December 9, 1969: On PL-cobordism

Set $M^*(X) = MPL^*(X)$ unoriented PL-cobordism and try to see what you can prove using power operations. One knows in any case that ~~[illegible]~~ there is a map $MO \longrightarrow MPL$ and hence $H^*(MPL)$ is a free $\mathcal{a}$-module. In principle ~~[illegible]~~ $M^*(pt)$ known via Milgram.

We have the geometric squaring operation

$$P: M^q(X) \longrightarrow M^{2q}(B\mathbb{Z}_2 \times X)$$

and as usual we put

$$P(x) = \sum_{i \geq 0} \underbrace{w^{i}}_{i} \cdot \underbrace{Sq^{q-i}x}_{2q-i} \qquad x \in M^q(X).$$

~~[illegible]~~ For geometric reasons the Sq^i should be stable cohomology operations satisfying the Cartan formula.

Recall the localization formula

$$w^{-n} P(f_* 1) = f_* e(\eta \otimes (\nu_f + n\varepsilon)) \qquad \text{in } M^*(B\mathbb{Z}_2 \times X)$$

So if E_n is the universal PL_n-bundle we can write

$$e(\eta \otimes E_n) = \sum_{i \leq n} \underline{w^{n-i}} \; a_i^{(n)}(E_n) \qquad \text{in } M^n(B\mathbb{Z}_2 \times BPL_n)$$

Consider the restriction $M^*(B\mathbb{Z}_2 \times BPL_{n+1}) \longrightarrow M^*(B\mathbb{Z}_2 \times BPL_n)$.

Then $e(\eta \otimes E_{n+1}) \longmapsto e(\eta \otimes (\varepsilon \oplus E_n)) = w e(E_n)$, so

$$\sum_{i\le n+1}' w^{n+1-i}\, a_i^{(n+1)}(E_{n+1}) \longmapsto \sum_{i\le n+1}' w^{n+1-i}\, a_i^{(n+1)}(\varepsilon\oplus E_n)$$

$$\| $$

$$w \sum_{i\le n}' w^{n-i}\, a_i^{(n)}(E_n)$$

We conclude that

$$a_i^{(n+1)}(\varepsilon\oplus E_n) = \begin{cases} a_i^{(n)}(E_n) & i\le n \\ 0 & i>n \end{cases}$$

Therefore we get a well-defined class

$$a_i = \{a_i^{(n)}(E_n)\}_{n\ge 0} \quad \text{in} \quad M^i(BPL) = \varprojlim_n M^i(BPL_n)$$

and we have the universal formula

<u>Prop</u>: $\quad e(\eta\otimes E) = \sum_{i\le n} w^{n-i} a_i(E) \quad$ in $M^n(BZ_2\times X)$

where $a_i \in M^i(BPL)$ for any PL_n-bundle E on X.

~~So we get our localization formula~~

$$\cancel{\boxed{x\in M^q(X) \Longrightarrow w^m(w^q Px - x) = \sum_{i>0}' w^{m-i}\cdot \hat{a}_i x.}}$$

~~Here~~ $\cancel{\hat{a}_i : M^*(X) \longrightarrow M^{*+i}}$

Returning to the localization formula

$$e(\eta \otimes (\nu_f + n\varepsilon))_{-q+n} = \sum_{i \leq m} w^{m-i} a_i(\nu_f)$$

So if we apply f_* we get

$$\boxed{w^{m+q}(Px) = \sum_{i \leq m} w^{m-i} \hat{a}_i(x).}$$

for m large and $x \in M^q(X)$. First conclusion we can draw from this formula is that $\hat{a}_i(x) = 0$ if $i > -q$. Actually this isn't of much help since the $\hat{a}_i$ are simply the Sq^j up to reindexing.

Seems necessary to generate new classes in $M^*(BPL)$ by some geometric method. One knows that there should be a decomposition of $M^*(X)$ into $M^*(pt) \otimes H^*(X)$ and hence the power operations ought to tell you when you have enough elements. The idea would be to produce geometrically elements $c_\alpha \in M^*(BPL)$ corresponding to a basis of elements of $H^*(BPL)$ such that

$$e(\eta \otimes E) = \sum_{\alpha \geq 0}' w^{n-\langle\alpha\rangle} a^\alpha c_\alpha(E)$$

Use the permutation representation of $\mathbb{Z}_2^n$ somehow proving that $U^{-2q}(\Sigma')$ is 2-divisible and hence an odd order group since its finitely generated.

~~Modifications for p-torsion, p odd~~

Improvement in preceding proof: Without assuming that each $U^q(pt)$ is finitely generated, it follows that an element of $U^*(pt)$ with zero Chern numbers is infinitely 2-divisible:

By induction one assumes that the kernel of the characteristic numbers map $U^{-2k}(pt) \longrightarrow (\mathbb{Z}[t_1, -])_{2k}$ is 2-divisible for $0 \leq k < q$. Again we work with

$$Q : U^{-2q}(pt) \longrightarrow U^{-4q}(\mathbb{R}P^{2N+1}) \qquad N \text{ very large}$$

and will use that w is a 2-torsion element. This follows from the Gysin sequence

$$U^*(\mathbb{C}P^N) \xrightarrow{F(x,x)} U^*(\mathbb{C}P^N) \longrightarrow U^*(\mathbb{R}P^{2N+1})$$

$$x \longmapsto w$$

which shows that

$$\Big(2 + \underbrace{w\, G(w,w)}_{\text{nilpotent}}\Big) w = 0$$

so on inverting 2 one kills w.

(Before going on we must fix up the imprecision of $U^*(B\mathbb{Z}_2)$ used in the preceding proof. We replace it by the inverse system $U^*_{\mathbb{Z}_2}(S^{2N-1})$, but it is now necessary to check that $\operatorname{Ker} w = U^*(pt)$? So consider the inverse system of

December 9, 1969:

Let X be a finite (pointed) complex. Then by Conner-Smith for n sufficiently large we can find a sequence of maps of finite complexes with basepoint

$$\Sigma^n X \longrightarrow Y_0 \longrightarrow Y_1 \longrightarrow Y_2 \longrightarrow \cdots \longrightarrow Y_r \longrightarrow * \longrightarrow *.$$

such that ~~[illegible]~~ composition of ~~[illegible]~~ consecutive maps is trivial and (i) each Y_q is torsion free

(ii) the sequence

$$0 \longleftarrow \tilde{U}^*(\Sigma^n X) \longleftarrow \tilde{U}^*(Y_0) \longleftarrow \cdots$$

is exact. Here's how the construction goes. One chooses generators for $U^*(X)$ and each of these can be realized by a map of $\Sigma^n X \longrightarrow$ ~~[illegible]~~ Thom space over a Grassmannian. This gives $\Sigma^{n_0} X \longrightarrow Y_0$ where Y_0 torsion $\ni$ $\tilde{U}^*(\Sigma^n X) \twoheadleftarrow \tilde{U}^*(Y_0)$ surjective. Then form the cone C_0 on this map and repeat. ~~[illegible]~~ If r is the homological dimension of $\tilde{U}^*(X)$, then $\tilde{U}^*(C_{r-1})$ ~~[illegible]~~ is free, hence $C_{r-1} = Y_r$ is torsion-free.

Next form the sequence

$$H^*(Y_0) \longleftarrow H^*(Y_1) \longleftarrow \cdots$$

cohomology modulo p. It is clear that the algebra of reduced powers acts on this complex, hence on the homology and the spectral sequence

$$E^2 = \mathrm{Tor}_p^{U^*}(U^*(X), \mathbb{Z}_p) \Longrightarrow H^*(\Sigma X).$$

problem: Can you put power operations to work on the spectral sequence?

~~The method is as follows.~~

$$P : U^{2q}(X) \longrightarrow U^{4q}(B\mathbb{Z}_2 \times X) \qquad \text{unstable}$$

however you might consider its effect

X replace by a gigantic suspension so you get your resolution

$$\Sigma^a X \longrightarrow Y_0 \longrightarrow Y_1 \longrightarrow \cdots \longrightarrow Y_q$$

are maps of finite complexes where each Y_j is torsion-free. now we get

$$H^*(Y_q, \mathbb{Z}_p)$$

$$\underbrace{\mathbb{Z}_2 \otimes_{U^*} U^*(B\mathbb{Z}_2)}\otimes_{U^*} U^*(Y_q) \xrightarrow{\sim} \mathbb{Z}_2 \otimes_{U^*} U^*(B\mathbb{Z}_2 \times Y_q)$$

$$H^*(B\mathbb{Z}_2, \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^*(Y_q, \mathbb{Z}_2)$$

therefore you get a map

$$P : \mathrm{Tor}_p^{U^*}(U^*(X), \mathbb{Z}_2)_{ev} \longrightarrow \mathrm{Tor}_p^{U^*}(U^*(X), \mathbb{Z}_2[w])_{ev}$$

A power operations map, which should be expressible using same formula

My question is whether we can conclude that the vanishing range for Tor_0 that we have extends to the higher Tor's. ~~Suppose we can introduce products in the spectral sequence, [illegible] so that the compatible with the natural ring structure on~~ ~~$\mathrm{Tor}_*^{U^*}(U^*(X), \mathbb{Z}_p)$. Then the fact $\mathrm{Tor}_*^{U^*}(U^*(X), \mathbb{Z}_p)$ is a module over Tor_0 implies~~ It's the question of whether the higher Tor's are unstable $\mathcal{A}^{(p)}$-modules.

~~$B(T) = B\mathbb{Z}_p$ is a circle bundle over $BT = BS^1$ with~~
~~... the zero Euler class~~
~~and the desired result is~~

~~The power operation:~~

question: $\boxed{\mathbf{Q} : U^{ev}(X) \longrightarrow U^{ev}(X \times B\mathbb{Z}_2)}$

can you find an explicit formula for this!!

The idea is that once we know that ~~$U^*(X \times B\mathbb{Z}_p)$~~ ~~$U^*(B\mathbb{Z}$~~ $U^*(pt)$ has no torsion, then

$$Q : U^*(pt) \longrightarrow U^*(B\mathbb{Z}_2)$$

$$\|$$

$$Q : U^*(pt) \longrightarrow U^*(pt)[[X]]\Big/\big(F(X,X)\big) = U^*(pt)[[w]]$$

$$Q(x+y) = Q(x) + Q(y) + xy \cdot \xi \qquad \xi = \frac{F(w,w)}{w} = 2 + w\, G(w,w)$$

still doesn't help

$$\begin{tikzcd}
 & & U^*(pt) \arrow[d, "Q"] \arrow[dr] & \\
0 \arrow[r] & U^*(pt) \arrow[r, "\xi"] \arrow[dr, "2"'] & U^*(B\mathbb{Z}_2) \arrow[r] \arrow[d] & U^*(B\mathbb{Z}_2)[w^{-1}] \\
 & & U^*(pt) &
\end{tikzcd}$$

$$Q(c_1(L)) = c_1(L)\, F(w, c_1(L)).$$

$$\boxed{U^*(B\mathbb{Z}_2) \hookrightarrow U^*(pt) \times U^*(B\mathbb{Z}_2)[w^{-1}]}$$

$$\begin{tikzcd}
\dfrac{U^*(pt)[[w]]}{F(w,w)} \arrow[r, hook] & \dfrac{U^*(pt)[[w]]}{(w)} \times \dfrac{U^*(pt)[[w]]}{\left(\frac{F(w,w)}{w}\right)} \\
U(pt) \arrow[u, "Q"] \arrow[ur]
\end{tikzcd}$$

(square, ring homom. you know in principle)

$U^*(B\mathbb{Z}_p)$ allready has 2 minimal prime ideals one where you invert Euler class and other note. Same as for cohomology

[now the ideal would be to prove the formula for

the characteristic Chern nos. U

To analyze the equation

$$X = x F(w, x)$$

~~where~~ over $\mathcal{U}(pt)[[w^{?}, X]] \Big/ F(w,w)=0$

and to decide whether this equation leads to a locally free extension.

Start with $\mathcal{U}(B\mathbb{Z}_2)[[X]]$ and consider equation

$$Y F(w, Y) = X$$

gives a map

$$\mathcal{U}(B\mathbb{Z}_2)[[X]] \longrightarrow \mathcal{U}(B\mathbb{Z}_2)[[Y]]$$

Question is whether this is locally-free of rank 2

examples. $wY + Y^2 = X$ OKAY since w ess. nilpotent

working modulo nilradical this is $Y^2 = X$, so it's OKAY

$$A \longrightarrow B$$

$$grA \qquad grB$$

$$L[w][[X]] \to L[w][[Y]]$$ standard filt.

$$\frac{X \qquad Y^2}{}$$

should be OKAY.

Calculation of

$$U(pt) \xrightarrow{Q_2} U(BZ_2):$$

$$\begin{tikzcd} x \arrow[d, phantom] & (xF(w,x),\, x) \\ U(\mathbb{P}^\infty) \arrow[r] \arrow[d] & F(\mathbb{P}^\infty) = U(BZ_2 \times \mathbb{P}^\infty) \dot{\otimes} U(\mathbb{P}^\infty) \arrow[d] \\ U(\mathbb{P}^\infty \times \mathbb{P}^\infty) \arrow[r, "Q"] & F(\mathbb{P}^\infty \times \mathbb{P}^\infty) \\ F(x,y) & (F(x,y)F(w, F(x,y)),\, F(x,y)) \end{tikzcd}$$

Thus

$$(QF)(Qx, Qy) = (F(x,y)F(w,F(x,y)),\, F(x,y))$$

now

$$Q(F(x,y)) = (Q_2 F(x,y),\, F(x,y))$$

and

$$Q_2 F(x,y) = (Q_2 F)(Q_2 x, Q_2 y) + \frac{F(x,y)^2 - F^{(2)}(x^2, y^2)}{2} \xi$$

Therefore

$$\boxed{(Q_2 F)(xF(w,x),\, yF(w,y)) + \frac{F(x,y)^2 - F^{(2)}(x^2,y^2)}{2}\xi = F(x,y)\,F(w, F(x,y)).}$$

Question: Does this determine $Q_2 F$?

$$xF(w,x) \equiv wx + x^2 \quad \text{mod deg } 3$$

~~Now introduce the change in variable~~

Question: Is the map sending

$$F(pt)[[X, Y]] \longrightarrow F(\mathbb{P}^\infty \times \mathbb{P}^\infty)$$

$$Y \longmapsto Qy$$
$$X \longmapsto Qx$$

injective?

$$(Q_2 F)(xF(w,x), yF(w,y)) = F(x,y)F(w,F(x,y)) - \frac{F(x,y)^2 - F^{(2)}(x^2,y^2)}{2}\xi$$

to determine what happens to ~~[illegible]~~

diff wrt y & set $y=0$.

$$(Q_2F)_2(xF(w,x), 0)(w) = F_2(x,0)F(w,x) + xF_2(w,x)F_2(x,0)$$
$$- \cancel{[illegible]}\, 2x\, F_2(x,0)\, \cancel{[illegible]}\, \xi$$

$$F_2(x,0)\cancel{x} + xF_2(0,x)F_2(x,0) - 2xF_2(x,0)$$

$$x + xF_2(0,x) - 2x$$

$$\cancel{F_2(0,x) =}$$

$$F_2(X,Y) = 1 + XG(X,Y) + XYG_2(X,Y)$$

$$F_2(0,Y) = 1.$$

$$\cancel{(Q_2F_2)(xF(w,x), 0) =}$$

Point is to determine Q modulo ξ first should be easy

$$0 \to U(pt) \xrightarrow{\xi} U(B\mathbb{Z}_2) \longrightarrow U(B\mathbb{Z}_2)[w^{-1}] \longrightarrow U_{*}^{odd}(B\mathbb{Z}_2) \to 0$$

$$Q(f_*1) = f_*\, c(\eta \otimes \nu_f).$$

December 15, 1969

I want to understand what localization at fixpoints means for power operations in K-theory. Recall the natural map $\Phi : U^{ev}(X) \longrightarrow K(X)$ given by periodicity carries $U^0(X)$ onto $K(X)$ and in fact there is an additive section given by $x \longmapsto \dim x - c_1(x^\vee)$; for a line bundle this $= 1-(1-L) = L$ and it's additive. Given $x \in K(X)$ we can therefore express it in the form $f_* 1 = x$ where $f : Z \longrightarrow X$ is proper complex-oriented of dimension 0 and we have

$$Px = f_* \, e(\eta \otimes \nu_f) \qquad \text{in } K(B\mathbb{Z}_2 \times X)[w^{-1}]$$

where $w = e(\eta)$. We know that

$$e(\eta \otimes L) = w + e(L) - w e(L)$$
$$= w + (1-w) e(L).$$

If $\dim E = r$

$$e(\eta \otimes E) = w^r + w^{r-1}(1-w) c_1(E) + \cdots + (1-w)^r c_r(E)$$

Therefore we find that

$$w^n Px = f_* \, e(\eta \otimes (n + \nu_f))$$

$$= f_* \left(w^n + w^{n-1}(1-w) c_1(n+\nu_f) + \cdots \right)$$

$$= w^n x + w^{n-1}(1-w) f_* c_1(\nu_f) + \cdots + (1-w)^n f_* c_n(\nu_f)$$

~~symplectic~~

$Sp^*(\mathbb{R}P^n)$

$$Sp^q(\mathbb{R}P^\infty) \longrightarrow Sp^{q+4}(\mathbb{R}P^\infty) \longrightarrow \underset{\text{0 if } q \geq 0}{Sp^{q+4}(\mathbb{R}P^3)}$$

so our first thing will be to understand this localization if that is possible. It is a theory modulo 4

Conjecture is that if $f : Y \longrightarrow X$ is a double covering with $e(Y \times_{\mathbb{Z}_2} H) = \sigma$ then

$$f_* 1 = \underbrace{1 + a_1}\, + a_2 \sigma + a_3 \sigma^2 + \ldots\ldots$$

so use same geometry

$$\begin{tikzcd} Y \arrow[r] \arrow[dr, bend right, "f"'] & \mathbb{P}(1+M) \arrow[d] \\ & X \end{tikzcd}$$

and it might work.

The other basic invariant is a map $\tilde{h}(\mathbb{R}P^2) \longrightarrow h(B\mathbb{Z}_2)$.
and a basic map

$$(\mathbb{R}P^2 \times \mathbb{R}P^2)/\mathbb{Z}_2 \xrightarrow{\sim} \mathbb{R}P^4$$

$$(az + b)(cz + d) \qquad acz^2 + (ad + bc)z + bd$$

so this gives a nice map

?

$$\begin{tikzcd} Sp(\mathbb{R}P^2) \arrow[r, "P"] & Sp(E\mathbb{Z}_2 \times_{\mathbb{Z}_2} (\mathbb{R}P^2)^2) \\ & Sp(\mathbb{R}P^4) \arrow[u] \end{tikzcd}$$

?

~~Sphere bundles:~~

$$S(S^\infty \times_{\mathbb{Z}_2} \mathbb{C}) \longrightarrow B\mathbb{Z}_2$$

$$0 \to \underline{S^1} \longrightarrow E \xrightarrow{g} B \longrightarrow 0$$

g principal S^1 bundle $\Rightarrow \exists\ g_*$

Therefore my formula reads

$$\rho^n Px =$$

$$w^2 = 2w \qquad \text{if } p=2$$

so we get the formula

$$\cancel{w\left\{2^n Px = 2^n x - 2^{n-1} f_* c_1(\nu_f) + \cdots\right.}$$

$$w\left\{-2^n P(f_* 1) + 2^n (f_* 1) - 2^{n-1} f_* c_1(\nu_f) + 2^{n-2} f_* c_2(\nu_f) - \cdots\right\} = 0$$

now we know that for a torsion-free space

$$\begin{tikzcd}
K(X) \arrow[r, "\cdot w"] \arrow[dr, "2\otimes id"'] & K(B\mathbb{Z}_2 \times X) \arrow[d] \\
& \mathbb{Q}_2 \otimes_{\mathbb{Z}} K(X)
\end{tikzcd}$$

is injective since

is the localization wrt w^{-1}!

so I conclude that

$$2^n P(f_* 1) = 2^n f_* 1 - 2^{n-1} f_* c_1(\nu_f) + \cdots$$

$$= \frac{(1+te(L))(1-\beta)}{1+te(L) \cancel{} - \beta(1+t)}$$

$$= \cancel{(1+te(L))(1-\beta)\left[1+te(L)+\beta(1+t)\right]}$$

~~So starting with K theory classes~~

If $x \in F_q K(X)$ write $x = f_* 1$ where $f: Z \to X$ represents an element of $U^{2q}(X)$ for $q > 0$. Then

$$2^n w \left\{ -P(f_* 1) + 2^q f_* 1 - 2^{q-1} f_* c_1(\nu_f) + \cdots \right\} = 0$$

so for a torsion free space we get

$$P(f_* 1) = 2^q f_* 1 - 2^{q-1} f_* c_1(\nu_f) + \cdots$$

The first idea I have is K-theoretic characteristic nos.

try to understand Steenrod operation in char 2. [they exist but one has some problem realizing $B\mathbb{Z}_2$ since real proj. space is $B\mu_2$.

char 2. want to embed $\mathbb{Z}_2 \longrightarrow G$. Artin-Shreier theory.

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{G}_a \xrightarrow{x^2 - x} \mathbb{G}_a \longrightarrow 0$$

December 19, 1969. On symplectic cobordism.

I want to compute the symplectic cobordism ring using my power operations technique. Only $p=2$ is interesting. Let η denote the non-trivial real character of $\mathbb{Z}_2$ and let $w = e(\eta \underset{\mathbb{R}}{\otimes} \mathbb{H}) \in Sp^4(B\mathbb{Z}_2)$. Then the localization formula reads

$$w^n P(f_* 1) = f_* \, e(\eta \otimes (\nu_f + n)) \qquad \text{in } Sp^*(B\mathbb{Z}_2 \times X)$$

if $f: Z \longrightarrow X$ is a proper quaternionic-oriented map. (I restrict this to mean dimension $f \equiv 0 \ (4)$. Here n is an integer large enough so that the quaternion orientation of f can be realized by a factorization $Z \xrightarrow{i} \mathbb{H}^n \times X \longrightarrow X$, and $\nu_f + n$ is the normal bundle of i. Now we know there are elements $a_k \in Sp^{4-4k}(B\mathbb{Z}_2)$ such that

$$e(\eta \otimes L) = w + \sum_{k \geq 1} a_k \, e(L)^k$$

whence as before we obtain the formula

$$\boxed{w^n P x = \sum_{\ell(\alpha) \leq n-q} w^{n-q-\ell(\alpha)} a^\alpha \cdot s_\alpha x \qquad \text{if } x \in Sp^{-4q}(X)}$$

The problem with using this formula stems from our ignorance of the a_k's.

Let L_1 and L_2 be two symplectic line bundles over X. Over the bundle $E = \mathrm{Hom}_{\mathbb{H}}(L_1, L_2)$, which is a real $\mathrm{Spin}(4) = S^3 \times S^3$ bundle, there is complex $\pi^* L_1 \longrightarrow \pi^* L_2$ of ~~¥~~ symplectic bundles which acyclic off the zero section. ~~This~~ defines an element (d) of $KSp(E, E-X)$ which we know is a Thom class, i.e.

$$KO(X) \xrightarrow{\sim} KSp(E, E-X)$$

I can form the first quaternionic Chern class of d; it is an element $cq_1(d) \in Sp^4(E, E-X)$. Is this element a Thom class?

Since $cq_1(d)$ is a global class it suffices to check when $X = pt.$

$$\begin{tikzcd}
cq_1(d) & Sp^4(E, E-X) \arrow[d] & Sp^4(BSp) \arrow[l] \arrow[d] & cq_1 \arrow[d, mapsto] \\
& Sp^4(\mathbb{H}, \mathbb{H}-0) \arrow[d, "\cong"] & H^4(BSp) \arrow[dl] & cq_1 \\
& H^4(\mathbb{H}, \mathbb{H}-0) & H^4(BU) \arrow[u] & c_2 \arrow[u, mapsto]
\end{tikzcd}$$

This diagram shows us that it suffices to show that the ~~bundle of relative relative~~ c_2 of the relative bundle over $\mathbb{H}$ constructed from the quaternion multiplication is the generator of $\tilde{H}^4(S^4)$. Forgetting the quaternionic structure we have the Bott generator for $\tilde{KU}^4(S^4)$. To see this we use that by Clifford theory the relative bundle just constructed is the generator for $\widetilde{KO}^4(S^4)$, hence already generates $\widetilde{KU}^4(S^4)$, since $\widetilde{KU}^4(S^4) \hookrightarrow \widetilde{KU}^4(S^4)$ in any case. But one knows that $c_n(\beta^n) = (n-1)!\cdot$ generator in complex K-theory, so we are OKAY for $n=2$. So we have proved

Proposition: Let L_1, L_2 be two symplectic line bundles over X and let $E = \underline{\mathrm{Hom}}_{\mathbb{H}}(L_1, L_2)$, which is a real $\mathrm{Spin}(4) = S^3 \times S^3$ bundle. Let $\lambda \in Sp^4(E, E-X)$ be the first quaternionic Chern class of the complex $\pi^* L_1 \to \pi^* L_2$ where $\pi: E \to X$ is the projection. Then λ is a Thom class, i.e.

$$Sp^q(X) \xrightarrow[\lambda\cdot]{\sim} Sq^{q+4}(E, E-X)$$

Questions: Might λ admit a nice geometric interpretation? Is E actually stably quaternionic? (NO by Hirzebruch)

The interest in the above proposition lies in the possibility of constructing elements in $Sp^*(pt)$ as follows. Consider the Wu classes $W_\alpha \in Sp^{4|\alpha|}(X)$ defined by

$$(*) \qquad \lambda \cdot w_{\underline{t}} = s_{\underline{t}}(\lambda)$$

Now universally we can write

$$w_{\underline{t}} = F_{\underline{t}}(e(L_1), e(L_2)) = \sum a_{jk}(t)\, e(L_1)^j e(L_2)^k$$

where $a_{jk\alpha} \in Sp^{4*}(pt)$. A basic question is whether the $a_{jk\alpha}$ might generate the non-torsion part of $Sp^*(pt)$. At least we ought to be able to decide whether we get enough interesting elements in the image of $Sp^*(pt) \longrightarrow U^*(pt)$.

~~[illegible]~~ The element ~~[illegible]~~ of the proposition is a ~~[illegible]~~ substitute for the Thom class on $\mathrm{Hom}_{\mathbb{H}}(L_1, L_2)$ coming from the quaternionic structure which isn't there. The following is related. According to Hirzebruch $\mathbb{H}P^1$ does not have a ~~[illegible]~~ weakly complex structure because this contradicts Todd genus being an integer. However we can produce an Sp^* Thom class for the tangent bundle as follows. Recall that the tangent bundle fits into an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{H}}(\mathcal{O}(-1), \mathcal{O}(-1)) \longrightarrow \mathrm{Hom}_{\mathbb{H}}(\mathcal{O}(-1), (n+1)) \longrightarrow T \longrightarrow 0$$

and by what we've seen $\mathrm{Hom}_{\mathbb{H}}(\mathcal{O}(-1), \mathcal{O}(-1))$ has a Sp^* Thom class, although by Hirzebruch it can't be stably symplectic. It is now necessary to understand the Hirzebruch proof carefully and to decide whether or not $\exists$ Gysin homomorphism in U^* for g in

$$(*) \qquad \begin{array}{ccc} U^*(\mathbb{C}P(E)) & \longrightarrow & \\ \downarrow & & U^*(X) \\ U^*(\mathbb{H}P(E)) & \xrightarrow{g_*} & \end{array}$$

$$\frac{x}{1-e^{-x}} = 1 + \frac{x}{2} - \frac{x^2}{12} \quad (\text{mod } x^3)$$

<u>Hirzebruch's proof</u> that $\mathbb{H}P^1$ is not weakly complex.
The Todd genus in low dimensions is

$$\text{Todd} = 1 + \frac{c_1}{2} + \frac{c_1^2+c_2}{12} + \cdots$$

and for an almost complex manifold is an integer. Also one knows that $\langle c_n(\tau_V), [V^n] \rangle =$ Euler characteristic of V. Thus

$$\text{Todd}(\mathbb{H}P^1) = \frac{1}{12}\langle c_1^2 + c_2, [\mathbb{H}P^1] \rangle = \frac{\text{Euler char}}{12} = \frac{1}{6}$$

is not an integer.

<u>Example</u> to show that (*), p. 4 cannot be used to define Gysin homomorphism for a quaternionic projective bundle.

$$\begin{tikzcd} \mathbb{C}P_4 \arrow[d, "\pi"'] \arrow[dr, "f"] & \\ \mathbb{H}P_2 \arrow[r, "g"'] & pt \end{tikzcd}$$

I will show that in K theory g_* cannot be defined so that $g_* \pi_* = f_*$. Let L be the sub-line bundle on $\mathbb{H}P_2$ so that

$$\pi^* L = \mathcal{O}(-1) + \mathcal{O}(1)$$

Then

$$K(\mathbb{C}P^4) = \mathbb{Z}[T]/(T-1)^4 \qquad T = [\mathcal{O}(-1)]$$

and we know that
$$f_*(T^i) = \begin{cases} 1 & i=0 \\ 0 & 0 < i \leq 3 \end{cases}$$

Also we know that

$$\pi_* 1 = 1 \quad , \quad \pi_* [\mathcal{O}(-1)] = 0.$$

So consider the element T^2+1

$$\pi_*(T^2+1) = \pi_*(\pi^*[L]\cdot T) = [L]\cdot\pi_* T = 0$$

$$f_*(T^2+1) = 1.$$

which provides the desired example!

The following calculation ~~example~~ shows that the Wu classes of (*) page 3 are ~~might~~ not ~~be~~ very interesting. ~~[illegible]~~ ~~[illegible]~~ Let λ be the first Chern class of the element of $KSp(E, E-X)$ represented by the complex $\pi^*L_1 \to \pi^*L_2$. Then as the first Chern class is additive we find

$$i^*(\lambda) = e(L_2) - e(L_1)$$

so ~~[illegible]~~

$$\pi^*\omega_{\underline{t}}\cdot\lambda = s_{\underline{t}}(\lambda)$$

$$\Rightarrow \quad \omega_{\underline{t}}\,(e(L_2)-e(L_1)) = s_{\underline{t}}\, e(L_2) - s_{\underline{t}}\, e(L_1)$$

$$\Rightarrow \quad \omega_{\underline{t}} = \sum_{j\geq 0} t_j \cdot \frac{e(L_2)^{j+1} - e(L_1)^{j+1}}{e(L_2) - e(L_1)}$$

This shows that we don't get ~~the~~ new elements in $Sp^*(pt)$.

what you are trying to do is to express

$$f_* 1 = \sum a_k \omega^{k-1}$$

to express in terms of elements in $Sp^{4*}(B\mathbb{Z}_2)$

but this is impossible unless cando already in $U^*(pt)$. The problem occurs

$$\frac{F(\omega,\omega)}{\omega} = 2 + \omega G(\omega,\omega) \equiv 2 - \omega P_1 \quad (\omega^2)$$

now $\quad F_2(\omega, 0) = 1 + \omega G(\omega, 0)$

$$= 1 - P_1\omega \qquad \text{mod}$$

$$\ell(X) = X + \frac{P_1}{2}X^2 +$$

$$F(X,Y) = \cancel{\frac{P_1}{2}X^2 + X - \frac{P_1}{2}Y^2 + Y}$$

$$X + \frac{P_1}{2}X^2 + Y + \frac{P_1}{2}Y^2 - \frac{P_1}{2}(X+Y)^2$$

$$\equiv X + Y - P_1 XY \qquad G(0,0) = -P_1$$

__further projects__ (after papers on $K_i(\mathbb{F}_q)$ and "cohomology of finite groups of rational points")

unitary group

do Brauer theory for ~~stable~~ families of finite groups as a monodromy-type study. (i.e. work over a henselian trait, etc.)

simplicial formulae for algebraic K-theory.

K-theory for the p-adic numbers; the basic exact sequences relating $K_*(\mathbb{F}_p)$ $K_*(\mathbb{Z}_p)$ $K_*(\mathbb{Q}_p)$.

$$\begin{array}{c} H^*(MF) \\ \uparrow \cong \\ H^*(BF) \end{array}$$

Them isom.

<u>Question!</u> Is there any chance that modulo F-isomorphism $H^*(BG)$ is generated by the Chern classes of the representations of G?

The point is that maybe D-L operations decrease cohomology dimension, while St. operations raise dimension

What are the geometric effect of the D-L operations on $H_*(BO)$ say. take E over X form $E \boxtimes E$ over X^2 equivariantly. Then one gets

$$H_*^{\mathbb{Z}_2}(BO^2) \longrightarrow H_*(BO)$$

$$\begin{tikzcd}
E\mathbb{Z}_2 \times_{\mathbb{Z}_2} (E^2) \arrow[r] \arrow[d] & EO \arrow[d] \\
E\mathbb{Z}_2 \times_{\mathbb{Z}_2} X^2 \arrow[r] & BO
\end{tikzcd}$$

$$\begin{tikzcd}
\tilde{H}^*_{\mathbb{Z}_2}(MO^2) & H^*(MO) \arrow[l] \\
H^*_{\mathbb{Z}_2}(BO^2) \arrow[u] & H^*(BO) \arrow[l] \arrow[u]
\end{tikzcd}$$

I am really quite confused. In any case I should be able to compute the ring homs.

$$H_*(BO) \longrightarrow H_*(BO)[w]$$

Coming by D-L.

must compute these out!

so work this way!

$$H_*(BO) \longrightarrow H^*(B\Sigma_2) \otimes H_*(BO)$$

$$x \longmapsto \sum w^i Q_i(x)$$

~~next suppose that k is independent of Z.~~

functorially:

$$\mathrm{Hom}_{\mathrm{rgs}}(H_*(BO), A) = \mathrm{Hom}_+(K, (H^*(?) \otimes A)^\times)$$

$$\uparrow$$

$$\mathrm{Hom}_{\mathrm{rgs}}(H^*(B\Sigma_2) \otimes H_*(BO), A)$$

thus given a ~~trans~~ char class from K with values in H^*_A and an element a of A one gets a new such ~~tra~~ transformation

---

$$E\Sigma_2 \times_{\Sigma_2} BO^2 \longrightarrow BO$$

$$H^*_{\Sigma_2}(BO^2) \longleftarrow H^*(BO)$$

so I have a basic operation.

$$H^*_{\Sigma_2}(BO^2) \cong \Gamma_2(H^*(BO)) + \sum w^i H^*(BO)^{(2)}$$

replace w_t by c_t and ~~departs~~

~~$x \otimes w$~~

Suppose $\quad \Psi(c_t) = \sum w^j a_t^{(j)}$ ~~qed~~

then
$$\sum w^{i+j} Sq^i a_t^{(j)} = w_t(E) \cdot w_t(\eta \otimes E)$$

---

what kind of transf. is

$$E \longmapsto \Psi_E(c_t) = \sum w^j a_t^{(j)}(E)$$

Θ ring homomorphism (ess. an isom.)

$$E \longmapsto \Theta \Psi_E(c_t) = c_t(E) \cdot c_t(\eta \otimes E)$$

sends sums of bundles into products.

hence $E \longmapsto \Psi_E(c_t)$ should send sums into products and should be determined by effect on line bundles.

$$\Theta \Psi_L(c_t) = c_t(L) \cdot w_t(\eta \otimes L)$$
$$= (1+te(L))(1+tw+te(L))$$

Suppose $\Psi_L(c_t) = 1 + b(w)t + c(w)t^2$

$\Longrightarrow$

Suppose that

$$\Psi_L(c_t) = \sum w^j t^k a_{jk}(L) \qquad k \leq 2$$

then

$$\sum w^{i+j} t^k Sq_i\, a_{jk} = (1+te(L))(1+tw+te(L))$$

$$\sum w^{i+j} Sq_i(a_{j2}) = e(L)^2 + we(L)$$

$$Sq_0(a_{02}) = e(L)^2 \qquad \Longrightarrow a_{02} = e(L).$$

$$Sq_1(a_{02}) + Sq_0(a_{12}) = e(L) \qquad Sq_0^* \text{ inj on } H^*(\mathbb{R}P^\infty).$$

$$a_{12} = 0$$

$$a_{j2} = 0 \quad j > 0.$$

$$\sum w^{i+j} Sq_i(a_{j1}) = e(L) + w + e(L) = w$$

$$Sq_0(a_{01}) = 0 \qquad a_{01} = 0$$

$$Sq_1(a_{01}) + Sq_0(a_{10}) = 1 \qquad a_{11} = 1$$

$$Sq_2(a_{01}) + Sq_1(a_{11}) + Sq_0(a_{21}) = 0$$

$$\therefore \quad \Psi_L(c_t) = 1 + wt + t^2 e(L)$$

$$\theta\, \Psi_L(c_t) = 1 + wt + t^2(we(L) + e(L)^2)$$

here's my idea for computing the effect of the D-L operations on $H_*(BO)$. Cohomologically what happens is we have a bundle E over X and we want formulas for ~~D-L operation~~ the classes of E^2 over X^2 (equiv.).

$$\begin{tikzcd} H^*(BO_{2n}) \arrow[r] \arrow[dr, bend right, "w_t"'] & H^*_{\Sigma_2}(X^2) \arrow[d, "\Delta^*"] \\ & H^*_{\Sigma_2}(X) = H^*(X)[w] \end{tikzcd}$$

$$w_t(\cancel{\ } \text{reg} \otimes E) = w_t(E) \cdot w_t(\eta \otimes E).$$

but we have ~~the map~~ that the map

$$\cancel{H^*_{\Sigma_2}} \quad H^*(X)[w] \longrightarrow H^*_{\Sigma_2}(X^2) \xrightarrow{\Delta^*} H^*_{\Sigma_2}(X)$$

$$x \otimes w^j \longmapsto Q(x)\cdot w^j \longrightarrow \sum_{i \geq 0} w^{i+j}\, Sq^i x.$$

$$\begin{tikzcd} & & H^*(X)[w] \arrow[dl] \arrow[ddl, "\Theta"] \\ H^*(BO) \arrow[urr, bend left, "\Phi"] \arrow[r] & H^*_{\Sigma_2}(X^2) \arrow[d, "\Delta^*"] & \\ & H^*(X)[w] & \end{tikzcd}$$

~~identically~~ an isomorphism
$e(L) \mapsto w e(L) + e(L)^2$
if one inverts w

$\sum w^i Sq^i \Phi(w_t)$ too many w's so it seems that the transf. $E \longmapsto \Phi_E(c_t)$ from bundles to $H^*(X)[\omega; t]$ sends

$$L \text{ to } \qquad 1 + t\omega + t^2 e(L).$$

In general ~~it takes~~ you must consider

$$E \longmapsto \Phi_E(c_{\underline{t}})$$

$$\Phi_L(c_{\underline{t}}) = \sum t^\alpha \, \Phi_L(c_\alpha) \qquad ?$$

So the idea is formally to write

$$1 + t\omega + t^2 e(L) = (1 + t\lambda_1)(1 + t\lambda_2)$$

and then

$$\Phi_L(c_{\underline{t}}) = \Big(\sum_{j \geq 0} \lambda_1^j t_j\Big)\Big(\sum_{j \geq 0} \lambda_2^j t_j\Big)$$

t^α where $\ell(\alpha) \leq 2$.

$\alpha =$

$$\sum_j t_j^2 \, e(L)^j + \sum_{j<k} t_j t_k \left(\lambda_1^j \lambda_2^k + \lambda_1^k \lambda_2^j\right).$$

This leads to a Wu type formula

examples: ① $H_*(BO) = \mathbb{Z}_2[a_1, a_2, \cdots]$

with $\Delta_{add}(a_n) = \sum_{i+j=n} a_i \otimes a_j$.

to work out this map carefully

② Suppose h is a GCT represented by a spectrum $M_n = \Omega M_{n+1} = \Omega^2 M_{n+2}$. Then we get the additive operations on

$$h^0(X) = [X, M_0]$$

so M_0 is an infinite loop space.

③ also want to work out the multiplicative version of ②. Big hope: Can you determine the homology of $M_0 = \varinjlim \Omega^{2k} MU(k)$ or equivalently (by duality) the set of all natural transf. from $U^0(X) \longrightarrow H^*(X)$

Problems to give students

1. mod p cohomology of $GL_n(\mathbb{F}_q)$.
   the exceptional case $l=2$, $q \equiv 3$ (4) by element. methods; coh. of that screwy dihedral group
   extra-special p groups, p odd e.g. Heisenberg group.

2. The splitting $G = \operatorname{Im} J \times \operatorname{Coker} J$ via the symmetric and the finite general linear groups. Any relation with Arf-invariant?

3. Stability thm: Are 2+3 cells required for getting $BGL(A)^+$ from $BGL(A)$ related to the Steinberg gen + relations?

4. You result that each embedding $N \hookrightarrow N$ acts trivially on $BGL(A)^+$, does it extend to a fibre-homotopy trivialization of $EM \times_M (BGL(A)^+) \longrightarrow BM$ ($M = \mathrm{Hom.inj.}(N,N)$). Analogues of contractibility of unitary group + relation to Karoubi theory of derived functors.

Questions:

What is the cobordism theory with structural group $Gl_\infty(\mathbb{F}_q)$?

$$\underset{\to}{S}(BU) \longrightarrow MU$$

$$BU \xrightarrow{\psi^q - 1} BU \qquad \text{all off } p$$

where $\underset{\to}{S}$ denotes suspension spectrum. Since $(\psi^q-1)x$ is sph. trivial one should be able to trivialize it over BU once and for all, thus obtaining a map of the suspension spectrum of BU into MU. Thus there should be an interesting (perhaps even canonical) map

$$\pi^*(\cancel{X}, BU) \longrightarrow U^*(X)\,! \qquad \underline{\text{off } p}.$$

Unitary group

start with $Gl_n(\mathbb{F}_{q^2})$

$$\mathbb{F}_q \xrightarrow[2]{} \mathbb{F}_{q^2}$$

so it has a conjugation — and I can form the unitary gp., i.e. $A \ni$

$$A\bar{A}^t = I$$

so diagonal matrices are fixed

Thus σ perhaps is transpose inverse together with

$$\bar{a} = a^q$$

so suppose $A^{(q)} = (A^t)^{-1}$

ie have $\theta(A) = (A^t)^{-1}$

and so as $\theta^2(\cancel{A}) = id$ $\quad F\theta = \theta F$

if $\theta(A) = F(A)$

then $F^2(A) = F(\theta A) = \theta^2 A = A.$

---

thus $\sigma = F\theta = \theta F.$

~~and on diagonal matrices~~ torus of diagonal matrices is stable. and on G_m one get

$$F\theta a = a^{-q} = (a^q)^{-1}$$

G(T1, T2) = a. q[(T1 + T2)q+1 - Tq+1 - Tq+1]

$$\omega_t(\psi^2 E - E) = \omega_t(\psi^2 E)/\omega_t(E)$$
$$= \frac{1}{\omega_t(E)}.$$

work with $\quad id - \psi^2$

and then $\quad (id - \psi^2)(\omega_t) = \omega_t$

so $\underbrace{id - \psi^{2^m}}_{\text{(fiber }F\text{ of}}$ has no 2-torsion $\quad H^*(F, \mathbb{Z}_2) = 0.$

so you ~~locally~~ localize ~~at 2~~ by adjoining $\frac{1}{2}$
do this also for Groth. gps + you kill mod 2 invariants.

Conjecture: $RO_K \otimes \mathbb{Z}[\frac{1}{2}]$ makes good sense in characteristic 2, also RSp and both coincide with self-conjugate theory.

The point is that this is obviously true in char $\neq 0$, and so what we are saying is that this non-deep aspect remains ~~to~~ true even in char 2.

(T + bεq Tq+1)*(T1 + T2) = T1 + T2 + bεq[(T1 + T2)q+1 - Tq+1 - Tq+1] generalizing to $O_n(\mathbb{F}_q)$, need to do the monodromy situation?

So the problem is that the standard representation of $O_n(\mathbb{F}_q)$ gives ~~[illegible]~~ an element of $RO(O_n(\mathbb{F}_q))$ which is stable under Ψ^{q^n} for n large (multiplicatively). Unfortunately it is not stable under Ψ^q?

$\mathbb{Z}/2$ cyclic of order.

$$\begin{array}{ccccc} [BG, J] & \longrightarrow & [BG, BO] & \xrightarrow{\psi^q - id} & [BG, BO] \\ \nearrow & & & & \\ [BG, SO] & & & & \end{array}$$

$KO^{-1}(BG)$ $\qquad$ G finite

$\widetilde{KO}(\Sigma(BG))$

$$\begin{array}{ccc} BG_\nu & \longrightarrow & BO \\ \downarrow & & \\ BG & & \end{array}$$

$$0 \to R^1 \underrightarrow{\lim} [BG_\nu, O] \to [BG, BO] \to \lim [BG_n, BO] \to 0$$

BG_ν

orthogonal group.

$$\begin{array}{c} J \\ \downarrow \\ BO \\ \downarrow {\scriptstyle \Psi^q - id} \\ BSO \end{array}$$

If l is odd, then one knows that $BSO \to BO$ is a mod l equivalence; these are H-spaces so there should be no real problem with the mod. l theory.

actually you replace J by SJ. Idea is that

Note since $O_n / O_n(\mathbb{F}_q) \not\simeq O_n^t$

you have to be careful in defining J. Thus J should be the fibre of

$$SO \longrightarrow J \longrightarrow BO \xrightarrow{\Psi} BSO$$

so can use E-M hopefully since base is connected. In any case can use Serre. So you get your bound on the cohomology of J. On other hand you have

$$\Phi(w_2), \ldots, \Phi(w_n), \ldots \in H^*(J)/H^*(BO)$$

so you want to determine restrictions to $BO_n(\mathbb{F}_q)$. So you get elements

$$\phi^*(w_1), \ldots, \phi^*(\Phi(w_2)) \ldots\ldots$$

which you want to understand and in particular compare with the elements of $H^*(BO_n(\mathbb{F}_q))$ I already know. Now my idea was to use the various $A_i \subset O_n(\mathbb{F}_q)$

$$\begin{tikzcd} EQ \times_Q Q^c \arrow[r] \arrow[d] & BQ^I \arrow[d] \\ BQ \arrow[r, "\Delta"] & BQ \times BQ \end{tikzcd}$$

$$\begin{tikzcd} BQ \times Q \arrow[r, Rightarrow] \arrow[d, "pr_1"] & J \arrow[d] \\ BQ \arrow[r] & BSO \end{tikzcd}$$

Take standard orthogonal representation of Q in O_n then over $BQ \times Q$ you must construct

for each $q \in Q$ can conjugate by q producing a different trivialization

---

$$\begin{tikzcd} H^*_G(G^t) \arrow[r, "res"] \arrow[d, equal] & H^*_Z(Z^t) = H^*_A(A) \\ H^*_{G^\sigma}(pt) & G/G^\sigma \end{tikzcd}$$

$$\begin{tikzcd} H^*_G(G^t) \arrow[r] & H_Z(Z^t) \\ H^*_{G^\sigma}(pt) \arrow[u, "\cong"] & \end{tikzcd}$$

G

Still have to describe

$$H^*(BSO_n(\mathbb{F}_8), \mathbb{Z}/2) \text{ multiplicatively.}$$

now somehow you must ~~...~~ use the various 2-tori sitting inside.

what we know is that

$$H^*(BSO_n(\mathbb{F}_8)) \twoheadrightarrow H^*(SO_n)$$

and that the former is free over $H^*(BSO_n) = \mathbb{Z}_2[w_2, \ldots, w_n]$

So how about Φ.

$$\Phi : H^*(BSO_n) \longrightarrow H^{*-1}(BSO_n(\mathbb{F}_8)) / H^{*-1}(BSO).$$

$$\downarrow$$

$$H^{*-1}($$

So if $A_1, \ldots, A_s$ are the different max. 2 abel gps. up to conjugacy how do I show that

$$H^*(BSO_n(\mathbb{F}_8)) \longrightarrow \prod_{i=1}^{s} H^*(BA_i)$$

is injective and how do I determine the image!

---

geometric Φ.

$$\begin{tikzcd}
BQ \times Q \arrow[r] \arrow[d] & J \arrow[d] \\
BQ \arrow[r, "f"] \arrow[d, shift right, "id"'] \arrow[d, shift left, "id"] & BSO \arrow[d, shift right, "id"'] \arrow[d, shift left, "\Psi^8"] \\
BQ \arrow[r, "f"] & BSO
\end{tikzcd}$$

orthogonal group and restriction to 2-torus

q odd. then ψ^q = identity on mod 2 cohomology of $H^*(BO)$.

and similarly for BSO.

$$\begin{array}{ccc} O_n/O_n(\mathbb{F}_q) & \longrightarrow & SO_n \\ \| & & \\ SO_n/SO_n(\mathbb{F}_q) & & \end{array}$$

$$\begin{array}{ccccc} H^* & \xrightarrow{\cdot w_1} & H^*(BO_n(\mathbb{F}_q)) & \longrightarrow & H^*(BSO_n(\mathbb{F}_q)) \\ & & \Uparrow & & \\ & & \mathrm{Ext}_{H^*(O_n)}(\mathbb{Z}/2, H^*(SO_n)) & \longrightarrow & \mathrm{Ext}_{H^*} \\ & & \| & & \end{array}$$

$$0 \longrightarrow BSO_n(\mathbb{F}_q) \longrightarrow BO_n(\mathbb{F}_q) \longrightarrow B\mathbb{Z}_2 \longrightarrow 0$$

$$H^p(\mathbb{Z}_2) \otimes H^q(BSO_n(\mathbb{F}_q)) \Longrightarrow H^{p+q}(BO_n(\mathbb{F}_q))$$

need to know

$$H^*(BO(\mathbb{F}_q)) \longrightarrow H^*(BSO_n(\mathbb{F}_q))$$

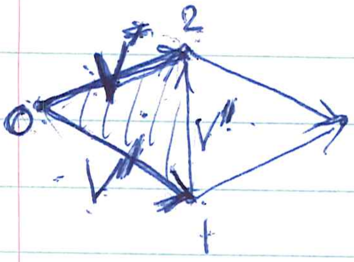
which will ~~follows~~ follow if you ~~...~~ determine ~~...~~ the generators of the latter.

$$H^*_{G^\sigma}(pt) \simeq H^*_G(G/G^\sigma) \xleftarrow{\simeq} H^*_G(G^t)$$

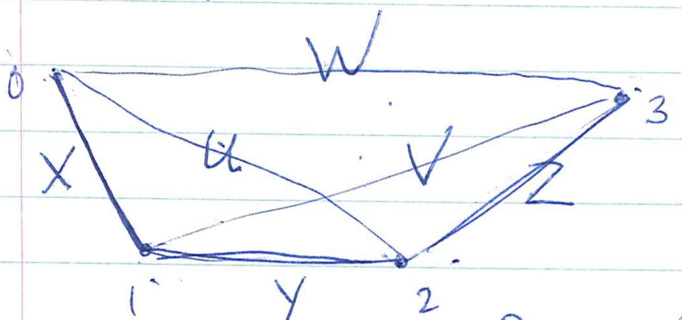
$$G/G^\sigma \longrightarrow G^t \longleftarrow A$$

thus all A_i are conj in G

1 cocycle



$$0 \to V' \to V \to V'' \to 0$$



$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X \arrow[r] \arrow[d, equal] & U \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & Z \arrow[r, equal] \arrow[d] & Z \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

$$0 \longrightarrow V_{i_0 i_1} \longrightarrow V_{i_0 i_2} \longrightarrow V_{i_1 i_2} \longrightarrow 0$$

$i_1 i_2 i_3$ $\quad V_{i_1 i_2} \otimes V'_{i_2 i_3}$

$i_0 i_2 i_3$ $\quad V_{i_0 i_2} \otimes V'_{i_2 i_3}$

$i_0 i_1 i_3$ $\quad V_{i_0 i_1} \otimes V'_{i_1 i_3}$

$i_0 i_1 i_2$ $\quad \not\cong \quad V_{i_0 i_1} \otimes V'_{i_1 i_2}$

$$\begin{tikzcd}
V_{i_0 i_1} \otimes V'_{i_1 i_2} \arrow[r] \arrow[d] & V_{i_0 i_2} \otimes V'_{i_1 i_2} \arrow[d] & V_{i_1 i_2} \otimes V' \\
V_{i_0 i_1} \otimes V'_{i_1 i_3} \arrow[r] \arrow[d] & {} & \\
V_{i_0 i_1} \otimes V'_{i_2 i_3} \arrow[r] & V_{i_0 i_2} \otimes V'_{i_2 i_3} \arrow[r] & V_{i_1 i_2} \otimes V'_{i_2 i_3}
\end{tikzcd}$$

~~And a 2-cocycle is the~~

and therefore a 2-cocycle should be a function associating to each (ordered) 2-simplex $i_0 i_1 i_2$ a vector space $V_{i_0 i_1 i_2}$ such that whenever $i_0 i_1 i_2 i_3$ form a three simplex one gets an exact sequence

$$0 \longrightarrow V_{i_1 i_2 i_3} \longrightarrow V_{i_0 i_2 i_3} \longrightarrow V_{i_0 i_1 i_3} \longrightarrow V_{i_0 i_1 i_2} \longrightarrow 0$$

and one still has to specify compatibility conditions $i_0 i_1 i_2 i_3 i_4$

$V_{i_1 i_2 i_3}$ $\qquad i_0 i_1 i_2 i_3 i_4$

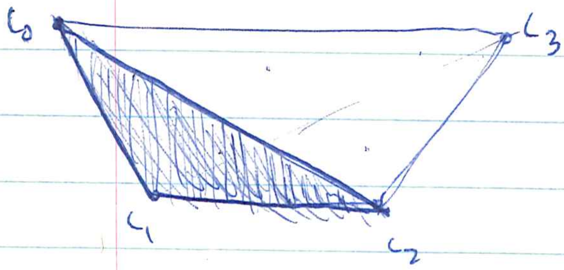
5

4

$\frac{5(5-1)}{2}$ $\quad 10$

$\boxed{\text{2 cocycle}}$ to every 2 simplex ${}^{\iota_0\iota_1\iota_2}$ get $V_{\iota_0\iota_1\iota_2}$

3 simplex $\iota_0 \cdots \iota_3$



$$0 \longrightarrow V_{\iota_1\iota_2\iota_3} \longrightarrow V_{\iota_0\iota_2\iota_3} \longrightarrow V_{\iota_0\iota_1\iota_3} \longrightarrow V_{\iota_0\iota_1\iota_2} \to 0$$

Euler: faces $-$ edges $+$ vertices $= \chi$

Cocycle: $(\delta f)(x_0, x_1, x_2) = f(x_1, x_2) - f(x_0, x_2) + f(x_0, x_1)$

SAVE

plus a higher homotopy connecting up

$$\mathrm{Aut}(P_0 \subset P_1 \subset P_2) \to \mathrm{Aut}(P_0) \times \mathrm{Aut}(P_1/P_0 \subset P_2/P_0)$$

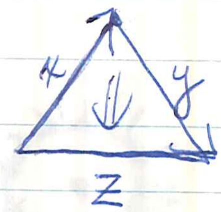
$$\mathrm{Aut}(P_0 \subset P_1 \subset P_2) \to \mathrm{Aut}(P_0 \oplus P_1/P_0 \subset P_2)$$

how about $P_0 \subset P_1 \subset P_2$

$$0 \to P_0 \to P_1 \to P_2 \to 0$$

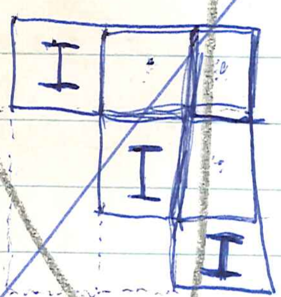
vertices P_1 and $P_0 \oplus P_2$

~~And I also h~~



$P_0 \to P_1$ 3

$$P_0 \subset P_1 \subset P_2$$



$$\Gamma_{mnp} \hookrightarrow G_{m+n+p}$$

I see at least 4 different maps corresponding to the modules

P_2

$P_1 + P_2/P_1$

$P_0 + P_2/P_0$

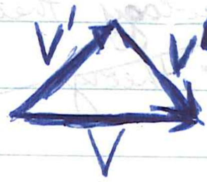
$P_2/P_1 + P_1/P_0 + P_0$

$P_1 \times_{P_1/P_0} P_2/P_0$

$$P_0 \subset P_1 \subset P_2$$

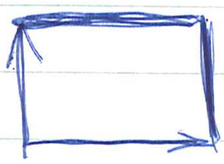
I have homotopies which I want to fill in with a square

to each one simplex one has a vector V
and two each 2-simplex an exact sequence



ordering is necessary

and ~~then~~ suppose one has another complex Y
with a k-vector space W for each 1-simplex giving an
exact sequence for each 2 simplex.
Then on the product $X \times Y$ what does one get



what we are trying to generalize is the cup product

$$Z^p_N(X, A) \times Z^q_N(Y, B) \longrightarrow Z^{p+q}_N(X \times Y, A \otimes B)$$
$$\|$$
$$\mathrm{Hom}(X, K(A,p)) \times \mathrm{Hom}(Y, K(B,q)) \longrightarrow \mathrm{Hom}(X \times Y, K(A \otimes B, p+q))$$

ie $$K(A,p) \overset{\times}{\otimes} K(B,q) \longrightarrow K(A \otimes B, p+q) \,!$$

thus given a 2 simplex in $X \times Y$ i.e.

Euler char.

$K(1)$ has 2 simplices $0 \to V' \to V \to V'' \to 0$

$K(1) \otimes K(1) \longrightarrow K(2)$

~~in dimension~~

put a ring structure on

V W $V \otimes W$

R S $R \otimes S$

suppose given a

2simplex $V_0 \to V_1 \to V_2$ 2-simplex $W_0 \to W_1 \to W_2$

get a 9x9 diagram

$$\begin{array}{ccccc}
V_0 \otimes W_0 & - & V_1 \otimes W_0 & \to & V_2 \otimes W_0 \\
\downarrow & & | & & | \\
V_0 \otimes W_1 & - & V_1 \otimes W_1 & - & V_2 \otimes W_1 \\
\downarrow & & | & & | \\
V_0 \otimes W_2 & - & V_1 \otimes W_2 & - & V_2 \otimes W_2
\end{array}$$

$\cancel{[V_0 \otimes W_0]}$

$\varphi(X) = [X, K(1)] = K^{+1}(X)$ $K^0(X) = \varphi(\Sigma X)$

think of assigning V to each 1 simplex

~~order~~

$$\sigma^*(c_i) = q^{-i} c_i$$

rather than the other way with

$$F^*(c_i) = q^i c_i$$

unitary gps!

$$1 \to \overline{B^\sigma} \to G^\sigma \to (G/B)^\sigma \to 1$$

$$\uparrow$$

$$(q-1)^n$$

zwischenzug

Suppose $H^*(G)$ ~~[illegible]~~ satisfies Borel's condition so that $H^*(BG) \cong S(P)$ where P stable under σ. Then

G reductive & linear $\quad H(BG) \hookrightarrow H(BT)$

so you must analyze the spectral sequence in an ~~[illegible]~~ ~~asinon~~ asinine way.

$H^*(G)$ satisfies Borel's condition, hence $H^*(G) = V(P)$ where P = primitive elements of $H^*(G)$, and $H^*(BG)$

$$H^*(BG) \longrightarrow H^*(BT) \twoheadrightarrow H^*(G/T)$$

so is it clear that $H^*(BG)$ is generated by universal algebraic cycles?

is it possible to prove the existence of invariants

So suppose that $x \in [QH^*(BG)]^\sigma$

Is there any chance that

$$H^*(BG)^\sigma \twoheadrightarrow QH^*(BG)^\sigma$$

point of your simplicial construction is that it replaces a functorial construction

Question: If A is an abelian monoid is it clear that $K(A,q+1) = BK(A,q)$ in the sense of homotopy theory

what is a $K(A,2)$? Answer:

$$\mathrm{Hom}(\Delta(q), K(A,2)) = Z^2_N(\Delta(q), A)$$

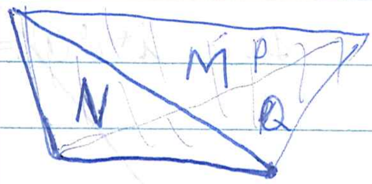
~~is a compatible family of 2 simplices on the faces of $\Delta(q)$~~
is a function which assigns to each 2 simplex of $\Delta(q)$ i.e. sequence $0 \leq i_0 < i_1 < i_2 \leq q$ an element of A such that the cocycle condition holds on 3 simplices. $a_{i_0 i_1 i_2}$

ie $$a_{i_1 i_2 i_3} + a_{i_0 i_1 i_3} = a_{i_0 i_2 i_3} + a_{i_0 i_1 i_2}$$

$$K(A, q+1) = \overline{W} K(A,q) \qquad \text{E-M.}$$

$\mathrm{Hom}(\Delta(q), K(A,2))$

~~true tri~~



$$N \oplus M \cong P \oplus Q$$

$$\begin{tikzcd} & P \arrow[d] & \\ N \arrow[r] & 0 \arrow[r] \arrow[d] & M \\ & Q & \end{tikzcd}$$

$\mathrm{Hom}(\Delta(q), K(A,2))$.

Think of as an Euler characteristic of length 4.

~~So I prefer the idea~~

Start with the idea of an $\boxed{\text{Euler characteristic}}$ as the basic animal.

So we ~~~~ started with 2 simplices of $K(A,1)$ ~~maps~~ as ~~simplices~~ exact sequences

$$V' \longrightarrow V \longrightarrow V''$$

$K(A,0)_0$ = Objects

$K(A,1)_1$ = Objects 2 simplex = 3 objects exact sequence

$K(A,2)_2$ = Objects

and the

so if $u \in H^2(Y \times Y, \mathbb{Z})$

$$\begin{tikzcd}
 & & Y \arrow[d, "\Delta"] \arrow[dr] & \\
 & B(\mathbb{Z}/l) \arrow[rr] \arrow[dd] & & S^\infty \arrow[dd] \\
X \arrow[rr, "(f,g)"] \arrow[dr] & & Y \times Y \arrow[dr, "mpr_1 - mpr_2"] & \\
 & BS^1 \arrow[rr, "l"] & & BS^1
\end{tikzcd}$$

Therefore we can assume that we have the square

$$\begin{tikzcd}
B(\mathbb{Z}/l) \arrow[r] \arrow[d] & S^\infty \arrow[d] \\
BS^1 \arrow[r, "l"] & BS^1
\end{tikzcd}
\qquad S^\infty / S^1$$

$\boxed{\text{Euler characteristics}}$ can't take into account the topology!

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$$

now try to take into account tensor product

Thus given a 1-cocycle ~~$f(i_0, i_1)$~~ $V_{i_0 i_1}$ on the ordered simplex $i_0 i_1$!! and another $W_{i_0 i_1}$ then we get a 2-cocycle

$$\cancel{V \otimes W} \qquad W_{i_0 i_1 i_2} = V_{i_0 i_1} \otimes V'_{i_1 i_2}$$

and if (i_0, i_1, i_2, i_3) is a 3 simplex then have

$$(i_1, i_2, i_3) \qquad W_{i_1 i_2 i_3} = V_{i_1 i_2} \otimes V'_{i_2 i_3}$$

$Gl_n(K)$ has an immeuble which is a simplicial complex whose simplices are the parabolic subgroups P of G $P \neq G$.

Let B be a Borel subgroup of G and consider the set of P containing B. I feel that such a P should correspond to the stabilizer of a flag

$$0 \subset V_1 \subset \cdots \subset V_\ell = V$$

idea is to consider P acting on G/B which will be a union of cosets PxB. Thus

$$P/B = \bigcup_{w \in W'} BwB$$

where W' is a subset of W. Therefore we see that there is a map

$$\{P\} \longrightarrow \text{subsets of } W!!!!$$

So we get an immeuble I, Serre claims that $I \sim \vee S_{r-1}$ $r =$ rank | paper of Tits-Solomon to appear

and that for $Gl_n(\mathbb{F}_q)$ the number of spheres is $q^{\frac{n(n-1)}{2}}$ and that the representation on

$$H^{r-1}(I)$$

is a canonical rep. constructed by Steinberg

$$\boxed{K_2(\mathbb{Q}) \longrightarrow K_1(\mathbb{Z}_p)}$$

Λ Dedekind domain, q.f. F

$$K_3(F) \longrightarrow \bigoplus K_2(\Lambda/\mathfrak{p}) \longrightarrow K_2(\Lambda) \longrightarrow K_2(F)$$
$$\longrightarrow \bigoplus K_1(\Lambda/\mathfrak{p}) \xrightarrow{\text{Mennicke}} K_1(\Lambda) \longrightarrow K_1(F)$$
$$\longrightarrow \bigoplus K_0(\Lambda/\mathfrak{p}) \longrightarrow K_0(A) \longrightarrow K_0(F) \to 0$$

exact sequence by Bass–Tate

completely exact for $\Lambda = \mathbb{Z}$

Bass–Tate hom.

$$K_q(F) \longrightarrow H^q(F, \mu_n^{\otimes q})$$

higher alg. K-theory.

$$\pi_i\left(\Omega B \coprod_{n\geq 0} B\,Gl_n(R)\right) = K_i(R) \qquad i \geq 1.$$

to make a theory one needs

a) ring structure if R- commutative

$\exists$ map

$$Gl_m(R) \times Gl_n(S) \longrightarrow Gl_{mn}(R\otimes S)$$

based on a ordering of $\{1,..,m\} \times \{1,..,n\}$.
It's effect must be understood.

b) ~~[illegible]~~ symmetric group operations

$$\Sigma_m \int Gl_n(R) \longrightarrow Gl_{n^m}(R)$$

$$V \longmapsto V^{\otimes m}$$

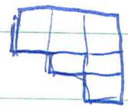
I better think of what goes on for bundles. ~~Thus we are given a bundle~~ $E\Sigma_m \times_{\Sigma_m} B\,Gl_n(R) \longrightarrow B\,Gl_{n^m}(R)$

which comes down to having a bundle E on X and forming the equivariant bundle $\bigoplus_{i=1}^{m} p_i^* E$ on X^m

---

~~Starting~~ with this ring R I form the ~~[illegible]~~ ultra-commutative H-space $\coprod_{n\geq 0} B\,Gl_n(R)$, which should give a trace theory and I wish to form the assoc. gen. coh. theory.

Perhaps you can reduce to line bundles. For line bundles $BG_1 = B(\mathbb{F}_q^*)$ and you know that mod l there is ~~only~~ exact one way of extending this to ~~$H_*(BU)$~~ higher bundles from your calculations. But can you make this geometric. Thus given a $Gl_n(\mathbb{F}_q)$ - bundle E_n do you have any idea how to split into line bundles?

Borel subgp has order  $q^{\frac{n(n-1)}{2}}(q-1)^n$ missing $n!$

Thus by Sylow theory I can suppose that E_n ~~is closed~~ has a system of imprimitivity. $E = f_* L$

$f : BT_n \longrightarrow BN_n$

and so the question is how to define

$\varphi(f_* L)$ except that it is simpler to obtain

How about $Gl_n(\mathbb{Q}_p)$ mod l

use that there is a subgroup G whose coh is trivial namely $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ mod G and there is no way to do this

---

how to get information on $Gl_n(\hat{\mathbb{Q}}_p)$ or $Gl_n(\hat{\mathbb{Z}}_p)$ ~~easy~~

$Gl_n(\hat{\mathbb{Z}}_p) \longrightarrow Gl_n(\mathbb{Z}_p)$ $\qquad Sl_2(\mathbb{Q}_p)$

all the cohomology Gl_n

so the idea is as follows free $\mathbb{Z}$-module

$Sl_2(\mathbb{Q}_p)$ - bundle is trivial if

$Sl_2(\mathbb{Q}_p) \hookleftarrow S_2(\mathbb{Z}_p)$

$\Gamma \backslash G / K$ ~~I understand them~~

$Gl_n(\mathbb{Z})$ at prime ℓ

want to examine elementary abelian ℓ-subgroups

$$Gl_n(\mathbb{Z}) \longrightarrow Gl_n(\mathbb{Z}[\ell^{-1}])$$

in $Gl_n(\mathbb{Z}[\ell^{-1}])$ $\exists$ exactly one elementary abelian ℓ-subgroup up to conjugacy!!

$$H^*\left(Gl_n(\mathbb{Z}[\ell^{-1}]), \mathbb{Z}_\ell\right)$$

$n=1$

$$Gl_{\ell-1}(\mathbb{Z}[\ell^{-1}]) \longleftarrow Gl_{\ell-1}(\mathbb{Z})$$

$$Gl_1(\mathbb{Z}[\zeta_\ell])$$

$[X, MB] \longrightarrow [X, B]$ $\quad 0$

~~in effect there is no other~~

$$\begin{array}{ccc} Y & \longrightarrow & B \\ \downarrow f & & \\ X & & B \end{array}$$

this idea is absurd unless Ψ^k fixes the standard repn. as an element of $R_{\mathbb{F}_q}(Gl_n(\mathbb{F}_q))$ and there's no reason unless $k = q^b$.

paradox: Suppose $v_2(q-1) = 2$. Consider the map

$$(*) \qquad BGl_n(\mathbb{F}_q) \longrightarrow BU$$

which one obtains from lifting the modular representation a la Brauer using an embedding $\phi: \bar{\mathbb{F}}_q^* \hookrightarrow \mathbb{C}^*$. Two such ϕ will differ by an element $k \in \left(\prod_{\ell \neq p} \hat{\mathbb{Z}}_\ell\right)^*$. Now the idea was that ψ^k would preserve the Brauer lifting provided k acts trivially on $\mathbb{F}_q^*$. In other words if $k \equiv 1 \mod (q-1)$ then ψ^k should leave $(*)$ invariant. (k acts on $\mathbb{F}_q^*$ by $\zeta \longmapsto \zeta^k$; it is thus mult. by k on a cyclic group of order $q-1$). But then ψ^k would be trivial on $K_{2i-1}(\mathbb{F}_q)$ ~~which ... is ...~~ on which ψ^k multiplies by k^i. Thus we would have to have $k^i \equiv 1 \mod (q^i - 1)$

This last condition of course gives trouble if $v_2(q-1) = 1$. In effect one ones that

$$v_\ell(q^i - 1) = v_\ell(i) + v_\ell(q-1)$$
$$v_\ell(k^i - 1) = v_\ell(i) + v_\ell(k-1)$$

except for $\ell = 2$ & $v_\ell(q-1) = 1$ or $v_\ell(k-1) = 2$. The example is to take $q \equiv -1 \mod 2^c$ and $v_2(k-1) = 2$. Then $c > 1 \Rightarrow v_2(q-1) = 1$ and yet

$$v_2(q^2 - 1) = c + 1$$

$$v_2(k^2 - 1) = 2 + 1 = 3$$

so we're in trouble if $c \geq 3$. (Note that k can be found such that $\quad k-1 \equiv 4 \ (8)$

$$k-1 \equiv 0 \ (q-1) \quad \text{i.e.} \ \left(\frac{q-1}{2}\right)$$

by Chinese remainder theorem in fact take

$$k-1 = 2(q-1) = 4\left(\frac{q-1}{2}\right) = 4\left(\underbrace{\frac{h2^c-2}{2}}_{\text{odd}}\right)$$

Take $q = 7$ and $k = 13$. Then $(q-1) \mid (k-1)$
and $(q^2-1) = 48 = 3 \cdot 16$ does not divide $k^2-1 = 168 = 8 \cdot 21$.

So what happens is that there isn't a canonical element of $R(Gl_n(\mathbb{F}_q))$ given a generator of $\mathbb{F}_q^*$. ~~More precisely we shall prove.~~

~~Proposition: If $v_2(q-1) \neq 1$, then given a generator of $\mathbb{F}_q^*$ there is a well-defined element of $R(Gl_n(\mathbb{F}_q))$ obtained by lifting the modular representation.~~

~~Proof.~~ Choose an alg. closure k of $\mathbb{F}_q$ and choose also an embedding $\phi : k^* \longrightarrow \mathbb{C}^*$ such that $\phi(\zeta) = \exp(2\pi i/(q-1))$. Then we can lift the modular representation V_k to a virtual representation $\phi(V_k) \in R(Gl_n(\mathbb{F}_q))$. If have another

$$\begin{tikzcd}
 & k^* \arrow[dd, dashed, "\theta"] \arrow[dr, "\phi"] & \\
\mathbb{F}_q^* \arrow[ur, hook] \arrow[dr, hook] & & \mathbb{C}^* \\
 & k'^* \arrow[ur, "\phi'"'] &
\end{tikzcd}$$

then $\exists$ field isomorphism $\theta \ni \quad (\phi'\theta)(x) = \phi(x)^b \quad x \in k^*$

where $b \in \varprojlim_j \mathbb{Z}^*_{(q^j-1)}$ and $b-1 \equiv 0 \quad (q-1)$.

θ permutes the eigenvalues of an element of G hence doesn't affect the sum of their liftings. Consequently $\Psi^b \phi(V_k) = \phi'(V_{k'})$. On the other hand ϕ commutes with Adams operations and

$$\Psi^b V = V \quad \text{in } R_k(G)$$

To see this last formula write $b = p^a \cdot b'$. Then b' is prime to $|G|$ hence

???

When is $\Psi^b V = V$ $\quad V =$ standard rep. of $Gl_n(\mathbb{F}_q)$ over $\mathbb{F}_q$

True if b is a power of q or if $b \equiv 1 \mod (q^j-1)$ all $1 \leq j \leq n$.

$\mathbb{G}_a \longrightarrow \mathrm{Aut}(V)$ can these be classified?
Jordan canonical form?

Work over Ω universal domain $/\mathbb{F}_2$.

Let $\xi \in \Omega$ be a generic element.

Then over Ω there is a change of coordinates such that

ξ

---

better take characteristic polynomial of ξ.

$$\xi^n - \mathbb{1} = 0.$$

$$0 \longrightarrow W \longrightarrow V \longrightarrow \mathbb{1} \longrightarrow 0$$

$$\begin{array}{ccccc} S^2W & \subset & VW & \subset & S^2V \\ & & \downarrow & & \downarrow \\ & & \mathbb{1}\otimes W & & \mathbb{1} \end{array}$$

$V = \mathbb{1} \oplus W$

$S^2V = \mathbb{1} \oplus \mathbb{1}\otimes W \oplus S^2W$

$$1 \longrightarrow W \longrightarrow S^2V/S^2W \longrightarrow \mathbb{1} \longrightarrow 0$$

any chance that ~~the same~~ split extension

$(e+w)^2 = e^2 + w^2$

so the point is clear

$$\varprojlim_{n} H^1(G(\mathbb{F}_{p^n}), V).$$

restrict to Borel subgroup $B = T \times N$

$$= \mathbb{G}_m \times \mathbb{G}_a$$

$$H^0(\mathbb{G}_m, H^1(\mathbb{G}_a, W))$$

so what can we say about

$$\cancel{H^1(\mathbb{G}_a, W) = 0}$$

$$H^1(\mathbb{F}_{2^d}, W)$$

as a representation of $(\mathbb{F}_{2^d})^*$

$$H^0(\mathbb{F}_{2^d}, V) \longrightarrow H^0(\mathbb{F}_{2^d}, 1) \xrightarrow{\delta} H^1(\mathbb{F}_{2^d}, W) \longrightarrow H^1(\mathbb{F}_{2^d}, V)$$

try casimir operator.

$V \longrightarrow$

v α

$$\begin{tikzcd}
0 \arrow[r] & S^2W \arrow[r] \arrow[d] & (V^{(2)}+S^2W) \arrow[r] \arrow[d] & \mathbb{1} \arrow[r] & 0 \\
0 \arrow[r] & VW \arrow[r] \arrow[d] & S^2V \arrow[r] \arrow[d] & \mathbb{1} \arrow[r] & 0 \\
 & W \arrow[r, equal] \arrow[d] & W \arrow[r] & 0 & \\
 & 0 & & &
\end{tikzcd}$$

I am hoping to do it in steps.

Suppose we find a good invariant f in S^2V. then do we have an invariant in $V^{(2)}+S^2W$

$$\frac{S^2V}{S^2W} \cong \underline{\mathbb{1} \oplus W}$$

hence we have an invariant in W $\ni$

$$f - \lambda w$$

$W_0 \subset W$

$W \longrightarrow V \longrightarrow \mathbb{1}$

$S^2(V/W_0)$

V ~~order~~ dim 2 base a, v

S^2V base a^2, av, v^2

invariants $\underline{a^2,\ v^2+av}$

so can't get it down into S^2W

$V^{(2)}+S_2W$

$0 \to W \to V \xrightarrow{\pi} \mathbb{1} \to 0$ choose $v \ni \pi v = 1$.

assume $G = \mathbb{Z}_2$ and $v \cdot v^\sigma \in S_2V \xrightarrow{p} W$

$v^2 + av$

$\sigma(v) = v + a$

$v = \sigma(v) + \sigma(a)$

$\sigma a = a.$

$$p(v \cdot v^\sigma)v = v \cdot v^\sigma - v^2 = av$$

$$p(v \cdot v^\sigma) = a$$

$$\cancel{\sum_a \left(\sum_{\substack{i \le \varepsilon \\ j > \varepsilon}} \gamma_{ija} \right) \geq (p-1)d}$$

$\cancel{H^i(GL_n \mathbb{F}_q) = 0}$

~~the simplicial side is now clear!!~~

$$\equiv \bigoplus_{n_1, n_2} H_*(G_{n_1 n_2}) \rightrightarrows \bigoplus H_*(G_n) \rightarrow k$$

N

$BN \longrightarrow X$

Mumford conj.

$0 \to W \to V \to 1 \to 0 \qquad H^1(G, W)$

Start with an alg. repn of $G = SL_2 \quad V$

Problem is to prove that

Can you show that $\quad H^1(G, V)$

$$\varprojlim_n H^1(G(F_{p^n}), V) = 0$$

✓

one-party, contiguous, ~~...~~
plain black phones and first + 2nd floors, 1 floor
wall-phone in kitchen if possible

---

triangular matrices. $H^*\left(\begin{smallmatrix} \infty & \\ & \infty \end{smallmatrix}\right) = 0$

This classifies characteristic classes for exact sequences.

$$0 \to E' \to E \to E' \to 0$$

$$0 \to E' \otimes \mathbb{F}_{q^r} \to E \otimes \mathbb{F}_{q^r} \to E' \otimes \mathbb{F}_{q^r} \to 0$$

zero for same reason.

$$\coprod_{n_1, n_2} BG_{n_1 n_2} \rightrightarrows \coprod_{n_1} BG_{n_1} \rightrightarrows pt$$

want universal covering:

$$\rightrightarrows \coprod_{n_0, n_1, n_2} \{n_0\} \times BG_{n_1 n_2} \rightrightarrows \coprod_{n_0, n_1} \{n_0\} \times BG_{n_1}$$

~~Also~~ Next I want to take homology. the point is simple enough: One takes

$$\begin{array}{c} \coprod_{n_1} BG_{n_1} \\ \downarrow \\ \mathbb{Z}^+ \end{array}$$

~~the idea~~

so we get a specific cobordism class which I want to complete!
any ideas at all? Idea is that I know what $U(G\times G)$ is very explicitly and the question becomes what happens to the F influence on $U(pt)$. For every space X $\exists$ endo.

$$\begin{tikzcd} X \arrow[rr, "F"] \arrow[dr] & & X \arrow[dl] \\ & pt & \end{tikzcd}$$

hence a map $F^*: U(X) \longrightarrow U(X)$ which leaves $U(pt)$ alone?

So now we have a basic duality for $U(G)$ ie we have

$$f_*: U(G) \longrightarrow U(pt).$$

and Poincaré duality. Thus we get a canonical element

$$\Gamma_* 1 \in U(G) \underset{U(pt)}{\otimes} U(G) \quad !!!$$
$$\parallel$$
$$U(G\times G)$$

hence a cobordism class of manifolds $X \longrightarrow G\times G$ of exactly half the dimension. So we get a map $X \longrightarrow G$ of degree 1 in cobordism. Now the idea is to make this as nice as possible

example: Find Shubert cycle representing $\Gamma_* 1$.

equivariant K. theory

set $F_G(\cancel{X}) = \underset{\sim}{K_G(X) \otimes \mathbb{Z}_p}$

~~C cyclic group R(C~~

A abelian compact Lie gp. $R(A) = \mathbb{Z}[\hat{A}]$

can go up to $U(n)$ down to T.
so I know that

$$F_G(X) \longrightarrow \varprojlim_{(A,\lambda)} F_A \qquad F\text{-isom}$$

if A runs over the category of ~~elementary~~ abelian groups.

Conjecture:

$$F_G(X) \overset{F}{=} \varprojlim_{(C,\lambda)} F_C$$

where C runs over all top. cyclic subgps $\ni [C:C_0]$ prime to p.

example G abelian $G = A \times T$

A finite abelian
T torus.

$$R(G)_p = R(A)_p \otimes \mathbb{Z}_p[t_1^{-1}, t_1, \ldots, t_n^{-1}, t_n]$$

In fact one knows that $R(A)_p = \prod\limits_{\text{cyclic subgroups prime to } p}$ ~~elements~~

Segal's theorem: If K is a ~~clsp~~ closed subgroup of a compact Lie group G and if $g \in G$ is not conjugate to an element of K, then $\exists$ $\chi \in R(G)$ with $\chi(K) = 0$ and $\chi(g) = 1$.

Special cases.

G finite and we want only $\chi(g) \neq 0$.

$$H \xrightarrow{i} G \xleftarrow{j} K$$

$$j^* i_* = \sum_{K g H} (i_g)_* j_g^*$$

$$\begin{tikzcd} K \cap gHg^{-1} \arrow[r, "j_g"] \arrow[d, "i_g"] & H \arrow[d, "i"] \\ K \arrow[r, "j"] & G \end{tikzcd}$$

So start ~~of~~ with $\chi_0 \in R(H)$, H = cyclic subgroup of G gen. by x_0 such that χ_0 is trivial on every proper subgroup of H. Then

$$j^*(i_* \chi) = 0, \quad \text{i.e.} \quad (i_* \chi)(K) = 0.$$

Now

$$i^* i_*(\chi_0) = \sum_{HgH} (i_g)_* j_g^* \chi_0$$

$$\begin{tikzcd} H \cap gHg^{-1} \arrow[r, "j_g"] \arrow[d, "i_g"] & H \arrow[d, "i"] \\ H \arrow[r, "j"] & G \end{tikzcd}$$

where

$$j_g^* \chi_0 = 0 \quad \text{if} \quad H \neq gHg^{-1}$$

Thus

$$i^* i_* (\chi_0) = \sum_{g \in N/H} \theta_{g^{-1}}^*(\chi_0) \quad \text{where } N = \text{normalizer of } H,$$

and $\theta_g : H \to H$ is conjugation by g. Now I have to check this sum.

Suppose H is cyclic of order n, and let χ_0 be $\ni$

$$\chi_0(h) = \begin{cases} M & h \text{ generates } H \\ 0 & h \text{ doesn't generate } H. \end{cases}$$

~~Thus~~ where M is an integer sufficiently large so that $\chi_0 \in R(H)$.
Then

$$\chi(g) = \left(\iota^*\iota_*(\chi_0)\right)(g) = [N:H]\,M \neq 0.$$

Now we want $\chi(g) = 1$. We consider

$$H \subset C \subset N$$

where C and N are the centralizer and normalizer of H, resp.
~~Then we see that~~

$$H \cong \mathbb{Z}/n\mathbb{Z} \qquad N/C \hookrightarrow (\mathbb{Z}/n\mathbb{Z})^*$$

and

$$\iota^*\iota_*(\chi_0) = [C:H] \sum_{x \in N/C} \theta_x^*(\chi_0)$$

We have to assign χ_0 on H_{gen} so this sum is nice?
Have to ~~Brauer's trick. Let P be a Sylow p-subgroup of C ... The point is that ...~~ think about characters of cyclic groups.

$R(\mathbb{Z}_n) = \mathbb{Z}[T]/(T^n - 1)$. I want elements of the form $\sum_{x \in \mathbb{Z}_n^*} x \cdot \chi$ with $\chi \in R(\mathbb{Z}_n)$. There is one interesting element

$$\sum_{x \in \mathbb{Z}_n^*} T^x \qquad\qquad T(g) = e^{\frac{2\pi i}{n}}$$

In fact one has a basis

$$\sum_{x \in \mathbb{Z}_n^*} T^{ax} \qquad \text{where } a \text{ runs over divisors of } n \quad ?$$

$H \subset G$ closed subgroup

X G manifold, X^H an N-manifold.

take embedding $Z \xrightarrow{i} X$

$$\begin{tikzcd} Z^H \arrow[r, "r"] \arrow[d, "i^H"'] & Z \arrow[d, "i"] \\ X^H \arrow[r] & X \end{tikzcd}$$

$$0 \longrightarrow \nu_{i^H} \longrightarrow \nu_i \longrightarrow \mu \longrightarrow 0$$

~~[illegible] the H-fixed sets, we can assume $X^H = X$~~

Look at $r^*\nu_i$ on Z^H. Then $r^*\nu_i$ is a N-bundle over Z^H and ν_{i^H} is the H-fixed set.

$$E^H \longrightarrow E \longrightarrow E/E^H$$

Z^H
(N/H)

N/H bundle without trivial components.

Therefore μ is an ~~[illegible]~~ N bundle on Z^H without H-trivial components and we get

$$r_X^*(i_* z) = (i^H)_*\left(\underline{e(\mu_i)}\, r_Z^* z\right)$$

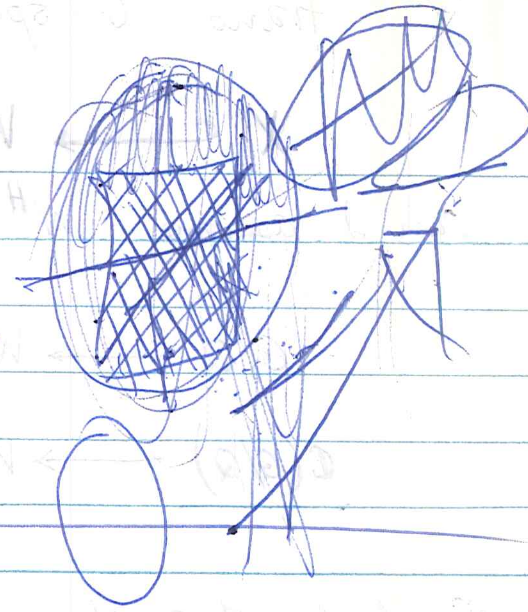
~~[illegible] can take~~

Situation ~~[illegible]~~ Classify N-~~[illegible]~~ bundles over an N/H space X.

(Classical is a proj. representation of N/H e.g. have N/H acting on PE, N acting on E H acts via scalars)

$$G/H \not\to G/Q$$

then can you find a $V \ni$



∇ $$G/Q \longrightarrow SV$$

but $$G/H \not\to SV.$$

$F(G/H)$. $\qquad$ $F(G/Q)$

$\neq 0$ $\qquad$ $= 0$

~~$F(G/H$~~

category of transitive G spaces

$F(G/H)$ $\qquad$ $F(G/Q)$

$G = H \times_\tau N$

to produce representations $V \ni V^N = 0$

take $W \in \hat{N} - 0$

Let H_W = stabilizer of W

$(H_W \times_\tau N)$

$$\begin{tikzcd} X \arrow[r, hook] & W \arrow[d] \\ & W/W^N \end{tikzcd}$$

If N normal in G, and if X is a G-space with no N-fixed points, then can embed

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, bend right, "f"'] & W \arrow[d] & G\text{-rep.} \\ & W/W^N & G\text{-rep.} \end{tikzcd}$$

Conclude f.D. for normal subgp N.

One sees that image of $X \xrightarrow{f} W/W^N$

doesn't meet 0 since $f(x) \in W^N$ then $N \cdot f(x) = 0$ so $N \cdot x = 0$

$X \overset{f}{\hookrightarrow} W = V_1 \oplus V_2$ X transitive G-space.

Then if $f(x) = (v, 0)$ then $f(X) \in V_1$.

Therefore take

$$\begin{tikzcd} & W' \arrow[d] \\ X \arrow[r, hook, "f"] & W \arrow[d, "p"] \\ & W'' \end{tikzcd}$$

take an equivariant embedding $X \overset{f}{\hookrightarrow} W$ and assume W generated by the image. Then take any irreducible part V of W and you have

$$\begin{tikzcd} W \arrow[r] & V \\ X \arrow[u] \arrow[ur, "\neq 0"'] & \end{tikzcd}$$

~~either~~ curious thing about a trans. G space if $X \hookrightarrow SW$, then equivariant cobordism problem.

$$\underbrace{F(X) \xrightarrow{\sim} F(X^H)}_{\substack{\text{not defined}\\ \text{as } G \text{ doesn't act on } X^H.}} \Longrightarrow \left\{ \begin{array}{l} e(V) \text{ invertible} \\ \text{for all } V \ni V^H = 0 \end{array} \right\}$$

If $(G/Q)^H = \phi \Longrightarrow V$ with $V^H = 0$ and $G/Q \to SV.$

try to induce from ~~H~~ Q to G.

$$\begin{tikzcd} H \cap gQg^{-1} \arrow[r, "\iota_g"] \arrow[d, "\iota_g"] & Q \arrow[d, "i"] \\ H \arrow[r, "j"] & G \end{tikzcd}$$

$(G/Q)^H = \phi \Longrightarrow HgQ = gQ$ imp.

$\Longrightarrow Hg \not\subset gQ$

$\Longrightarrow H \not\subset gQg^{-1}$

$\Longrightarrow H \cap gQg^{-1} < Q.$

$G \times_N X^H \longrightarrow X$

$(G/Q)^N = 0$

$F(X) \longrightarrow F(G \times_N X^H)$

$NQ \supset Q$

$N \supset Q \cap N$

$H \triangleleft G$

$F(X) \xrightarrow{\sim} F(X^H)$

$e(V)$ inv. all $V \in \hat{G} \ni V^H = 0.$

$H \triangleleft G$

$$\begin{tikzcd} H \cap Q \arrow[r, hook] \arrow[d, "g"] & Q \arrow[d] \\ H \arrow[r] & G \end{tikzcd}$$

Λ

Assume $Q \cap N < N$

can you produce a V with

$G/Q \longrightarrow S(V)$

and $V^N = 0$

?

$0 \longrightarrow N/_{Q \cap N} \to G/Q \longrightarrow G/\underline{QN}$

~~example: $N \neq H$. Then N acts trivially on X so~~

$$\cancel{K_N(X) = R(N) \otimes K(X)}.$$

~~and so $\mu = \bigoplus_{V \in \hat{N} - 0} V \otimes \mathrm{Hom}_N(V, \mu)$~~

example: $N = H$: then N ~~[illegible]~~ acts triv. on X so

$$K_N(X) = R(N) \otimes K(X)$$

and you have tom Dieck description for μ. But ~~you must kill all elements~~ to insure that restriction to H fixpoint set is an isom. I must kill ~~all~~ 1 on

G/Q with no H fixpoints

thus must kill the induction elements $\quad f^*1 = 0. \Rightarrow f_*1 = 0.$

$G/Q \longrightarrow pt.$

You invert all Euler classes $e(V)$ such that $S(V)^H = \phi$. or $V^H = 0$.

~~If~~ $F(X) \xrightarrow{\sim} F(X^H) \implies \left\{ \begin{array}{l} e(V) \text{ invertible all } V \in \hat{G} \\ \ni V^H = 0. \end{array} \right\}$

~~Also~~ The converse would be true if one knew that

$(G/Q)^H = \phi \implies \exists$ ~~[illegible]~~ V ~~[illegible]~~ with $V^H = 0$ and a map $G/Q \longrightarrow SV$.

ask Segal how to prove separation of ch.

$$\left\| \begin{array}{l} H \subset G \\ S \text{ conj set} \quad S \cap H = \phi \end{array} \right) \Rightarrow \exists\, \chi \text{ char } \ni \chi(H) = 0 \quad \chi(S) \neq 0.$$

December 23, 1969:

On the symmetric group. Recall the basic formula

$$H^*_G(X^k) \cong H^*(BG, H^*(X^k)) \qquad (\text{mod } p)$$

ie, the spectral sequence degenerates because of a canonical isomorphism. ~~[illegible]~~ The reason this is so is that there is an isomorphism in the derived category $H^\cdot(X) \simeq C^\cdot(X)$ hence also for $H^\cdot(X)^{\otimes k} \longrightarrow C^\cdot(X)^{\otimes k} \simeq C^\cdot(X^k)$ where we use a cell decomposition.

$G = \mathbb{Z}_2$: Then we get

$$(*) \qquad H^*_G(X^2) = \Gamma_2\{H(X)\} \oplus \bigoplus_{i>0} \omega^i \otimes H(X)^{(2)}$$

Now this formula should be stated with a bit more precision as follows. Let

$$Q: H^q(X) \longrightarrow H^{2q}_G(X^2)$$

be the external square. Then given $\lambda \in H^k_G(X^2)$ $\exists!$ elements $x_0 \in (\Gamma_2 H(X))^k$ and $x_i \in H^i(X)$

$$\lambda = x_0 + \sum_{2i<k} \omega^{k-2i} Q x_i$$

and moreover the elements x_i are independent. ~~Recall~~ ~~that~~

Recall that

$$\Delta^*: H^*_G(X^2)[\omega^{-1}] \xrightarrow{\sim} H^*_G(X)[\omega^{-1}]$$

by the Smith theorem argument. This means that (any element of) the kernel of $\Delta^*: H^*_G(X^2) \longrightarrow H_G(X)$ is killed by a power of ω. However it is clear ~~that~~ by $(*)$ that ~~only~~ $\Lambda_2 H(X) \subset \Gamma_2 H(X)$ is the kernel of multiplication by ω.

<u>Proposition:</u> There is an exact sequence

$$0 \longrightarrow H_G(X^2) \xrightarrow{(\iota^*, \Delta^*)} \Gamma_2 H(X) \oplus H_G(X) \cancel{\xrightarrow{F-\varphi} H(X)^{(2)} \to 0}$$

where ι^* is forgetting the G-action, ~~and~~ ~~$F: \Gamma_2 H(X) \to H(X)$ sends $x \otimes x$ to $x^{\otimes 2}$ and φ is some mysterious map.~~

~~<u>Proof:</u>~~

$$\begin{array}{ccccccccc}
0 \longrightarrow & (\Lambda^2 H(X))^q & \longrightarrow & H^q_G(X^2) & \xrightarrow{\Delta^*} & H^q_G(X) & \longrightarrow 0 & \text{wrong, take } X = (S^n, pt) \\
& \| & & \downarrow & & \downarrow \varphi & & \\
0 \longrightarrow & (\Lambda^2 H(X))^q & \longrightarrow & (\Gamma^2 H(X))^q & \longrightarrow & (H(X)^{(2)})^q & \longrightarrow 0 & \\
& & & \| & & \| & & \\
& & & H^q(X^2)^G & & \begin{cases} H^{q/2}(X)^{(2)} & \text{if } q \text{ even} \\ 0 & \text{if } q \text{ odd} \end{cases} & &
\end{array}$$

What is the map φ? Start with $x \in H^q(X)$, $Qx \in H^{2q}_G(X^2)$. Then in $H(X)^{(2)}$ you get $x^{\otimes 2}$, while $\Delta^* Qx = \sum \omega^{q-i} Sq^i x$.
Thus $\varphi(\sum \omega^{q-i} Sq^i x) = x^{\otimes 2}$?

<u>Corollary 1:</u> A mod 2 cohomology class of $\Sigma(n)$ is determined by its restrictions to the elementary 2-abelian subgroups. In particular $H^*(B\Sigma(n))$ has no nilpotent elements.

<u>Proof:</u> Restriction to ~~the~~ a Sylow subgroup is injective. If $n = \sum_{i=0}^{k} \varepsilon_i 2^i$ is the dyadic expansion, one knows that the Sylow subgroup is a product of those of symmetric groups of degree 2^i. If $n = 2^i$ then the Sylow subgroup T_i is a wreath product $(T_{i-1} \times T_{i-1}) \times_\sigma \mathbb{Z}_2$. There are two kinds of elementary 2-abelian subgroups: those of the form $\Delta_A \times \mathbb{Z}_2$ and otherwise $A \times B$. However by the proposition the map

$$H^*(BT_i) \longrightarrow H^*(BT_{i-1} \times BT_{i-1}) \times H^*(BT_{i-1} \times B\mathbb{Z}_2)$$

is injective. So done by induction

Situation for the prime p: One computes that

$$H^{2a+\varepsilon}(B\mathbb{Z}_p, H^*(X)^{\otimes p}) = \begin{cases} \eta^\varepsilon w^a \, H^*(X)^{(p)} & 2a+\varepsilon > 0 \\ \left(H^*(X)^{\otimes p}\right)^{\mathbb{Z}_p} & 2a+\varepsilon = 0. \end{cases}$$

In effect if e_i is a basis for V, then $e_{i_1} \otimes \cdots \otimes e_{i_p}$ is a basis for $V^{\otimes p}$ permuted by $\mathbb{Z}_p$. The orbits which are non-trivial give rise to induced $\mathbb{Z}_p$-modules, hence have trivial cohomology. The only part that counts therefore is the diagonal part. Again we find that

$$H^*_G(X^p) \xrightarrow{(i^*, \Delta^*)} H^*(X^p) \oplus H^*_G(X)$$

is injective. By the same induction as before we find that a mod p cohomology class of $\Sigma(n)$ is determined by its restrictions to the elementary p-abelian subgroups.

December 26, 1969.

Let us recall the basic power operation

$$P : H^q(X) \longrightarrow \prod_{k\geq 1}' H^{kq}(B\Sigma_k \times X)$$

(all mod 2)

$$\overset{\text{defn}}{=} \left\{ (\alpha_k)_{k\geq 1} \;\middle|\; \operatorname{res}^{\Sigma_{k+l}}_{\Sigma_k\times\Sigma_l} \alpha_{k+l} = \alpha_k \otimes \alpha_l \text{ for all } k,l \geq 1 \right\}$$

By means of Künneth we may view $\alpha_k \in H^*(B\Sigma_k \times X)$ as a homomorphism from $H_*(B\Sigma_k) \longrightarrow H^*(X)$. The conclusion is that

$$\prod_{k\geq 1}' H^*(B\Sigma_k \times X) = \operatorname{Hom}_{\text{rings}}\Big(\mathbb{Z}_2 + \bigoplus_{k\geq 1} H_*(B\Sigma_k), H^*(X)\Big)$$

where

$$R = \mathbb{Z}_2 + \bigoplus_{k\geq 1} H_*(B\Sigma_k)$$

has the product structure given by the natural map

$$H_*(B\Sigma_k) \otimes H_*(B\Sigma_l) = H_*(B(\Sigma_k * \Sigma_l)) \longrightarrow H_*(B\Sigma_{k+l}).$$

The problem is to determine the ring R.

Note that if σ is a generator of $H_0(B\Sigma_1) = \mathbb{Z}_2$ then

$$R/\sigma - 1 = H_*(B\Sigma_\infty).$$

Here is how one can construct generators of R. Observe that in addition to the ~~[illegible]~~ map $\Sigma_k \times \Sigma_\ell \hookrightarrow \Sigma_{k+\ell}$ corresponding to the union of sets, there is also the map $\Sigma_k \times \Sigma_\ell \longrightarrow \Sigma_{k\ell}$ given by the product of sets. In particular the iterate

$$(1) \qquad \underbrace{\Sigma_2 \times \cdots \times \Sigma_2}_{r\text{-times}} \longrightarrow \Sigma_{2^r}$$

gives a maximal elementary 2-abelian subgroup of Σ_{2^r}. So now take a dyadic partition, which we choose to write as a polynomial $\sum_{i>0} \varepsilon_i x^i$, where the ε_i are non-negative integers. Now associate to the monomial x^i the map

$$\delta_i : H_*(B\Sigma_2)^{\otimes i} \longrightarrow H_*(B\Sigma_{2^i})$$

given by the above map (1).

<u>Proposition:</u> R is generated (as an algebra) by the images of δ_i for $i \geq 0$ ($\delta_0 = \sigma$ by definition).

<u>Proof</u>: Given an integer n we know that $H_*(B\Sigma_n)$ is the sum of the images of $H_*(BA)$ where A runs over the elementary 2-abelian subgroups. Given such an A it comes from a (dyadic) partition $n = \sum_{i>0} \varepsilon_i 2^i + \varepsilon_0$ ($\varepsilon_0 = 1$ or 0) and hence by the union of the Klein groups (1) in Σ_{2^i} for the summands

in this partition. Summing in the partition corresponds to products in R, so we are done.

Review of work of Nakaoka:

1) $H^*(\Sigma_n) \longrightarrow H^*(\Sigma_k)$ surjective if $n \geq k$ and an isomorphism in a kind of stable range.

2) $H^*(\Sigma_\infty)$ is a Hopf algebra (comm. & cocomm.) and is a polynomial ring (mod 2) (false mod p)

3) $H_*(\Sigma_\infty)$ is a polynomial ring with known generators (free comm. alg. mod p). These generators are related to the sequences $Sq^{i_k} \cdots Sq^{i_1}$ that generate the cohomology of E-M spaces.

Nakaoka's method consists in using the Steenrod isom.

$$H^*_i(\Sigma_k) \xrightarrow{\cong} H^{nk-i}(S^{n\cdot k}/\Sigma_k) \qquad i<n$$

$$\lambda \longmapsto u^k/\lambda$$

as well as the Dold-Thom theorem and the known calculation of cohomology of E-M spaces.

From your point of view what's important is that 1) and 3) imply that R is a polynomial ring. Indeed $R/(\sigma-1) = H_*(\Sigma_\infty)$ is a polynomial ring and $\sigma-1$ is a non-zero-divisor. Indeed given $\sum_{i=0}^{n} x_i$, $x_i \in H_*(\Sigma_i)$ killed by $\sigma-1$, then $\sigma x_n = 0$ so $x_n=0$, etc. Since R is a poly.

ring ~~with~~ hopefully with an explicit set of generators; we might be able to describe $\prod'_{k\geq 1} H^*(B\Sigma_k \times X)$ as a kind of Witt ring of $H^*(X)$.

December 29, 1969: If V is a vector space over a field k, let

$$F(V) = \prod_{k\geq 1}' H^*(B\Sigma_k, V^{\otimes k})$$

Then $V \mapsto F(V)$ is an additive functor and moreover is endowed with a transformation

$$F(V)\otimes F(W) \longrightarrow F(V\otimes W)$$

compatible with associativities, etc. In fact

$$F(V) \cong F(k) \otimes_k V$$

where the k-module structure comes from

$$\lambda\cdot(\alpha_k) = (\lambda^k \alpha_k).$$

Now given a space X we have

(check this later)

$$P_{ext} : H^*(X) \longrightarrow \prod_{k\geq 1}' H^*_{\Sigma_k}(X^k) \cong \prod_{k\geq 1}' H^*(B\Sigma_k, H^*(X)^{\otimes k})$$

(is this composite $x \mapsto 1\otimes x$?)

$$\cong F(k)\otimes_k H^*(X)$$

This is a ring homomorphism.

The above is a bit too surprising; it is necessary to proceed cautiously.

$$F(V) = \prod_{k\geq 1}{}' H^*(B\Sigma_k, V^{\otimes k})$$

$$= \left\{ (\alpha_k)_{k\geq 1} \;\middle|\; \alpha_k \in H^*(B\Sigma_k, V^{\otimes k}) \right.$$

$$\left. \operatorname{res}^{\Sigma_{k+l}}_{\Sigma_k \times \Sigma_l} \alpha_{k+l} = \alpha_k \otimes \alpha_l \right\}$$

$$\text{if } k, l \geq 1$$

For example, suppose that we look at 0-dimensional classes

$$\alpha_k \in \Gamma_k V \longrightarrow \Gamma_i V \otimes \Gamma_{k-i} V$$

$$\alpha_k \longmapsto \alpha_i \otimes \alpha_{k-i}$$

$$\Longrightarrow \quad \alpha_k = \alpha_1 \otimes \cdots \otimes \alpha_1 \quad \in \Gamma_k V$$

$F(V)$ is evidently a functor of V. ~~Given~~ ~~Make~~ Define

$$(\alpha_k) + (\beta_k) = (\gamma_k) \quad \text{where}$$

$$\gamma_k = \alpha_k + \sum_{i=1}^{k-1} \operatorname{ind}^{\Sigma_k}_{\Sigma_i \times \Sigma_{k-i}} \alpha_i \otimes \beta_{k-i} + \beta_k$$

It is clear that $F(V)$ is an additive functor of V, in fact k-linear if we define

$$\lambda(\alpha_k) = (\lambda^k \alpha_k).$$

Define a basic map

$$\cancel{F(k) \otimes F(V) \longrightarrow F(k) \otimes F(V) \cong F(k \otimes V) \cong F(V)}.$$

$$F(k) \otimes V \longrightarrow F(V)$$

by using $\forall v \in V$ the unique k-linear map $k \to V$ sending 1 to v.

Now $F(k) = \prod'_{k \geq 1} H^*(B\Sigma_k) = \mathrm{Hom}_{\text{rings}}(R, k)$, so a basis for this should correspond to a system of polynomial generators for R.

<u>Question:</u> Let E be a (PL, maybe Top, G) bundle over X, say oriented, so that there is a Thom isomorphism

$$\iota_* : H^*(X) \xrightarrow{\sim} H_c^{*+d}(E)$$

in cohomology mod p. One defines the Wu classes of E by the formula

$$\iota_*\left(w_t(E)\right) = Sq_t(\iota_* 1).$$

Can one by use of the power operations procedure define the exotic classes?

The idea is that given $X \xrightarrow{i} E$ then we get an equivariant embedding $X^k \xrightarrow{\iota^k} E^k$ and hence an embedding

$$\iota_G^k : EG \times_G X^k \longrightarrow EG \times_G E^k$$

where $G \to \Sigma(k)$ is given. Now I want to take

$$\begin{array}{ccc} (\iota_G^k)_* 1 & \in & H^{2gk}_{p/BG}(EG \times_G E^k) \\ \downarrow & & \downarrow \Delta^* \\ \Delta^*(\iota_G^k)_* 1 & \in & H^{2gk}_{p/BG}(BG \times E) \end{array}$$

and then expand out using Künneth

$$\Delta^*(\iota_G^k)_* 1 = \sum a_i \otimes i_*(u_i)$$

where a_i is a basis for $H^*(BG)$, and where u_i are well-defined elements in $H^*(X)$. The question I am asking comes down to whether ~~or not~~ the elements u_i are exotic, that is, not obtainable from the Wu classes.

It seems unlikely that you get something new because

(?) $$(\iota_G^k)_* 1 = P_{ext}(i_* 1)$$

by the definition of P_{ext}. More carefully now suppose we work with cohomology modulo 2; then P_{ext} may be defined as follows: Given $x \in H_c(E)$, represent x by a ~~an~~ diagram

$$\begin{array}{ccc} & & Z \\ & & \downarrow f \\ E & \longrightarrow & Y \end{array}$$

where f is a smooth map of smooth manifolds. By PL-transversality, I may form the pull-back of f and this is a PL-submanifold $j: W \to E$ having the same ~~channel~~ class $\iota_*^H 1 = j_*^H 1$, where $i: X \to E$ is the zero-sections. Now we have maps

$$\begin{tikzcd} N^*(?) \arrow[r, "\alpha"] \arrow[rr, bend right] & MPL^*(?) \arrow[r, "\beta"] & H^*(?) \end{tikzcd}$$

(?) which should be compatible with the power operations This is certainly true for α. Suppose so, then as $\beta(\iota_*^M 1) = \beta(j_*^M 1)$ it follows that

$$P_{ext}(\iota_*^H 1) = P_{ext}(\beta \iota_*^M 1)$$
$$= \beta P_{ext}(\iota_*^M 1)$$
$$= \beta (\iota_G^k)_*^M 1 = (i_G^k)_*^H 1$$

The above is terribly confusing, but what you are ultimately up to is this. You want to know that

(?) $(*)$
$$\begin{array}{ccc} M^*(E) & \xrightarrow{\beta} & H^*(E) \\ \downarrow{\scriptstyle P^M} & & \downarrow{\scriptstyle P^H} \\ M^*(BG\times E) & \xrightarrow{\beta} & H^*(BG\times E) \end{array}$$

commutes, because then one knows that for the Thom class $\iota_*^H 1 = \beta \iota_*^M 1$ we have

$$P^H(\iota_*^H 1) = \beta P^M(\iota_*^M 1) \qquad (\text{by } *)$$
$$= \beta(\Delta^*(\iota_G^k)_*^M 1) \qquad (\text{defn. of } P^M)$$
$$= \Delta^*(\iota_G^k)_*^H 1 \qquad (\text{since } \beta \text{ commutes with Gysin})$$

This formula implies that when you expand $\Delta^*(\iota_G^k)_*^H 1$ you get the same as if you expanded $P(\iota_* 1)$, hence you get the squares applied the Thom class, and therefore you only get Wu classes.

The weakness with this argument is that there is no geometric reason for (*) to commute since the ~~[illegible]~~ Steenrod power operation in cohomology is defined quite differently from the geometric one of M^*. To understand this clearly suppose we decompose M^* with respect to N^* and use N^* throughout. So from now on

$$H^*(X) = \mathbb{Z}_2 \otimes_N N^*(X)$$

where $N = N^*(pt)$. The first task will be to understand the Thom homomorphism

$$M^*(X) \xrightarrow{\varepsilon} H^*(X).$$

In other words if E is a PL-bundle over X I have to construct a Thom class $\mu_E \in H^d(E, E-X)$, $d = \dim E$. However one knows that there is a unique such class restricting to a generator over each point. This one can prove by Mayer-Vietoris, the key point being that things happen only in one dimension.

From this point of view there is no reason why ε should commute with P_{ext}. In fact the construction of power operations in gen. coh. theories is mysterious. My hope now is that the use of power operations will provide me with enough invariants to generate the cobordism groups $M^*(pt)$.

December 27, 1969

Let M^* denote the unoriented symplectic cobordism functor and let $\beta : M^* \longrightarrow H^*$ be the Thom homomorphism, the unique transformation of gen. coh. theories preserving Thom classes for PL-bundles. I claim that ~~if E is a PL bundle over X, then~~ β commutes with power operations, i.e

$$\begin{array}{ccc} M^*(X) & \xrightarrow[P^M_{ext}]{} & M^*(EG \times_G X^k) \\ \downarrow \beta & & \downarrow \beta \\ H^*(X) & \xrightarrow[P^H_{ext}]{} & H^*(EG \times_G X^k) \end{array}$$

commutes for any space X. Start with $x \in$ ~~[illegible]~~ $M^*(X)$ and represent it in the form $f_* 1$ where $f : Z \longrightarrow X$ is a proper map of PL-manifolds. Then $P^M_{ext} x$ is represented by $EG \times_G f^k : EG \times_G Z^k \longrightarrow EG \times_G X^k$, and $\beta P^M_{ext} x$ is represented by $(EG \times_G f^k)_* 1$ in cohomology. We have to show that this is the same as $P^H_{ext}(f_* 1)$, $f_* 1$ being taken in cohomology. It is enough to consider the case of an embedding. ~~[illegible]~~ Indeed, factor f: $Z \xrightarrow{i} V \underset{p}{\overset{j}{\leftrightarrows}} X$. Then

$$j_* f_* 1 = i_* 1 \quad \text{in } M^*(V \text{ prop}/X). \text{ and also } H^*(V \text{ prop}/X).$$

so assuming true for an embedding, we have

$$\beta P^M i^M_* 1 = P^H i^H_* 1 = P^H(j^H_* 1 \cdot p^* f^H_* 1) = P^H j^H_* 1 \cdot p^* P^H f^H_* 1$$

$$\| $$

$$\beta P^M j^M_* f^M_* 1 = \beta P^M(j^M_* 1 \cdot p^* f^M_* 1) = \beta P^M(j^M_* 1) \cdot p^* \beta P^M f^M_* 1$$

~~Now $= EG \times_G P^H j^H_* \ldots = \beta P^M f^M_* 1$ by definition $\beta P^M(j^M_* 1)$~~

Now by definition $\beta P^M(j^M_* 1) = (EG \times_G j^k)^H_* 1$ is a Thom claim, which by assumption is equal to $P^H j^H_* 1$, whence we conclude from the string of equalities on the last page that

$$P^H f^H_* 1 = \beta P^M f^M_* 1.$$

Thus we are reduced to the case where f is an embedding, in fact to the **inclusion** $i: X \to E$ given by the zero section of a PL-bundle, and we have to show that $(EG \times_G \iota^k)^H_* 1 = P^H \iota^H_* 1$. But the former is the Thom class in H^* for the bundle $EG \times_G (E^k)$ over $EG \times_G X^k$, and it is characterized by restricting to a generator over each point of $EG \times_G X^k$. Now a point lifts to $EG \times X^k$ and as EG is contractible, it is enough to check that $P^H \iota^H_* 1$ restricts to the Thom class of E^k over X^k when G-action is forgotten. But when you forget G-action, $P^H \iota^H_* 1 = (\iota^H_* 1)^{\otimes k}$ which is clearly the Thom class. QED.

<u>Conclusion:</u> Use of power operations on the Thom class of a PL-bundle will only give you products of Wu classes.

December 31, 1969 (still groggy; since Dec 18)

I want to understand Nakaoka's calculation of the ring

$$R = \mathbb{Z}_2 + \bigoplus_{k\geq 1} H_*(B\Sigma_k) \qquad \text{(coeffs. mod 2)}$$

Now we know that for any ~~[illegible]~~ $\mathbb{Z}_2$-algebra A

(1) $$\prod_{k\geq 1}' H^*(B\Sigma_k)\otimes A = \mathrm{Hom}_{\text{rings}/\mathbb{Z}_2}(R, A)$$

inherits a natural ring structure, hence Spec R is a ring scheme. I recall that the addition in (1) is defined by

$$(\alpha_k) + (\beta_k) = (\gamma_k)$$

where

$$\gamma_k = \sum_{i=0}^{k} \mathrm{ind}_{\Sigma_i\times\Sigma_{k-i}}^{\Sigma_k} \alpha_i \otimes \beta_{k-i}.$$

Thus given maps $\alpha_k, \beta_k : H_*(B\Sigma_k) \longrightarrow A$ for $k\geq 0$ constituting ring homomorphisms, the sum γ is given by

$$H_*(B\Sigma_k) \xrightarrow{\Delta} \bigoplus_{i=0}^{k} H_*(B\Sigma_i \times B\Sigma_{k-i}) \xrightarrow{\sum \alpha_i\otimes\beta_{k-i}} A$$

This shows what the additive Δ must be. The multiplicative Δ is just the usual diagonal

$$H_*(B\Sigma_k) \longrightarrow H_*(B\Sigma_k)\otimes H_*(B\Sigma_k)$$

Proposition: R is a polynomial ring.

<u>Proof:</u> By Borel's theorem R is a tensor product of truncated polynomial rings. Now given any space X and class $x \in H^*(X)$ we get a ring homomorphism

$$P_x : R \longrightarrow H^*(X)$$
$$\lambda \in H_i(\Sigma_k) \longmapsto x^k/\lambda$$

Now by Steenrod's theorem

$$\begin{array}{ccc} H_i(\Sigma_k) \xrightarrow{\sim} H^{nk-i}(SP_k(S^n)) & & i<n \\ \lambda \longmapsto u^k/\lambda \qquad\qquad \downarrow & & \\ H^{nk-i}(K(\mathbb{Z},n)) & & i<n \end{array}$$

In other words ~~the map~~ if $u \in H^n(K(\mathbb{Z},n); \mathbb{Z}_2)$ is the fundamental class, then

$$Pu: R \longrightarrow H^*(K(\mathbb{Z},n))$$

is injective on $H_i(\Sigma_k)$ for $i<n$. Now one knows by Serre, that $H^*(K(\mathbb{Z},n))$ is a polynomial ring, hence one sees that no element of R can be nilpotent. By Borel, it follows that R is a polynomial ring.

A special case concerns the element $\sigma \in H_0(B\Sigma_1)$ which generates. Taking $x = 1 \in H^0(pt)$ one has that $\sigma \mapsto 1$. Now ~~the~~ Hopf algebra theory implies that R is free as a $\mathbb{Z}_2[\sigma]$-

module, but in fact one can carry this argument out even with coefficients which are not a field or even with H^* replaced by U^*, etc. (Dold, Decomp. theorems for $S(n)$-complexes Ann. of Math. 1962).

<u>January 1, 1969</u>: The following observation may be important. Recall that if V is a k-vector space we set

$$F(V) = \prod_{k\geq 1}{}' \, H^*(B\Sigma_k, V^{\otimes k})$$

(work mod 2 so as to avoid sign problems)

Then $F(V)$ is an additive functor from k-vector spaces to k-vector spaces, hence there is a canonical isomorphism

$$\varphi : F(k) \otimes_k V \xrightarrow{\sim} F(V)$$

Let X be a space. I claim that the following diagram commutes

$$\begin{tikzcd}
H^*(X) \arrow[r, "P_{ext}"] \arrow[d, "1 \otimes id"'] & \prod'_{k\geq 1} H^*_{\Sigma_k}(X^k) \arrow[d, "\Phi", "\wr"'] \\
F(\Gamma) \otimes H^*(X) \arrow[r, "\varphi", "\cong"'] & \prod'_{k\geq 1} H^*(B\Sigma_k, H^*(X)^{\otimes k})
\end{tikzcd}$$

where Φ is the canonical isomorphism constructed by Nakaoka and used by him to define the wreath product of cohomology classes. To see this note that the isomorphism φ is compatible with induction and restriction homomorphisms (see L. Evens paper) and consequently does give rise to a ring isomorphism. On the other hand by construction the diagram

$$\begin{tikzcd}
H^0(B\Sigma_k, H^*(X)^{\otimes k}) \arrow[r, "\psi_k"] \arrow[d, equal] & H^*_{\Sigma_k}(X^k) \\
\Gamma_k H^*(X) & \\
H^*(X) \arrow[u, "\gamma_k"] \arrow[uur, "P_{k,ext}"'] &
\end{tikzcd}$$

commutes. (More precisely, assume X is a cell complex so that $H^*_{\Sigma_k}(X^k)$ is the equivariant cohomology of $C^*(X)^{\otimes k}$. Then ψ_k is given by means of a quasi-iso. $\rho : C^*(X) \to H^*(X)$. If $f \in C^*(X)$ then $f^{\otimes k}$ in $C^*(X)^{\otimes k}$ represents $P_{k,ext}(\rho f)$ while $(\rho f)^{\otimes k}$ represents $\gamma_k(\rho f)$. It is now clear that the diagram commutes)

<u>Remark</u>: $\psi_k : H^*(B\Sigma_k, H^*(X)^{\otimes k}) \simeq H^*_{\Sigma_k}(X^k)$ is <u>not</u> compatible with passage to the diagonal. Indeed for $k = 2$ and $u \in H^*(X)$ we have

$$\begin{tikzcd}
H^*(B\Sigma_2, H^*(X)^{\otimes 2}) \ni u\otimes u \arrow[r] \arrow[d, "\psi_2", "\cong"'] \arrow[dr, phantom, "\text{not comm.}"] & H^*(B\Sigma_2, H^*(X)) \arrow[d, "\cong"] & u^2 \arrow[d, mapsto] \\
H^*_{\Sigma_2}(X^2) \arrow[r, "Q(u)"] & H^*_{\Sigma_2}(X) & \sum w^i Sq_i u
\end{tikzcd}$$

Here's how to produce a system of polynomial generators for R. Note that the indecomposable quotient space of R is dual to the primitive subspace of $R^{\vee} = \prod_{k\geq 1} H^*(B\Sigma_k)$. Thus for each k we want

$$PH^*(B\Sigma_k) = \bigcap_{i=1}^{k-1} \mathrm{Ker}\left\{H^*(B\Sigma_k) \xrightarrow{\text{res}} H^*(B\Sigma_i \times B\Sigma_{k-i})\right\}$$

for this to be non-zero it must be so that $k = 2^a$. Moreover the composition

$$(*) \qquad PH^*(B\Sigma_{2^a}) \hookrightarrow H^*(B\Sigma_{2^a}) \xrightarrow{\text{res}} H^*(B\mathbb{Z}_2^a)$$

is injective since the other maximal elementary 2-abelian subgroups of Σ_{2^a} are contained in subgroups of the form $\Sigma_i \times \Sigma_{2^a-i}$ $\quad 0<i<2^a$. (rest. to all 2-abel. subgps inj. by Dec 23.)

~~Proposition~~ Recall that the restriction homomorphism from Σ_{2^a} to $\mathbb{Z}_2^a$ has for image the invariants of $H^*(B\mathbb{Z}_2^a) = S(\mathbb{Z}_2^{a*})$ under $Gl(\mathbb{Z}_2\, a)$, which by Dickson's theorem is the poly. subring generated by $w_{2^a-2^i}(\text{reg}\, \mathbb{Z}_2^a)$ $\quad 0 \leq i < a$.

<u>Proposition</u>: The map $(*)$ has for its image the ideal in $\mathbb{Z}_2[w_{2^a-1}(\text{reg}), \ldots, w_{2^a-2^{a-1}}(\text{reg})]$ generated by $w_{2^a-1}(\text{reg})$.

<u>Proof</u>: Let Δ_{2^a} be the standard representation of Σ_{2^a}, so that

$$\text{res}^{\Sigma_{2^a}}_{\mathbb{Z}_2^a} \Delta_{2^a} = \text{reg}\, \mathbb{Z}_2^a$$

~~Then~~

$$\text{res}^{\Sigma_{2^a}}_{\Sigma_i \times \Sigma_{2^a-i}} \Delta_{2^a} = \Delta_i \boxplus \Delta_{2^a-i}\,.$$

Now Δ_k has a trivial 1-dimensional subrepresentation hence $res^{\Sigma_k}_{\Sigma_i \times \Sigma_{k-i}} \Delta_k$ has two trivial " " and hence its top two Whitney classes vanish. Thus

$$w_{2^a-1}(\Delta_{2^a}) \in PH(B\Sigma_{2^a})$$

and hence any multiple does also. Let $p \in PH(B\Sigma_{2^a})$. Then we can write uniquely

$$(**) \qquad res^{\Sigma_{2^a}}_{\mathbb{Z}_2^a}\, p = A_1(reg) + A_2(reg)\, w_{2^a-1}(reg)$$

where A_1 is a poly in $w_{2^a-2^{a-1}}, \dots, w_{2^a-2}$ and A_2 is a poly in $w_{2^a-2^{a-1}}, \dots, w_{2^a-1}$. I am going to show that A_1 is zero. Subtracting $A_2(\Delta)\, w_{2^a-1}(\Delta)$ from p we may suppose that $A_2 = 0$. Now let's recall how $\mathbb{Z}_{2^a}$ is embedding in Σ_{2^a}. Start with an embedding $\mathbb{Z}_{2^{a-1}} \xhookrightarrow{\iota_{a-1}} \Sigma_{2^{a-1}}$ and form the composite

$$\begin{tikzcd}
\Sigma_{2^{a-1}} \times \Sigma_2 \arrow[r, "\Delta \times id"] & (\Sigma_{2^{a-1}})^2 \times_{semi} \Sigma_2 \arrow[r, hook, "\text{wreath prod.}", "\text{embedding}"'] & \Sigma_{2^a} \\
\mathbb{Z}_{2^{a-1}} \times \mathbb{Z}_2 \arrow[u, "\iota_{a-1} \times id"] \arrow[urr, bend right, "\iota_a"'] & &
\end{tikzcd}$$

This implies that the following square is commutative

$$\begin{tikzcd}
\mathbb{Z}_{2^{a-1}} \arrow[r, "(\iota_{a-1}, \iota_{a-1})"] \arrow[d, hook] & \Sigma_{2^{a-1}} \times \Sigma_{2^{a-1}} \arrow[d, hook] \\
\mathbb{Z}_{2^a} \arrow[r, "\iota_a"] & \Sigma_{2^a}
\end{tikzcd}$$

and consequently that Δ_{2^a} restricted to $\mathbb{Z}_2{}^{a-1}$ is $\mathrm{reg}(\mathbb{Z}_2{}^{a-1}) + \mathrm{reg}(\mathbb{Z}_2{}^{a-1})$. Hence restricting $(**)$ to $\mathbb{Z}_2^{a-1}$

$$A_1(2 \cdot \mathrm{reg}\, \mathbb{Z}_2{}^{a-1}) = 0$$

But $\quad w_{2^a - 2^i}(2 \cdot \mathrm{reg}\, \mathbb{Z}_2{}^{a-1}) = w_{2^{a-1} - 2^{i-1}}(\mathrm{reg}\, \mathbb{Z}_2{}^{a-1})^2$ if $i \geq 1$. In other words ~~[illegible]~~ we have a polynomial $A_1(z_1, \ldots, z_k)$ with $A_1(x_1^2, \ldots, x_k^2) = 0$, where the x_i's are indeterminates. Thus $A_1 = 0$, which proves the proposition.

Corollary: $\quad PH^*(\Sigma_{2^a}) = H^*(\Sigma_{2^a}) \cdot w_{2^{a-1}}(\Delta_{2^a})$

Proof: Given a primitive element p we can remove a multiple of $w_{2^a - 1}(\Delta_{2^a})$ ~~which~~ so that the difference restricts to zero on $\mathbb{Z}_2{}^a$. The difference then restricts to zero on every elementary abelian 2-subgroup, hence is zero.