{Re}\,\lambda > 1$. ~~and~~ and then analyt. continue.

$$\chi(t)^{-1} g(-\tfrac{1}{t}+\gamma)$$

Assume $\operatorname{Re}\lambda > 0$ then $\lim_{t\to 0} \chi(t)^{-1} \longrightarrow \infty$ fast and so $g(x) \longrightarrow 0$ as $x \to \infty$.

In fact for $\operatorname{Re}\lambda = \frac{1}{2}$ we get principal values for both $t=0$ and ∞.

$$\int \chi(t)^{-1} g(-\tfrac{1}{t}+\gamma)\,d\gamma = \chi(t)^{-1} \int g.$$

First integrate with respect to t, then with resp to γ.

$$\lim_{R\to\infty} \int_{-R}^{R} \chi(t)^{-1} g(-\tfrac{1}{t}+\gamma)\,dt = ?$$

If were in strip, then $g(\infty) =$

$$\int_{-\infty}^{\infty} \underbrace{\chi(t)^{-1}}_{(\operatorname{sgn} t)^{2\nu}\boxed{|t|^{-\lambda/2}}} g(-\tfrac{1}{t}+\gamma)\,dt$$

is well defined for $\operatorname{Re}\lambda > 1$

$$\int_0^\infty |t|^{-\lambda/2} g\left(-\frac{1}{t}+\gamma\right) dt + \int_{-\infty}^{0} (-1)^{2\nu} |t|^{-\lambda/2} g\left(-\frac{1}{t}+\gamma\right) dt$$

$$- \int_0^\infty (-1)^{-2\nu} |t|^{-\lambda/2} g\left(\frac{1}{t}+\gamma\right) dt$$

$$= \int_0^\infty |t|^{-\lambda/2} \left(g\left(\gamma - \frac{1}{t}\right) - (-1)^{2\nu} g\left(\gamma + \frac{1}{t}\right)\right) dt$$

Assume $\nu = 0$

But

$$\int_{-\infty}^{\infty} d\gamma \int_0^\infty |t|^{-\lambda/2} \left(g\left(\gamma - \frac{1}{t}\right) - g\left(\gamma + \frac{1}{t}\right)\right) dt$$

Want to calculate

$$\int_J dy_1 \int_J dy_2 \; f(\alpha_s^{-1} y_1, \alpha_s y_2)$$

$$\parallel$$

$$\int d\gamma \int dt \; f\left(\begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix}\begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix}\right)$$

where

$$f\left(\begin{bmatrix} t & a \\ 0 & t^{-1} \end{bmatrix} Z\right) = |t|^{\lambda/2} f(Z). \qquad \nu = 0$$

hopefully this integral makes sense now

$$\begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix}\begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{t} & -1 \\ & t \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{t}+\gamma \\ 0 & 1 \end{bmatrix}$$

$$f\left(\quad '' \quad \right) = |t|^{-\lambda/2} \underbrace{f\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{t}\gamma \\ 0 & 1 \end{bmatrix}\right)}_{g(-\frac{1}{t}+\gamma)}$$

$$\int_{-\infty}^{\infty} |t|^{-\lambda/2} g\left(\gamma - \tfrac{1}{t}\right) dt$$

$$\parallel$$

$$\int_0^{\infty} |t|^{-\lambda/2} g\left(\gamma - \tfrac{1}{t}\right) + \int_{+\infty}^{0} |t|^{-\lambda/2} g\left(\gamma + \tfrac{1}{t}\right)(-dt)$$

$$= \int_0^{\infty} |t|^{-\lambda/2} \left[g\left(\gamma - \tfrac{1}{t}\right) + g\left(\gamma + \tfrac{1}{t}\right) \right] dt$$

Also try $\displaystyle\int_{-\infty}^{\infty} (\mathrm{sgn}\,t)\,|t|^{-\lambda/2}\, g\!\left(-\tfrac{1}{t}+\gamma\right) dt$

$$= \int_0^{\infty} |t|^{-\lambda/2} g\!\left(-\tfrac{1}{t}+\gamma\right) dt + \int_{+\infty}^{0} (-1)\, g\!\left(+\tfrac{1}{t}+\gamma\right)(-dt)$$

$$= \int_0^{\infty} |t|^{-\lambda/2}\left[g\!\left(\gamma-\tfrac{1}{t}\right) - g\!\left(\gamma+\tfrac{1}{t}\right)\right] dt$$

---

we know that $\displaystyle\lim_{t\to 0} (\mathrm{sgn}\,t)^{-2\nu}|t|^{-\lambda/2} g\!\left(-\tfrac{1}{t}+\lambda\right) = f(id)$

Stuck: ~~try~~ for $sl(2)$.

I need the ~~integral~~ product expansion for the γ function.

$$\frac{1}{\Gamma(s)} = s e^{\gamma s} \prod_{n=1}^{\infty} \left(1+\frac{s}{n}\right) e^{-s/n}$$

Back to $sl(2,\mathbb{R})$.

$$\frac{c_\sigma}{c_{\sigma+1}} = \frac{\lambda+\sigma}{1-\lambda+\sigma}$$

Generically to construct a map $I(\lambda,\nu) \longrightarrow I(1-\lambda,\nu)$ and find a formula for it.

$$\sigma \in \frac{1}{2\pi i} \log \nu$$

can proceed as long as $\left.\begin{matrix} \lambda + \sigma \\ 1 - \lambda + \sigma \end{matrix}\right\} \notin \mathbb{Z}.$

$$e^{2\pi i \lambda} \nu \neq 1$$

$$e^{-2\pi i \lambda} \nu \neq 1$$

$\boxed{e^z - 1 \text{ [redacted] has a simple zero at } z = 2\pi i n.}$

$$\Gamma(z)\Gamma(-z) = \frac{\pi z}{\sin \pi z} \quad ?$$

~~Write up your results~~

Question: $\mathfrak{g}$ semi-simple, $\mathfrak{b}$ Borel, λ weight, when is

$\mathcal{U}(\mathfrak{g}) \otimes_{\mathfrak{b}} \lambda$ irreducible?

unsolved problems	maximal ideals
structure of Ω_Λ.	
induced dominant weight reps.	
bruhat maps, duality	
Generalized Kostant thm.	
reducibility of $I(\zeta)$.	

functions on $\boxed{S^1 = P_1(\mathbb{R})}$

I have $\underline{\alpha_s}$.

Bruhat

Want to do everything carefully for $sl(2,\mathbb{R})$. Discovered a mistake, namely

$$F(f)(g) = \int_J f(\alpha_s^{-1} j g)\, dj$$

may not be a convergent integral. Example of $sl(2,\mathbb{R})$ - take ρ = trivial rep. so that ~~we are in~~ f is a function on $G/B = S^1$. Then $\alpha_s^{-1} J B$ = complement of ∞, so if f is a function on ~~B~~ G/B which is non-zero at B the above integral is infinite. Thus the distribution first examines values at B and removes enough so that $\int$ converges.

The problem is to calculate this for $sl(2,\mathbb{R})$.

$$G = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{matrix} a, b, c, d \text{ real} \\ ad - bc = 1 \end{matrix}$$

$$B = \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \qquad N = J = \left\{ \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \right\}$$

$$\alpha_s = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$G/B \xrightarrow{\sim} \mathbb{P}^1(\mathbb{R})$$

G acts as proj. transf.
B = fixpt of ∞.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \frac{a}{c}$$

~~ab~~ $\infty \mapsto \frac{a}{c}$

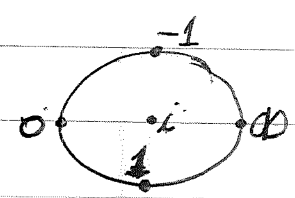
$c = 0$ ~~$a \neq 0$~~.

now want the standard map

$$\mathbb{P}^1(\mathbb{R}) \longrightarrow S^1$$

compatible with G.

$$x \longmapsto \frac{x - i}{x + i}$$



How does G act on S^1 ?

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \frac{\alpha z+\beta}{\gamma z+\delta} \qquad z \in \text{unit disk.}$$

$$\frac{\bar\alpha z^{-1}+\bar\beta}{\bar\gamma z^{-1}+\bar\delta} = \overline{\frac{\alpha z+\beta}{\gamma z+\delta}} = \frac{\gamma z+\delta}{\alpha z+\beta}$$

$$\frac{\bar\alpha+\bar\beta z}{\bar\gamma+\bar\delta z}$$

$\therefore \ \bar\alpha = \delta$

$\gamma = \bar\beta$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \frac{\alpha z+\beta}{\bar\beta z+\bar\alpha} \qquad \underline{z=1}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{x-i}{x+i} = \frac{\alpha\left(\frac{x-i}{x+i}\right)+\beta}{\bar\beta\left(\frac{x-i}{x+i}\right)+\bar\alpha} = \frac{\frac{ax+b}{cx+d}-i}{\frac{ax+b}{cx+d}+i}$$

$$\frac{(\alpha+\beta)x+i(-\alpha+\beta)}{(\bar\alpha+\bar\beta)x+i(\bar\alpha-\bar\beta)} = \frac{(a-ic)x+(b-id)}{(a+ic)x+(b+id)}$$

$\therefore$

$\alpha+\beta = a-ic$

$-\alpha+\beta = (b-id) =$

$2\beta = a+b-i(c+d)$ $\qquad \beta = \frac{a+b}{2} - i\left(\frac{c+d}{2}\right)$

$\alpha = \frac{a-b}{2} -$

$$\alpha = \frac{a-b}{2} - i\left(\frac{c-d}{2}\right)$$

$$\beta = \frac{a+b}{2} - i\left(\frac{c+d}{2}\right)$$

Now take $x=0$ $\quad z=-1$

$$\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}(z) = \frac{[(1-t)+i]z+[(1+t)-i]}{[(1+t)+i]z+[(1-t)-i]}.$$

$$\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}(-1) = \frac{-(1-t+i)+(1+t-i)}{-((1+t)+i)+((1-t)-i)} = \frac{2(t-i)}{-2(t+i)}$$

$$= \frac{i-t}{i+t} =$$

$$\alpha+\beta = a-ic$$

$$+\alpha-\beta = bi+d$$

$$\alpha = \frac{a+d}{2}+i\left(\frac{b-c}{2}\right) = 1+\frac{i}{2}t$$

$$\beta = \frac{a-d}{2}-i\left(\frac{b+c}{2}\right) = -\frac{i}{2}t$$

$$\begin{bmatrix}1 & t\\ & 1\end{bmatrix}(z) = \frac{\left(1+\frac{i}{2}t\right)z-\frac{i}{2}t}{\frac{i}{2}tz+\left(1-\frac{i}{2}t\right)} = \frac{1+\frac{i}{2}t(z-1)}{1+\frac{it}{2}(} = 1\,?$$

$$=$$

$$\begin{bmatrix}1 & t\\ & 1\end{bmatrix}(-1) = \frac{t-i}{t+i}$$

$$0 \longmapsto -1$$

$$\begin{pmatrix}1 & t\\ & 1\end{pmatrix} 0 = t \longmapsto \frac{t-i}{t+i}$$

Therefore: we calculate that

$$\int_{-\infty}^{\infty} f(z)\, dt$$

will not make sense unless $f(z) \equiv O(z-1)^2$...

Thus our distribution will have to take some linear fn. of $f(1)$ and $f'(1)$ before it will make sense.

So I must now work these into the formula

Conjecture: $\mathrm{End}_{\mathfrak{g}}(U(\mathfrak{g})\otimes_k \Lambda) \simeq U(\mathfrak{a})^W \otimes \mathrm{Hom}_M(\Lambda,\Lambda)$

Definition of the map:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathfrak{g}}(U(\mathfrak{g})\otimes_k \Lambda, U(\mathfrak{g})\otimes_k \Lambda) \arrow[r, "1\otimes_{\mathfrak{n}}"] \arrow[ddd, "\gamma"] & \mathrm{Hom}_{M,\mathfrak{a}}(U(\mathfrak{a})\otimes \Lambda, U(\mathfrak{a})\otimes \Lambda) \arrow[d, equal] \\
 & U(\mathfrak{a}) \otimes \mathrm{Hom}_M(\Lambda,\Lambda) \arrow[d, "\beta\otimes id"] \\
 & \\
U(\mathfrak{a})^W \otimes \mathrm{Hom}_M(\Lambda,\Lambda) \arrow[r, hook] & U(\mathfrak{a}) \otimes \mathrm{Hom}_M(\Lambda,\Lambda)
\end{tikzcd}$$

where $\beta : U(\mathfrak{a}) \longrightarrow U(\mathfrak{a})$ is the homomorphism defined by
$\beta(A) = A \pm g(A)$ $\qquad g = \frac{1}{2} \sum'_{\alpha\in\Sigma'} \alpha$

Corroboration the the map is correct. Take Casimir operator of $\mathfrak{g}$.

$$\mathfrak{g} = k + \mathfrak{a} + \mathfrak{n}$$
$$\mathfrak{h} = \mathfrak{h}_k + \mathfrak{a}$$
$$\mathfrak{n} = \sum'_{\alpha\in\Sigma'} e_\alpha$$

$$Cas = \sum' H_i^2 + \sum_{\alpha\in\Sigma} e_\alpha e_{-\alpha} + e_{-\alpha} e_\alpha$$

since
$$\langle \alpha, H_\alpha\rangle = \langle [e_\alpha, e_{-\alpha}], H_\alpha\rangle = + \langle e_\alpha, \alpha(H_\alpha) e_\alpha\rangle \Rightarrow \langle e_\alpha, e_{-\alpha}\rangle = 1$$

$$Cas. = \sum H_i^2 + \sum_{\alpha\in\Sigma'} 2e_\alpha e_{-\alpha} - [e_\alpha, e_{-\alpha}]$$

module $\mathfrak{n}\,U(\mathfrak{g})$

Cas. $\underbrace{\sum H_i^2 + \sum_{\alpha\in\Sigma''} e_\alpha e_{-\alpha} + e_{-\alpha} e_\alpha}_{\text{Laplacian in } \mathfrak{h}_k} + \sum_{\alpha\in\Sigma'} 2 e_\alpha e_{-\alpha} - H_\alpha + \underbrace{\sum H_i^2}_{\text{Laplacean in } \mathfrak{a}}$

$$\boxed{\mathrm{Cas}_{\mathfrak{g}} \equiv \mathrm{Cas}_{\mathfrak{m}}} + \mathrm{Lap}_{\mathfrak{a}} - \sum_{\alpha\in\Sigma''} H_\alpha$$

to get something that is invariant under W consider this quadratic function

$$\langle \mathrm{Lap}\,\mathfrak{a} - \sum_{\alpha\in\Sigma'} H_\alpha, \lambda \rangle = \sum_{\alpha\in\Sigma'} \alpha(\lambda)^2 - \sum_{\alpha\in\Sigma'} \alpha(\lambda)$$

$$= |\lambda - g|^2 - |g|^2 \qquad ?$$

Actually you are not certain where the image lies.

$$\mathrm{Cas}_{\mathfrak{g}} \equiv \mathrm{Cas}_{\mathfrak{m}} + \sum_{\alpha\in\Sigma'} (H_\alpha^2 - H_\alpha) + \varepsilon\,\mathrm{Lap}\,\mathfrak{h}_k \quad (\mathrm{mod}\ \mathfrak{n}U(\mathfrak{g}))$$

Thus to get a W invariant element you want to send H_α to $H_\alpha + \frac{1}{2}$

$$\left(H_\alpha - \tfrac{1}{2}\right)^2 - \tfrac{1}{2}$$

If ~~eigen~~value is $|\lambda - g|^2 - |g|^2$ at λ

send A to $A + g(A)$ then

~~B. Calculation of $S_\xi N$.~~

$sl(2,\mathbb{R})$ calculations.

$$k = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \alpha = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \qquad n = \begin{pmatrix} & 1 \\ & \end{pmatrix}.$$

I will work with the adjoint group so $M = 0$.

H, X, Y

$$\begin{cases} [H,X] = X \\ [H,Y] = -Y \\ [X,Y] = H \end{cases} \qquad \begin{array}{l} k = \mathbb{C}H \\ p = \mathbb{C}X + \mathbb{C}Y \end{array}$$

$$A = \frac{1}{\sqrt{2}}(X+Y) \qquad \begin{array}{l} \langle H,H\rangle = 2 \\ \langle X,Y\rangle = 2 \end{array}$$

$$\mathrm{ad}X\,\mathrm{ad}Y\ H = [X,Y] = H$$
$$\mathrm{ad}X\,\mathrm{ad}Y\ X = \mathrm{ad}X(-H) = -[X,H] = X.$$

$$\therefore\ \langle A,A\rangle = \tfrac{1}{2}(2+2) = 2$$

$$N = H - \frac{1}{\sqrt{2}}(X-Y) \qquad \langle N,N\rangle = 2 + \tfrac{1}{2}(-2-2) = 0.$$

$$[A,N] = N.$$

Take a reps. Λ_0 of K given by $Hv = \lambda_0(H)v$.
Calculate the map

$$
\begin{tikzcd}
\mathrm{Hom}_{\mathfrak{g}}(U(\mathfrak{g})\otimes_k \Lambda_0,\ U(\mathfrak{g})\otimes_k \Lambda_0) \arrow[r] & U(\mathfrak{a}) \otimes \mathrm{Hom}(\Lambda_0, \Lambda_0) \\
\mathrm{Hom}_k(\Lambda,\ U(\mathfrak{g})\otimes_k \Lambda_0) \arrow[u, Rightarrow] \arrow[d] & \\
\mathrm{Hom}(\Lambda,\ U(\mathfrak{a}+\mathfrak{n})\otimes \Lambda_0) \arrow[d] & \\
\mathrm{Hom}(\Lambda,\ U(\mathfrak{a})\otimes \Lambda) &
\end{tikzcd}
$$

$$\boxed{U(\mathfrak{a}) \otimes_{U(\mathfrak{a})+\mathfrak{n}} U(\mathfrak{a}+\mathfrak{n})}$$

$$U(\mathfrak{g})\otimes_k \Lambda = U(\mathfrak{a})\otimes U(\mathfrak{n})\otimes \Lambda$$
$$= \boxed{\mathbb{C}[A, N] \otimes \Lambda}$$

$$
\begin{tikzcd}
U(\mathfrak{a}) \otimes_{U(\mathfrak{a},\mathfrak{n})} U(\mathfrak{a}+\mathfrak{n}) \otimes \Lambda \arrow[d] \\
U(\mathfrak{a}) \otimes \Lambda
\end{tikzcd}
$$

$P(A)\otimes N Q(A,N)\otimes \Lambda$

"

Thus ~~g~~ we kill $\mathfrak{n}U(\mathfrak{g})$

Thus given

$$\begin{tikzcd} \Lambda \arrow[r, "\varphi"] \arrow[drr, bend right, "F_\varphi"'] & U(\mathfrak{g})\otimes_k \Lambda \arrow[r, equal] & \bigoplus_{i,j} N^i A^j \Lambda \arrow[d, "\text{kill all with } i>0"] \\ & & \bigoplus_j A^j \Lambda \end{tikzcd}$$

The problem for you is to decide when an F_φ comes from a k homomorphism φ.

Λ is 1-dimensional so we are looking at polynomials in A. What polys. do we get?

Take ye old Casimir operator in the K, A, N form.

$$H^2 + XY + YX \qquad (2\ \text{Casimir})$$

have to take Casimir and find its image in

$$U(\mathfrak{g}) / \mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})(H-\lambda)$$

$$H = H$$

$$A = \frac{1}{\sqrt{2}}(X+Y)$$

$$N = H - \frac{1}{\sqrt{2}}X + \frac{1}{\sqrt{2}}Y$$

$$[H,A] = H - N$$

$$[H,N] = -A$$

$$[A,N] = N$$

$$N - H + A = \sqrt{2}\,Y$$

$$-(N-H) + A = \sqrt{2}\,X$$

$$2XY = [A-(N-H)][A+N-H].$$

$$XY \equiv \cancel{\tfrac{1}{2}[A^2 + HA + AN - AH}$$

$$= \frac{1}{2}\left\{ \begin{array}{l} A^2 + AN - AH \\ -NA - N^2 + NH \\ HA + HN - H^2 \end{array} \right\}$$

$$\equiv \frac{1}{2}\left\{ \begin{array}{l} A^2 - AH \\ \cancel{N} - N^2 + 2NH \\ H - \cancel{N} + (-A) - H^2 \end{array} \right.$$

$$= \frac{1}{2}\left\{ \begin{array}{l} A^2 - A + H - H^2 \\ -N^2 + 2NH \end{array} \right\}.$$

$$2C = \cancel{\tfrac{1}{2}}\left\{ A^2 - A + \cancel{H - H} - N^2 + 2NH \right\}.$$

<u>Check calculation.</u>

$$H, A, \quad H-N$$

$$\langle A, H-N\rangle = \tfrac{1}{2}\langle X+Y, X-Y\rangle$$

$$= \tfrac{1}{2}(2-2) = 0.$$

$$\langle H-N, H-N\rangle = \tfrac{1}{2}\langle X-Y, X-Y\rangle = -2.$$

Thus Casimir is

$$\tfrac{1}{2}\left(H^2 + A^2 - (H-N)^2\right)$$

$$2C = \cancel{H^2} + A^2 - \cancel{H^2} + [H,N] + 2NH - N^2$$

$$= A^2 - A - N^2 + 2NH$$

which gives $A^2 - A$ as its image.

Iwasawa theory:

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$
$$= \mathfrak{h} + \bigoplus_{\alpha \in \Delta} \mathfrak{g}^{\alpha}$$

Assume $\mathfrak{h}$ stable under θ, $\mathfrak{h} = \mathfrak{h}_k + \mathfrak{h}_p$.

One may choose a system Π of simple root vectors H_i $i = 1, \dots, l$ such that

$$H_i \in \mathfrak{h}_p \qquad 1 \leq i \leq m$$
$$H_i \in \mathfrak{h}_k \qquad m < i \leq l$$

~~In addition one may obtain~~

If Σ is the set of positive roots $\Sigma = \Sigma' \cup \Sigma''$ where

$$\alpha \in \Sigma'' \iff \alpha \circ \theta = \alpha \iff \alpha(\mathfrak{h}_p) = 0$$ (then $\theta e_\alpha = e_\alpha$)

$$\alpha \in \Sigma' \iff \alpha \circ \theta < 0.$$ (then $\theta e_\alpha = e_{\theta\alpha}$)

$$\mathfrak{m} = \mathfrak{h}_k + \sum_{\alpha \in \Sigma''} \mathbb{C} e_\alpha + \sum_{\alpha \in \Sigma''} \mathbb{C} e_{-\alpha}$$

$$\mathfrak{a} = \mathfrak{h}_p$$

$$\mathfrak{n} = \sum_{\alpha \in \Sigma'} \mathbb{C} e_\alpha$$

$$\mathfrak{k} = \mathfrak{h}_k + \sum_{\alpha \in \Sigma'} \mathbb{C}(e_\alpha + e_{\theta\alpha}) + \mathfrak{m}$$

$\mathrm{Hom}_k(\Lambda_\lambda, U(\mathfrak{g})\otimes_k \Lambda_{\lambda+1})$

$$Hm = \lambda m.$$

$$HYm = (YH - Y)m = (\lambda - 1)Ym.$$

So $\mathrm{Hom}_k(\Lambda_\lambda, U(\mathfrak{g})\otimes_k \Lambda_{\lambda+1}) \ni (v_\lambda \xrightarrow{\psi} Y\otimes v_{\lambda+1})$

Checked

$$\psi(Hv) = \psi(\lambda v) = \lambda Y\otimes v$$

$$H\psi(v) = H(Y\otimes v) = (YH - Y)\otimes v$$
$$= (\lambda+1-1)Y\otimes v = \lambda(Y\otimes v)$$

Calculate ~~image~~ map

$$\begin{tikzcd}
\Lambda_\lambda \arrow[r, "\psi"] & U(\mathfrak{g})\otimes_k \Lambda_{\lambda+1} \arrow[r, "pr\otimes"] \arrow[d] & U(\mathfrak{a})\otimes \Lambda_{\lambda+1} \arrow[d, equal] \\
& \bigoplus \Lambda^i A^j \Lambda_{\lambda+1} \arrow[r] & \bigoplus_j A^j \Lambda_{\lambda+1}
\end{tikzcd}$$

where pr is given by

But $\quad Y = \frac{1}{\sqrt{2}}(N - H + A)$

So

$$v_\lambda \overset{\psi}{\longmapsto} Y \otimes v_{\lambda+1}$$
$$\parallel$$
$$\frac{1}{\sqrt{2}}(N + A - H)\otimes v_{\lambda+1} \xrightarrow{\text{proj}} \frac{1}{\sqrt{2}}(A - (\lambda+1))\otimes v_{\lambda+1}$$

Thus its

$$v_\lambda \longmapsto \frac{1}{\sqrt{2}}(A - \lambda - 1)\otimes v_{\lambda+1}.$$

$$\therefore\ \psi \longmapsto \underbrace{\frac{1}{\sqrt{2}}(A-\lambda-1)\otimes(v_\lambda \mapsto v_{\lambda+1})}_{}.$$

this is a basis for ~~Hom~~ image in $U(\mathfrak{g})\otimes \mathrm{Hom}(\Lambda_\lambda, \Lambda_{\lambda+1})$

basis for
$\mathrm{Hom}_k(\Lambda_\lambda, U(\mathfrak{g})\otimes\Lambda_{\lambda+1})$.

<u>Next calculation:</u> Here we have an element ψ and we can pre and post multiply. Then if $z = 2C = A^2 - A - N^2 + 2NH$ we have

$$z\psi = \psi z$$

$\underline{Z\psi}$: $1 \otimes v_\lambda \longmapsto Z(Y \otimes v_{\lambda+1})$

$\underline{\psi Z}$: $1 \otimes v_\lambda \longmapsto$ ~~[illegible]~~ $\psi(Z \otimes v_\lambda) = Z(Y \otimes v_{\lambda+1})$

$$Z\, Y \otimes v_{\lambda+1} = (A^2 - A - N^2 + 2NH)(N - H + A)\frac{1}{\sqrt{2}} \otimes v_{\lambda+1}$$

$$= (A^2 - A)(N - \lambda - 1 + A)\frac{1}{\sqrt{2}}$$

$$- N^2(N - \lambda - 1 + A)\frac{1}{\sqrt{2}}$$

$$+ 2NH(\qquad\qquad)\frac{1}{\sqrt{2}}$$

This behaves correctly!

$$X = \frac{1}{\sqrt{2}}(A - N + H)$$

So get

$\varphi \in \mathrm{Hom}_k(\Lambda_{\lambda+1}, U(\mathfrak{g}) \otimes_k \Lambda_\lambda)$ $\qquad \varphi(v_{\lambda+1}) = X \otimes v_\lambda$

$$\varphi \xmapsto{\text{pr}} \frac{1}{\sqrt{2}}(A + \lambda) \otimes (v_{\lambda+1} \to v_\lambda).$$

So

$$\varphi \circ \psi \longmapsto \left[\frac{1}{\sqrt{2}}(A+\lambda) \otimes (v_{\lambda+1} \to v_\lambda)\right] * \left[\frac{1}{\sqrt{2}}(A - \lambda - 1) \otimes (v_\lambda \to v_{\lambda+1})\right]$$

But $\quad (\varphi \circ \psi)(v_\lambda) = \varphi(Y \otimes v_{\lambda+1}) = Y\,\varphi(1 \otimes v_{\lambda+1})$

$$= YX \otimes v_\lambda$$

$$YX = \tfrac{1}{2}(N - H + A)(A - N + H)$$

$$= \tfrac{1}{2}\big(\cancel{NA} - N^2 + NH$$

$$- [H,A] - \cancel{AH} + [H,N] + NH - H^2$$

$$+ A^2 - \cancel{NA} - [A,N] + \cancel{AH}$$

$$= \tfrac{1}{2}\big(-N^2 + 2NH - A + H^2 + A^2 - \cancel{N}\big)$$

$$- H + \cancel{N}$$

$$= \tfrac{1}{2}\big(-N^2 + 2NH - H^2 - H + A^2 - A\big)$$

$$\text{proj}_\lambda YX = \tfrac{1}{2}(-\lambda^2 - \lambda + A^2 - A).$$

~~Summary~~ $\qquad$ Set $\varphi^\lambda_{\lambda+1} : v_\lambda \to v_{\lambda+1}$

$\cancel{(A+\lambda) \otimes (v_{\lambda+1} \to v_\lambda)}$

$$\cancel{(A+\lambda) \otimes \varphi^{\lambda+1}_\lambda \circ (A - \lambda - 1) \otimes \varphi^\lambda_{\lambda+1} = (A^2 - A - \lambda^2 - \lambda)}$$

$$\psi : \mathcal{U}(\mathfrak{g}) \otimes_k \Lambda_\lambda \longrightarrow \mathcal{U}(\mathfrak{g}) \otimes_k \Lambda_{\lambda+1}$$

$$\psi(x \otimes v_\lambda) = xY \otimes v_{\lambda+1}$$

wanted to calculate image of ψ in

$\mathcal{U}(\mathfrak{a}) \otimes \mathrm{Hom}(\Lambda_\lambda, \Lambda_{\lambda+1})$.

$$\frac{1}{\sqrt{2}}(A - \lambda - 1) \otimes \varphi^\lambda_{\lambda+1}$$

$$\varphi : \mathcal{U}(\mathfrak{g}) \otimes_k \Lambda_{\lambda+1} \longrightarrow \mathcal{U}(\mathfrak{g}) \otimes_k \Lambda_\lambda$$

$$\varphi(x \otimes v_{\lambda+1}) = xX \otimes v_\lambda$$

Image of φ in

$\mathcal{U}(\mathfrak{a}) \otimes \mathrm{Hom}(\Lambda_{\lambda+1}, \Lambda_\lambda)$ is

$$\frac{1}{\sqrt{2}}(A + \lambda) \otimes \varphi^{\lambda+1}_\lambda$$

Thus ~~image of~~ $\varphi \circ \psi \in \mathrm{End}_{\mathfrak{g}}(\mathcal{U}(\mathfrak{g}) \otimes \Lambda_\lambda)$ is

$$\varphi\psi(x \otimes v_\lambda) = \varphi(xY \otimes v_{\lambda+1}) = xYX \otimes v_\lambda$$

with image

$$\frac{1}{2}(A^2 - A - \lambda - \lambda^2) \otimes \varphi^\lambda_\lambda$$

$$A(A-\lambda-1) + \lambda(A-\lambda-1) = A^2 - \cancel{\lambda A} - A + \cancel{\lambda A} - \lambda^2 - \lambda$$
$$= A^2 - A - \lambda^2 - \lambda$$

Anyway I get a map

$$\mathrm{Hom}_{\mathfrak{g}}\big(U(\mathfrak{g})\otimes_k \Lambda_1,\ U(\mathfrak{g})\otimes_k \Lambda_2\big) \longrightarrow U(\mathfrak{a})\otimes \mathrm{Hom}(\Lambda_1,\Lambda_2)$$

which is compatible with ~~the image~~ composition
and the problem is to identify the image.

The $sl(2,\mathbb{R})$ case suggests maybe that the map $Z \to U(\mathfrak{a}) \otimes id$ might have its image independent of Λ.

Calculate Casimir operator for general Iwasawa decomp.

$$\mathfrak{g} = \underline{k} + \mathfrak{a} + \mathfrak{n}$$

given $e_\alpha \in \mathfrak{n}$ ~~then~~ $\alpha \in \Sigma'$ then $e_\alpha - e_{\theta\alpha} \in \mathfrak{p}$.

$$\sum_{\alpha\in\Delta} e_\alpha e_{-\alpha} + \sum H_i K_i$$

Try $\alpha \in \Sigma'$

$$\sum_{\alpha\in\Sigma'} 2 e_\alpha e_{-\alpha} \overset{-[e_\alpha, e_{-\alpha}]}{\cancel{+ e_{-\alpha} e_\alpha}} + \overset{\in C_{\mathfrak{m}}}{\boxed{\sum_{\alpha\in\Sigma''} e_\alpha e_{-\alpha} + e_{-\alpha} e_\alpha + \sum_{i>m} H_i K_i}}$$

$$+ \sum_{1\le i\le m} H_i K_i$$

$$\underset{\mathfrak{n}U(\mathfrak{g})}{\equiv} \underset{\in\mathfrak{a}}{-\sum_{\alpha\in\Sigma'} H_\alpha} + \underbrace{\sum_{1\le i\le m} H_i K_i + C_{\mathfrak{m}}}_{\in\mathfrak{a}\ \text{because}\ \theta\Sigma' = -\Sigma'}.$$

Conclusion is that I get something in $\mathfrak{A}(\alpha)$ + casimir operator in $\mathfrak{m}$ which of course gives an interesting operator in $\mathrm{Hom}(\Lambda, \Lambda)$, in fact ~~interesting~~ in $\mathrm{Hom}_M(\Lambda, \Lambda)$. Therefore in studying the irreducible principal series this will be constant. i.e. recall fmla

$$\boxed{\bigoplus \Lambda \otimes \mathrm{Hom}_M(\Lambda, \nu).}$$

for the K module structure of principal series.

~~Heuristic Conclusion: Choose a max ideal in $\mathfrak{U}(\alpha)$ and an algebraic (for~~

✓ It's becoming clear that the important thing is the reductive group MA = centralizer of A, and that K doesn't play much of a role!

Comments:

A. On computation of Ω_Λ. Still difficult.

~~Question. We know that Ω_Λ is a module over Z which has~~

Suppose V_λ an irred rep of $\mathfrak{g}$ with dominant wgt. λ.
Whats eigenvalue of Casimir?

$$\text{Casimir} = \sum_{\alpha \in \Sigma^+} e_\alpha e_{-\alpha} + e_{-\alpha} e_\alpha + \sum H_i K_i$$

but $$e_\alpha e_{-\alpha} + e_{-\alpha} e_\alpha = [e_\alpha, e_{-\alpha}] + 2 e_{-\alpha} e_\alpha = H_\alpha + 2 e_{-\alpha} e_\alpha$$

Thus

$$C v_\lambda = \sum_{\alpha \in \Sigma^+} H_\alpha v_\lambda + \sum \lambda(H_i) \lambda(K_i)$$

$$\boxed{= \langle g, \lambda \rangle + \langle \lambda, \lambda \rangle}$$

Thus Casimir operator has eigenvalue $\langle g, \lambda \rangle + \langle \lambda, \lambda \rangle$

$$= |\lambda + g|^2 - |g|^2$$

in the irreducible representation with dominant weight λ.

What is the trace of the Casimir operator on V_λ? something like the dimension!

$$= (\dim V_\lambda) \cdot (|\lambda + g|^2 - |\lambda|^2) = \prod_{\alpha \in \Sigma^+} \frac{\langle \lambda + g, \alpha \rangle}{\langle g, \alpha \rangle} \cdot (|\lambda + g|^2 - |\lambda|^2)$$

B.

In the case of $sl(2,\mathbb{R})$ we know that $\mathrm{Hom}_{\mathfrak{g}}(j_!\Lambda_1, j_!\Lambda_2)$ is a free module of rank 1 over Z, ~~[illegible]~~ and that the projection image is a free module of ranke 1 over $U(\alpha)^W$. Question: Can you select for each ~~element~~ pair Λ_1, Λ_2 an element $\varphi_{\Lambda_2}^{\Lambda_1}$ such that

$$\varphi_{\Lambda_3}^{\Lambda_2}\varphi_{\Lambda_2}^{\Lambda_1} = \varphi_{\Lambda_3}^{\Lambda_1} \quad ?$$

Every element in $U(\mathfrak{g})$ is uniquely expressible in the form $\underline{X^i y^j H^k}$ and the k-weight is $i-j$. Hence

$$U(\mathfrak{g})^{\lambda} = \sum_{i-j=\lambda} X^i y^j P(H).$$

$$U(\mathfrak{g})^{0} = \sum_{i=j=\lambda} X^i y^i P(H)$$

$$= \bigoplus C^i P(H)$$

$$\boxed{U(\mathfrak{g})^{k} = Z \otimes U(k).}$$

here

and
$$U(\mathfrak{g})^{\lambda} = \begin{cases} X^{\lambda} Z \otimes U(k) & \lambda \geq 0 \\ Y^{-\lambda} Z \otimes U(k) & \lambda \leq 0. \end{cases}$$

free module of rank 1 over $U(\mathfrak{g})^k$.

Conclusion: since a poly ring has no non-trivial units the generators of $\mathcal{U}(\mathfrak{g})^\lambda$ are unique up to scalars.

$$\mathcal{U}(\mathfrak{g})\otimes_k \Lambda_\lambda \simeq \bigoplus_{i\geq 0} Z\cdot X^i v_\lambda + \bigoplus_{i<0} Z Y^i v_\lambda$$

$$\Big\downarrow \text{proj}$$

$$\bigoplus_{i\geq 0} P(A^2-A)\,\widetilde{(A-N+H)}^i v_\lambda$$

$$\bigoplus_{i>0} P(A^2-A)(A+N-H)^i v_\lambda$$

Conclusion:

~~proj (Hom$_k$ (Λ_λ~~

$$\text{pr}\,(v_{\lambda+i} \mapsto X^i v_\lambda).$$

$$\text{pr}\left(\text{Hom}_k(\Lambda_{\lambda+i}, \mathcal{U}(\mathfrak{g})\otimes_k \Lambda_\lambda)\right) = \mathbb{C}[A^2-A] \; \cancel{\text{proj}(X^i v_\lambda)}$$

Idea: For $sl(2,\mathbb{R})$ each $(\Lambda_1,\Lambda_2)=\text{Hom}(\mathfrak{g}; \Lambda_1, \mathfrak{g}; \Lambda_2)$ is free of rank 1 over Z. Z is a poly ring in 1 vbl hence has only trivial units, so generator of (Λ_1, Λ_2) is unique up to scalar. $\varphi^{\Lambda_1}_{\Lambda_2}$. Note that

$$\varphi^i_{i+1}(v_i) = X\cdot v_{i+1}$$

$$\varphi^{i+1}_i(v_{i+1}) = X\, v_i$$

So

$$\left(\varphi_i^{i+1}\circ\varphi_{i+1}^{i}\right)(v_i) = \varphi_i^{i+1}\left(\cancel{\varphi_{i+1}}\, Y v_{i+1}\right) = \underline{YX}\, v_i$$

$$2C v_i = \left(H^2 + XY + YX\right) v_i = \left(H^2 + \overset{[X,Y]}{H} + 2YX\right) v_i$$

$$= \left(i^2 + i + 2YX\right) v_i$$

Therefore

$$(YX)\, v_i = \left(C - \frac{i^2+i}{2}\right) v_i$$

$$\therefore \quad \boxed{\varphi_i^{i+1}\circ\varphi_{i+1}^{i} = \left(C - \frac{i^2+i}{2}\right)\varphi_i^i}$$

This proves the impossibility of finding the category $\mathrm{Hom}_M(\cdot,\cdot)$ within differential operators on G/K

$H^1(G,M)=0$

all M

$H^*(G,\mathbb{Z})=0$

$H^*(G,\mathbb{Z}/p\mathbb{Z})=0$

$G \longrightarrow G_p$

$G \quad \underline{V}$

$$c_i(V) \in H^{2i}(G,\mathbb{Z})$$

$$c:\ R(G) \longrightarrow H^{\text{even}}(G,\mathbb{Z})$$

$$c(x+y) = c(x)\cdot c(y)$$

Question: Is it possible to enlarge $U(\alpha)^W$ so as to realize $\mathrm{Hom}_M(\cdot,\cdot)$ within

Can I find

$$\varphi^j_k \circ \varphi^i_j = a(k,j,i)\,\varphi^i_k$$

Thus $a(k,j,i)$ is a 2 cocycle, e.g.

$$\begin{array}{c} \varphi^k_\ell\, \varphi^j_k\, \varphi^i_j = \varphi^k_\ell\, \overset{\text{center}}{\overset{\downarrow}{a(k,j,i)}}\, \varphi^i_k = a(k,j,i)\, a(\ell,k,i) \\ \| \\ a(\ell,k,j)\, \varphi^j_\ell\, \varphi^i_j = a(\ell,k,j)\, a(\ell,j,i). \end{array}$$

$$a(k,j,i)\, a(\ell,j,i)^{-1}\, a(\ell,k,i)\, a(\ell,k,j)^{-1} = 1.$$

In other words we find ourselves with a 2 cocycle with values in the center Z, which we want to make a 2 coboundary but can't, as things stand.

So can we enlarge Z?

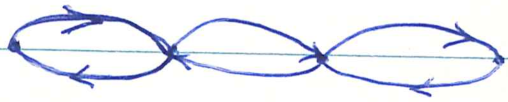
~~The~~

I have a groupoid, namely the integers for objects and at ~~most~~ exactly one map from one to the other. Thus this cocycle sits on the want

$$b_k^{\,j} b_j^{\,i}\, a(k,j,i) = b_k^{\,i}$$

i.e.

$$(\delta b)(k,j,i) = b_j^{\,i}(b_k^{\,j})^{-1} b_k^{\,j} = a(k,j,i)^{-1}.$$



This leads to the following point of view

<u>M normalizes N ?</u>

The important group is MA which is the centralizer of A. This we complete to form a parabolic group MAN. ~~8~~ Thus $N \triangleleft MAN$ so M normalizes N? i.e.

$$[e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta} \qquad \alpha \in \Sigma'', \beta \in \Sigma'$$

assume $\alpha+\beta \in \Sigma_i'$. does $\alpha+\beta \in \Sigma''$ ie is ~~$\theta(\alpha+\beta)$~~ $(\alpha+\beta)\circ\theta = \alpha\circ\theta + \beta\circ\theta = \alpha + \beta\circ\theta < 0$?

The point is that $\alpha+\beta \in \Sigma_i'$ and $(\alpha+\beta)(h_p) = \beta(h_p) \neq 0$ so $\alpha+\beta \in \Sigma_i''$. Thus $\mathfrak{m}+\mathfrak{a}$ normalizes $\mathfrak{n}$; but MA is connected and so MA normalizes N.

Corollary: The mapping is compatible with M.

Proposition: The image of the map

$$\mathrm{Hom}_{\mathfrak{g}}(U(\mathfrak{g}) \otimes_{\mathfrak{k}} \Lambda_1, U(\mathfrak{g}) \otimes_{\mathfrak{k}} \Lambda_2) \longrightarrow U(\mathfrak{a}) \otimes \mathrm{Hom}_{\mathfrak{k}}(\Lambda_1, \Lambda_2)$$

lies in $\quad U(\mathfrak{a}) \otimes \mathrm{Hom}_M(\Lambda_1, \Lambda_2)$.

Proof: Let $m \in M_0$ act on $\mathfrak{g}, \mathfrak{k}, \Lambda_1, \Lambda_2$ through the K_0 action (here shall assume M_0 generates M). It then preserves everything in sight so the mapping is compatible with the M action; so have to show it acts trivially on the left. ~~But this is clear~~

So given $\varphi : \Lambda_1 \longrightarrow U(\mathfrak{g}) \otimes_k \Lambda_2$ comp. with k hence K action since both integrate, hence φ invariant under m.

Conclusion: There is a canonical map

$$F : \mathrm{Hom}_{\mathfrak{g}}\left(U(\mathfrak{g}) \otimes_k \Lambda_1, U(\mathfrak{g}) \otimes_k \Lambda_2\right) \longrightarrow U(\mathfrak{a}) \otimes \mathrm{Hom}_M(\Lambda_1, \Lambda_2).$$

compatible with compositions.

A calculation of irreducible $sl(2,\mathbb{C})$ modules which decompose into finite dimensional modules over k = Cartan subalg. $\qquad$ A 1

$\mathfrak{g} = sl(2,\mathbb{C}) \qquad k = \mathbb{C}H.$ $\qquad$ usual relations

$$[H,X] = X \qquad [H,Y] = -Y \qquad [X,Y] = H$$

$$H = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad X = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad Y = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

<u>Problem:</u> Classify irreducible $\mathfrak{g}$ modules which are inductive limits of finite dimensional k modules.

Let M be such a $\mathfrak{g}$ modules so that $M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$ where $M_\lambda = \{ m \in M \mid \exists n \ (H-\lambda)^n m = 0 \}$. Commutation formula

$$P(H)X = X P(H+1)$$
$$P(H)Y = Y P(H-1)$$

P is a polynomial

~~Therefore~~ $\cancel{XM_\lambda \subset M_{\lambda+1}}$ $\cancel{(H-\lambda-1)^n Xm = X(H-\lambda)^n m}$.
$\cancel{YM_\lambda \subset M_{\lambda-1}}$
~~and X~~

Therefore

$$M_\lambda = \{ m \mid Hm = \lambda m \}$$
$$XM_\lambda \subset M_{\lambda+1}, \quad YM_\lambda \subset M_{\lambda-1}$$
$$\{ \lambda \in \mathbb{C} \mid M_\lambda \neq 0 \} = \lambda_0 + \mathbb{Z}$$

This has to be justified but can be

Next let $m \in M_{\lambda_0}$ be an eigenvector for $XY : M_{\lambda_0} \to M_{\lambda_0}$ say

$$XYm = \alpha m \qquad m \neq 0.$$

Claim that subspace of M spanned by $X^k m, Y^k m$ $k \geq 0$ is a ~~g~~ submodule. Proof:

$$\alpha + \lambda_0(k-1) - \frac{k(k-1)}{2} \quad \checkmark$$

$$XY^k m = \left(\alpha + \frac{(2\lambda_0 - k)(k-1)}{2}\right) Y^{k-1} m \qquad k \geq 1$$

$$YX^k m = \left(\overset{\alpha - \lambda_0}{\cancel{\alpha}} - \frac{(2\lambda_0 + k)(k-1)}{2}\right) X^{k-1} m \qquad k \geq 1$$

$$\alpha - k\lambda_0 - \frac{k(k-1)}{2}$$

as one sees by induction on k.

Since M is irreducible ~~we have the following possibilities:~~ therefore see that each M_λ is at most one dimensional and we have the following possibilities:

a) bdd. above λ_0 $(= q)$ such that $X \cdot M_{\lambda_0} = 0$ in which case $\alpha = \lambda_0$ and the eigenvalues of XY are

$$qk - \frac{k(k-1)}{2} \qquad \text{for} \quad k \geq 0 \qquad \checkmark$$

b) bdd. below λ_0 $(= p)$ such that $Y \cdot M_{\lambda_0} = 0$ in which case $\alpha = 0$ and the eigenvalues of XY are

$$-pk - \frac{k(k-1)}{2} \qquad \text{for} \quad k \geq 0$$

c) unbounded in which case eigenvalues of XY ~~are~~ in M_{λ_0+k} is

$$\alpha - k\lambda_0 - \frac{k(k-1)}{2} \qquad k \in \mathbb{Z}$$

which can never be zero.

Casimir operator for $\mathfrak{sl}(2,\mathbb{C})$

$$C = \tfrac{1}{2}(H^2 + XY + YX)$$

Its eigenvalues in the representation constructed is

$$\frac{1}{2}\left(\lambda_0^2 - \lambda_0 + 2\alpha\right)$$

(Not the only invariant of the representation, e.g. in the Harish-Chandra case $\alpha = 0$ + then ~~the~~ reps corresponding to λ_0 and $1-\lambda_0$ have same character.)

Question: We have reduced the $\mathfrak{g}, k$ modules to pairs (λ_0, α) under equivalence relation

$$(\lambda_0, \alpha) \sim \left(\lambda_0 + k,\ \alpha - k\lambda_0 - \frac{k(k-1)}{2}\right) \qquad k \in \mathbb{Z}$$

ⓐ do all such pairs occur.

Start with m and let

$$M_{\lambda_0 + k} = \begin{cases} X^k m & k \geq 0 \\ y^{-k} m & k \leq 0 \end{cases}$$

and define

$$X(X^k m) = (X^{k+1} m) \qquad k \geq 0$$

$$X(y^{-k} m) = \left\{ \left(\alpha + (k+1)\lambda_0 - \frac{(k+1)(k+1)}{2}\right) y^{-k-1} m \qquad k < 0 \right.$$

Problem: Classify irreducible $\mathfrak{g}, k$ modules.

There are two situations to understand

(i) $sl(2,\mathbb{R})$

(ii) complex case.

Review $sl(2,\mathbb{C})$ calculation

X, Y, H

$[H,X] = X$

$[H,Y] = -Y$

$[X,Y] = H.$

M irreducible $\mathfrak{g}$ module

$M = \bigoplus_{n\in\mathbb{Z}} M_{\lambda_0+n} \qquad \lambda_0 = 0, \frac{1}{2}$

(i) bdd above. Then $Xe_{\lambda_0} = 0$ so

$$M = \underline{\bigoplus_{k\geq 0} \mathbb{C}Y^k e_{\lambda_0}}$$

$$(XY)Y^k e_{\lambda_0} = (YX + H)Y^k e_{\lambda_0}$$

$$\varphi(k)Y^k e_{\lambda_0} = (\varphi(k-1) + \lambda_0 - k)Y^k e_{\lambda_0}$$

$$\varphi(k) - \varphi(k-1) = \lambda_0 - k$$

$$\varphi(k) = (k+1)\lambda_0 - \frac{k(k+1)}{2}$$

$$XYe_{\lambda_0} = 0 + \lambda_0 \qquad \varphi(0) = \lambda_0$$

$$(XY)(Ye_{\lambda_0}) = \lambda_0 + \lambda_0 - 1 = 2\lambda_0 - 1.$$

Summary

If bdd above at λ_0 so

$X e_{\lambda_0} = 0$ basis $Y^k e_{\lambda_0}$ $k \geq 0$

$$\boxed{(XY)(Y^k e_{\lambda_0}) = \left[(k+1)\lambda_0 - \frac{k(k+1)}{2}\right] Y^k e_{\lambda_0}} \checkmark$$

$H^2 + XY + YX = H^2 \mp H + 2XY$ has eigenvalue

$$\lambda_0^2 \mp \lambda_0 + 2\lambda_0$$
$$= \boxed{\lambda_0^2 + \lambda_0}$$

(ii) bdd below $Y e_{\lambda_0} = 0.$ basis $X^k e_{\lambda_0}$

$$XY(X^k e_{\lambda_0}) = X(XY \mp H) X^{k-1} e_{\lambda_0}$$
$$\|$$
$$\varphi(k) X^k e_{\lambda_0} = X \ \varphi(k-1) X^{k-1} e_{\lambda_0} - (\lambda_0 + (k-1)) X^{k-1} e_{\lambda_0}$$

$$\varphi(k) = \varphi(k-1) - \lambda_0 \mp (k-1).$$ $XYe_{\lambda_0} = 0.$

$$\varphi(k) = -k\lambda_0 \mp \frac{k(k-1)}{2} \checkmark$$ $XYXe_{\lambda_0} = -\lambda_0 X e_{\lambda_0}$

Summary:

If bdd below at λ_0 so $X e_{\lambda_0} = 0$ then basis $X^k e_{\lambda_0}$ $k \geq 0$

$$\boxed{(XY)(X^k e_{\lambda_0}) = \left[-k\lambda_0 \mp \frac{k(k-1)}{2}\right] X^k e_{\lambda_0}}$$

~~$H^2 + XY + YX$~~

$H^2 + XY + YX = H^2 - H + 2XY$ has eigenvalues $\boxed{\lambda_0^2 - \lambda_0}$

Suppose bdd. then

$$\underbrace{\lambda_0^2+\lambda_0}_{\text{bdd. above}} = \underbrace{\lambda_1^2-\lambda_1}_{\text{bdd below}}$$

$$\underbrace{M_{\lambda_1} \text{ - - - - - - - - } M_{\lambda_0}}$$

$$\Longrightarrow \quad \lambda_0 - \lambda_1 \in \mathbb{Z}.$$

so let $\lambda_0 = \lambda_1 + \ell$

Then

$$\lambda_0 = \text{eigenvalue of } XY \text{ on } M_{\lambda_0}$$

$$\lambda_1 + \ell = -\ell\lambda_1 - \frac{\ell(\ell-1)}{2}$$

$$(1+\ell)\lambda_1 = -\frac{\ell(\ell-1)}{2} - \ell = -\frac{\ell(\ell+1)}{2}$$

$$\lambda_1 = -\frac{\ell}{2} \qquad\qquad \lambda_0 = +\frac{\ell}{2}.$$

Conclusion:

The representation bdd above at λ_0 is f.d. $\Longleftrightarrow$ $\lambda_0 = \frac{\ell}{2}$ where ℓ is an integer ≥ 0 in which case the representation is the same as the representation bdd below at $-\frac{\ell}{2}$.

(iii) <u>unbounded</u>. Choose λ_0 + let α be eigenvalue of XY in M_{λ_0}. Then eigenvalue $\varphi(k)$ of XY in M_{λ_0+k} satisfies

$$\varphi(k) = \varphi(k-1) - \lambda_0 - (k-1)$$

so
$$\boxed{\varphi(k) = \alpha - k\lambda_0 - \frac{k(k-1)}{2}}$$

$$XY\,Xe_{\lambda_0} = X(XY-H)e_{\lambda_0}$$
$$= (\alpha-\lambda_0)\,Xe_{\lambda_0}$$

$$\cancel{\psi(\lambda_0+k) = \varphi(k) = \alpha - k\lambda_0 - \frac{k(k-1)}{2}}$$

$$\cancel{\lambda = \lambda_0 + k}$$

$$\cancel{\psi(\lambda) = \alpha - k(\lambda-k) - \frac{k(k-1)}{2}}$$

$$\cancel{= \alpha - k\lambda + k^2 - \frac{k(k-1)}{2} = \alpha - k\lambda}$$

$$\cancel{\alpha - \frac{(\lambda-\lambda_0)(\lambda-1)}{2}}$$

eigenvalue of $H^2 + XY - XY$ = $\boxed{\lambda_0^2 - \lambda_0 + 2\alpha}$

Results: The following is a complete list of irreducible $\mathfrak{g}$ modules M, $\mathfrak{g} = sl(2,\mathbb{C})$ which decompose into sums of finite dimensional representations under $\mathbb{C}H$; $M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$. $M_\lambda = \{m \in M \mid \exists n\ (H-\lambda)^n m = 0\}$

(Always M_λ is 1-dimensional $XM_\lambda \subset M_{\lambda+1}$ $YM_\lambda \subset M_{\lambda-1}$)

(i) bdd ones. $M_\lambda \neq 0$ for $\lambda = k - \frac{\ell}{2}$ $k = 0, \dots, \ell$ ℓ integer ≥ 0

eigenvalue of $\mathcal{C} = H^2 + XY + YX = \frac{\ell^2}{4} + \frac{\ell}{2}$

(ii) bdd. below $M_\lambda \neq 0$ for $\lambda = \lambda_0 + k$ $k = 0, 1, \dots$

$\lambda_0 \notin \{-\frac{\ell}{2} \mid \ell = 0, 1, 2, \dots\}$

eigenvalues of $2\mathcal{C}$ = $\lambda_0^2 - \lambda_0$

(iii) bdd. above $M_\lambda \neq 0$ for $\lambda = \lambda_0 - k$ $k = 0, 1, 2, \dots$

$\lambda_0 \notin \{\frac{\ell}{2} \mid \ell = 0, 1, 2, \dots\}$

eigen of $2\mathcal{C} = \lambda_0^2 + \lambda_0$

(iv) unbdd. $M_\lambda \neq 0$ for $\lambda \in \lambda_0 + \mathbb{Z}$ $0 \leq \operatorname{Re} \lambda_0 < 1$

$\alpha \notin \{k\lambda_0 + \frac{k(k-1)}{2} \mid k \in \mathbb{Z}\}$

eigenvalues of $2\mathcal{C} = \lambda_0^2 - \lambda_0 + 2\alpha$

Now restrict to case where eigenvalues are to be half integers.
Restricts λ_0 to be a half integer in (ii) and (iii) and
$\lambda_0 = 0$ or $\frac{1}{2}$ in (iv). [This probably is wrong because ~~so~~ $Sl(2, \mathbb{R})$ has a contractible covering group, so all λ_0 should occur.

To calculate orbits of K on p.

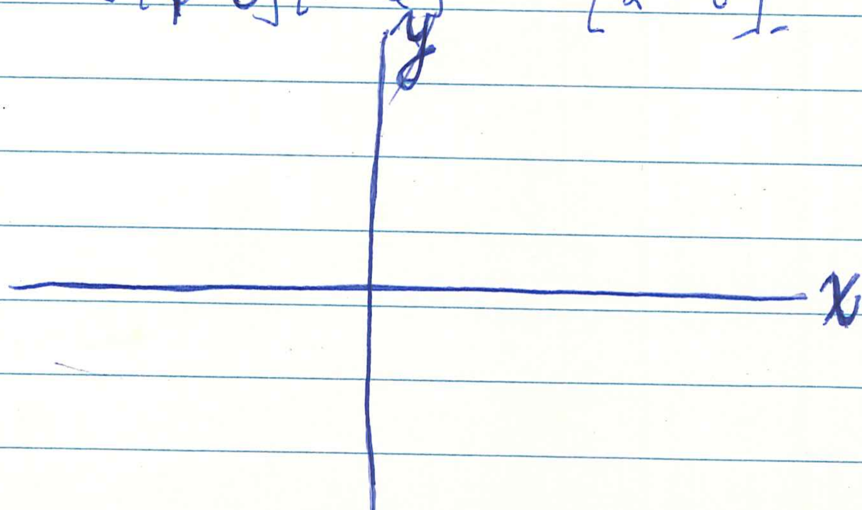
$$K = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix}, \ \alpha \in \mathbb{C}^* \right\}.$$

$$H = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad X = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad Y = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$p = \mathbb{C}X + \mathbb{C}Y$$

$$\begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \alpha^{-1} & \\ & \alpha \end{bmatrix} = \begin{bmatrix} 0 & \alpha^2 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \alpha^{-1} & \\ & \alpha \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \alpha^{-2} & 0 \end{bmatrix}.$$



orbits of plane under $x, y \mapsto (\alpha^2 x, \alpha^{-2} y)$ are
various hyperbolas ~~Ex~~ $xy = C$ + lines $y = 0, x \neq 0$
and origin $(x, y = 0, 0)$. $x = 0, y \neq 0$

Concerning the fact that λ_0 should be a half integer, this is incorrect. In effect $Sl(2,\mathbb{R})$ is a $K(\mathbb{Z},1)$ so has contractible universal covering group; thus taking finite covering groups we can achieve $\lambda_0 \in \mathbb{Q}$ and it seems reasonable that any $\lambda_0 \in \mathbb{R}$ will occur for the covering group.

It is perhaps worthwhile to note that if G_0 is a real semi-simple group with compact subalgebra k_0 ~~[illegible]~~ and with K_0 the corresponding subgroup of G_0, then any irreducible unitary representation of G_0 decomposes into a direct sum of finite dimensional K_0 representations. In effect $\tilde{K}_0 \simeq$ compact group $\times$ euclidean group and a generating lattice ~~maps~~ of euclidean group maps into the center of G_0 which is represented by a scalar. Thus only have to worry about compact part and thats OKAY.

Invariant equation approach still works ~~[illegible]~~

<u>Recalculate</u>

$$N = H - \frac{1}{\sqrt{2}}(X - Y) \qquad \langle H, H\rangle = 2$$

$$A = \frac{1}{\sqrt{2}}(X + Y) \qquad \langle A, A\rangle = 2$$

thus H and A are conjugate

$$[A, N] = H + \frac{1}{\sqrt{2}}(-X + Y) = N.$$

want to calculate the induced representation

$$U(\mathfrak{g}) \underset{U(\mathfrak{a}+\mathfrak{n})}{\otimes} \mathbb{C}$$

coming from character $\chi : \mathfrak{a} + \mathfrak{n} \longrightarrow \mathbb{C}$ given by

$$\chi(A) = \beta$$
$$\chi(N) = 0$$

From what Bert + Rallis tells me, this as a K module should be isomorphic to Harmonics i.e. $\sum_{n \geq 0} \mathbb{C}X^n + \mathbb{C}Y^n$, hence should be a representation of type (iv) with $\lambda_0 = 0$. So I try to find a map

$$\varphi : U(\mathfrak{g}) \underset{U(\mathfrak{a}+\mathfrak{n})}{\otimes} \mathbb{C} \longrightarrow M$$

which will be an isom. Thus I look for an element

$$m = \sum m_k \in \bigoplus_{k \in \mathbb{Z}} M_k$$

Such that

$$Am = \beta m \qquad X m_{k-1} + Y m_{k+1} = \sqrt{2}\beta m_k$$

$$Nm = 0 \qquad X m_{k-1} - Y m_{k-1} = \sqrt{2}\, k\, m_k$$

$$X m_{k-1} = \frac{\beta+k}{\sqrt{2}} m_k$$

$$Y m_{k+1} = \frac{\beta-k}{\sqrt{2}} m_k$$

But m is a finite sum, hence if $m_{k-1}=0$, $m_k \neq 0$ have $\beta+k=0$. Thus for p least $\ni$ $m_p \neq 0$ have $\beta+p=0$ and for q greatest $\ni$ $m_q \neq 0$ have $\beta=q$, which shows that $\beta=q=-p$ and so as $p \leq q$ we have $\beta = q \geq 0$. Thus β is an integer ≥ 0 and

$$XY m_0 = X\left(\frac{\beta+1}{\sqrt{2}}\right) m_{-1} = \frac{(\beta+1)\beta}{2} m_0$$

Thus

$$\alpha = \text{eigenvalues of } XY \text{ in dim } 0 = \frac{\beta(\beta+1)}{2}$$

Conclusion: The induced representation picture is not the correct finitely generated $U(\mathfrak{g})$ module picture.

In this case we get a map

$$U(\mathfrak{g}) \underset{U(\alpha+n)}{\otimes} \mathbb{C} \longrightarrow M^\wedge$$

where the formula relating

α = eigenvalue of XY on M_0

β = eigenvalue of A on inducing element of $M^\wedge$

is

$$\alpha = \frac{\beta(\beta+1)}{2}$$

Example of a max. left ideal not containing a max ideal.

Take an irred. inf. ~~d~~representation of $sl(2,\mathbb{R})$ having same character as a finite dimensional one. e.g. take principal series repn ~~[illegible]~~ containing 0 and with eigenvalue of $2C = \left(\frac{l}{2}\right)^2 + \frac{l}{2}$ l int ≥ 0. Then this representation contains ~~[illegible]~~ a finite dimensional repn, ~~and~~ an unbdd, and a bdd representation all with same character. Thus either infinite piece gives a max. left ideal ~~[illegible]~~ not containing a maximal ideal.