see Smith thm. at end.

Nostalgia: these papers represent original research for the paper on spectrum, begun October 15, 1969 at I.A.S., all ideas complete Nov. 5, 1969, paper finished Feb 22, 1971

November 5, 1969

Let G be a compact Lie group acting on a nice G-space X and let $\mathcal{I}_G(X)$ be the category whose objects are pairs (A,λ), where A is an elementary abelian p-subgroup of G and $\lambda \in [G/A, X]$, and where $\mathrm{Hom}((A,\lambda),(B,\mu))$ is the set of $\varphi \in [G/A, G/B]$ such that

$$\begin{tikzcd} G/A \arrow[d, "\varphi"] \arrow[dr, "\lambda"] & \\ G/B \arrow[r, "\mu"'] & X \end{tikzcd}$$

is homotopy commutative.

Define a functor (contravariant) from $\mathcal{I}_G(X)$ to (rings) by associating to (A,λ) the ring

$$H_G(G/A) \simeq H_A(pt)$$

and to $\varphi: (A,\lambda) \longrightarrow (B,\mu)$ the induced homomorphism

$$\varphi^*: H_G(G/B) \longrightarrow H_G(G/A)$$

It is clear that the collection of maps

$$\lambda^*: H_G(X) \longrightarrow H_G(G/A),$$

as (A,λ) runs over $\mathcal{I}_G(X)$, gives rise to a ring homomorphism

(1) $$H_G(X) \longrightarrow \varprojlim_{(A,\lambda)} H_A(pt)$$

We shall call a homomorphism $u: R \longrightarrow R'$ of rings of characteristic p an F-isomorphism if any element of ~~[illegible]~~ the kernel and cokernel of u is killed by a sufficiently high iterate of the Frobenius endomorphism. Note if this is the case, then

~~Theorem: The map (1) is an~~

$$u^*: \operatorname{Spec}(R') \longrightarrow \operatorname{Spec}(R)$$

is a homeomorphism, in fact, a universal homeomorphism (ref. [illegible], converse ?).

Theorem: The map (1) is finite and is an F-isomorphism.

The finiteness is clear, since $I_G(X)$ is equivalent to a finite category, and since ~~[illegible]~~ $H_A(pt)$ is already a finite module over the noetherian ring $H_G(pt)$ as $A \subset G$.

One knows that both sides do not change if we replace G by a larger group U and X by $U \times_G X$. Hence we may suppose G is a unitary group.

Let F be the flag manifold of a faithful representation. We must check compatibility with the descent situation

$$X \times F \times F \rightrightarrows X \times F \longrightarrow X$$

For $H_G(X)$ we have that

$$H_G(X) \longrightarrow H_G(X\times F) \rightrightarrows H_G(X\times F\times F)$$

is exact. Next we want to check that

$$\varprojlim_{G/A\to X} H_A(pt) \longrightarrow \varprojlim_{G/A\to X\times F} H_A(pt) \rightrightarrows \varprojlim_{G/A\to X\times F\times F} H_A(pt)$$

is exact. Injectivity is clear at the left since $F^A \neq \emptyset$ for any A. Now suppose we give a cohomology class $u_{A,\lambda}$ in $H_A(pt)$ for each $\lambda\in[G/A, X\times F]$ and elem. p-ab A, and that $u_{A,\lambda}=u_{A,\lambda'}$ if $pr_2\lambda = pr_2\lambda'$ where pr_2 is the projection $[G/A, X\times F]\to[G/A,X]$. Then clearly we get a $v_{A,\lambda}$ well-defined for each $\lambda\in[G/A,X]$. To check compatibility note that if $(A,\lambda)\to(B,\mu)$ is a map in $I_G(X)$, then it can be lifted to a map in $I_G(X\times F)$ since all we have to do is to choose the component in $\pi_0(F^A)$ to come from one in $\pi_0(F^B)$.

So in intelligible terms we are looking at

$$\begin{tikzcd}
0 & 0 \arrow[d] & 0 \arrow[d] \\
\prod_{\pi_0X^A} H_A \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \prod_{\pi_0X^A\times\pi_0F^A} H_A \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \prod_{\pi_0X^A\times\pi_0F^A\times\pi_0F^A} H_A \arrow[d, shift left] \arrow[d, shift right] \\
\prod_{\pi_0(G/B)^A\times\pi_0X^A} H_A \arrow[r] & \prod_{\pi_0(G/A)^B\times\pi_0X^A\times\pi_0F^A} H_A \arrow[r, shift left] \arrow[r, shift right] & \prod_{\pi_0(G/A)^B\times\pi_0X^A\times\pi_0F^A\times\pi_0F^A} H_A
\end{tikzcd}$$

vertical arrows exact by defn of $\varprojlim$

lower two horizontal arrows exact since $\pi_0(F^A)\neq\emptyset$.

So now we are reduced to the case of a torus G. Taking the product with $G/_pG$ is OK since

$$H^*_G(X \times G/G_p) \longleftrightarrow H^*_G(X)$$

is clearly an ~~exterior~~ algebra hence an F-isom. On the other hand $\pi_0((G/_pG)^A) = \pi_0(G/_pG) = 0$ since $A \subset_p G$ and G is connected.

Thus I may assume that all isotropy groups of X are elementary p-abelian. Now I claim that $\exists$ isom

$$\begin{tikzcd}
H^0(X/G, \mathcal{O} \longmapsto H^*_G(\mathcal{O})) \arrow[rr, "\cong"] & & \varprojlim_{G/A \to X} H^*_A(pt) \\
& H_G(X) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

such that the triangle commutes. Indeed ~~element of the left is a function assign to each orbit $\mathcal{O}$ an element a cohomology class observe that the left side is functorial in X since~~ what is the left side? It is the sheaf associated to the presheaf $V \subset X/G \longmapsto H^*_G(V)$, and the stalk of the sheaf on an orbit $\mathcal{O}$ is $H^*_G(\mathcal{O})$ ([ ]). An element $f \in H^0(X/G, \mathcal{O} \longmapsto H^*_G(\mathcal{O}))$ is thus a function assigning to each orbit $\mathcal{O}$ an element $f(\mathcal{O}) \in H^*_G(\mathcal{O})$, which is continuous is the sense that $\forall \mathcal{O}\ \exists\ V \supset \mathcal{O}$ and $g \in H_G(V) \ni$

$$f(\mathcal{O}') = g|\mathcal{O}' \text{ for all } \mathcal{O}' \in V.$$

Now note that the left side is functorial in X. Thus given $G/A \xrightarrow{g} X$ we get a map

$$g^*: H^0(X/G, \mathcal{O} \mapsto) \longrightarrow H^*_G(G/A)$$

and moreover this map ~~is in~~depends ~~of~~ only on the homotopy class of g since

$$H^0((G/A \times I)/G, \mathcal{O} \longmapsto\ ) = H^0(I, H^*_G(G/A)).$$

E_2^{0*} should satisfy the homotopy axiom.

More concisely denote by $E_2^{0*}(X) = H^0(X/G, \mathcal{O} \longmapsto H^*_G(\mathcal{O}))$. Then $E_2^{0*}(X)$ is a functor of X and

$$pr_1^* : E_2^{0*}((G/A) \times I) \xleftarrow{\sim} E_2^{0*}(G/A)$$

whence by a standard argument $E_2^{0*}(g)$ depends on the homotopy class of g.

Consequently there is a ~~map~~ diagram

$$\begin{tikzcd}
E_2^{0*}(X) \arrow[r, "\#"] & \varprojlim_{I_G(X)} E^{0*}(G/A) \\
H^*_G(X) \arrow[u] \arrow[r] & \varprojlim_{I_G(X)} H^*_G(G/A) \arrow[u, "\simeq"']
\end{tikzcd}$$

Lemma: $\#$ is an isomorphism. ~~More generally~~

Proof: ~~Injectivity is obvious since~~ Define a map in the ~~More generally~~ reverse direction by given $\mathcal{O}$ choose an isom $G/A \simeq \mathcal{O}$ and check independence of choices. Ultimately one uses that

~~[illegible]~~ around an O there is a U which deformation retracts strongly to O. This concludes the checking of the theorem.

Let $\mathfrak{p}$ be a minimal prime ideal of $H_G(X)$. Then

$$H_G(X)_{\mathfrak{p}} \longrightarrow \varprojlim_{(A,\lambda)} H_A(pt)_{\mathfrak{p}}$$

must also be an F-isomorphism. But

$$\begin{aligned} H_A(pt)_{\mathfrak{p}} \neq 0 &\iff \lambda^*(\text{complement of } \mathfrak{p}) \cap \mathfrak{r}_A = 0 \\ &\iff \mathfrak{p} \supset (\lambda^*)^{-1}\mathfrak{r}_A = \mathfrak{p}_{A,\lambda} \\ \text{so by minimality} \quad &\iff \mathfrak{p} = \mathfrak{p}_{A,\lambda}. \end{aligned}$$

Now we know that if ~~[illegible]~~ $f^*: (A,\lambda) \longrightarrow (B,\mu)$ is an map, then ~~[illegible]~~ $\mathfrak{p}_{B,\mu} \subset \mathfrak{p}_{A,\lambda}$ with equality iff f is an isomorphism. (this follows from the fact that ~~the~~ $\dim \mathfrak{p}_{B,\mu} = \text{rank } B$, hence rank A = rank B) Thus the right side

$$\varprojlim_{(A,\lambda)} H_A(pt)_{\mathfrak{p}} = \prod_{\mathfrak{p}_{(A,\lambda)} = \mathfrak{p}} \left\{ H_A(pt)_{\mathfrak{p}} \right\}^{N_{A,\lambda}}$$

and so there is exactly one (A,λ) with $\mathfrak{p}_{(A,\lambda)} = \mathfrak{p}$. Thus I find that minimal primes ~~[illegible]~~ correspond to ~~[illegible]~~ minimal connected strata.

Now we know by Cohen-Seidenberg and the essential injectivity of

$$H_G(X) \longrightarrow \prod_{A,\lambda} H_A(pt)$$

that the number of invariant primes is finite. This means we obtain a stratification of Spec $H_G(X)$ using the irreducible invariant subvarieties. I want now to check my conjecture for the structure of a strata for the maximal strata. So let ~~[illegible] choose an $f \notin p$~~ suppose the maximal stratum has generic point $p_{A,\lambda}$ and let f be an element of $H_G(X)$ not in $p_{A,\lambda}$ which is contained in all of the other invariant prime ideals. Thus $D(f)$ is a typical open affine in the maximal stratum in question. So again localizing with respect to f, we kill all $H_B(pt)$ where $p(B,\mu) \neq p(A,\lambda)$ and we get ~~that~~ an F isomorphism

$$H^*_G(X)_f \xrightarrow{\lambda^*} H_A(pt)_f^{N_{A,\lambda}} .$$

Thus at any point of the maximal stratum the ~~answer is~~ stalk is what it should be.

Let $\mathfrak{a}$ be any ~~prime~~ ideal in $H_G(X)$. I claim that

$$H_G(X)/\mathfrak{a} \longrightarrow \varprojlim_{(A,\lambda)\in I_G(X)} H_A(pt)/\mathfrak{a} H_A(pt)$$

is an F isomorphism. ~~Indeed it suffices to show that~~

$$\cancel{\varprojlim_{(A,\lambda)} H_A(pt) \otimes H_G(X)/\mathfrak{a} \longrightarrow \varprojlim_{A,\lambda} H_A(pt)/\mathfrak{a} H_A(pt)}$$

~~is an F-isomorphism, and this reduces to showing that if~~

This follows from

Lemma: Let $R \xrightarrow{f} R' \underset{q}{\overset{p}{\rightrightarrows}} R''$ be F-exact, where the maps are finite and R is noetherian. If $\mathfrak{a}$ is any ideal in R, then

$$R/\mathfrak{a} \xrightarrow{\bar{f}} R'/\mathfrak{a}R' \underset{\bar{q}}{\overset{\bar{p}}{\rightrightarrows}} R''/\mathfrak{a}R''$$

is also F-exact.

Proof: First we show the kernel of $\bar{f}$ consists of nilpotent elements. Let $\bar{f}(x+\mathfrak{a}) = 0$, then

$$f(x) = \sum f(a_i)\, r_i' \qquad a_i \in \mathfrak{a},\ r_i' \in R'$$

as f is F-surj ? $r_i'^{\,p^N} \in \operatorname{Im} f$ so

$$f(x^{p^N}) = \sum f(a_i^{p^N}) f(z_i) = f(a) \qquad a \in \mathfrak{a}$$

Thus $x^{p^N} - a \in \operatorname{Ker} f$ is nilpotent and so $x \bmod \mathfrak{a}$ is nilpotent.

Next suppose $y + \mathfrak{a}R'$ is equalized by $\bar{p}$ and $\bar{q}$. Thus $py - qy \in \mathfrak{a}R''$. By Artin-Rees $\exists\, N$ such that

$$\mathfrak{a}^N R'' \cap \operatorname{Im}(p-q) \subset (p-q)(\mathfrak{a}R')$$

hence as $(py-qy)^{p^r} = p(y^{p^r}) - q(y^{p^r}) \in \mathfrak{a}^{p^r} R''$, we have

$$p(y^{p^r}) - q(y^{p^r}) = (p-q)\, z \qquad z \in \mathfrak{a}R'$$

$$p(y^{p^r} + z) = q(y^{p^r} + z)$$

thus

$$y^{p^{r+s}} + z^{p^s} = f(x)$$

so $(y + \mathfrak{a}R')^{p^{r+s}} \in \operatorname{Im} \bar{f}$. qed.

Now let $\mathfrak{p}$ be a prime ideal in $H_G(X)$. Then we find that by dividing out by $\mathfrak{p}$ and then localizing that

$$k(\mathfrak{p}) \longrightarrow \varprojlim_{(A,\lambda)\in I_G(X)} H_A(pt)\otimes k(\mathfrak{p})$$

is an F-isomorphism. Now $H_A(pt)\otimes k(\mathfrak{p})$ is a finite $k(\mathfrak{p})$ algebra and so modulo F-isomorphism is a product of finite separable extensions of $k(\mathfrak{p})$. Form the flat base extension by $k(\mathfrak{p}) \longrightarrow \overline{k(\mathfrak{p})}$. Then we get an F-isomorphism

$$\overline{k(\mathfrak{p})} \longrightarrow \varprojlim_{(A,\lambda)\in I_G(X)} H_A(pt)\otimes_{H_G(X)} \overline{k(\mathfrak{p})}$$

and now up to F-isomorphism $H_A(pt)\otimes_{H_G(X)}\overline{k(\mathfrak{p})}$ is ~~the~~ ~~set of geometric points lying over~~ a product of copies of $\overline{k(\mathfrak{p})}$. In fact since $\overline{k(\mathfrak{p})}$ is perfect after dividing out by the nilpotent elements we have

$$\overline{k(\mathfrak{p})} \xrightarrow{\sim} \varprojlim_{(A,\lambda)} \overline{k(\mathfrak{p})}^{S(A,\lambda)}$$

where $S(A,\lambda) = \mathrm{Hom}_{k(\mathfrak{p})}\left(H_A(pt)\otimes_{H_G(X)}\overline{k(\mathfrak{p})},\ \overline{k(\mathfrak{p})}\right)$. Thus

$$\overline{k(\mathfrak{p})} = \mathrm{Hom}_{sets}\left(\varinjlim_{(A,\lambda)} S(A,\lambda),\ \overline{k(\mathfrak{p})}\right)$$

and so

$$\varinjlim_{(A,\lambda)} S(A,\lambda) = 1.$$

Put another way this calculation shows that as sets

$$\varinjlim_{(A,\lambda)} \mathrm{Hom}_{\mathbb{Z}_p\text{-alg}}(H_A(pt), \Omega) = \mathrm{Hom}_{\mathbb{Z}_p\text{-alg.}}(H_G(X), \Omega)$$

if Ω is an algebraically closed field

November 7, 1969

Let $\mathfrak{p}$ be a prime ideal in $H_G(X)$ with residue field $k(\mathfrak{p})$. If $k(\mathfrak{p}) \longrightarrow \Omega$ is an embedding into an algebraically closed field, then we have proved that

$$\Omega \xrightarrow{\sim} H_G(X) \otimes_{H_G(X)} \Omega \longrightarrow \varprojlim_{(A,\lambda)} H_A(pt) \otimes_{H_G(X)} \Omega$$

is an F-isomorphism. Moreover dividing out by nilpotent elements we have

$$\Omega \longrightarrow \varprojlim_{(A,\lambda)} \Omega^{\mathrm{Hom}_{H_G(X)}(H_A(pt),\, \Omega)}$$

is an F-isomorphism and therefore that

$$\varinjlim_{(A,\lambda)} \mathrm{Hom}_{H_G(X)}(H_A(pt), \Omega) = e$$

or equivalently that

(A) $$\varinjlim_{(A,\lambda)} S(A,\lambda) = e$$

where $S(A,\lambda)$ is the set of geometric points $\eta : H_A(pt) \longrightarrow \Omega$ lying over $\xi : H_G(X) \longrightarrow \Omega$.

Proposition. Let ξ be a geometric point of $H_G(X)$ with values in Ω. Then there exist a pair (A,λ) ~~such~~ and a

$\xi : H_A(pt) \longrightarrow \Omega$ lying over (via λ) ξ which is minimal in the sense that ~~A can't be replaced by a smaller group a subgroup~~ the triple (A, λ, ξ) dominates any other such triple.

Proof: We know that the fiber over ξ is non-empty by the Cohen-Seidenberg theorem. Indeed the dotted arrows exist in

$$\begin{tikzcd}
 & \coprod_{(A,\lambda)} \operatorname{Spec} H_A(pt) \arrow[d, "\text{finite surjective}"] \\
\operatorname{Spec}\{\Omega\} \arrow[ur, dotted] \arrow[r] & \operatorname{Spec}\{H_G(X)\}
\end{tikzcd}$$

$$\begin{tikzcd}
 & H_A(pt)/q \arrow[dl, dotted] \\
\Omega & H_G(X)/p \arrow[l, hook'] \arrow[u, "\text{finite}"']
\end{tikzcd}$$

Choose such an (A,λ) which is minimal, i.e. ξ doesn't come from a proper subgroup of A. I claim this dominates any other (A', λ', ξ')

~~We start with the formula (★) which tells us that the arrows~~

$$\cancel{\{(B, x, A, \lambda, \xi)\} \rightrightarrows \{(A, \lambda, \xi)\}}$$

~~generate the indiscrete equivalence relation. Here $x : G/B \to G/A$ and the bottom arrow is~~

$$\cancel{(B, x, A, \lambda, \xi) \longmapsto (B, x^*(\lambda), x^*(\xi))}$$

~~Where~~

$$\begin{tikzcd} G/B \arrow[d, "x"'] \arrow[dr, "x^*(\lambda)"] & \\ G/A \arrow[r, "\lambda"'] & X \end{tikzcd} \qquad \begin{tikzcd} H_G(G/B) & \\ H_G(G/A) \arrow[u, "x^*"] \arrow[r, "\xi"] & \Omega \end{tikzcd}$$

We start with the formula (*) on page 11, which tells us that we can get from (A, λ, ξ) to (A', λ', ξ') by a finite sequence of arrows in the category ~~$I_G(X)$. This~~ of triples. It suffices to show that the objects of this category of triples which are dominated by (A, λ, ξ) form a full subcategory. ~~Thus~~ The critical thing to show is that there is always a dotted arrow in a diagram

$$\begin{tikzcd} (A, \lambda, \xi) \arrow[dr] \arrow[dd, dotted] & \\ & (A', \lambda', \xi') \\ (A'', \lambda'', \xi'') \arrow[ur] & \end{tikzcd}$$

Without loss of generality we may suppose that the maps φ, φ''

$$\begin{tikzcd} G/A \arrow[dr, "\varphi"] \arrow[drr, bend left, "\lambda"] & & \\ & G/A' \arrow[r, "\lambda'"] & X \\ G/A'' \arrow[ur, "\varphi''"] \arrow[urr, bend right, "\lambda''"'] & & \end{tikzcd}$$

are given by inclusions ~~in~~ $A \subset A'$, $A'' \subset A'$. By assumption $\varphi_*(\xi) = \xi' = \varphi''_*(\xi'')$. Thus we have a ~~square~~ diagram

$$\begin{tikzcd}
 & & H_A(pt) \arrow[lld, "\xi"'] & \\
\Omega & & & H_{A'}(pt) \arrow[lu, "\mathrm{res}^{A'}_{A}"'] \arrow[lll, "\xi''"'] \arrow[ld, "\mathrm{res}^{A'}_{A''}"] \\
 & & H_{A''}(pt) \arrow[llu, "\xi'"] &
\end{tikzcd}$$

and hence a map

$$H_{A''}(pt) \otimes_{H_{A'}(pt)} H_A(pt) \simeq H_{A \cap A''}(pt) \longrightarrow \Omega.$$

By our minimality assumption ~~of~~ on (A, λ, ξ) where have $A \cap A'' = A$ ie $A \subset A'$, whence the required dotted arrow.

<u>Remark</u>: We have just proved that the ~~[illegible]~~ category of triples (A, λ, ξ) satisfies the filtering axiom

$$\begin{tikzcd}
 & \bullet \arrow[rd] & \\
\bullet \arrow[ru, dotted] \arrow[rd, dotted] & & \bullet \\
 & \bullet \arrow[ru] &
\end{tikzcd}$$

We note in addition that the same argument works if instead of ξ we took a prime ideal $\mathfrak{p}$ of $H_A(pt)$. In effect one knows that $\operatorname{Spec}(A \otimes_k B) \to \operatorname{Spec} A \times_{\operatorname{Spec} k} \operatorname{Spec} B$ is surjective (here this is trivial since the restriction homomorphisms are surjective).

<u>Corollary</u>: If (A,λ) is minimal supporting ξ, then the set of $\jmath$ is an orbit under the action of

$$N_{A,\lambda} = \{x \in G \mid x^*(\lambda) = \lambda\}.$$

<u>Proof</u>: If $(A,\lambda,\jmath)$ is minimal, then it isomorphic to any other $(A,\lambda,\jmath')$ and such an isomorphism comes from a map $G/A \longrightarrow G/A$, that is, an element of $N_{A,\lambda}$.

<u>Proposition</u>: Let $\mathfrak{p}$ be a prime ideal of $H_G(X)$. Then a minimal pair (A,λ) supporting $\mathfrak{p}$ is unique up to G-conjugation. Two primes in $H_A(pt)$ restricting to $\mathfrak{p}$ are conjugate under $N_{A,\lambda}$.

<u>Proof</u>: It suffices to prove that

$$\varinjlim_{(A,\lambda)} \mathfrak{P}(A,\lambda) = e$$

where $\mathfrak{P}(A,\lambda)$ is the set of primes of over $\mathfrak{p}$. So take $\xi : H_G(pt) \to \Omega$ to be an algebraic closure of $k(\mathfrak{p})$. Then the Galois gp Γ of $\Omega / k(\mathfrak{p})$ acts on the geometric points $S(A,\lambda)$ of $H_A(pt)$ over ξ. In fact there are maps

$$S(A,\lambda)_\Gamma \twoheadleftarrow P(A,\lambda) \twoheadleftarrow S(A,\lambda)$$

Indeed given $\mathfrak{q}$ over $\mathfrak{p}$ we have existence of a ζ over ξ giving $\mathfrak{q}$

$$\begin{tikzcd}
H_G(X) \arrow[r] \arrow[d, "\text{finite}"] & k(\mathfrak{p}) \arrow[r, "\xi"] \arrow[d, "\text{finite}"] & \Omega \\
H_A(pt) \arrow[r] & k(\mathfrak{q}) \arrow[ur, dashed, "\zeta"'] &
\end{tikzcd}$$

while ~~and~~ if ζ and ζ' are two ~~primes of $H_A(pt)$ containing $\mathfrak{p}$~~ ~~arising from~~ geometric points over $\mathfrak{q}$ then they are conjugate over $\mathrm{Gal}\{\Omega/k(\mathfrak{q})\}$, whence give the same ~~image~~ element in $S(A,\lambda)_{\mathfrak{p}}$. Consequently

$$\begin{tikzcd}
\varinjlim S(A,\lambda) \arrow[r, two heads] \arrow[d, phantom, "\cong" rotate=90] & \varinjlim B(A,\lambda) \arrow[r, two heads] & \varinjlim S(A,\lambda)_{\mathfrak{p}} \arrow[d, phantom, "\cong" rotate=90] \\
C & & C_{\mathfrak{p}}
\end{tikzcd}$$

which implies the desired result.

Corollary: If $\mathfrak{p}$ and $\mathfrak{p}'$ are two primes in $H^*_G(X)$ ~~with supports (A, λ) and~~ supported by (A,λ) and (A',λ') respectively, then if $\mathfrak{p} \supset \mathfrak{p}'$ there is a diagram

$$\begin{tikzcd}
\mathfrak{q} \arrow[rr, mapsto] & & \mathfrak{p} \\
G/A \arrow[r, "\lambda"] \arrow[d, "\alpha"] & X & \\
G/A' \arrow[ur, "\lambda'"'] & & \\
\mathfrak{q}' \arrow[rr, mapsto] & & \mathfrak{p}'
\end{tikzcd}$$

such that $\mathfrak{q}' \subset x^{*-1}(\mathfrak{q})$. (Geometrically if $\mathfrak{p}'$ specializes to $\mathfrak{p}$ ~~then its support (A,λ)~~ we can choose (A',λ') such that $\mathfrak{p}'$ comes from $\mathfrak{q}'$ in $H_{A'}(pt)$ and $\mathfrak{p}$ comes from $\mathfrak{q}$ in $H_{A'}(pt)$ and such that $\mathfrak{q}'$ specializes to $\mathfrak{q}$.)

~~Remark: This implies that $\operatorname{Spec} H_G(X)$ is the quotient topological space of $\coprod_{(A,\lambda)} \operatorname{Spec}\{H_A(pt)\}$~~

~~Suppose that U is an open set in $\operatorname{Spec} H_A(pt)$. Then I claim the image of U under the map $\lambda : \operatorname{Spec} H_A(pt) \longrightarrow \operatorname{Spec} H_G(X)$ is open. Indeed as it is constructible (thm. of Chevalley) it suffices to show that $\lambda(U)$ is closed under generization. So let $\mathfrak{q} \in U$, let $\mathfrak{p} = \lambda(\mathfrak{q})$ and let $\mathfrak{p}'$ specialize to $\mathfrak{p}$, i.e. $\mathfrak{p}' \subset \mathfrak{p}$~~

false e.g.

Corollary: Invariant primes in $H_G(X)$ are in 1-1 correspondence with pairs (A,λ).

Proof: If $\mathfrak{p}$ is invariant with support (A,λ), I claim that $\mathfrak{p} = \lambda(\mathfrak{r}_A)$. In effect $\mathfrak{p} = \lambda(\mathfrak{q})$ for some $\mathfrak{q}$ and by earlier lemma $\mathfrak{q}$ is invariant hence comes from a subgroup B of A. ~~Thus $\mathfrak{q} = \mathfrak{r}_A$~~ By definition of support $B = A$, so $\mathfrak{q} = \mathfrak{r}_A$

Consider the set $T_{A,\lambda}$ of all primes in $H_G(X)$ whose support is (A,λ). Then $T_{A,\lambda}$ is a locally closed subset of $\operatorname{Spec}\{H_G(X)\}$, since there are only a finite number of invariant primes.

Theorem: Let $w = e(\overline{\mathrm{reg}}\, A)$ be the product of the Euler classes of the non-trivial characters of A. Then there is an F-isomorphism

$$\mathrm{Spec}\,\{H_A(pt)\,[w^{-1}]\}_{N_{A,\lambda}} \longrightarrow T_{A,\lambda}$$

Proof: ~~Let $\mathfrak{p}'$ be a prime in $H_G(X)$ with support $\mathfrak{p} = \mathfrak{p}_{A,\lambda}$~~ Let $\mathfrak{p} = \mathfrak{p}_{A,\lambda}$. Then we know that

$$H_G(X)/\mathfrak{p} \longrightarrow \varprojlim_{B,\mu} H_B(pt)/\mathfrak{p}\, H_B(pt)$$

is an F-isomorphism. Now $H_B(pt)/\mathfrak{p}\, H_B(pt)$ is the ring of algebraic functions on the subvariety of B_Ω which is the union of the ~~subvarieties corresponding to~~ subvarieties corresponding to $\mathfrak{q}$ over $\mathfrak{p}$. Such a $\mathfrak{q}$ is invariant hence corresponds to a subgroup A' and as $(A', \mu|A', \nu_{A'})$ is minimal ~~with support~~ restricting to $\mathfrak{p}$ it follows that $(A', \mu|A', \nu_{A'}) \cong (A, \lambda, \nu_A)$. Thus $\mathfrak{q}$ comes from a map of (A,λ) to (B,μ). ~~Therefore the $\mathfrak{q}$'s in $H_A(pt)$ over $\mathfrak{p}$ are in 1-1 correspondence with subgroups A' of A such that ... A~~ Therefore an element of $\varprojlim_{B,\mu} H_B(pt)/\sqrt{\mathfrak{p}\, H_B(pt)}$ may be viewed as a function which assigns to each (B,μ) a function on the union of the subgroups of B which are the images of the maps $G/A \xrightarrow{x} G/B$ with $x(\lambda) = \mu$ and this should be compatible with maps $(B,\mu) \longrightarrow (B',\mu')$. Thus we see that such a function is determined by its restriction to $(A,\lambda) = (B,\mu)$. ~~Thus~~ Moreover

$$\varprojlim_{(B,\mu)} H_B(pt)/\sqrt{_p H_B(pt)} = \left\{ f \in H_A(pt)/n_A \;\middle|\; \begin{array}{l} \forall x \in G \ni \\ (A, xAx^{-1}) = 1, \\ f = f^x \text{ on } A \cap xAx^{-1} \\ \text{\& } \lambda \text{ descends to } \langle A, xAx^{-1} \rangle \end{array} \right\}$$

Indeed once one gives a function on A in order that we may extend it ~~to supp~~ over all subgroups of any (B,μ) under (A,λ) we must know that ~~given xAx^{-1}~~ the two definitions of f on xAx^{-1} and $yAy^{-1} \subset B$ are the same. Up to conjugacy we may as well suppose $y = 1$ and B is generated by A and xAx^{-1}.

(This isn't quite correct, but is OKAY up to F-isom.)

Now let us localize with respect to a prime p' in $H_G(X)$ with support p. Then we get

$$\varprojlim_{(B,\mu)} \left(H_B(pt)/\sqrt{p H_B(pt)}\right)_{q} = \left\{ f \in \left(H_A(pt)/n_A\right)_{q} \;\middle|\; \forall x \in N_{A,\lambda},\ f = f^x \right\}$$

since the localization with respect to q will kill all the proper subgroups of A. Thus we find that

$$\left(H_G(X)/p\right)_{q} \longrightarrow \left(H_A(pt)/n_A\,[w^{-1}]\right)_{q}^{N_{A,\lambda}}$$

is an F-isomorphism ~~which~~ which proves the theorem.

Application to the ζ function of $H^*_G(X)$: One knows that the ζ function is multiplicative with respect to stratifications. So let V be a vector space over $\mathbb{Z}_p = \mathbb{F}_p$

and let G be a group of linear transformations of V. Then G acts freely on $V_\Omega - \bigcup_{W \subset V} W_\Omega$ and we shall denote by

$$\Phi(V^\circ; G)$$

the zeta function of $\left(V_\Omega - \bigcup_{W \subset V} W_\Omega\right)/G$. Then we have

$$\boxed{\zeta(H_G(X)) = \prod_{A,\lambda} \Phi(A^\circ; N_{A,\lambda}/C_A)}$$

It's conceivable that we could determine $\Phi(V^\circ; G)$ since

$$\zeta(V) = \frac{1}{1 - p^N z}$$

should factor into a product of L functions associated to the characters of G, one of which is essentially $\Phi(V^\circ; G)$.

symmetric gps
Dickson thm.

November 29, 1969

Let $\Sigma(n)$ be the symmetric group on n letters and let A be a maximal elementary abelian p-subgroup of $\Sigma(n)$. Think of $\Sigma(n)$ as the permutations of a set S and decompose S into its A-orbits

$$S = \coprod_{i=1}^{k} S_i$$

Now for each i there is a subgroup A_i of A such that $S_i = A/A_i$ as sets with A-action. In fact since A is abelian A_i is unique. Now the inclusion $A \hookrightarrow \Sigma(n)$ factors into

$$A \longrightarrow \prod_{i=1}^{k} A/A_i \hookrightarrow \prod_{i=1}^{k} \mathrm{Aut}\, S_i \hookrightarrow \mathrm{Aut}(S)$$

As A is a maximal elementary abelian p-group it follows that

$$A \xrightarrow{\sim} \prod_{i=1}^{k} A/A_i$$

In other words we have decomposed S into subsets S_i and given each S_i an affine space structure over $\mathbb{Z}_p$ and A is the product of the group of translations of the A_i. As two affine space structures on the set of the same size are isomorphic, we conclude

<u>Theorem:</u> <u>To each</u> p-<u>adic partition</u> $\pi = (p^{i_1}, p^{i_2}, \ldots)$

$p^{i_1} \geq p^{i_2} \geq \dots \quad \sum_j p^{i_j} = n$ of n corresponds a conjugacy class of maximal elementary p-subgroups of $\Sigma(n)$ as follows: Write $S = \coprod_i S_i$ where card $S_j = p^{i_j}$ and choose a bijection between S_i and $(\mathbb{Z}_p)^{i_j}$. Let A_j be the elementary abelian p-group of $\mathrm{Aut}\, S_j$ given by the translations for its $\mathbb{Z}_p$-vector space structure and let $A_\pi = \prod A_j$ considered as a subgroup of $\mathrm{Aut}\, S$. ~~The conjugacy class of maximal element~~ Then A_π is a maximal elementary abelian p-subgroup of $\Sigma(n)$ and every such one is conjugate to A_π. for a unique π. The normalizer of A_π is

$$N_\pi = \prod_j Gl(i_j, \mathbb{Z}_p).$$

According to Milgram there is the following old theorem of Dicksen:

Theorem: Let G be the group of automorphisms of a vector space V of dimension n over the field $\mathbb{F}_q$. Then the ring of invariants $S(V^*)^G$ is a polynomial ring with n-generators of degrees $q^n - q^i \quad 0 \leq i < n$. In fact the generators are the coefficients of the polynomial

$$c_t = \prod_{\lambda \in V^*} (t - \lambda)$$

<u>Examples:</u> If $n = 1$ then $G = (\mathbb{F}_q)^*$ is a cyclic group of order $q-1$ acting by multiplications on $V = \mathbb{F}_q$. In this case $S(V^*) = \mathbb{F}_q[z]$ where $z \mapsto \xi^{-1} z$ is the transformation induced by $\xi \in (\mathbb{F}_q)^*$. The invariant subring is

$$S(V^*)^G = \mathbb{Z}_q[z^{q-1}]$$

$$\prod_{\lambda \in \mathbb{F}_q} (t - \lambda z) = t^q - t z^{q-1}$$

which checks.

<u>Proof of Dickson's theorem:</u> We know already that e_t is $\mathbb{F}_q$-linear hence only has power t^{q^ν} $\nu = 0, \ldots, n$. This gives exactly n non-zero coefficients $w_{q^n - q^i}$ $i = 0, \ldots, n-1$ for e_t. Every $\lambda \in V^*$ satisfies $e_\lambda = 0$ hence $S(V^*)$ is ~~an~~ integral extension of $\mathbb{F}_q[w_{q^n-1}, \ldots, w_{q^n - q^{n-1}}]$. This means that these w's form a regular sequence and that $S(V^*)$ is a free module of rank

$$\prod_{i=0}^{n-1} (q^n - q^i) = |G|$$

over $\mathbb{F}_q[w \ldots]$. Now argue that by Galois theory that $\mathbb{F}_q[w \ldots] \longrightarrow S(V^*)^G$ is birational and finite, whence an isomorphism as $\mathbb{F}_q[w, \ldots]$ is integrally closed in its quotient field.

~~(Same argument works for non-degenerate quadratic functions over a finite field of char 2 over $\mathbb{F}_2$. Namely~~

(To see if same argument might answer our old question about orthogonal invariants in $S(V^*)/J$. Note that $S(V^*)$ is a free module of rank

$$\prod_{j=0}^{n-1} (1+q^j)(q^n-q^j)$$

over the subring generated by $Q(x), \dots, B(x, x^{q^{n-1}})$, $w_{q^n-1}, \dots, w_{q^n-q^{n-1}}$. and hence $S(V^*)/J$ is the same rank over the poly ring generated by the w's. But the order of the orthogonal group is

$$\underbrace{\left[\prod_{j=0}^{n-1} (1+q^j)\right]}_{\substack{\text{no. of max} \\ \text{isot. subspaces } W}} \cdot \underbrace{q^{\frac{n(n-1)}{2}}}_{\substack{\text{no. of isot. complements} \\ \text{to } W = \text{no. of alt. maps} \\ W \to W^*}} \cdot \underbrace{\left[\prod_{j=0}^{n-1} q^n-q^j\right]}_{\substack{\text{no. of frames} \\ \text{in } W}}$$

and these numbers <u>don't agree</u> because of the factor $q^{\frac{n(n-1)}{2}}$?)

So for each p-adic partition π of n we get a *restriction* homomorphism

$$H^*(B\Sigma(n)) \longrightarrow \bigotimes_j H^*(B(\mathbb{Z}_p)^{i_j})_{reg}^{Gl(i_j, \mathbb{Z}_p)}$$

into a polynomial ring. To make this more precise let π be

$$n = \sum_j p^{i_j}$$

whence the polynomial on the left has generators

$$\underline{w}^{(j)}_{k} = w^{(j)}_{p^{i_j} - p^k} \qquad 0 \leq k < i_j$$

Next see how this behaves with respect the induction pairing

$$H^*(B\Sigma(m)) \otimes H^*(B\Sigma(n)) \longrightarrow H^*(B\Sigma(m+n)).$$

The idea is that if $A_1 \subset \Sigma(m)$ and $A_2 \subset \Sigma(n)$ are maximal elementary abelian p-subgroups, then ~~so is~~ $A_1 \times A_2 \subset \Sigma(m) \times \Sigma(n) \subset \Sigma(m+n)$ is also ~~[illegible]~~ a maximal elementary abelian p-subgroup. Unfortunately

$$\begin{array}{ccc} H^*(B\Sigma(m)) \otimes H^*(B\Sigma(n)) & \longrightarrow & H^*(BA_1) \otimes H^*(BA_2) \\ \big\downarrow \text{ind} & & \big\downarrow \\ H^*(B\Sigma(m+n)) & \xrightarrow{\text{res}} & H^*(B(A_1 \times A_2)) \end{array}$$

doesn't commute, ~~in fact it commutes with factor $\frac{(m+n)!}{m!\,n!}$~~ ~~[illegible]~~ and the res oind formula involves sum over m, n shuffles.

old 1969

<u>Localization theorem (general form)</u>: Let Y be a G-space, ~~and~~ let (A,λ) be an object of $I_G(Y)$, and let $\mathfrak{p}$ be a prime ideal in $H_G(Y)$ with support (A,λ). Let $f: X \to Y$ be a G-space over Y and let $X^{A,\lambda,f}$ be the ~~components~~ union of the components of X^A which lie over the component λ of Y^A. Then the restriction hom.

$$H_G(X) \longrightarrow H_G(G \cdot X^{A,\lambda,f})$$

induces an isomorphism

$$H_G(X)_{\mathfrak{p}} \xrightarrow{\sim} H_G(G\cdot X^{A,\lambda,f})_{\mathfrak{p}}$$

<u>Proof</u>: suffices to show that if $X^{A,\lambda,f} = \phi$, then

$$H_G(X)_{\mathfrak{p}} = 0$$

So if not there is a $\mathfrak{q}$ over $\mathfrak{p}$ and the support of $\mathfrak{q}$ is (A,μ). Thus there is a homot comm.

$$\begin{tikzcd} & X \arrow[d, "f"] \\ G/A \arrow[ur, "\mu"] \arrow[r, "\lambda"'] & Y \end{tikzcd}$$

hence $X^{A,\lambda,f} \neq \phi$.

<u>Localization theorem (useful form)</u>: suppose $\mathfrak{p}$ is a prime ideal in H_G with support A. Then

$$H_G(X)_{\mathfrak{p}} \xrightarrow{\sim} H_G(G\cdot X^A)_{\mathfrak{p}} \quad !!$$

<u>Relation with the Gysin homomorphism</u> (generalization of the fixpoint formula).

Given $f: X \to Y$ a proper oriented (if p is odd) map of G-manifolds. Then there is a Gysin homomorphism

$$f_*: H^*_G(X) \longrightarrow H^*_G(Y).$$

Now recall that we have a much simpler description of elements of $H_G(Y)$ if we ignore nilpotents. Thus given $x \in H^*_G(X)$ we might want to know the "value" of $f_* x$ at a prime ideal $\mathfrak{p}$ of $H_G(Y)$. Such a prime has for support a pair (A, λ), $\lambda: G/A \to Y$, and we know that $\lambda^*: H_G(Y) \longrightarrow S(A^*)[e^{-1}]$ detects the value of $f_* x$ at $\mathfrak{p}$ in the sense that there is a prime ideal $\mathfrak{q}$ in $S(A^*)[e^{-1}]$ (unique up to conjugation by $N(A,\lambda)$) such that $(\lambda^*)^{-1}\mathfrak{q} = \mathfrak{p}$. Thus we want to compute $\lambda^* f_* x$.

Now it is necessary to localize here because the maps

$$\begin{tikzcd} & X \arrow[d, "f"] \\ G/A \arrow[r, "\lambda"] & Y \end{tikzcd}$$

are not transversal nor can they be made so. In fact there is an obstruction to transversality; replacing λ by an inclusion one gets that the obstruction is the same for

$$\begin{tikzcd} & G \times_A Z \arrow[d] \\ G/A \arrow[r] & G \times_A V \end{tikzcd}$$

where

$$\begin{tikzcd} & Z \arrow[d, "g"] \\ pt \arrow[r] & V \end{tikzcd}$$

is a map of A-manifolds. This last diagram has the usual obstruction which roughly is the denominator of the Euler class $e(\mu_g)$ where as usual

$$\nu_g | Z^A = \nu_{g^A} \oplus \mu_g$$

is the decomposition into A-invariant and non-trivial parts respectively.

This shows that to obtain a formula for $\lambda^* f_* x$ we are forced to invert the Euler classes $e(G \times_A V) \in H_G(G/A)$ where V is a representation of A with $V^A = 0$. With this basic algebraic step the best result seems to be the following.

<u>Theorem:</u> Let $S \subset H_N(Y^{A,\lambda})$ be the set of Euler classes of oriented representations (unoriented ~~real~~ if $p=2$) V of $N = N(A,\lambda)$ such that $V^A = 0$. Then in the diagram

$$\begin{tikzcd} S^{-1}H_N(X^{A,\lambda}) \arrow[d, "f_*^{A,\lambda}"'] \arrow[dr, phantom, "e(\mu_{f,A,\lambda})"] & H_G(X) \arrow[l, "r_{X,A,\lambda}"'] \arrow[d, "f_*"] \\ S^{-1}H_N(Y^{A,\lambda}) & H_G(Y) \arrow[l, "r_{Y,A,\lambda}"'] \end{tikzcd}$$

we have the formula

$$\boxed{r^*_{Y,A,\lambda} f_* x = f_*^{A,\lambda}\left(e(\mu_{f,A,\lambda}) \cdot r^*_{X,A,\lambda} x\right)}$$

where $X^{A,\lambda}$ is the part of X^A situated over $Y^{A,\lambda} \subset Y^A$ and

$$\nu_f \mid X^{A,\lambda} = \nu_{f^{A,\lambda}} \oplus \mu_{f,A,\lambda}$$

is the decomposition into trivial and non-trivial parts. (It is necessary to suppose that $\nu_{f^{A,\lambda}}$ oriented invariantly under N when p is odd).

Proof: Choose a factorization of f into $X \xrightarrow{i} Y \times V \xrightarrow{p} Y$ where i and V are G-oriented. For the embedding i (resp. the zero section j of p) we have the clean intersection formula

$$r^*(i_* x) = i^A_*(e(\mu_{i,A}) \cdot r^* x)$$

$$r^*(j_* y) = j^A_*(e(V/V^A) \cdot r^* y) \qquad \text{in } H^*_{N,c}(Y^A \times V^A)$$

where the r maps refer to the restrictions $H^*_G(X) \longrightarrow H^*_N(X) \longrightarrow H^*_N(X^A)$ and similarly for $Y \times V$ and Y. ~~...~~ The restriction from G to N commutes with Gysin and from X to X^A introduces the $e(\mu)$ factor. Taking $y = f_* x$ and putting these formulas together we get

$$e(V/V^A) \cdot r^* f_* x = f^A_*(e(\mu_{i,A})\, r^* x) \qquad \text{in } H^*_N(Y^A)$$

Now restrict everything to $Y^{A,\lambda}$, whence $e(V/V^A) \in S$, and so this becomes the desired formula since

$$e(\mu_{f,A,\lambda}) = e(\mu_{i,A})\, e(V/V^A)^{-1} \mid X^{A,\lambda}.$$

Leaving out the generalizations, this is what we get:

<u>Theorem:</u> Let X be a compact G-manifold (oriented if p is odd). Let A be an elementary abelian p subgroup of G ~~and suppose that if μ is the normal bundle of X^A in X, has~~ and let μ be the normal bundle of X^A in X. Then in $H_A^*[e^{-1}]$ we have

$$\operatorname{res}_A^G (f_* x) = f_*^A \left(e(\mu) \cdot \operatorname{res}_{X^A}^X x \right).$$

Here if p is odd we suppose that μ is oriented invariantly under A and X^A is compatibly oriented; $f: X \longrightarrow pt$ and $f^A: X^A \longrightarrow pt$ are the canonical maps.

<u>Question:</u> Is μ necessarily orientable?

~~necessarily if X is allowed to be non-compact. Take $A = \mathbb{Z}_p$ and let θ be the isomorphism of η and η^{-1} given by the conjugation on $\mathbb{C}$. Then using θ we get a real 2-plane bundle E over S^1 with $\mathbb{Z}_p$ action such that S^1 is the fixed set. Unfortunately E is not oriented.~~

Take $\mathbb{Z}_p = A$ and let E be an A-real vector bundle over a trivial A-space X such that $E^A = X$. Then there is a unique complex structure on E_x such that $1 \in \mathbb{Z}_p$ acts by roots of unity in the upper half plane. Hence E is a complex bundle so orientable

~~Next take $A = \mathbb{Z}_p \times \mathbb{Z}_p$ with generators g_1, g_2~~

~~Now in general consider $\operatorname{Hom}_A(E, E)$ which is a bundle~~

Now (with a general A) assume that E is isotypical for A, i.e. that over each $x \in X$ the representation is a direct sum of the same irreducible real representation of A. Such a representation comes from a **fixed** cyclic quotient $A \twoheadrightarrow \mathbb{Z}_p$, thus we are in the preceding case. Conclusion: If A is an elementary abelian p-group, p odd, acting trivially on X, and if E is a real A-bundle over $X \ni E^A = 0$; then E admits a complex structure. Consequently in the above theorem μ is always orientable.

The general assertion is this:

Proposition: Let G be a finite group of odd order and let E be a real G-bundle over a trivial G-space X. Assume that $E^G = 0$. Then E admits an equivariant complex structure.

Proof: Recall that each irreducible non-trivial real representation W of G admits a complex structure, since G is of odd order. More precisely: let V be an irreducible complex representation of G supposed an arbitrary compact Lie group. Then for $V \cong V^*$ means that $(V \otimes V)^G = (\Lambda^2 V)^G + (S^2 V)^G \neq 0$. By Schur's lemma $\dim (V \otimes V)^G = 1$. Thus

$$\langle 1, \psi^2 V \rangle = \langle 1, S^2 V \rangle - \langle 1, \Lambda^2 V \rangle$$

$$= \begin{cases} 1 & \text{if } V \text{ has a symm. non-deg. bil. form} \\ 0 & \text{if } V \not\cong V^* \\ -1 & \text{if } V \text{ has a skew-symm non-deg bilinear form.} \end{cases}$$

Note that there exists a unique inv. hermitian form on V, so the first ~~case~~ case implies ~~an underlying~~ $V = W \otimes \mathbb{C}$ where W is a real representation and the last case implies V is a quaternionic representation. The second case is where V is irreducible as a real representation and has $\mathbb{C}$ for endo. ring.) Moreover ψ^{-1} ~~[illegible]~~ defines a free $\mathbb{Z}_2$-action on the non-trivial irred. complex representations. Let S be a set of representatives for this action. Then if E is ~~[illegible]~~ a real representation of G we know that

$$E \otimes_{\mathbb{R}} \mathbb{C} = F \oplus F^*$$

where F is the ~~[illegible]~~ part of E which transforms according to the representations in S. ~~[illegible]~~ Moreover

$$E \xrightarrow{id\otimes 1} E \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\simeq} F \oplus F^* \xrightarrow{pr_1} F$$

is an isomorphism, hence E has a canonical complex structure given S. This obviously works over a space X so the proposition follows.)

Proposition: Let N be a group of odd order which acts on a group G of odd order. Then the set S of representatives for the non-trivial irreducible complex representation under conjugate can be chosen to be invariant under N.

Proof: Let $T = \hat{G} - \{1\}$. Then $N \times \mathbb{Z}_2$ acts on T with $\mathbb{Z}_2$ acting freely. Let O be an orbit of the action. Then card O is even hence N can't act transitively. Thus $O = Nx + N\tau x$ (τ generates $\mathbb{Z}_2$). ~~[illegible]~~ Choosing an x in each $N\mathbb{Z}_2$ orbit the propn. follows.

Corollary: Let N be a group of odd order, ~~and~~ let ~~let X be a N/H spaces~~ H be a normal subgroup, and let X be a N/H-space. If E is a real N-bundle over X such that $E^H = 0$, then E admits an equivariant complex structure.

Proof: Let S be a set of complex conjugation representatives for the irred. non-trivial complex representations of H. Can assume S stable under N/H by the preceding proposition. Then can write

$$E \otimes_{\mathbb{R}} \mathbb{C} \cong F \oplus F^*$$

where F is the subbundle transforming via the representations in S. Note that F is stable under N. (In effect let V be an irred. subrep. of F, then $n \cdot V \cong V^n$ which again is in S). Also $E \longrightarrow E \otimes_{\mathbb{R}} C \cong F \oplus F^* \longrightarrow F$ is an isom. q.e.d.

1969

§8. Localization theorem and the fixpoint formula

The present section is a variation of the ~~ideas~~ ideas of [] with K_G replaced by H_G^*.

Let $\mathfrak{p}$ be a prime ideal in H_G and let A be an elementary abelian p-subgroup of G which is a support for $\mathfrak{p}$. Let X be a G-space and $G\cdot X^A$ be the ~~the sm~~ smallest G-invariant subspace of X containing the fixpoints of A. Then we have the following localization theorem.

Theorem 8.1: If $\mathfrak{p}$ is a prime ideal of H_G with support A, then the inclusion $G\cdot X^A \subset X$ induces ~~isomorphism~~ ~~iso~~ an isomorphism

$$H_G^*(X)_{\mathfrak{p}} \xrightarrow{\sim} H_G^*(G\cdot X^A)_{\mathfrak{p}}$$

Consider the long exact sequence of the pair $(X, G\cdot X^A)$. It suffices to show that $H_G^*(X, G\cdot X^A)_{\mathfrak{p}} = 0$, and as this is a module over $H_G^*(X - G\cdot X^A)_{\mathfrak{p}}$, it suffices to show that if $X^A = \phi$, then $H_G^*(X)_{\mathfrak{p}} = 0$. ~~By the~~ ~~theorem If $H_G^*(X)_{\mathfrak{p}} \neq 0$, then there is a prime ideal~~

~~Consider the map~~

$$\cancel{\operatorname{Spec} H_G(X) \longrightarrow \operatorname{Spec} H_G}$$

~~which is closed by Cohen-Seidenberg. If $H_G^*(X)_{\mathfrak{p}} \neq 0$, then there is a prime ideal of in $H_G(X)$ over $\mathfrak{p}$ and such a.~~

~~prime ideal has for support a map $G/B \xrightarrow{\lambda} X$ for some B. Clearly B is also support for $\mathfrak{p}$, hence~~

~~Consider the map~~

$$\cancel{\coprod_{(B,\lambda)} \operatorname{Spec} S(B^*) \longrightarrow \operatorname{Spec} H_G(X) \longrightarrow \operatorname{Spec} H_G}$$

~~where (B,λ) runs over $I_G(X)$.~~

If $H_G(X)_{\mathfrak{p}} \neq 0$, then ~~[illegible]~~ $\mathfrak{p}$ is in the image of the map $\operatorname{Spec} H_G(X) \longrightarrow \operatorname{Spec} H_G$, hence by ... in the ~~[illegible]~~ image of the map $\operatorname{Spec} S(B^*) \xrightarrow{\lambda_*} \operatorname{Spec} H_G(X) \longrightarrow \operatorname{Spec} H_G$ for some pair (B,λ) in $I_G(X)$. Can assume B minimal whence B is a support for $\mathfrak{p}$ and so B is conjugate to A by 5.9. Then $X^B \neq \phi$ and so we have a contradiction.

Of course 8.1 is really a statement about the G-map $X \longrightarrow pt$. We leave to the reader the task of generalizing it to a G-map $f : X \longrightarrow Y$ and a prime $\mathfrak{p}$ in $H_G(Y)$.

(The localization theorem amounts to proving that if $K \subset G$ is a subgroup $\ni$ $(G/K)^A = \phi$, then $H_G(G/K)_{\mathfrak{p}} = 0$. In other words we have produced an element of H_G not in $\mathfrak{p}$ ~~which~~ goes to zero in ~~[illegible]~~ H_K. In more elementary terms what we have done is to ~~[illegible]~~ consider the exact for Spec $H_G(X)$ paper 1969

<u>localization theorem:</u> Let $\mathfrak{p}$ be a prime ideal in $H_G(X)$ with support (A,λ). Let $f: Y \to X$ be a G-map and let Z be the smallest G-subspace of Y containing the inverse image of $(X^A)_\lambda$.

Let $X^{A,\lambda}$ = the component of X^A corresponding to $\lambda \in \pi_0(X^A)$.
Let $Z = Gf^{-1}(X^{A,\lambda})$

~~A,λ~~ $Y^{A,\lambda}$ the part of Y^A lying over the component of X^A corresponding to λ. Then

$$H_G(Y)_\mathfrak{p} \xrightarrow{\sim} H_G(G\cdot Y^{A,\lambda})_\mathfrak{p}$$

suffices to show that

$$\cancel{H_G(X)_\mathfrak{p} \xrightarrow{\sim} H_G(G\cdot X^{A,\lambda})_\mathfrak{p}}$$

It suffices to show that $H_G(Y)_\mathfrak{p} = 0$ if $Y^{A,\lambda} = \phi$

~~It suffices to show that $H_G(Y)_\mathfrak{p} \neq 0$ simply show its spectrum is empty. A prime ideal is of course~~

(A,μ)

$$\begin{tikzcd} G/A \arrow[r, "\mu"] \arrow[dr, "\lambda"'] & Y \arrow[d, "f"] \\ & X \end{tikzcd}$$

$\boxed{\text{Localization theorem}}$ $\mathfrak{p}$ prime ideal in H_G with support A. A elementary abelian subgroup of G.

$$H_G(X)_\mathfrak{p} \xrightarrow{\sim} H_G(G\cdot X^A)_\mathfrak{p}$$

~~ ~~ $(G/B)^A = \phi.$ use restriction to fixpoint theorem to conclude that some localization is zero.

$$(G/B)^{\mathbb{Z}} = \coprod_{A \times B} A / A \cap x B x^{-1}$$

so take e

---

fixpoint formula: If A is an elementary abelian group, ~~then~~ and let X be an oriented A-manifold compact. Then have Gysin homomorphism $f_*: H_A(X) \longrightarrow H_A(pt)$ and a restriction to the fixpoint ~~con~~ formula

$$r(f_* x) = f_*(e(\nu_f)\, r(x)).$$

in other words if we localize with respect to e, then

$$\int_X x \;\; = \sum_i \int_{X_i} \frac{x}{e(\nu_i)}$$

---

radical, so if $x^n \in I$, then $(P_t x)^n = P_t(x^n) \in \sqrt{I}[[t]]$, and so $P_t x \in \sqrt{I}[[t]]$. ~~Consequently the~~ Taking $I = 0$, we see that ~~the~~ $a \otimes B$ induces an action on $H_G(X)_{red}$. We propose now to determine the invariant prime ideals in $H_{A,red} = S(A^*)$.

Theorem 6.3: Let $\mathfrak{p}$ be an invariant prime ideal in $S(A^*)$

~~prime ideals I know th~~

I ~~fel~~ think I can prove that conj. classes of maximal A are same as dyadic part.

Claim: Let B act faithfully on S and let A, A' be two maximal [p]-gps containing B. Then A, A' are conjugate ~~under~~ in the centralizer of B if they have same p-adic partition

Proof: Decompose S into B orbits

$$S = \coprod_{i \in I} S_i$$

and let B_i be the ~~stab~~ stabilizer of S_i so that after choosing a basepoint $x_i \in S_i$ get $B/B_i \xrightarrow{\sim} S_i$.

$$[B_i] \longmapsto x_i$$

Let $S = \coprod_{j \in J} T_j$ be the decomposition into A, orbits

~~Theorem 6.3. Let $\mathfrak{p}$ be an invariant prime ideal in A_q where A is an elementary abelian p-group. Then there is a unique subgroup B of A such that $\mathfrak{p}$~~

~~Let $S = \coprod_{j \in J} A_j$~~ Then

$$I = \coprod_{j \in J} I_j$$

$$\ni \qquad T_j = \coprod_{i \in I_j} S_i$$

Let A_j be the stabilizer (in A) of every element of T_j. Then

$$A_j \cap B = B_i \qquad \text{for} \quad i \in I_j$$

hence all B_i are the same for $i \in I_j$.

The S_i are the cosets of B/B_i in A/A_j.

More precisely recall that we have chosen $x_i \in S_i$

I_j becomes a $A/B+A_j$ space trans. faithful

~~so the situation is clear.~~

<u>Proposition:</u> Let G be a compact Lie group and let X be a nice G-space. Then

$$H_G(X,\mathbb{Z})\otimes\mathbb{Z}_p \longrightarrow H_G(X,\mathbb{Z}_p)$$

is an F-isomorphism.

One knows that it is injective and the cokernel is the elements of (odd, if p is odd) order p in $H_G^{*}(X,\mathbb{Z})$. Since $H_G^*(X,\mathbb{Z})$ is a finite $H^*(BU(n),\mathbb{Z})$-module one knows that the ~~[illegible]~~ ideal of p-torsion elements is finitely generated, hence killed by p^n for some n. This implies that for the **Bockstein spectral sequence** of X_G $E_r = E_\infty$ for $r \gg n$. Given $f \in H_G(X,\mathbb{Z}_p)$ one knows that ~~[illegible]~~ $F^\nu f$ ~~[illegible]~~ survives to $E_{\nu+2}$ so comes from an element of $H_G(X,\mathbb{Z})\otimes\mathbb{Z}_p$ ~~[illegible]~~ for ν large.

~~More precisely [illegible] consider the spectral sequence~~

More precisely ~~[illegible]~~ change p to l

$$E_1^{pq} = H^{p+q}(X_G, l^p\mathbb{Z}_{l^n}/l^{p+1}\mathbb{Z}_{l^n})$$

and this spectral sequence lives in the strip $0 \le p < n$. So by our old arguments I know that the image of

$$H(X_G,\mathbb{Z}_{l^n}) \longrightarrow H(X_G,\mathbb{Z}_l)$$

contains all ℓ^ν powers for ν large. However as we have that all ℓ-torsion is $H_G(X, \mathbb{Z})$ is of order ℓ^n, we know that

$$\operatorname{Im}\left\{H(X_G, \mathbb{Z}) \longrightarrow H(X_G, \mathbb{Z}_\ell)\right\}$$

$$\shortparallel$$

$$\operatorname{Im}\left\{H(X_G, \mathbb{Z}_{\ell^n}) \longrightarrow H_G(X, \mathbb{Z}_\ell)\right\}$$

In effect

$$\begin{tikzcd}
 & & {}^{y}H(\mathbb{Z}) \arrow[d, "\ell^n"] & \\
H(\mathbb{Z}) \arrow[r] \arrow[d, "\rho"] & H(\mathbb{Z}_\ell)^{x} \arrow[r] \arrow[d, equal] & {}^{x'}H(\mathbb{Z}) \arrow[r, "\ell"] \arrow[d, "\rho"] & H(\mathbb{Z}) \arrow[d, "\rho"] \\
H(\mathbb{Z}_{\ell^n}) \arrow[r] & H(\mathbb{Z}_\ell)^{x} \arrow[r, "\delta"] & {}^{0}H(\mathbb{Z}_{\ell^n}) \arrow[r, "\ell"] & H(\mathbb{Z}_{\ell^n})
\end{tikzcd}$$

But $\ell^{n+1} y = 0 \implies \ell^n y = 0 \implies x' = 0.$

Thus argument works for K_G theory also.

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G finite group of order $p^k > 2$

$c_i(\text{reg } G) = 0$ for $i \neq p^k - p^s$

$s = 0, \dots, k-1$

and moreover higher ones are determined from $c_{p^k - p^{k-1}}(\text{reg } G)$ by Steenrod operations

also $H(G) \longrightarrow H(P) \neq 0$ in dim $\frac{n}{p^k}(p^k - p^{k-1})$

Vol 33, 7391 rest. hom.

P_σ cyclic order p.

$i : P_\sigma \longrightarrow G$

R.G. Swan, Proc. Amer. Math. Soc. 11 (1960), 885-887

$i^* c_k(\omega) =$ image of $\lambda^k \omega$

image of i^* starts ~~as a~~ with a power of 2

Venkov B.B. Cohomology algebras for some classifying spaces. Dokl. Akad. Nauk SSSR 127 (1959), 943-944.

§. $\boxed{p=0}$.

cohomology coefficients in $\mathbb{Q}$.

Theorem: Let $\mathcal{F}$ be a family of subtori of G closed under conjugation and containing ~~the~~ family $\mathcal{F}_0$ maximal tori of the isotropy groups of X. Assume either ~~that~~ i) X is compact or ii) ~~that~~ $cd_{\mathbb{Q}}(X)<\infty$, $H^*(X)$ finite diml, and the family $\mathcal{F}_0$ ~~of~~ has finitely many conjugate classes. Then the ~~map~~ homomorphism

$$H_G^{ev}(X) \longrightarrow Q_{\mathcal{F}}^{ev}(X)$$

is integral (finite in case ii)) and induces a bijection of points with values in any field.

The induced map

$$\operatorname{Spec} Q_{\mathcal{F}}^{ev}(X) \longrightarrow \operatorname{Spec} H_G^{ev}(X)$$

is thus a homeomorphism, ~~and~~ in fact a universal homeomorphism, and the induced maps of residue fields are ~~~~~~ isomorphisms.

Case i) follows from ii) by passage to limit.

Proof: ~~$Q_{\mathcal{F}}^*(X) \to Q_{\mathcal{F}_0}^*(X)$~~

First step consists of identifying

$$\Gamma(X/G, \mathcal{H}_G^t) \xrightarrow{\sim} Q_{\mathcal{F}}^t(X)$$

As $Q_{\mathcal{F}}^t(X) = Q_{\mathcal{F}_0}^t(X)$ enough to worry about $\mathcal{F}_0$ which is finite up to conjugacy by hypotheses. So can use (ref) to ~~[illegible]~~ reduce to case $X = Gx$ ~~[illegible]~~ where $\mathcal{F}$ = conjugates of G_x. Then must show

$$H_G^* = \left(H_T^*\right)^{N/T}$$

T maximal torus in G, and N = normalizer of T, which is well-known.

Now look at spectral sequence

$$E_2^{st} = H^s(X/G, \mathcal{H}_G^t) \Longrightarrow H_G^{s+t}(X)$$

As $cd_{\mathbb{Q}}(X) < \infty$, one knows that Kernel of edge hom

$$H_G^*(X) \longrightarrow \Gamma(X/G, \mathcal{H}_G^*)$$

is the nilideal, and so we must show that

$$E_{r+1}^{0,*v} \hookrightarrow E_r^{0,*v}$$

as E_{r+1}^{0*} is the kernel of a derivation d_r is finite and induces a bijection on geometric. This follows from lemma in appendix once we show that E_2^{0*} is finite ~~module~~ over $H_G^*(X)$, ~~[illegible]~~ and $H_G^*(X)$ is noetherian. But this follows from hyp. $H^*(X)$ f.d.

Lemma: Suppose $\mathcal{F}$ family of subtori closed under conjugacy with finitely many conj classes. Let $\mathcal{F}'$ be the family consisting of the identity components of all ~~finite~~ intersections $K_1 \cap \cdots \cap K_r$, $K_i \in \mathcal{F}$. Then $\mathcal{F}'$ has finitely many conjugacy classes.

Proof: Suppose the intersection $K_1 \cap \cdots \cap K_r$ is irredundant, i.e. the dimension increases if any K_i is removed. Then

$$\dim K_1 > \dim K_1 \cap K_2 > \cdots$$

and so $r \leq \dim$ max. torus of G. But now the result follows because there are only finitely many ~~conjugacy~~ orbits types among the G-spaces $G/A_{i_1} \times \cdots \times G/A_{i_q}$ with $1 \leq i_1, \dots, i_q \leq N$, $q \leq m$.

~~General fact that if a G-space X has finitely many orbit types, same is true for all the products $\{X^n\}$.~~

Assume $\begin{cases} X \text{ compact or } cd_p(X) < \infty \\ H^*(X) \text{ f.d.} \\ \mathcal{F}_0 \text{ finite no. of classes.} \end{cases}$

Suppose now $\mathcal{F}$ contains finitely many classes and is closed under intersection. Then we can form category of pairs (A,λ), $A \in \mathcal{F}$, $\lambda \in \pi_0(X^A)$, call it $I(X)$ and we know

$$H^*_G(X) \longrightarrow \varprojlim_{(A,\lambda)} H^*_A$$

is a universal homeomorphism. The category $I(X)$ being essentially finite one gets

$$\operatorname{Spec}\{H^*_G(X)\} = \varinjlim \{\operatorname{Spec} H^*_T\}$$

and as $\mathcal{F}$ is closed under intersection the filtering property ( ) holds. Thus one sees that $\operatorname{Spec} H^*_G(X)$ is stratified into primes with support = a given conjugacy class in $\mathcal{F}$.

Since closed subgroups of G satisfy d.c.c. any prime $\mathfrak{p}$ is supported by a minimal (A,λ) and as two A such A could have been included in in $\mathcal{F}$ one sees that supports are unique up to conjugacy.

Proposition: If $\mathfrak{p}$ prime in $H^*_G(X)$, then two minimal pairs (A,λ), (A',λ') supporting $\mathfrak{p}$ are conjugate. Minimal primes in $H^*_G(X)$ are in 1-1 correspondence with maximal pairs (A,λ) up to conjugacy.

A_i are conj. classes of abelian subgps.

G

$$\chi(\cancel{X}) = \sum_i{}' \, [G:A_i]\, \chi(X^{(A_i)}/N_i)$$

$$\chi(\cancel{X}/G) = \sum_i{}' \, \chi(X^{(A_i)}/N_i)$$

the point is that these are equal & might give useful information.

Thus ~~[scribbled]~~ we want to worry about the set of abelian subgroups B $\quad A \subset B \subset G$ so can of course restrict to $B \subset \mathrm{Cent}(A)$

So given A consider ~~[scribbled]~~ $\mathrm{Cent}(A)$ and from this we should be able to understand

$$\chi(X^{(A_i)}/N_i) \qquad A$$

~~$\chi(X^{(A_i)}$~~

problem: suppose

$\chi(X^{(A)})$ depends only on centralizer of A.
$B \subset \mathrm{Cent}\, A$.

$$\chi(F) = \sum_i [G : N_i] \frac{\chi(X^{(A_i)})}{}$$

$$\chi(F/G) = \sum_i |N_i| \; \chi(X^{(A_i)})$$

G group

F flag manifold

A^c

F^G X

$$X^{(A)} = X^A - \bigcup_{B > A} X^B$$

$X^{(A)}$ same for X + Centralizer of A.

$X^A = X$

C/A acts?

$$\boxed{X^{(A)} = X^A - \bigcup_{B > A} X^B}$$

~~[illegible]~~

doesnt depend on normalizer

replace G by centralizer.

so A in center | G preserves various V_χ

what about $\chi(X^{(A)})$? so again $X^A = \frac{\coprod}{\varphi}$

$\prod_\chi F(V_\chi)$ 'φ' fixed. χ fixed. / a fixed point for A must be one

~~[illegible]~~

~~so take for any abelian gp.~~

$$\chi(F^A) = n! \qquad A \text{ abelian}$$
$$= 0 \qquad A \text{ not abel.}$$

now question becomes to see if the fact that

$$\chi(F/G) = \chi(F)$$
$$\|$$
$$\sum^{\text{over}} \text{conjugacy classes of } A\text{'s}.$$

idea might be to take an abelian group A and look at $GX^{(A)}$ which is N/A free
$$= G\times_N X^{(A)}$$
hence

$$\chi(GX^{(A)}) = \chi(G\times_N X^{(A)})$$
$$= [G:N]\,\chi(X^{(A)})$$

Thus
$$\chi(F) = \sum_i [G:N_i]\cdot\chi(X^{(A_i)})$$

Θ

$$= \sum_i [G:A_i]\,\chi(X^{(A_i)}/N_i)$$

and
$$\chi(F/G) = \sum_i \chi(X^{(A_i)}/N_i)$$

$A=0$ free part.

Let G be finite, F the flag manifold of a faithful complex representation. Then we have integration map

$$f_* : H_G^*(F) \longrightarrow H_G^* \qquad \text{degree } -(n^2-n)$$

and a canonical element $z \in H_G^{+n^2-n}(X)$ such that $f_*(z) = 1$. On the other hand if A is a maximal elementary abelian [p]-group, say normal, then

$$\begin{array}{c} H_G^*(F)_{p_A} = H_G^*(F^A)_{p_A} \\ \| \\ H^*_{G,p_A} \otimes_{H^*(BU_n)} H^*(BT_n) \end{array}.$$

Have to understand F^A. If F is the flag manifold of V, then we can split up V according to the eigenspaces of A

$$V = \bigoplus_{\lambda \in A^\vee} V_\lambda$$

and G/A will act on these eigenspaces. Suppose for simplicity there is a single orbit. Now any flag $L_1 \oplus \cdots \oplus L_n$ invariant under A will have to have its lines contained in these eigenspaces. Hence F^A has the following form: ~~form~~ it will be a disjoint union:

$$\coprod \prod_{\lambda \in \mathcal{O}} F(V_\lambda).$$

The disjoint union being taken over cosets $\Sigma_n / (\Sigma_d)^m$, $m = \text{card } \mathcal{O}$, $d = \dim V_\lambda$. Check the Euler characteristic

$$\chi\,(U_n / T_n) = n! \qquad (\text{order of } W)$$

$$\chi \left\{ \prod_{\lambda \in \mathcal{O}} F(V_\lambda) \right\} = (d!)^m$$

$m = \text{card } \mathcal{O}$
$d = \dim V_\lambda$ all $\lambda \in \mathcal{O}$.

Can you say anything about $H_G^*(F^A)$? If A is in the center of G then F^A breaks up G-invariantly into a disjoint union of products: $\prod_\lambda F(V_\lambda)$, and A acts trivially. So best one has is a reduction to

$$H_G^*(Y) \qquad Y = Y^A, \ Y \text{ free for } G/A$$

the action of G on Y maybe tor$\mathbb{Z}$.

$$H_G^*(Y) = H_G^* \otimes H^*(Y) \quad \text{additively}$$

$$H_G^*(Y) \Leftarrow H^*(Y/G) \otimes H^*(A)$$

$$G/A \longrightarrow Y \longrightarrow Y/G$$

$$H_G^*(Y) \leftarrow H^*(Y/G)$$

$$B < A$$

norm $\quad H_B^{\alpha} \xrightarrow{N} H_A^{\alpha}$ $\qquad$ work modulo 2.

first suppose $\quad [A:B] = 2$

$$\boxed{A = \mathbb{Z}_2 \times B}$$

Recall have external Steenrod operation

$$\begin{tikzcd}
H(X) \arrow[r] \arrow[dr] & H_{\mathbb{Z}_2}(X^2) \arrow[d, "\Delta"] \\
& H_{\mathbb{Z}_2}(X) =
\end{tikzcd}$$

On other hand the norm is defined in terms of

$$H^*(B) \longrightarrow H^*(\mathbb{Z}_2 \tilde{\times} B^2) \longrightarrow H^*(A) \; !!!!$$

so it would seem that in this case the ~~norm~~ satisfies

$$\mathrm{Norm}(\xi) = (\omega + \xi)\xi \qquad \xi \in H^1(B)$$

and in general Norm is a ring homomorphism $\quad \xi$

$$\mathrm{Norm}(\xi) = \prod_{\omega \in (A/B)^\#} (\omega + \xi)$$

$$= \xi^{2^r} + c_{2^r - 2^{r-1}}(\mathrm{reg}\, B/A)\, \xi^{2^{r-1}} + \cdots$$

$$+ \, c_{2^r - 1}(\mathrm{reg}\, B/A)\, \xi$$

$\boxed{\text{question:}}$ G pro-p-gp $\ni$ $G' < G$ finite index $H^*(G')$ f.d. $\Longrightarrow$ $\chi(G)$ defined.

<u>conjecture</u>: $\chi(G)$ can be determined for a compact Lie group from the Poincaré series of $H^*(G, \mathbb{Z}/p)$ any p. ?

<u>question</u>: the missing integral formula.

example of impossible groups $=$ S^1 $\chi(S^1) = 0$ and $\chi(BS^1) = \infty$, ~~[illegible]~~ similarly for all connected compact Lie groups.

~~next problem: consider the~~

Serre problem: G pro-p ~~profinite~~ gp $[G:G'] < \infty$

G no torsion, ~~H^* f.d. $\overset{?}{\Rightarrow}$ G is~~ $cd_p(G') < \infty \overset{?}{\Longrightarrow} cd_p(G) < \infty$

Serre's proof must work namely let L be a free G' complex resolving $\mathbb{Z}/p$ by free modules, and form ~~[illegible]~~

$$\mathrm{Norm}_{G' \to G}(L)$$

must consult paper by Serre in Topology to understand why he computes Steenrod invariant prime ideals.

Let A be an elementary abel. p-group. Then I want to compute $\chi(X^B)$ $B \subset A$ from $H_A^*(X)$ as an H_A^*-module.

Form $H_A^*(X) \otimes \Lambda^*(A/B)^\vee$ double complex.

$$\begin{array}{ccc} \Lambda^n(A/B)^\vee \otimes H_A^i(X) & \longrightarrow & \Lambda^{n-1}(A/B)^\vee \otimes (A/B)^\vee \otimes H_A^i(X) \\ & & \downarrow \\ & & \Lambda^{n-1}(A/B) \otimes H_A^{i+1}(X). \end{array}$$

e.g. codim $B = 1$. not correct.

Point is

$$H_B^*(X)\,[e_B^{-1}] = H_B^*(X^B)[e_B^{-1}]$$

not clear

$$H_A^*(X - X^A) \longrightarrow H_A^*(X) \longrightarrow H_A^*(X^A)$$

now $H_A^*(X - X^A)$ has support on all $B \subset A$ so what perhaps I should do is to partition and use the spectral sequence of a filtered space.

Suppose $X = X^{(B)}$ i.e. every point has isotropy gp B. Then

$$H_A^*(X) \cong H^*_{B \times C}(pt \times X) = H_B^* \otimes H_C^*(X)$$

Riemann-Roch

X algebraic variety$/\mathbb{C}$, G finite group

X/G $\quad$ F sheaf on X/G get

$F \longmapsto \chi(X/G, F)$

actually let ~~$H^*(X/G, F)$~~ ~~if F is a G~~

$$f : X \longrightarrow X/G$$

and if F is a G-sheaf on X, then

$$H^i_G(X, F) = H^i(X/G, (f_* F)^G)$$

$$= H^i(X, F)^G$$

$H^*_G(X, F) = H^*(X, F)^G$ $\qquad$ G finite so H^*_G trivial.

$= H^*(X/G, f_* F)^G$ $\qquad$ because f finite

$= H^*(X/G, (f_* F)^G)$ $\qquad$ (decompose $f_* F = \bigoplus_{\chi \in \hat{G}} V_\chi \otimes \cdots$

~~now the idea was to compute rational Pontryagin classes.~~

What is to be computed:

idea is that $F \mapsto \chi(X/G, F)$

X/G should have Todd class $\in H^*(X/G, \mathbb{Q}) = H^*(X, \mathbb{Q})^G$

$$\begin{array}{ccc} K(X/G) & \xrightarrow{\chi} & \mathbb{Z} \\ \cong \downarrow ch & \nearrow & \\ H^*(X/G) & & \end{array}$$

$$\boxed{\chi(X/G, E) = \int_{X/G} ch(E)\, Todd(X/G).}$$

this example might lead one to believe that

$$H^*_G(X) \longrightarrow H^0(X/G, H^*_N) \qquad F \text{ isomorphism}$$

so $(\beta u) \in H^2_N \ni \boxed{(\beta u)^{p^n}}$ does comes from $H^*_G(X)$.

so localize with respect to this element back in $H^*_G(X)$

---

my idea is to see if I can prove that maybe

Euler char of X/G and ?

$H^*_G(X)$ built up out of $H^*(X/G) \otimes H^*(N)$

hence maybe $H^*_G(X)$ is perfect over H^*_G and its $\chi = \chi(X/G)$.

---

obvious method is to take Poincaré series and ~~reuter~~

If G/N acts freely on X
one should be able to say something
about $H^*_G(X)$.

obstructions local. coeff. system

$$0 \longrightarrow \pi_1(X) \longrightarrow \pi_1(X/G) \longrightarrow G/N$$

and G/N acts on H^*_N. So first must assume
G/N acts trivially on H^*_N.

---

main problem is to show that

$$H^*_G(X) \longrightarrow H^*_N \qquad X/G \text{ conn.}$$

is surjective, if so

$$H^*_G(X) \underset{\text{add}}{\cong} H^*(X/G) \otimes H^*_N.$$

first example $G = \mathbb{Z}/p^2$, $N = p\mathbb{Z}/p^2\mathbb{Z}$.
Then H^*_N generated by $u: N \xrightarrow{\sim} \mathbb{Z}/p\mathbb{Z}$ in H^1_N.
so problem is whether $H^1_G(X) \twoheadrightarrow H^1_N$

$$0 \to H^1(X/G) \to H^1_G(X) \to H^0(X/G, H^1_N) \xrightarrow{d} H^2(X/G)$$
$$\downarrow$$
$$H^2_G(X)$$

~~$H^*_A(X^{(0)})$~~

A maximal in G finite, can you find an additive function constructed from ~~the~~

$$H^*_G(X)_p \qquad \text{as a } (H^*_G)_p \text{ module}$$

suppose G cyclic order p^e

A min. subgroup.

X^A set of non-free points

$X - X^A$ free.

what is $H^*_G(X^A)$ like.

suppose G/A acts freely on X.

what might $H_G(X)$ be.

spec. seq. $H^*_{G/A}(H^*_A(X)) \Rightarrow H^*_G(X)$

$\qquad\qquad\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad\qquad\qquad H^*($

orbit space spectral sequence

$$E_2 = H^*(X/G, H^*(A)) \qquad \text{unknown.}$$

~~cohomological~~

~~Some basic situations~~

~~A~~ General case

A suppose $B \subset A$ codim 1

how to compute $\chi(X^B)$

$$H^*_A(\cancel{\ }) \quad \Lambda A^\vee \otimes S A^\vee$$

B defines an element $\lambda \in A^\vee$

which acts as a derivation

$A = B \times C$

$$H^*_A(X) \longrightarrow H^*_A(X^B) = H^*_{A/B}(X^B) \otimes H^*(B)$$

~~solvalld~~ solvald.

$$\begin{tikzcd} M_{n-1} \arrow[r] \arrow[d] & M_n \arrow[r] \arrow[d] & {} \\ M_n \arrow[r] \arrow[d] & M_{n+1} \arrow[r] \arrow[d] & {} \\ M_{n+1} \arrow[r] & M_{n+2} \end{tikzcd}$$

$$H^*_A(X^B) \cong H^*_{A/B}(X^B) \otimes H^*_B$$

depends maybe on splitting

$$0 \to B \to A \to A/B \to 0$$

prolongation

$$S_n A^\vee \longrightarrow S$$

$$S_n(A^\vee) \otimes \Lambda(A^\vee)$$

~~Question~~

$$\overset{K}{H_A^*(X-X^A)} \xrightarrow{I} H_A^*(X) \xrightarrow{J} H_A^*(X^A) \xrightarrow{\delta} H_A^{*+1}(X-X^A)$$

$*+1$
K

$$\to H_u^*(I) \to H_u^*(H_A X) \to H_u^*(J) \xrightarrow{\delta}$$

$$H_u^*(J) \xrightarrow{\sim} H_u^{*}(K)$$

$$H_u^*(K) \to H_u^*(H_A(X-X^A)) \to H_u^*(J) \to \cdots$$

---

u, v independent

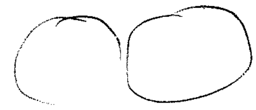
$u(m) = 0$

$v(m) = u(m')$

$$(\beta v)(m) = \beta(u(m')) \pm v(\beta m)$$
$$= (\beta u)\, m' \pm u \cdot \beta m' \pm v(\beta m)$$

$$(\beta v)(m) + u \cdot \beta m' = v(\beta m) + (\beta u)\, m'$$

$u(\beta m)$
$= \beta u \cdot m$



$L_1 - \cdots - L_m$

G group. vague idea that Euler char
vague idea these measures

<u>idea:</u> $A = \mathbb{Z}/p\mathbb{Z}$

Then use mod p equiv. coh. get formulas for

$\chi(X^A)$

$\chi_c((X-X^A)/A)$

but $\chi(X) = \chi_c(X-X^A) + \chi(X^A) = p\,\chi_c(X/A - X^A) + \chi(X^A)$

$\chi(X/A) = \chi_c(X/A - X^A) + \chi(X^A)$

but if A acts trivially on the rational cohomology of X, then

$$\chi(X) = \chi(X/A)$$

so one gets interesting relation between the Euler character of $X - X^A$ which is a fine invariant of the equivariant cohomology mod p and the repn of A on $H^*(X,\mathbb{Q})$

every element of A of order p look at X = flag manifold of a faithful representation.

$$
\begin{tikzcd}
K^\circ(X/G) \arrow[r] \arrow[d] & K_\cdot(X/G) \arrow[dl, "\pi^*"'] \\
K_G(X) \arrow[r, "\pi_*"] & K_\cdot(X/G) \otimes R(G)
\end{tikzcd}
$$

commutes

direct summand

point is that we can define character for elements of $K_\cdot(X/G)$ resolve them on X $\qquad (f_* f^* F)^G = F$

Actually suppose F sheaf on X/G. Then $f^* F$

mod 2 cohomology G odd.

$$
\begin{tikzcd}
\mathcal{H}(X)^G \arrow[r] \arrow[d, shift left, "\cong"] & \mathcal{H}(Y)^G \arrow[d] \\
\mathcal{H}(X/G) \arrow[r, "(f/G)_*"] \arrow[u, shift left] & \mathcal{H}(Y/G)
\end{tikzcd}
$$

$\pi_* \pi^* f = f$

now this whole square projects onto homology mod 2

thing to understand

X manifold open

~~$K \subset X$ compact~~ $\qquad f : X \longrightarrow \mathbb{Z}_2$ very constructible function which is harmonic. Then can define Stiefel-Whitney-Sullivan classes for f.

characteristic classes for orbit spaces

$$X \xrightarrow{f} Y$$

$$X/G \qquad Y/G$$

Atiyah situation X/G complex algebraic variety, hence have a map $K_\cdot(X/G) \xrightarrow{f_!} K_\cdot(Y/G)$
on other hand have

$$\begin{array}{ccc} K^\cdot(X/G) & & \\ \downarrow & & \\ K_G(X) & \xrightarrow{f_!} & K_G(Y) \end{array}$$

So given a vector bundle E over X isotropy repns. are all trivial. Then $R^q f_*(E)$ now a sheaf on Y, in fact a G-sheaf. But there is a possibility that the complex Rf_* OKAY. Yes because if you factor first by a closed embedding it's OKAY, the point is that nonsingularity of X implies K_G formed out of coherent or loc. free sheaves is the same.

So the idea should be the same
Analogue should be now the assertion that $f_!$ maps $K^\cdot(X/G)$ ~~$K_\cdot(X/G) \to K_G$~~ to $K^\cdot(Y/G)$. I think of $K^\cdot(X/G)$ as the K_G harmonic fns.

other possibility is to dualize. Thus in old theory ΛT^* module C^* dual to $S(T)$-module M and correct formulas where $\mathrm{Ext}_{\Lambda T^*}(\ ,k)$

Then one takes

$$\underset{h}{\beta}(\underset{d}{u}m) = (\beta u)m + \underset{d}{u}\cdot\underset{h}{\beta} m$$

$$A^\vee \otimes M \longrightarrow M$$

and dualizes

$$A \otimes M'_n \longleftarrow M'_{n+1}$$

and forms $S(A)\otimes M'$ with these differentials like the above Γ-complex

$$\longrightarrow \Gamma_n A^\vee \otimes M_p \longrightarrow \Gamma_{n-1} A^\vee \otimes M_{p+1} \longrightarrow \cdots$$

This thing has advantage of giving finite complexes.

---

go back to $A = \mathbb{Z}/p$ and see what the difference is. ~~easier~~. M free.

$$\Gamma_n A^\vee \otimes \Lambda_p A^\vee$$

$$\Gamma_\cdot A^\vee \otimes (\Lambda A^\vee) \qquad \text{free resolution of } k$$

and hence this computes

$$\mathrm{Tor}_\cdot^{\Lambda A^\vee}(k, M)$$

~~MB~~

$\mathbb{Z}_2$-graded modules $M^{ev} \oplus M^{odd}$ over H_A^* together with β of degree -1 derivation

$$\boxed{\beta(fm) = \beta f \cdot m + (-1)^{\deg f} f \cdot \beta m}$$

$H^*(u, M)$ β work here?

$$u(\beta m) = \beta(um) \pm (\beta u) m$$

$um = 0 \overset{?}{\Longrightarrow} u(\beta m) = 0$ NO $u(\beta m) = \beta u \cdot m$

<u>Problem</u>: work with iterated cohomology.

~~so the still need to see why $\beta u = 0$.~~

so I still want to see why iterated homology is finite. : One possibility would be to show

u gen $(\cancel{A}/B)^\vee$

$$H_A^*(X - X^A) \longrightarrow H_A^* X \longrightarrow H_A^*(X^A)$$

$$\longrightarrow HH_A^i \longrightarrow H^i I \longrightarrow$$

X compact space

$$\begin{tikzcd} H_c^i(X_r) \arrow[r] & H_c^i(X_{r-1}) \\ & H_c^i(X_r - X_{r-1}) \arrow[ul] \end{tikzcd}$$

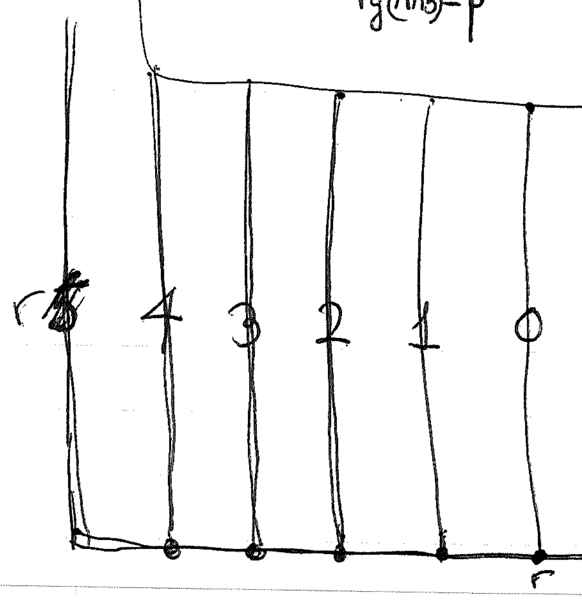
H^i

$$\begin{tikzcd} \arrow[d] & & \\ H^{i-1}(X_{r-1}) \arrow[r] \arrow[d] & H^{i}(X_r - X_{r-1}) \arrow[r] & H^i(X_r) \arrow[r] \arrow[d] & \\ H^{i-1}(X_{r-2}) & & H^i(X_{r-1}) \end{tikzcd}$$

$$H^i(X_r - X_{r-1}) \longrightarrow H^{i+1}(X_{r+1} - X_r)$$

$$E_1^{pq} = H_A^{p+q}(X_p - X_{p-1}) \Longrightarrow H_A^{p+q}(X).$$

$$\boxed{E_1^{pq} = \bigoplus_{rg(A/B)=p} H_A^{p+q}(X^{(B)}) \Longrightarrow H_A^{p+q}(X)}$$



This is completely clear.

dimension r.

~~no p~~ clear

$S \subset S(A^\vee)$ be the elements which don't vanish in $S(B^\vee)$.
Then localization wrt. S kills $H^*_{B'}$ for all $B' \not\supset B$
since $\exists\ \lambda \in A^\vee$, $\lambda|B \neq 0$, $\lambda|B' = 0$

enough to do additively

$$S^{-1}H^*_A(X) = S^{-1}H^*_A(X^B)$$

$$\cong (S^{-1}H^*_B) \otimes H^*_{\cancel{B/A}\ A/B}(X^B)$$

Compute iterated cohomology wrt basis of $(A/B)^\vee$

$$H^*_A(X) \longmapsto \mathcal{H}((A/B)^\vee, H^*_A(X))$$

assert last is a finite module over H^*_B

~~localized~~ Can localize at generic point

$u \in (A/B)^\vee$

$u \cdot x = 0$

$\Longrightarrow (\beta u)x = -u(\beta x)$

$\Longrightarrow \beta u$ acts trivially on $\mathcal{H}(u, H^*_A(X))$

so obvious induction reduces one to proving that procedure works with $B = 0$.

must show that iterated homology of $H^*_A(X)$ wrt $A^\vee$ yields a finite module (clear) with χ

$u \cdot x = 0$

$(\beta u)x = u \cdot \beta x$

use the spectral sequence of a filtered space.

characterize the type of modules obtained.

Suppose $H_A^*(X)$

$$M^+ \underset{e_i}{\overset{\longleftarrow}{\longrightarrow}} M^- \qquad e_i^2=0 \qquad e_ie_j+e_je_i=0.$$

I can take iterated homology with respect to a basis. Steenrod modules.

question 1. Suppose get finite homology one way. do you get also any other way.

Let

$$X_\nu \subset X$$

subspace whose isotropy groups are order $\geq p^{r-\nu}$
rank $\geq r-\nu$

$X^A = X_r$

$$\phi \subset X_r \subset X_{r-1} \subset \cdots \subset X_0 = X$$

<u>closed</u>.

$\cancel{H_A^*(X^{(0)})}$

A maximal in G finite, can you find an additive function constructed from ~~the~~

$$H_G^*(X)_p \quad \text{as a } (H_G^*)_p \text{ module}$$

suppose ~~G~~ cyclic order p^a.

A min. subgroup.

X^A set of non-free points

$X - X^A$ free.

what is $H_G^*(X^A)$ like.

suppose G/A acts freely on X.
what might $H_G(X)$ be.

spec. seq. $H^*_{G/A}(H_A^*(X)) \Rightarrow H_G^*(X)$

$\qquad\qquad\qquad\qquad\qquad\; \shortparallel$

$\qquad\qquad\qquad\qquad\qquad H^*($

orbit space spectral sequence

$$E_2 = H^*(X/G, H^*(A)) \quad \text{unknown.}$$

If G/N acts freely on X
one should be able to say something
about $H^*_G(X)$.

obstructions local. coeff. system

$$0 \longrightarrow \pi_1(X) \longrightarrow \pi_1(X/G) \longrightarrow G/N$$

and G/N acts on H^*_N. So first must assume
G/N acts trivially on H^*_N.

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main problem is to show that

$$H^*_G(X) \longrightarrow H^*_N \qquad\qquad X/G \text{ conn.}$$

is surjective, if so

$$H^*_G(X) \underset{\text{add}}{\cong} H^*(X/G) \otimes H^*_N .$$

first example $G = \mathbb{Z}/p^2$, $N = p\mathbb{Z}/p^2\mathbb{Z}$.
Then H^*_N generated by $u: N \xrightarrow{\sim} \mathbb{Z}/p\mathbb{Z}$ in H^1_N.
so problem is whether $H^1_G(X) \twoheadrightarrow H^1_N$

$$\begin{array}{c} 0 \to H^1(X/G) \to H^1_G(X) \to H^0(X/G, H^1_N) \xrightarrow{d} H^2(X/G) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad H^2_G(X) \end{array}$$