not very important
(i) char nos. in H^*
(ii) Cartier curves
(iii) equiv. K– idea that can define char. if you assume $P_n \to 0$.

February 1, 1969:

Let $\theta : \Omega(pt) \longrightarrow H_*(BU, \mathbb{Z})$ be the chern numbers map.

$$\begin{array}{ccc} \pi_n(MU) & \xrightarrow{\text{Hurewicz}} & H_n(MU) \\ & & \Big\updownarrow \wr \;\text{Thom.} \\ & & H_n(BU) \;=\; \mathrm{Hom}_{\mathbb{Z}}(H^n(BU), \mathbb{Z}) \end{array}$$

Thus given a weakly cx manifold Z^n

$$\begin{array}{ccc} Z^n & \xrightarrow{\;f\;} & BU(N) \\ \downarrow \iota & & \downarrow \iota' \\ S^{n+N} & \xrightarrow{\;f'\;} & MU(N) \end{array}$$

Recall Hurewicz given by

$$\alpha\left(f'_*[S^{n+N}]\right) = (f'^*)\alpha\,[S^{n+N}].$$

$$\therefore\; \iota'_*(c^\beta) f'_*[S^{n+N}] = (f'^*)(\iota'_*)\, c^\beta\, [S^{n+N}]$$

$$= \iota_* f^* c^\beta\, [S^{n+N}]$$

$$= \int_{Z^n} f^* c^\beta = \int_{Z^n} c^\beta(\nu).$$

Thus $\theta : \Omega_n(pt) \longrightarrow H_*(BU, \mathbb{Z})$ is given by sending

$$[Z^n] \longmapsto \left(c^\beta \mapsto \int_{Z^n} c^\beta(\nu)\right).$$

Now we have the convenient description

$$H_*(BU) = \mathbb{Z}[b_1, b_2, \ldots]$$

where the b_i comes by the map $H_*(BU(1)) \longrightarrow H_*(BU)$ from the elements $b_i \in H_{2i}(BU(1))$ with $\langle b_i, c_1^i \rangle = 1$. Moreover the diagonal on $H_*(BU)$ is given by

$$\Delta b_i = \sum_{j+k=i} b_j \otimes b_k \qquad b_0 = 1.$$

<u>Problem:</u> What is $\theta(P_n)$?

We therefore have to calculate Chern nos. of P_n. It is convenient instead of using the monomials $c^\alpha = c_1^{\alpha_1} \cdots$ in the Chern classes to use the basis c_α dual to b^α e.g.

$$\langle b_\alpha, c_\beta \rangle = \delta_{\alpha\beta}.$$

Then

$$\Delta c_\alpha = \sum_{\beta+\gamma=\alpha} c_\beta \otimes c_\gamma \qquad \text{e.g. } c_\alpha(E+F) = \sum_{\beta+\gamma=\alpha} c_\beta(E)\, c_\gamma(F)$$

and the c_α with $|\alpha| = \alpha_1 + 2\alpha_2 + \cdots = q$ form a base for $H^{2q}(BU)$.

Now

$$\nu_{P^n} = -(n+1)\,\mathcal{O}(1)$$

$\therefore$ If $\underline{t} = (t_1, t_2, \cdots)$ and $c_{\underline{t}} = \sum c_\alpha t^\alpha$, then

$$c_{\underline{t}}(\nu_{P^n}) = \frac{1}{c_{\underline{t}}(\mathcal{O}(1))^{n+1}}$$

But for a line bundle L we have

$$c_\alpha(L) = \sum_{i=0}^{\infty} \langle b_i, c_\alpha \rangle\, c_1(L)^i$$

and $\langle b_i, c_\alpha \rangle = 0$ unless $\alpha = \delta_i = (0, \ldots, 1, \ldots)$ ith place

Thus

$$c_{\underline{t}}(\mathcal{O}(1)) = \sum_i t_i \, c_1(\mathcal{O}(1))^i .$$

$$c_1(\mathcal{O}(1)) = H$$

so

$$c_{\underline{t}}(\nu_{\mathbb{P}^n}) = \frac{1}{\left(\sum_i t_i H^i\right)^{n+1}}$$

Thus $c_\alpha(\nu_{\mathbb{P}^n})$ = coefficient of t^α in $c_t(\nu_{\mathbb{P}^n})$.

Hence the linear functional on $H^n(BU)$ represented by $\mathbb{P}^n$ is given by

$$c_\alpha \quad |\alpha| = n \longmapsto \int_{\mathbb{P}^n} c_\alpha(\nu_{\mathbb{P}^n})$$

$$= \text{coefficient of } H^n \cdot t^\alpha \text{ in } \frac{1}{\left(\sum t_i H^i\right)^{n+1}}$$

But this linear functional written out as a linear combination of b_α is

$$\sum b_\alpha \text{ coeff of } H^n \text{ in } c_\alpha(\nu_{\mathbb{P}^n}) = \text{coeff of } H^n \text{ in } c_{\underline{b}}(\nu_{\mathbb{P}^n})$$

$$= \text{coeff. of } H^n \text{ in } \frac{1}{\left(\sum b_i H^i\right)^{n+1}}$$

$$\therefore \theta(P_n) = \operatorname{res}\left\{ \frac{dH}{\left(H \sum_{i \geq 0} b_i H^i\right)^{n+1}} \right\}$$

Now repeat old argument: $\bar{H} = H \sum_{i \geq 0} b_i H^{i+1}$ $\quad H = Q(\bar{H})$

$$dH = Q'(\bar{H})\, d\bar{H}.$$

$$\theta(P_n) = \operatorname{res}\left\{ \frac{Q'(\bar{H})\, d\bar{H}}{\bar{H}^{n+1}} \right\}.$$

$$\Rightarrow \quad Q'(\bar{H}) = \sum{}' \theta(P_n)\, \bar{H}^n$$

$$\Rightarrow H = Q(\bar{H}) = \sum_{n\geq 0}{}' \theta(P_n)\, \frac{\bar{H}^{n+1}}{n+1}.$$

<u>Conclusion</u>: $\theta(P_n)$ is recursively determined by the equation

$$H = \sum_{n\geq 0}{}' \theta(P_n) \frac{H^{n+1}}{n+1} \left[\sum_{i\geq 0}{}' b_i H^i\right]^{n+1}$$

In other words

if $\chi(H)$ is the inverse to $\sum_{i\geq 0}{}' b_i H^{i+1}$, then

$$\boxed{\begin{gathered} \sum_{n\geq 0}{}' \theta(P_n)\, \frac{x^{n+1}}{n+1} = \chi(x). \\ \text{where} \quad \chi\Big(\sum_{i\geq 0} b_i H^{i+1}\Big) = H \end{gathered}}$$

In view of our conjectures that $\Omega(pt.)$ is the ground ring for the universal formal commutative group law in 1 variable it is useful to interpret the map

$$\Omega(pt) \longrightarrow H_*(BU) = \mathbb{Z}[b_1, b_2, \ldots]$$

as representing the morphism of functors

$$\Lambda(A) = \{1 + a_1 t + \cdots \mid a_i \in A\} \longrightarrow \mathcal{F}(A) = \text{formal gp laws over } A$$

given by sending a power series

$$f(t) = \sum_{i \geq 0} a_i t^i \qquad a_0 = 1$$

into the formal group law

$$X * Y = g(g^{-1}(X) + g^{-1}(Y))$$

where $g(t) = t f(t)$.

On the theorems of Cartier:

A formal group of dimension $\underbrace{n}_{\text{with coordinates}}$ over a ring A may be identified with a comult.

$$\Delta : A[[X_1, \cdots, X_n]] \longrightarrow A[[X_1 \otimes 1, \cdots, X_n \otimes 1, 1 \otimes X_1, \cdots, 1 \otimes X_n]]$$

$$\Delta \underline{X} = F(\underline{X}, \underline{Y})$$

which is associative and such that $F(\underline{X}, 0) = F(0, \underline{X}) = \underline{X}$.

A curve in the formal group is a power series $\underline{q}(t) = (q_1(t), \cdots q_n(t))$ with $q(0) = 0$. Cartier asserts that ~~there is a unique~~ given a curve c in formal $\overbrace{\text{group}}^{\text{comm}}$ $\overbrace{}^{\text{over } A}$ there is a unique morphism

$$A[[c_1, c_2, \cdots\cdots]] \xleftarrow{\ \Phi\ } A[[\underline{X}]]$$

compatible with the comultiplication such that

$$\begin{cases} c_1 \rightsquigarrow t \\ c_q \rightsquigarrow 0 \qquad q > 1. \end{cases}$$

i.e. $\Phi(X_i)(t, 0, \cdots) = q_i(t)$.

<u>Special case:</u> $\left| \begin{array}{l} n=1 \\ g(t)=t \end{array} \right.$ $\quad A[[X]] \xrightarrow{\Delta} A[[X,Y]]$, $\quad X \longmapsto F(X,Y)$

According to Cartier $\exists!$ map $A[[X]] \xrightarrow{\Phi} A[[c_1, c_2, \cdots]]$, $X \longmapsto f(c_1, c_2, \cdots)$

such that

(i) compatible with Δ e.g.

$$F(f(c_1, c_2, \cdots), f(c_1', c_2', \cdots)) = f(c_1 + c_1', c_2 + c_1 c_1' + c_2', \cdots)$$

(ii) $f(t, 0, \cdots\cdots) = t$

In fact the power series f is determined by the formulas for all n

$$f(c_1, \cdots, c_n, 0, \cdots\cdots) = X_1 * \cdots * X_n$$

where as usual $\quad c_t = \prod_{i=1}^{n} (1 + tX_i)$

~~Gen~~ $\left| \begin{array}{l} n=1 \\ g(t) \text{ arbitrary} \ni g(0)=0 \end{array} \right.$

$$f(c_1, \cdots, c_n, 0, \cdots) = g(X_1) * \cdots * g(X_n)$$

This formula even works for $n \geq 1$.

For example there is a canonical morphism

$$\Omega[[H]] \xrightarrow{\ \Phi\ } \Omega[[c_1, c_2, \cdots]] = \mathrm{Hom}(\tilde{K}, \Omega)$$

$$H \longmapsto f(c_1, c_2, \cdots)$$

given by

$$f(c_1, c_2, \cdots c_n, 0, \cdots) = X_1 * \cdots * X_n$$

where $\quad c_t = \prod (1 + tX_i)$

such that

$$\begin{cases} f(c(E+F)) = F(f(cE), f(cF)) \\ f(c(L)) = c_1(L). \end{cases}$$

It seems that $\quad f(c(E)) = \underline{c_1(\det E)}.$

Not very interesting

Feb 2, 1969

On Equivariant cobordism theory

$$\begin{tikzcd} \Omega_{S_1}(pt) \arrow[r, "\rho"] & \Omega(BS^1) = \Omega[[H]] \\ R(S^1) = \mathbb{Z}[T, T^{-1}] \arrow[u, "c_1"] & \end{tikzcd}$$

where $\rho c_1(T) = H$

$$\rho c_1(T^k) = H * H * \cdots * H \qquad k \text{ times}$$
$$= \psi^{-1}(k\psi(H))$$

where $\psi(X) = \sum_{n\geq 0} P_n \dfrac{H^{n+1}}{n+1}$

Question: Are the elements $\rho c_1(T^k) \in \Omega[[H]]$ algebraically independent over Ω? It seems likely since an algebraic relation involves only finitely many P_j in the coefficients. ~~If so then how can you construct~~

In any case can you construct a cohomology theory for S^1 manifold with ground ring $\Omega[X_k]_{k\in\mathbb{Z}}$ where $X_k = c_1(T^k)$?

Remark: If you replace Ω by $\hat{\Omega} = \prod_{q\geq 0} \Omega_q$. (this has effect of making the P_i topologically nilpotent), then

$$ch_q L = \frac{1}{q!}\psi(c_1(L))^q$$

is a restricted power series in $c_1(L)$ over $\hat{\Omega}$, hence one might be able to define an equivariant cohomology theory $V_G(X)$

as the universal recipient for maps

$$ch_q : K_G(X) \longrightarrow V_G(X) \qquad\qquad q \geq 0$$

such that

$$\left.\begin{cases} ch_q(x+y) = ch_q(x) + ch_q(y) \\ ch_q(xy) = \sum\limits_{i+j=q} ch_i x \cdot ch_j(y) \\ ch_q(\psi^k x) = k^q \, ch_q x. \\ ch_0(x) = rg(x). \end{cases}\right\} \Rightarrow V_G(X) = \hat{\Omega} \otimes gr K_G(X)$$

and in turn define

$$c_1(L) = \psi^{-1}(ch_1 L).$$

alg. gp interp of Hopf algebra $\mathbb{Z}[S_\infty]$
how to invert a power series

February 3, 1969

Operations in cobordism theory (after Novikov + Adams).

Let F be a cohomology theory on manifolds endowed with a gysin homomorphism for U-oriented proper maps. Then we have

$$\begin{array}{ccl}
\bar\beta \quad \varphi & & \mathrm{Hom}(\tilde K, F) = \{\varphi : \tilde K(X) \to F(X) \text{ all } X \text{ compatible with } f^*\} \\
\uparrow \quad\ \downarrow & & \quad \wr\| \\
\beta \quad \hat\varphi & & \mathrm{Hom}^{st}(\Omega, F) = \{\varphi : \Omega(X) \to F(X) \text{ compatible with } f^* \text{ all } f,\ f_* \text{ for all } f \text{ with } \nu_f = 0\}
\end{array}$$

$$\boxed{\hat\varphi(f_* 1) = f_* \varphi(\nu_f).} \qquad \boxed{\bar\beta(E) = \iota_*^{-1} \beta \iota_* 1 \text{ where } \iota : X \to E \text{ zero section}}$$

Note that if F has products, then

$$\begin{array}{c} \varphi(x+y) = \varphi(x)\varphi(y) \\ \varphi(0) = 1. \end{array} \iff \begin{array}{c} \hat\varphi(u \cdot v) = \hat\varphi(u) \cdot \hat\varphi(v) \\ \hat\varphi(1) = 1 \end{array}$$

So now applying this to $F(X) = \Omega(X) \otimes_{\Omega(pt)} R$ where R is an algebra over $\Omega(pt)$, we find that

$$\mathrm{Aut}^{\otimes}(\Omega \otimes_{\Omega(pt)} R / R) \cong \left\{ \varphi : \tilde K \to \Omega \otimes_{\Omega(pt)} R \ : \ \begin{array}{l} \varphi(x+y) = \varphi(x)\varphi(y) \\ \varphi(0) = 1 \end{array} \right\}$$

By the splitting principle ~~such a φ is determined by a power series $\sum_{i \ge 0} a_i X^i$ $a_0 = 1$, $a_i \in R$ by the rule~~

$$\varphi(L) = \sum_{i \ge 0} a_i\, c_1(L)^i.$$

for line bundles L. ~~Therefore~~

Applying this to $F = \Omega(X)\otimes_{\mathbb{Z}} R$ where R is a $\mathbb{Z}$-algebra, we have

$$\mathrm{Autst}^{\otimes}(\Omega\otimes_{\mathbb{Z}} R) = \left\{\hat{\varphi} : \Omega \longrightarrow \Omega\otimes_{\mathbb{Z}} R \;\middle|\; \hat{\varphi}(u\cdot v) = \hat{\varphi}(u)\cdot\hat{\varphi}(v)\right\}.$$

Homst (not autos)

$$= \left\{\varphi : \tilde{K} \longrightarrow \Omega\otimes_{\mathbb{Z}} R \;\middle|\; \begin{array}{l}\varphi(x+y) = \varphi(x)\varphi(y)\\ \varphi(0) = 1\end{array}\right\}.$$

By the splitting principle such a φ is determined by a power series $\sum_{i\geq 0} a_i X^i$ $\quad a_0 = 1$, $a_i \in$ ~~$R\otimes$~~ $\Omega(pt)\otimes_{\mathbb{Z}} R$ by

$$\varphi(L) = \sum a_i\, c_1(L)^i \qquad \text{for line bundles } L.$$

Hence as functors of R (to sets) we have

Homst (not autos) $\mathrm{Autst}^{\otimes}(\Omega\otimes_{\mathbb{Z}} R) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[b_1, b_2, \cdots], \Omega(pt)\otimes_{\mathbb{Z}} R)$.

~~Next we determine the composition~~ To a power series $\varphi(X) = \sum_{i\geq 0} a_i X^i$ $a_0 = 1$, one associates the operation

$$\tilde{\varphi}(f_* 1) = f_* \overset{+}{\varphi}(\nu_f)$$

where $\quad \overset{+}{\varphi}(L) = \varphi(c_1(L)) \quad + \quad \overset{+}{\varphi}(E+F) = \overset{+}{\varphi}(E)\,\overset{+}{\varphi}(F)$.

Consequently given two power series $\varphi_1(X)$ and $\varphi_2(X)$ we have

$$\begin{aligned}\tilde{\varphi}_1(\tilde{\varphi}_2 f_* 1) &= \tilde{\varphi}_1\, f_*\, \overset{+}{\varphi}_2(\nu_f)\\ &= f_*\{\overset{+}{\varphi}_1(\nu_f)\, \tilde{\varphi}_1\, \overset{+}{\varphi}_2(\nu_f)\} = f_*\, \overset{+}{\varphi}_3(\nu_f) = \tilde{\varphi}_3(f_* 1)\end{aligned}$$

~~But~~ where $\quad \overset{+}{\varphi}_3(L) = \overset{+}{\varphi}_1(L)\cdot\tilde{\varphi}_1\,\overset{+}{\varphi}_2(L)$

Now $\quad \tilde{\varphi}_1 \, \overset{\perp}{\varphi}_2(L) = \tilde{\varphi}_1 \sum a_i^{(2)} \, c_1(L)^i$

$$= \sum_i \tilde{\varphi}_1(a_i^{(2)}) \left[\tilde{\varphi}_1(c_1(L))\right]^i$$

and $\quad \tilde{\varphi}_1 \, c_1(L) = \tilde{\varphi}_1 \, \iota^* \iota_* 1 \qquad \iota : X \longrightarrow L$ zero section

$$= \iota^* \iota_* \overset{\perp}{\varphi}_1(L) = c_1(L) \overset{\perp}{\varphi}_1(L)$$

$$= c_1(L) \sum a_i^{(1)} c_1(L)^i$$

$$\therefore \; c_1(L) \overset{\perp}{\varphi}_3(L) = \sum_{i \geq 0} \tilde{\varphi}_1(a_i^{(2)}) \left\{ c_1(L) \overset{\perp}{\varphi}_1(L) \right\}^{i+1}$$

In other words if instead (we consider) the power series

$$\psi_j(X) = \sum_{i \geq 0} a_i^{(j)} X^{i+1} \qquad j = 1, 2, 3$$

and write $\tilde{\varphi}$ for the operation, then we have that

$$\boxed{\psi_3(X) = \psi_2^{\tilde{\varphi}_1}(\psi_1(X))}$$

where

$$\psi_2^{\tilde{\varphi}_1}(X) = \sum_{i \geq 0} \tilde{\varphi}_1(a_i^{(2)}) X^i$$

Conclude: For any $\mathbb{Z}$-algebra R

Hamst$^{\otimes}$

$$\mathrm{Aut}^{\otimes}(\Omega \otimes_{\mathbb{Z}} R / R) = \mathrm{Hom}\left(\mathbb{Z}[b_1, b_2, \ldots\ldots], \Omega \otimes_{\mathbb{Z}} R\right)$$

with composition given above. Unfortunately this seems to make $\Omega' \otimes \mathbb{Z}[b_1, b_2, \ldots]$ into a Hopf algebra. (?).

So consider the subfunctor of $\mathrm{Aut}^{\otimes}(\Omega \otimes_{\mathbb{Z}} R/R)$ consisting of operations given by power series $\sum_{i \geq 0} a_i X^{i+1}$, $a_0 = 1$ with $a_i \in R$. Then this is closed under composition. Denote it

$$\mathrm{Aut}'^{\otimes}(\Omega \otimes_{\mathbb{Z}} R/R) = \mathrm{Hom}(\mathbb{Z}[b_1, \ldots], R)$$

$$= \left\{ \sum_{i \geq 0} a_i X^{i+1} \;\middle|\; a_0 = 1,\ a_i \in R \right\}$$

$$\mathrm{Hom}(\mathbb{Z}[b], R) \times \mathrm{Hom}(\mathbb{Z}[b], R) \longrightarrow \mathrm{Hom}(\mathbb{Z}[b], R)$$

$$\psi_1(X) \times \psi_2(X) \longmapsto \underline{\psi_2(\psi_1(X))}.$$

$$\sum_i \Delta b_i X^{i+1} = \sum_j 1 \otimes b_j \left(\sum_i (b_i \otimes 1) X^{i+1} \right)^{j+1}$$

~~In Adams notation where one puts X = 1 and considers b_i as of weight i~~

or

$$\sum_{i\geq 0}(\Delta b_i)X^i = \sum_{j\geq 0}\Big(\sum_{i\geq 0} b_i X^i\Big)^{j+1}\otimes(b_j X^j)$$

or finally in Adams' notation where one sets $X=1$ and regards b_i as of degree i this becomes

$$\Delta b = \sum_{j\geq 0} b^{j+1}\otimes b_j$$

where $b = \sum_{i\geq 0} b_i$.

Clearly

$\mathrm{Aut}'^{\otimes}(\Omega\otimes_{\mathbb{Z}}R/R)$ is the subfunctor of $\mathrm{Aut}^{\otimes}(\Omega\otimes_{\mathbb{Z}}R/R)$ generated by the operations $\hat{c}_a = s_a$ corresponding to the Chern class

$$c_a : \tilde{K} \longrightarrow \Omega\otimes_{\mathbb{Z}}R$$

$$c_a(x+y) = c_a(x)\,c_a(y) \qquad c_a(0) = 1$$

$$c_a(L) = \sum_{i\geq 0} a_i\, c_1(L)^i \qquad a_0 = 1,\quad a_i \in R.$$

In other words

$$\mathrm{Hom}(\mathrm{Spec}\,R, \underline{\mathrm{Aut}}'^{\otimes}\Omega) \cong \{c_a : \tilde{K} \longrightarrow \Omega\otimes_{\mathbb{Z}}R\}_{a=(a_1,a_2,\dots)\in R^{\mathbb{N}}} \cong \{\hat{s}_a : \Omega \longrightarrow \Omega\otimes_{\mathbb{Z}}R\}_{a=(a_1,\dots)\in R^{\mathbb{N}}}$$

(What relation has above general nonsense to the fact that the coordinate ring of $\underline{\mathrm{Aut}}'^{\otimes}\Omega$ is the dual of the Hopf algebra of Novikov operations?

Let H be the dual of $\coprod_\alpha \mathbb{Z} s_\alpha$. Then an ~~co~~algebra map $H \longrightarrow R$ is same as elements $r_\alpha \in R \ni \varphi = \sum r_\alpha s_\alpha$ satisfies $\Delta\varphi = \varphi \otimes \varphi$. $\Longrightarrow$ $r_\beta r_\gamma = r_{\beta+\gamma} \Longrightarrow r_\alpha = r^\alpha$ where $r = (r_1, r_2, \cdots) \in R^I$. Thus $H = \mathbb{Z}[b_1, b_2, \cdots\cdots]$.)

A more pedestrian point of view:

$$\mathcal{a} = \mathrm{Homst}(\Omega, \Omega) \simeq \mathrm{Hom}(\tilde{R}, \Omega) = \Omega[[c_1, c_2, \cdots]]$$

Thus any element in $\mathrm{Hom}(\tilde{R}, \Omega)$ of the form $\sum a_\alpha c^\alpha$ $\quad a_\alpha \in \Omega$ or better $\sum a_\alpha c_\alpha$ (infinite sums). Every element of $\mathcal{a}$ is of the form $\sum a_\alpha \cdot s_\alpha$ infinite sums. To determine the algebra structure of $\mathcal{a}$ we must know how to compute $s_\alpha(a)$ and $s_\alpha s_\beta$.

$$s_t(f_* 1) = f_* \, c_t(\nu_f)$$

$$f : P^n \longrightarrow pt. \qquad \nu_f = -(n+1)\mathcal{O}(1)$$

$$s_t(f_* 1) = f_* \frac{1}{c_t(\mathcal{O}(1))^{n+1}} = f_* \frac{1}{\left(\sum t_i H^i\right)^{n+1}}$$

But

$$\frac{1}{\left(\sum t_i H^i\right)^{n+1}} = \sum_\alpha \left(c_\alpha, b^{-n-1}\right) t^\alpha H^{|\alpha|}$$

$$\therefore \qquad s_t(P_n) = \sum_\alpha \left(c_\alpha, b^{-n-1}\right) t^\alpha P_{n-|\alpha|}$$

or

$$\boxed{s_\alpha P_n = \left(c_\alpha, b^{-n-1}\right) P_{n-|\alpha|}} \qquad \text{(Adams–Novikov)}$$

How to invert a power series.

Proposition: If $\bar{H} = \sum_{i \geq 0} a_i H^{i+1}$ $\qquad a_0 = 1$, $a_i \in R$,

then

$$H = \sum_{\alpha} \frac{1}{1+|\alpha|} (c_\alpha, b^{-1-|\alpha|})\, a^\alpha\, \bar{H}^{1+|\alpha|}$$

and

$$H^{g} = \sum_{\alpha} \frac{g}{g+|\alpha|} (c_\alpha, b^{-g-|\alpha|})\, a^\alpha\, \bar{H}^{g+|\alpha|},$$

where $\quad (c_\alpha, b^{-N}) =$ coefficient of t^α in $\left(\sum_{i \geq 0}^{\infty} t_i\right)^{-N}$, $\quad t_0 = 1$.

In particular $\qquad \frac{g}{g+|\alpha|} (c_\alpha, b^{-g-|\alpha|}) \in \mathbb{Z}$.

Proof: Enough to prove the formula over $\mathbb{Q}$ as it will then follow by extension of algebraic identities. Introduce new variables P_n, $P_0 = 1$. As

$$\left(\sum_{i \geq 0} t_i H^i\right)^{-(n+1)} = \sum (c_\alpha, b^{-n-1})\, t^\alpha H^{|\alpha|},$$

we have

$$\sum_{\alpha} (c_\alpha, b^{-n-1})\, t^\alpha P_{n-|\alpha|} = \operatorname{res} \left[\frac{\sum_{i \geq 0} P_i H^i \, dH}{\left(H \sum_{i \geq 0} t_i H^i\right)^{n+1}} \right]$$

$$= \operatorname{res} \left(\frac{\sum P_i\, \varphi(\bar{H})^i\, \varphi'(\bar{H})\, d\bar{H}}{\bar{H}^{n+1}} \right) \qquad \text{for all } n \text{ where}$$

$$\varphi(\bar{H}) = H.$$

Thus

$$\sum_{i\geq 0} P_i\, \varphi(\bar{H})^i\, \varphi'(\bar{H}) = \sum_n \sum_\alpha (c_\alpha, b^{-n-1}) t^\alpha P_{n-|\alpha|}\, \bar{H}^n$$

Integrate wrt $\bar{H}$

$$\sum_{i\geq 0} P_i \frac{H^{i+1}}{i+1} = \sum_n \sum_\alpha (c_\alpha, b^{-n-1}) t^\alpha P_{n-|\alpha|} \frac{\bar{H}^{n+1}}{n+1}$$

Comparing coefficients of P_{g-1} we have $\qquad n-|\alpha| = g-1$

$$n = g+|\alpha|-1$$

$$\frac{H^g}{g} = \sum_\alpha (c_\alpha, b^{-g-|\alpha|})\, t^\alpha \frac{\bar{H}^{g+|\alpha|}}{g+|\alpha|}$$

QED.

$$\left(\sum_{i\geq 0} t_i\right)^N = \left(1+\sum_{i\geq 1} t_i\right)^N = \sum_\beta \frac{N!}{\beta!\,(N-\langle\beta\rangle)!}\, t^\beta$$

where $\langle\beta\rangle = \sum_{i\geq 1} \beta_i$ $\qquad \beta! = \beta_1!\,\beta_2!$

$|\beta| = \sum_{i\geq 1} i\,\beta_i$

$$\left(\sum_{i\geq 0} t_i\right)^N = \sum_\beta \frac{N(N-1)\cdots(N-\langle\beta\rangle+1)}{\beta!}\, t^\beta$$

This formula should hold for N negative too. Thus

$$\boxed{(c_\alpha, b^{-g-|\alpha|}) = \frac{(-g-|\alpha|)(-g-|\alpha|-1)\cdots(-g-|\alpha|-\langle\alpha\rangle+1)}{\alpha!}}$$

$$= (-1)^{\langle\alpha\rangle} \frac{(|\alpha|+\langle\alpha\rangle)\cdots(|\alpha|+1)}{\alpha!} \qquad \text{if } g=1$$

Therefore

$$\boxed{\bar{H} = \sum_{i\geq 0} a_i H^i \quad a_0 = 1 \implies H = \sum_{\alpha} \frac{(|\alpha| + \langle\alpha\rangle)\cdots(|\alpha|+2)}{\alpha!} (-a)^{\alpha} \bar{H}^{|\alpha|+1}}$$

$$(c_\alpha, b^{-n-1}) = \frac{(-n-1)\cdots(-n-\langle\alpha\rangle)}{\alpha!}$$

$$= (-1)^{\langle\alpha\rangle} \frac{(n+1)\cdots(n+\langle\alpha\rangle)}{\alpha!}$$

$$\boxed{s_\alpha(P_n) = (-1)^{\langle\alpha\rangle} \frac{(n+1)\cdots(n+\langle\alpha\rangle)}{\alpha!} P_{n-|\alpha|}}$$

Preceding proof shows that

$$\sum_{n=0}^{\infty} s_t(P_n) \frac{\left(H \sum_{i\geq 0} b_i H^i\right)^{n+1}}{n+1} = \sum_{n=0}^{\infty} P_n \frac{H^{n+1}}{n+1}$$

According to our conjecture $\Omega(pt)$ with $F(X,Y)$ given by $F(c_1L, c_1L') = c_1(L\otimes L')$ is the universal formal group law in one variable. On the other hand the Novikov algebra $\sum' \mathbb{Z} S_\alpha$ acts on $\Omega(pt.)$ or if one prefers there is an action of $\underline{Aut}'^{\otimes}\Omega$ on $Spec\,\Omega(pt)$. The obvious action is to make power series $\xi(x) = \sum' a_i X^{i+1}$, $a_0 = 1$ act on the group law F by

$$(\xi\cdot F)(X,Y) = \xi\{F(\xi^{-1}X, \xi^{-1}Y)\}$$

But

$$\Omega(pt) \xrightarrow{\tilde{\xi}} \Omega(pt)$$

$$F(X,Y) \longmapsto F^{\tilde{\xi}}(X,Y)$$

$$\cancel{\psi(x) \longmapsto (\tilde{\xi}\psi)(x)}$$

$$\cancel{\psi^{\xi}(X) = \sum{}' \xi(P_n)\cdot\frac{X^{n+1}}{n+1}}$$

~~But~~ $\cancel{\tilde{\xi}\psi(c_1(L)) = \psi^{\tilde{\xi}}(\tilde{\xi}c_1(L)) = \psi^{\tilde{\xi}}(\xi(c_1(L)))}$

~~Hence new group law given by the power series~~

$$\cancel{\therefore \psi^{\tilde{\xi}}(c_1(L)) = \tilde{\xi}\psi(\xi^{-1}(c_1(L)))}$$

What is $F^{\tilde{\xi}}(X,Y) = \psi^{\tilde{\xi}\,-1}(\psi^{\tilde{\xi}}(X) + \psi^{\tilde{\xi}}(Y))$, where

$$\psi^{\tilde{\xi}}(X) = \sum \tilde{\xi}(P_n)\cdot\frac{X^{n+1}}{n+1}$$

But

$$\tilde{\xi}(P_n) = f_*^n\left\{\left(\frac{\xi(H)}{H}\right)^{-n-1}\right\} = \mathrm{res}\left(\frac{\sum P_i H^i\, dH}{\xi(H)^{n+1}}\right) \quad \text{all } n$$

$\Rightarrow$ as usual

$$\sum \tilde{\xi}(P_n)\,\frac{\xi(H)^{n+1}}{n+1} = \psi(H)$$

$$\therefore \quad \psi^{\tilde{\xi}}(\xi(H)) = \psi(H) \quad \text{or} \quad \boxed{\psi^{\tilde{\xi}} = \psi\circ\xi^{-1}}$$

$$\Longrightarrow \quad F^{\tilde{\xi}}(X,Y) = \xi\,\psi^{-1}(\psi\,\xi^{-1}X + \psi\,\xi^{-1}Y)$$
$$= \xi\,F(\xi^{-1}X, \xi^{-1}Y).$$

which is what we conjectured!

Thom isomorphism

Novikov operations:

Given $\varphi : \tilde{K} \longrightarrow \Omega$ natural set map for f^*.

Given X and an element $u \in \Omega(X)$ represent u by

$$u = f_* 1 \qquad f : Z \longrightarrow X \quad \text{proper oriented}$$

and consider

$$f_* \, \varphi(\nu_f) \in \Omega(X).$$

where $\nu_f = f^*\theta_X - \theta_Z \in \tilde{K}(Z)$. Claim that this element depends only on u and not the choice of f. In effect given another representation $u = f'_* 1$ $f : Z' \longrightarrow X$, we may form usual diagram.

$$\begin{tikzcd}
Z \arrow[rr, "j_0"] \arrow[dd, "f"'] & & W \arrow[dd, "h"] \\
& Z' \arrow[ur, "j_1"] \arrow[dl, "f'"] & \\
X \arrow[rr, shift left, "i_0"] \arrow[rr, shift right, "i_1"'] & & X \times S^1
\end{tikzcd}$$

Then $\quad j_0^* \nu_h = \nu_f, \quad j_1^* \nu_h = \nu_{f'} \quad$ so

$$f_* \, \varphi(\nu_f) = f_* \varphi(j_0^* \nu_h) = f_* j_0^* \varphi(\nu_h) = i_0^* h_* \varphi(\nu_h)$$

$$\| $$

$$f'_* \, \varphi(\nu_{f'}) = f'_* \varphi(j_1^* \nu_h) = f'_* j_1^* \varphi(\nu_h) = i_1^* h_* \varphi(\nu_h)$$

Thus we ~~suggest~~ obtain

$$\hat{\varphi} : \Omega(X) \longrightarrow \Omega(X)$$

given by

$$\boxed{\hat{\varphi}(f_* 1) = f_*(\varphi(\nu_f)).}$$

<u>Properties</u>: (i) $\hat{\varphi}(u+v) = \hat{\varphi}(u) + \hat{\varphi}(v)$

(ii) If $f : X \to Y$ is (proper oriented) such that $\nu_f = 0$, then

$$\hat{\varphi} f_* = f_* \hat{\varphi} \qquad \text{(stability)}$$

(iii) $\widehat{\varphi + \varphi'} = \hat{\varphi} + \hat{\varphi'}$ where $(\varphi+\varphi')(x) = \varphi(x) + \varphi'(x)$.

(iv) If $f : X \to Y$ is arbitrary, then

$$\hat{\varphi} f^* = f^* \hat{\varphi}.$$

<u>Proof</u>: (i) Given $u, v \in \Omega(X)$ represent them as $f_* 1$, $g_* 1$ where $f : Z \to X$, $g : Z' \to X$. Then $f_* 1 + g_* 1$ is represented by $f+g : Z \vee Z' \longrightarrow X$ and

$$(f+g)_*(\varphi(\nu_{f+g})) = (f+g)_*(i_*\varphi(\nu_f) + i_*\varphi(\nu_g))$$

$$= f_* \varphi(\nu_f) + g_* \varphi(\nu_g).$$

(ii) Given $u \in \Omega(X)$ represented as $g_* 1$, $g : Z \to X$ and given $f : X \to Y$ with $\nu_f = 0$, we have

$$\hat{\varphi}(f_* u) = \hat{\varphi}((fg)_* 1) = f_* g_* \varphi(\nu_{fg}) = f_* g_* (\varphi(\nu_g + g^* \nu_f))$$
$$= f_*(g_* \varphi(\nu_g)) = f_* \hat{\varphi}(u)$$

(iii) clear

(iv) Assume $u = g_* 1$ and g transversal to f. Then

$$\hat{\varphi} f^*(u) = \hat{\varphi} f^*(g_* 1) = \hat{\varphi}(g'_* 1) = g'_*(\varphi(\nu_{g'})) = g'_* \varphi f'^* \nu_g$$
$$= g'_* f'^* \varphi(\nu_g) = f^* g_* \varphi(\nu_g) = f^* \hat{\varphi}(u).$$

<u>Example</u>: Let $\varphi : \tilde{K} \longrightarrow \Omega$ be $\ni$ $\varphi(x) = 1_X$ for all $x \in \tilde{K}(X)$. Then $\hat{\varphi} = id$ on Ω.

<u>Remark</u>. If F is any cohomology theory with Thom isom for complex bundles, then can generalize above and define

$$\mathrm{Map}(K, F) \longrightarrow \mathrm{Hom}^*(\Omega, F)$$
$$\varphi \longmapsto \hat{\varphi}$$

$\boxed{\leftarrow \text{ functorial for maps } F \to F' \text{ compatible with Gysin}}$

defined by $\hat{\varphi}(f_* 1) = f_*(\varphi(\nu_f))$.

Given $\beta : \Omega \longrightarrow F$ set map compatible with f^*, set

$$\varphi(x) = \beta(1) \qquad \text{for all } x \in K(X)$$

Then $f^*(\varphi(x)) = \varphi(f^* x)$ so we get $\varphi \in \mathrm{Hom}(K, F)$.

<u>Remark</u>: Let F range over the category of cohomology functors with Thom isomorphism for complex bundles and define by $\mathrm{Hom}\,\mathrm{Gys.}(F_1, F_2)$ those additive natural transformation compatible with both f^*, f_* and $\mathrm{Hom}(F_1, F_2)$ those compatible with just f^*. Let Map denote not necessarily additive natural transformations. Then generalizing the above we have defined

$$\boxed{\mathrm{Map}(\tilde{K}, F) \xrightarrow{\sim} \mathrm{Homst}(\Omega, F)}$$

$$\varphi \longmapsto \hat{\varphi}\,, \quad \hat{\varphi}(f_* 1) = f_*(\varphi(\nu_f))\,,$$

where Homst denotes natural transformations satisfying (i) (ii) + (iv) on page 2.

<u>Proposition</u>: The above map is an isomorphism inverse map being given by $\beta \longmapsto \bar{\beta}$ where

$$\bar{\beta}(E) = (\pi^E)_* \, \beta \, \iota^E_* 1$$

where $\pi^E : E \longrightarrow X$ and $\iota_E : X \longrightarrow E$ are standard maps.

<u>Proof</u>: This is just the standard Thom isomorphism

$$F(BU) \simeq F(MU).$$

But can give direct proof. Thus

$$\bar{\hat{\varphi}}(E) = \pi^E_* \hat{\varphi} \iota^E_* = \pi^E_* \iota^E_* \varphi(\nu_i) = \varphi(E).$$

and given $f: Z \longrightarrow X$ factor it

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[dr, "f"'] & V \arrow[d, "p"] \\ & X \end{tikzcd}$$

where i is zero section of a vector bundle and p open in $X \times S^n$ hence $\nu_p = 0$. Then

$$\hat{\bar{\beta}}(f_* 1) = f_*(\bar{\beta}(\nu_f)) = f_*(\bar{\beta}(\nu_i))$$

$$= f_* \pi_* \beta \iota_* 1$$

$$= p_* \iota_* \pi_* \beta \iota_* 1$$

$$= p_* \beta \iota_* 1 \qquad \text{since } \beta i_* 1 \text{ is proper} + \iota\pi \sim id$$

$$= \beta(p_* \iota_* 1) \qquad \text{since } \beta \text{ stable.}$$

$$= \beta(f_* 1)$$

QED.

This shows that if β satisfies (ii) + (iv) on page 2 it satisfies (i) also.

Terminology:

$\varphi \longmapsto \hat{\varphi}$ is the "char no." associated to the function φ.

$\beta \longmapsto \bar{\beta}$ is the ~~Wu-class associated~~ the operation β.

For Chern classes of vector bundles in H^* we have the formulae

$$c_1(xy) = (\mathrm{rg}\,x)\,c_1(y) + (\mathrm{rg}\,y)\,c_1(x)$$

$$\begin{aligned} c_2(xy) = {} & \binom{\mathrm{rg}\,x}{2} c_1(y)^2 + \binom{\mathrm{rg}\,y}{2} c_1(x)^2 \\ & + (\mathrm{rg}\,x)\,c_2(y) + (\mathrm{rg}\,y)\,c_2(x) \\ & + (\mathrm{rg}\,x \cdot \mathrm{rg}\,y - 1)\,c_1(x)\,c_1(y) \end{aligned}$$

showing that we need the binomial structure on $H^0(X,\mathbb{Z})$ to express the universal formula for $\tilde{c}(xy)$ in terms of $\tilde{c}(x)$ and $\tilde{c}(y)$.

February 5, 1969 obselete see June 5, 69 1

Characteristic numbers (revisited):

Let F be a cohomology theory with products and a Gysin homomorphism for ~~[illegible]~~ U-oriented maps. For each ring R we can define functors

$$\mathrm{Homst}_{\otimes}(\Omega, F\otimes R) = \left\{ \varphi : \Omega(X) \longrightarrow F(X)\otimes_{\mathbb{Z}} R \;\middle|\; \begin{array}{l} \varphi f^* = f^*\varphi \\ \varphi f_* = f_*\varphi \text{ if } \nu_f = 0 \\ \varphi(u\cdot v) = \varphi(u)\cdot\varphi(v) \\ \varphi(1) = 1 \end{array} \right\}$$

φ ring homom.

$$\mathrm{Map}_+(\tilde{K}, F\otimes R) = \Big\{ \alpha : \tilde{K}(X) \longrightarrow F(X)\otimes_{\mathbb{Z}} R \;\Big|$$

$$\alpha(x+y) = \alpha(x)\,\alpha(y)$$
$$\alpha(0) = 1$$
$$\alpha f^* = f^*\alpha \qquad \Big\}.$$

and we have that these functors are isomorphic by rules

$$\alpha \longmapsto \hat{\alpha} \qquad \hat{\alpha}(f_*1) = f_*(\alpha(\nu_f))$$

$$\varphi \longmapsto \bar{\varphi} \qquad \bar{\varphi}(E) = \iota_*^{-1}(\varphi\, \iota_* 1)$$

where $\iota : X \to E$ zerosection.

The splitting principle allows us to conclude that

$$\mathrm{Map}_+(\tilde{K}, F\otimes R) \cong \left\{ \sum_{i\geq 0} a_i X^i ,\ a_0 = 1 \;\middle|\; a_i \in F(pt)\otimes R \right\}$$

$$\alpha(L) = \sum_{i\geq 0} a_i\, c_1(L)^i$$

Hence

$$Map_+(\tilde{K}, F\otimes R) \cong Hom(\mathbb{Z}[b_1, b_2, \cdots], F(pt)\otimes R).$$

If $\alpha(X) = \sum a_i X^i$ ~~[illegible]~~ where multiplication $a_i \in F(pt)\otimes R$, let $\alpha_{\underline{a}}(E)$ be the operation on bundles given by

$$\alpha_{\underline{a}}(L) = \sum_{i\geq 0} a_i (c_1(L))^i$$

and let $\hat{\alpha}_{\underline{a}}$ be the corresponding operation $\Omega \to F\otimes R$. When $\underline{a} = \underline{b}$ in $R = \mathbb{Z}[b_1, b_2, \cdots]$ we get the universal operations

$$\alpha_{\underline{b}} : \tilde{K} \longrightarrow F\otimes \mathbb{Z}[b_1, b_2, \cdots]$$

$$\hat{\alpha}_{\underline{b}} : \Omega \longrightarrow F\otimes \mathbb{Z}[b_1, \cdots].$$

Then we can form the characteristic numbers map

$$(*) \qquad \hat{\alpha}_{\underline{b}} : \Omega(pt) \longrightarrow F(pt)\otimes \mathbb{Z}[b_1, \cdots]$$

given by

$$\hat{\alpha}_{\underline{b}}(P_n) = \sum_{\alpha} (c_\alpha, b^{-n-1})\, b^\alpha \rho(P_{n-|\alpha|})$$

where $\rho : \Omega(pt) \longrightarrow F(pt)$ is $\rho(f_*^{\Omega}1) = f_*^{F}1$, or equivalently by

$$(**) \qquad \sum_{n\geq 0} \hat{\alpha}_{\underline{b}}(P_n) \frac{\left(\sum_{i\geq 0} b_i X^{i+1}\right)^{n+1}}{n+1} = \sum_{n\geq 0} \rho(P_n) \frac{X^{n+1}}{n+1}$$

~~Ex~~ *Examples*:

1) $F = H^*(\ , \mathbb{Z})$. Then for any ring R and $a_1, \cdots \in R$ we can form the $\underline{a}$-characteristic numbers

$$\Omega(pt) \longrightarrow R$$

$$(f_* 1) \longmapsto f_*[\alpha_{\underline{a}}(\nu_f)]$$

$$P_n \longmapsto \sum_\alpha (c_\alpha, b^{-n-1})\, a^\alpha\, \rho(P_{n-|\alpha|})$$

$$= \sum_{|\alpha|=n} (c_\alpha, b^{-n-1})\, a^\alpha .$$

In particular for $\underline{a} = (b_1, \ldots$ where $R = \mathbb{Z}[b_1, b_2, \cdots]$ $= H_*(BU, \mathbb{Z})$. We get

$$\Omega(pt) \longrightarrow \mathbb{Z}[b_1, b_2, \cdots] = H_*(BU, \mathbb{Z}).$$

$$P_n \longmapsto \sum_{|\alpha|=n} (c_\alpha, b^{-n-1})\, b^\alpha$$

or

$$P_n \longmapsto \sum_{|\alpha|=n} \frac{(n+1)\cdots(n+\langle\alpha\rangle)}{\alpha!} (-b)^\alpha$$

where $\langle\alpha\rangle = \alpha_1 + \cdots$

2) $F = K$. Then ~~we~~ have $\rho(P_n) = 1$ all n

$$\Omega(pt) \longrightarrow \mathbb{Z}[b_1, \cdots] = \cancel{\mathrm{Hom}}^{cont}_{\mathbb{Z}}(K(BU), \mathbb{Z})$$

$$P_n \longmapsto \sum_{|\alpha| \le n} \frac{(n+1)\cdots\cdots(n+\langle\alpha\rangle)}{\alpha!} (-b)^\alpha$$

Stong–Hattori theorem:

version 1: The image of $\Omega_{2n}(pt) \longrightarrow H_{2n}(BU, \mathbb{Q})$ consists of all elements z such that

$$(z,\ ch\,\alpha \cdot Todd^{-1}) \in \mathbb{Z}$$

for all $\alpha \in K(BU)$ and where $ch\,\alpha \cdot Todd^{-1} \in H^*(BU, \mathbb{Q})^\wedge$.

version 2: The map $\cancel{\Omega_{2n}(pt) \longrightarrow Hom^{cont}_{\mathbb{Z}}(K(BU), \mathbb{Z})}$ $K(BU) \longrightarrow Hom_{\mathbb{Z}}(\Omega_{2n}(pt), \mathbb{Z})$ given by $\alpha \longmapsto (f_*1 \longmapsto f_*\alpha(\nu_f))$ is surjective.

The versions are related by the diagram

$$\begin{tikzcd}
\alpha \arrow[d, mapsto] & K(BU) \arrow[r, "\alpha \longmapsto ch\,\alpha \cdot Todd^{-1}"] \arrow[d, dashed] & H^*(BU, \mathbb{Q})^\wedge \arrow[d] & \varphi \arrow[d, mapsto] \\
(f_*1 \mapsto f_*\alpha(\nu_f)) & Hom(\Omega_{2n}(pt), \mathbb{Z}) \arrow[r, hook] & Hom(\Omega_{2n}(pt), \mathbb{Q}) & ([M] \mapsto \int_M \varphi(\nu_M))
\end{tikzcd}$$

In effect version 1 by duality is equivalent to saying that elements φ of $H^{2n}(BU, \mathbb{Q})$ for which $\int_M \varphi(\nu_M) \in \mathbb{Z}$ all the same as 2k dimensional components of something of the form $ch\,\alpha \cdot Todd^{-1}$. This is the same by the diagram as the dotted arrow being surjective.

Observe that though

$$K(BU) \longrightarrow Hom(\Omega_{2n}(pt), \mathbb{Z})$$

is surjective for each n, it ~~does~~ does certainly not follow that

$$K(BU) \longrightarrow \mathrm{Hom}(\Omega(pt), \mathbb{Z})$$

is surjective.

<u>Remark</u>: The characteristic numbers map $*$ has the following interpretation in view of the conjecture $\hat{\alpha}_{\underline{b}}$ carries the group law ~~[illegible]~~ with logarithm

$$\psi(X) = \sum_{n \geq 0} P_n \frac{X^{n+1}}{n+1}$$

into the group law over $F(pt) \otimes \mathbb{Z}[\underline{b}]$ with logarithm $\sum \hat{\alpha}_b(P_n) \frac{X^{n+1}}{n+1}$. In virtue of $**$, this means ~~[illegible]~~ the group law is the one with logarithm $\sum \rho(P_n) \frac{X^{n+1}}{n+1}$ modified by the substitution $X \mapsto \sum b_i X^{i+1}$. Thus if the map $\Omega(pt) \longrightarrow \mathrm{Hom}(K(BU), K(pt))$ were an isomorphism the group law over $\Omega(pt)$ would be equivalent to $\mathbb{G}_m$ which is impossible.

February 6, 1969. (Proof that $\Omega(pt)$ is the Lazard ring)
(low dimensions of universal group law)

Stong-Hattori thm. Consider diagram

$$\begin{array}{ccc}
\Omega(pt) & \xrightarrow{A} & \mathrm{Hom}^{\mathrm{cont}}_{A}(K(BU), \mathbb{Z}) \\
\Big\downarrow B & & \Big\downarrow \\
& & \mathrm{Homcont}(K(BU), \mathbb{Q}) \\
& & \Big\uparrow C \ \wr \\
\mathrm{Homcont}(H^{**}(BU), \mathbb{Z}) & \hookrightarrow & \mathrm{Homcont}(H^{**}(BU, \mathbb{Q}), \mathbb{Q})
\end{array}$$

$$A(f_* 1) = \left(\alpha \in K(BU) \longmapsto f_! \alpha(\nu_f)\right)$$

$$B(f_* 1) = \left(\beta \in H^{**}(BU) \longmapsto f_* \beta(\nu_f)\right)$$

$$C(\varphi) = \left(\alpha \in K(BU) \longmapsto \varphi(ch\,\alpha \cdot \mathrm{Todd}^{-1})\right)$$

The diagram commutes by RR thm.

$$f_! \alpha(\nu_f) = f_*\left[ch\, \alpha(\nu_f) \cdot (\mathrm{Todd}\,\nu_f)^{-1}\right]$$

Next identify

$$\mathbb{Z}[a_1, a_2, \cdots] \simeq \mathrm{Homcont}(K(BU), \mathbb{Z})$$

$$\mathbb{Z}[b_1, b_2, \cdots] \simeq \mathrm{Homcont}(H^{**}(BU), \mathbb{Z})$$

where (a_i, α) is determined by $\alpha(L) = \sum_{i \geq 0} (a_i, \alpha)\, c_1^K(L)^i$

(b_i, β) is " " $\beta(L) = \sum_{i \geq 0} (b_i, \beta)\, c_1^H(L)^i$

and diagram becomes

$$\begin{tikzcd} \Omega(pt) \arrow[r, "A"] \arrow[d, "B"'] & \mathbb{Z}[\underline{a}] \arrow[d, "T"] \\ \mathbb{Z}[\underline{b}] \arrow[r, hook] & \mathbb{Q}[b] \end{tikzcd}$$

where

$$A(f_* 1) = f_* \, \varphi_{\underline{a}}(\nu_f)$$
$$B(f_* 1) = f_* \, \varphi_{\underline{b}}(\nu_f)$$

where $\varphi_{\underline{a}} : K \longrightarrow K \otimes \mathbb{Z}[\underline{a}]$ $\qquad \varphi_{\underline{b}} : K \longrightarrow H \otimes \mathbb{Z}[b]$

are the unique mult. operations given by

$$\varphi_{\underline{a}}(L) = \sum a_i (1-L^{-1})^i$$
$$\varphi_{\underline{b}}(L) = \sum b_i \, c_1^H(L)^i$$

One knows that

$$\sum A(P_n) \frac{\left(\sum a_i H^{i+1}\right)^{n+1}}{n+1} = -\log(1-H) \quad , \text{ the}$$

logarithm for the law $(AF)(X,Y) = X+Y-XY$

$$\sum B(P_n) \frac{\left(\sum b_i H^{i+1}\right)^{n+1}}{n+1} = H \quad , \text{ the}$$

logarithm for the law $(BF)(X,Y) = X+Y$. Thus A (resp. B) as a morphism of functors from powerseries $f(x) = \sum a_i X^{i+1}$ to group laws is

$$A: \quad f(X) \longmapsto F(X,Y) = f\left(f^{-1}X + f^{-1}Y - f^{-1}X \cdot f^{-1}Y\right)$$

$$B: \quad f(X) \longmapsto F(X,Y) = f\left(f^{-1}X + f^{-1}Y\right)$$

and I calculated that T is given by

$$\boxed{\sum_{i\geq 0}(Ta_i)H^i = \sum_{i\geq 0} b_i H^i\Big(\sum_{n\geq 0}\frac{H^n}{n+1}\Big)^{i+1}}$$

Showing that

$$Ta_i = b_i + \lambda_{i-1} b_{i-1} + \cdots + \lambda_i$$

where the $\lambda_i \in \mathbb{Q}$.

Stong-Hattori thm: $\Longrightarrow$ $\Omega(pt)$ is the largest homogeneous subring of $\mathbb{Q}[b]$ contained in $T(\mathbb{Z}[a])$. ($T(\mathbb{Z}[a])$ is the Atiyah-Hirzebruch subring of $H_*(BU, \mathbb{Q})$ consisting of elements z such that $(z, ch(\alpha)\cdot \mathrm{Todd}^{-1}) \in \mathbb{Z}$ for all $\alpha \in K(BU)$.)

The method of Stong consists of showing that (for each prime p) the image of $\Omega(pt) \xrightarrow{A} \mathbb{Z}[\underline{a}] \xrightarrow{\rho_p} \mathbb{F}_p[a]$ is a polynomial ring with a generator coming from $M_i^{(p)} \in \Omega_{2i}(pt)$ for each $i > 0$. Then by an algebraic lemma it follows that $\Omega(pt)/\text{torsion}$ is a polynomial ring and (in fact is) the largest homogeneous subring of $\mathbb{Q}[b]$ contained in $\mathbb{Z}[a]$.

$$
\begin{aligned}
M_i^{(p)} &= P_i \qquad \text{if} \quad i+1 \equiv 0 \quad (p)\\
&= H_{m,n} = c_1(\mathcal{O}(1)\times\mathcal{O}(1)) \text{ in } P^n \times P^m\\
&\qquad \text{if} \quad i+1 \text{ not } p^s \quad \text{where } m,n \text{ are something}\\
&= P_{p-1} \qquad i = p-1\\
&= c_1(\underbrace{\mathcal{O}(1)\times\cdots\times\mathcal{O}(1)}_{p\text{ times}}) \text{ in } P_{p^{s-1}}\times\cdots\times P_{p^{s-1}} \quad \text{if } i+1 = p^s \quad s>0
\end{aligned}
$$

It follows that $\Omega(pt)/\text{torsion}$ (hence $\Omega(pt)$ by Milnor) is generated by the Chern classes of canonical line bundles in products of projective spaces. This means we can prove our conjecture.

Theorem: The group law over $\Omega(pt)$ ~~coming from~~ determined by

$$c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2))$$

is ~~the~~ the universal (formal commutative 1-variable) group law in the sense that given any ~~the~~ ring A and group law F' over A $\exists!$ map $\Omega(pt) \longrightarrow A$ sending F to F'.

Proof: Let ~~the~~ (Laz, F_0) be the ~~[illegible]~~ universal group law of Lazard, and $\varphi: Laz \longrightarrow \Omega(pt)$ that homomorphism sending F_0 to F. $\varphi \otimes_{\mathbb{Z}} \mathbb{Q}$ is an isomorphism because

$$Laz \otimes \mathbb{Q} = \mathbb{Q}[a_1, \dots]$$

where the logarithm for F_0 is

$$l_0(X) = \sum_{n \geq 0} a_n \frac{X^{n+1}}{n+1}$$

and because (Thom)

$$\Omega(pt) \otimes \mathbb{Q} = \mathbb{Q}[P_1, \dots]$$

and because

$$\varphi(a_i) = P_i \qquad \text{(Myshenko)}$$

But Lazard has shown that Laz is a polynomial ring over $\mathbb{Z}$ hence torsion free. Thus φ is injective.

We now show the generators of $\Omega(pt)$ comes from Laz. Given $Z = c_1(\mathcal{O}(1) \times \cdots \times \mathcal{O}(1))$ on $(\mathbb{P}_{p^s})^p$.

$$\begin{tikzcd}
 & \Omega(pt)[H_1, \ldots, H_p] / (H_1^{p^s+1}, \ldots H_p^{p^s+1}) \arrow[d, "f_*"] \\
\mathrm{Laz} \arrow[r, hook] & \Omega(pt)
\end{tikzcd}$$

Then $Z = H_1 * H_2 * \cdots * H_p \in \mathrm{Laz}[H_1, \ldots, H_p] / (H_1^{p^s+1}, \ldots, H_p^{p^s+1})$

so $f_*(Z) \in \mathrm{Laz}$.

QED.

Formulas:

If
$$\psi(X) = X + \frac{P_1}{2}X^2 + \frac{P_2}{3}X^3 + \cdots$$

$$F(X,Y) = X + Y + aXY + b(X^2Y + XY^2) + c(X^3Y + XY^3) + d(X^2Y^2) + \cdots$$

and $\psi(F(X,Y)) = \psi(X) + \psi(Y)$, then

$$a + P_1 = 0$$
$$b + aP_1 + P_2 = 0$$
$$c + bP_1 + aP_2 + P_3 = 0$$
$$d + \frac{a^2}{2}P_1 + 2bP_1 + 2aP_2 + \frac{3}{2}P_3 = 0$$

$$\begin{cases} a = -P_1 \\ b = P_1^2 - P_2 \\ c = -P_1^3 + 2P_1P_2 - P_3 \\ d = \dfrac{-5P_1^3 + 8P_1P_2 - 3P_3}{2} \end{cases}$$

showing that in dimension 6 $\Omega_6(pt)$ not generated by the monomials in the P_i.

Conner-Floyd version of Stong's calculation

ω denotes a partition $\{i_1, i_2, \dots, i_k\}$ $i_1 \geq i_2 \geq \cdots \geq i_k$ with 1 repeated α_1 times, 2 repeated α_2 times, etc.; $\underline{s}_\omega$ is the characteristic no. in K-theory

$$\underline{s}_\omega[P_n] = \frac{(n+1)!}{\alpha!\,(n+1-\alpha_1-\cdots-\alpha_s)}\binom{n}{\alpha_1+2\alpha_2+\cdots+s\alpha_s}.$$

N dual to $\mathcal{O}(1)\boxtimes\cdots\boxtimes\mathcal{O}(1)$ in $\mathcal{C}(P_{p^k})^p$. Then

$\underline{s}_{i_1,\dots,i_\ell}[N] = 0 \bmod p$ if $|\alpha| = \alpha_1+2\alpha_2+\cdots+s\alpha_s > p^{k+1}-p$

if $|\alpha| = p^{k+1}-p$, then

$s_{i_1\cdots i_\ell}[N] = 0\ (p)$ if $\ell < p$

$$s_{i_1\cdots i_p}[N] = \begin{cases} 0\ (p) & \text{if unless } i_j = p^k-1 \text{ all } j \\ 1\ (p) & \text{otherwise.} \end{cases}$$

Given a partitions $\omega = \{i_1 \geq i_2 \geq \cdots\}$ set $d(\omega) = \text{degree} = i_1+\cdots+i_k$

$n(\omega) = k$. If $\omega' = \{j_1 \geq \cdots \geq j_\ell\}$ set

$\omega' > \omega$ if $d(\omega') > d(\omega)$

or if $=$ & $n(\omega') < n(\omega)$

or if $=$ $=$

and $j_1 = i_1, \dots, j_s = i_s, j_{s+1} > i_{s+1}$.

{Proof that a typical law of height ∞ over a ring of char p is equal to $X+Y$}

February 9, 1967:

Cartier's theory of curves in formal groups.

K fixed ground ring. The formal line D over K is the functor $(K\text{-algs}) \longrightarrow (\text{sets})$ associating to each K-alg its set of nilpotent elements. This functor is pro-representable

$$D(R) \cong \varinjlim_{N} \mathrm{Hom}_{K_{alg}}(K[X]/(X^N), R)$$

$$(\text{or less precisely}) \cong \mathrm{Homcont}_{K\text{-alg}}(K[[X]], R).$$

A formal group G over K of dimension n is a functor $(K\text{-alg}) \longrightarrow (Ab)$ isomorphic to D^n as a functor to sets. ~~The isomorphism is a system of coordinates for G.~~

$$\cancel{G(R) \cong \mathrm{Homcont}(K[[X_1, \dots, X_n]], R)}$$

~~with group law given by~~

$$\cancel{\Delta : K[[\underline{X}]] \longrightarrow K[[\underline{X}]] \hat{\otimes} K[[\underline{X}]]}$$

$$\cancel{\Delta \underline{X} = \underline{F}(\underline{X} \otimes 1, 1 \otimes \underline{X})}$$

Such an isomorphism $\theta : G \to D^n$ is called a system of coordinates for G. ~~Ordinary~~ The origin $\theta(0) \in D^n(K)$ is a sequence of nilpotent elements of K, hence by a translation of D^n we may assume the coordinates are centered at 0, i.e. $\theta(0) = 0 \in D^n(K)$. Then the system of coordinates gives an isomorphism

$$G(R) \simeq \mathrm{Homcont}_{k\text{-alg}}(K[[X_1, \dots, X_n]], R)$$

with group law given by

$$\Delta : K[[\underline{X}]] \longrightarrow K[[\underline{X}]] \hat{\otimes} K[[\underline{X}]] \qquad \underline{X} = (X_1, \dots, X_n)$$

$$\Delta \underline{X} = \underline{F}(\underline{X} \otimes 1, 1 \otimes \underline{X}) \qquad \underline{F} = (F_1, \dots, F_n)$$

A curve in a formal group G is a morphism $\gamma : D \to G$ of functor to sets such that $\gamma(0) = 0$. The set of curves in G forms an abelian group $C(G)$ under addition. ~~with the following operations.~~

~~decalage~~

Given a coordinate system in G a curve is a homom. of rings.

$$\gamma : K[[\underline{X}]] \longrightarrow K[[t]]$$

$$X_i = \gamma_i(t) \quad \text{power series} \quad \gamma_i(0) = 0$$

and

$$(\underline{\gamma} + \underline{\gamma}')(t) = \mathbf{F}(\underline{\gamma}(t), \underline{\gamma}'(t)) \overset{\text{my notation}}{=} \underline{\gamma}(t) * \underline{\gamma}'(t)$$

Operations on curves:

<u>decalage</u>: $\quad (V_n \gamma)(t) = \gamma(t^n) \qquad n \geq 1$

<u>Frobenius</u>: $\quad (F_n \gamma)(t) = \gamma(t_1) * \cdots * \gamma(t_n) \qquad n \geq 1$

where $\prod_{i=1}^{n} (X - t_i) = X^n - t$.

<u>homothety</u>: If $c \in K \quad ([c]\gamma)(t) = \gamma(ct)$

Basic identities:

$$[V_n, V_m] = [F_n, F_m] = 0. \quad \text{better } V_nV_m = V_{nm} \quad F_nF_m = F_{nm}$$

$$[F_n, V_m] = 0 \quad \text{if } (n,m) = 1$$

$$\underline{F_n V_n} = n \cdot \mathrm{id}_C$$

<u>Theorem</u>: The functor $G \longmapsto C(G)$ is representable by the formal group $\hat{W}_K$ of infinite dimension given by

$$\hat{W}_K(R) = \{ 1 + r_1 t + \cdots \in R[[t]] \mid \exists N \text{ with } r^\alpha = 0 \quad |\alpha| \geq N \}$$

under multiplication.

More precisely given a curve $\gamma : D \longrightarrow G$ ~~[illegible]~~ there is a unique homomorphism of group functors $u : \hat{W}_K \longrightarrow G$ such that ~~[illegible]~~ for any $r \in D(R)$ we have $u(1 - rt) = \gamma(r)$. In concrete terms u is given by

$$K[[\underline{X}]] \longrightarrow K[[\underline{a}]]$$

$\underline{a} = (a_1, a_2, \ldots$

$\Delta \underline{a} = \underline{a} \otimes \underline{a}$

$$u : \underline{X}_i \longrightarrow \underline{f}(\underline{a})$$

where

$$\underline{f}(a_1, a_2, \ldots, a_n, 0 \ldots 0) = \underline{\gamma}(z_1) * \cdots * \underline{\gamma}(z_n)$$

where $\displaystyle 1 + \sum_{i=1}^{n} a_i Z^i = \prod_{i=1}^{n} (1 - z_i Z)$.

<u>Remark</u>: One can then carry over the operations F_n, V_n to $\hat{W}_K$. This theorem may eventually be important to me because of the fact that the coordinate ring of $\hat{W}_K$, i.e. $K[[\underline{a}]]$ is the completed Hopf algebra $K(BU)$, in such a way that $a_i \leftrightarrow \lambda^i$.

as λ^i don't go to zero.

Now suppose K is an algebra of $\mathbb{Z}_{(p)}$, the integers localized at p. Then one define $CT(G) \subset C(G)$ "typical curves" as those $\gamma \ni F_q \gamma = 0$ for all q prime to p. Cartier's projection operator

$$\sum_{(n,p)=1} \frac{\mu(n)}{n} V_n F_n = \prod_{\substack{\text{all primes} \\ q \neq p}} \left(1 - \frac{V_q F_q}{q}\right)$$

projects $C(G)$ onto $CT(G)$. Every curve is uniquely expressible in the form

$$\gamma = \sum_{(n,p)=1} V_n \gamma_n \qquad \text{with } \gamma_n \in CT(G)$$

~~in such a way that~~ so that there is a canonical isomorphism

$$C\cancel{T}(G) \simeq \prod_{(n,p)=1} CT(G)$$

This corresponds to the standard decomposition

$$\hat{W}_K \cong \prod_{(n,p)=1} \hat{W}_{p^\infty K}.$$

Let $F(R)$ = the set of formal group laws ~~of~~ of dimension 1 over R, ~~a formal group law~~ that is, power series $F(X,Y) \in R[[X,Y]]$ such that

$$F(F(X,Y),Z) = F(X,F(Y,Z))$$
$$F(X,Y) = F(Y,X)$$
$$F(X,0) = F(0,X) = X.$$

Such a formal group law is a formal group G of dimension 1 over R endowed with a coordinate, that is, a curve $\gamma: D \longrightarrow G$ whose derivative at the origin $\gamma': D \longrightarrow \mathfrak{g}$ is an isomorphism. We say that a formal group law (R over $\mathbb{Z}_{(p)}$) is typical if the coordinate $\gamma: D \longrightarrow G$ is a typical curve. Cartier's projector carries γ into a curve

$$\gamma' = \sum_{(n,p)=1} \frac{\mu(n)}{n} V_n F_n \gamma$$

which is again a coordinate for G since $(V_n F_n \gamma)(t)$ is a power series in t^n. ~~for~~ Thus we have a functor $FT(R)$ = typical formal group laws, which being a retract of a representable functor is again representable. Let N be the group scheme over $\mathbb{Z}$ given by

$$N(R) = \left\{ \sum_{i\geq 0} a_i X^{i+1} \;\middle|\; a_0 = 1,\ a_i \in R \right\}$$

with group law given by composition

$$(f \cdot g)(X) = f(g(X))$$

Then $N(R)$ acts on $F(R)$ by

$$(f\cdot F)(X,Y) = f(F(f^{-1}X, f^{-1}Y)).$$

Denoting ~~[illegible]~~ by N also the base change of the N over $\mathbb{Z}$ to $\mathbb{Z}_{(p)}$, Cartier's projector gives us a map s

$$\boxed{\begin{array}{l} i: FT \hookrightarrow F \\[1em] N\times FT \underset{\pi}{\overset{s}{\rightleftarrows}} F \end{array}}$$

$$\pi(n,\alpha) = n\cdot i(\alpha).$$

$$\pi s = id$$

If p is not a divisor of zero in K, and $L = K[\frac{1}{p}]$, then we can define $\log: G_L \longrightarrow g_L$ and a curve $\gamma(t)$ is typical iff

$$\log \gamma(t) = \sum \frac{m_j t^{p^j}}{p^j} \qquad \text{with } m_j \in g_L$$

In fact $m_j \in g_K$. ~~[illegible]~~ The logarithm l satisfies

$$l(F(X,Y)) = l(X) + l(Y)$$

$$\text{or} \quad l'(X)\cdot F_y(X,0) = 1$$

$$l'(X) = \frac{1}{F_y(X,0)}$$ should ~~[illegible]~~ contain only powers of X^{p^h-1}. ~~[illegible]~~

[Side remark: On a ~~[illegible]~~ formal group G over K an arbitrary ring ~~[illegible]~~, there is a unique invariant differential form ω on G with values in $\mathfrak{g}$ such that at the identity $\omega_0 : \mathfrak{g} \longrightarrow \mathfrak{g}$ is the identity. For one variable $\mathfrak{g} \simeq \tilde{G}_a$ by means of dX, so we want

$$\omega = f(X)\,dX \qquad f(0) = 1$$

which is invariant i.e. remains same under $X \longmapsto Y * X$
e.g.

$$f(Y*X)\,d(Y*X) = f(Y*X)\,F_2(Y,X)\,dX = f(X)\,dX$$

$$\xRightarrow{(X=0)} \qquad f(Y)\,F_2(Y,0) = 1 \quad \text{or that}$$

$$\omega = \frac{dX}{F_y(X,0)}\,.$$

ω and l are related by the formula

$$\boxed{\omega = dl}$$

showing that in <u>characteristic zero</u> l always exists.)

<u>Proposition</u>: If $F(X,Y)$ is a typical law over K, a $\mathbf{F}_p$-algebra, and if $\underbrace{X * \cdots * X}_{p\ \text{times}} = 0$, then $F(X,Y) = X+Y$.

<u>Proof</u>: (Using machinery of Lazard-Cartier). It is enough to consider the problem universally. Thus consider the functor R: ($\mathbb{Z}_{(p)}$-algebras) to (**sets**) associating to R the set of formal group laws

---

Proof when K has no nilpotent elements: Then $x \mapsto x^p$ from K to K is injective so by enlarging K we may assume it is perfect. Then $W(K) = R$ is a torsion-free ring with $R/pR \xrightarrow{\sim} K$. By Lazard the group lifts to R and so we obtain a group law F over R with $X * \cdots * X$ (p times) $\equiv 0 \mod pR$. Let l be the logarithm of F over R

$$l(X) = \sum_{n \geq 1} a_n \frac{X^n}{n} \qquad a_1 = 1, \quad a_n \in R.$$

Then

$$p \cdot l(X) = l(X^{*p}) = \sum_{n \geq 1} a_n \frac{(X^{*p})^n}{n}.$$

so

$$l(X) = \sum_{n \geq 1} a_n \frac{(p \cdot g(X))^n}{np}.$$

But

$$\frac{p^n}{np} = \frac{p^{n-1}}{n} \in \mathbb{Z}_{(p)}, \quad \text{so } l(X) \in R[[X]]$$

$\Rightarrow$ $l(F(X,Y)) = l(X) + l(Y)$ over R and hence over K. By Cartier's projection we could have assumed that F was typical over R, hence that l has only X^{p^n}. Then over K we have $l(F(X,Y)) = l(X) + l(Y)$ where

$$l(X) = \sum a_n X^{p^n}$$

Thus l is an automorphism of $\hat{\mathbb{G}}_a$, hence l^{-1} is also so

$$F(X,Y) = l^{-1}(l(X) + l(Y)) = X + Y.$$

Proof in general: suppose ~~[illegible]~~ $F(X,Y) \neq X+Y$ and let n be the ~~[illegible]~~ order of $F(X,Y)-X-Y$. By Lazard (Bull Math Soc France 83 (1955) 251-274 ~~[illegible]~~ proposition 2) there is $\lambda \in K$ such that

$$F(X,Y) = X+Y+\lambda C_n(X,Y)$$

where $C_n(X,Y) = c\{(X+Y)^n - X^n - Y^n\}$

$\left\{ \begin{array}{l} c = 1 \text{ if } n \text{ not a power of a prime} \\ c = \frac{1}{q} \text{ if } n = q^a, \text{ some prime } q \end{array} \right.$

Case 1: $n = p^a$. Then we will show that $\lambda \neq 0 \Longrightarrow X^{*p} \neq 0$. We may calculate X^{*p} by means of formulas in torsion free rings using the logarithm, here given by

$$\ell(X) \equiv X - \lambda c X^n \quad \text{mod deg } n+1$$

observe c not a unit hence this doesn't make sense in char p

$$\left(X+Y+\lambda c\{(X+Y)^n - X^n + Y^n\} - \lambda c(X+Y)^n \equiv X - \lambda c X^n + Y - \lambda c X^n \right)$$

$$p\ell(X) = \ell(X^{*p}) \equiv X^{*p} - \lambda c (X^{*p})^n$$

$$\equiv pX - p\lambda c(X^n)$$

$$\Longrightarrow \quad X^{*p} \equiv pX - p\lambda c(X^n) + \lambda c\, p^n X^n \quad \text{(mod deg } n+1)$$

$$X^{*p} \equiv pX + \lambda c\{p^n - p\}X^n \quad \text{(mod deg } n+1)$$

This formula is valid in characteristic p. As $n = p^a$, $c = \frac{1}{p}$ and it becomes

$$X^{*p} \equiv \lambda \underline{(p^{n-1}-1)} X^n \neq 0 \quad \text{if } \lambda \neq 0$$

This contradicts assumption that $X^{*p} = 0$.

<u>Case 2</u>: n not a power of p. Then we show that the law is not typical. Set

$$\ell(X) = X - \lambda c X^n \qquad \text{(defined since } c \in (\mathbb{F}_p)^*\text{)}$$

so that

$$\ell(F(X,Y)) \equiv \ell(X) + \ell(Y) \qquad (\text{mod deg } n+1).$$

Write $n = p^a k$ where $(k,p) = 1$, $k > 1$. Then we show that $F_k X \neq 0$.

$$F_k X = t_1 X * \cdots * t_k X \qquad \text{where } \prod_{i=1}^{k} (Z - t_i) = Z^k - 1$$

$$\ell(F_k X) \equiv \sum_{i=1}^{k} \ell(t_i X) \equiv \sum_{i=1}^{k} (t_i X - \lambda c X^n)$$

$$\equiv -\lambda c k X^n \qquad (\text{mod deg } n+1)$$

$$\neq 0 \quad \text{if } \lambda \neq 0.$$

QED

~~Actually the proof gives a slightly better result:~~

~~**Proposition:** Let F be a typical formal group law over a $\mathbb{Z}_{(p)}$ algebra R. If F is of infinite height, i.e. $X^{*p} \equiv 0$ (mod pR), then $F(X,Y) = X + Y$.~~

February 10, 1969:

Application of typical (formal group laws) in cobordism theory.

Notation: We work over $\mathbb{Z}_{(p)}$, the integers localized at p. $\Omega(X)$ denotes the complex cobordism of X tensored with $\mathbb{Z}_{(p)}$. If F is a cohomology theory with products and Gysin-Thom homomorphism for complex bundles we have isomorphisms

$\mathrm{Homst}^{\otimes}(\Omega, F) = \{$ natural transformations α from Ω to F compatible with products and f_* for f proper with $\nu_f = 0$. (such a $\alpha : \Omega(X) \to F(X)$ is a ring homomorphism)

$\cong$

$\mathrm{Map}^{\otimes}(\tilde{K}, F) =$ natural transf. $\beta : \tilde{K} \longrightarrow F$ such that $\beta(x+y) = \beta(x)\beta(y)$, $\beta(0) = 1$

$\cong$

$\mathrm{Hom}_{\mathbb{Z}_{(p)}\text{-alg}}(\mathbb{Z}_{(p)}[a_1, a_2, \dots], F(pt)) = \{$ power series $\varphi(X) \in F(pt)[[X]]$ with $\varphi(0) = 1\}$

Given a power series $\varphi(X) = \sum_{i \geq 0} a_i X^i$ $a_0 = 1$, $a_i \in F(pt)$ we shall denote by

$\bar{\varphi}$ the power series $\bar{\varphi}(X) = \sum_{i \geq 0} a_i X^{i+1}$

$\tilde{\varphi}$ the (multiplicative) operation $\tilde{K} \longrightarrow F$ given on line bundles by
$$\tilde{\varphi}(L) = \sum a_i (c_1^F L)^i$$

$\hat{\varphi}$ the (stable) natural transformation $\Omega \longrightarrow F$ given by
$$\hat{\varphi}(f_* 1) = f_* \tilde{\varphi}(\nu_f),$$ or equivalently the stable

natural ring homomorphism such that

$$\hat{\varphi}\left(c_1^{\Omega}(L)\right) = \bar{\varphi}\left(c_1^{F}(L)\right)$$

If R is a $\mathbb{Z}_{(p)}$ algebra let $\underline{E}(R)$ be the set of formal group laws (commutative 1 variable) over R and let $\underline{ET}(R)$ be the subset of typical laws, that is laws such that the curve in the associated formal group defined by the coordinate is typical. The functors $\underline{L}$ and $\underline{LT}$ are representable by $\mathbb{Z}_{(p)}$-algs. L and LT and there is a surjection

$$\pi : L \longrightarrow LT$$

corresponding to the inclusion $\underline{LT} \longrightarrow \underline{L}$. In addition there is a canonical section of π constructed by Cartier as follows. An element of $\underline{L}(R)$ is the same as a formal group G endowed with a curve $\gamma : D \longrightarrow G$ (D = formal affine line) such that $d\gamma_0 : D \longrightarrow \mathfrak{g}$ is an isomorphism. Let $C\gamma$ be the typical curve given by the Cartier projector

$$C\gamma = \prod_{\substack{q \text{ prime} \\ q \neq p}} \left(1 - \frac{V_q F_q}{q}\right)\gamma = \sum_{(n,p)=1} \frac{\mu(n)}{n} \sum_{\zeta^n = 1} \gamma(\zeta\,\cdot)$$

Then $C\gamma$ is a typical coordinate for G. Let $\underline{N}$ be the group scheme given by

$\underline{N}(R)$ = power series $\sum a_i X^{i+1}$ $\quad a_i \in R$, $a_0 = 1$

under composition and let $\underline{N}$ act on $\underline{L}$ by

$$f \in \underline{N}(R) \qquad F \in \underline{L}(R) \longmapsto (f * F)(X, Y) = f(F(f^{-1}X, f^{-1}Y)).$$

Then Cartier's projector defines a map

$$\mathcal{C} : \underline{L} \longrightarrow \underline{N}$$

such that

(i) $\mathcal{C}(F) = e$ identity of $\underline{N}$ iff F typical

(ii) $\mathcal{C}(F) * F$ is typical.

In particular $\mathcal{C}$ gives a retraction of $\underline{L}$ onto $\underline{LT}$ by $F \mapsto \mathcal{C}(F) * F$, to which corresponds a sections

$$i : LT \longrightarrow L$$

of π.

Let $F^{\Omega}(X,Y) \in \Omega(pt)[[X,Y]]$ be the formal group law such that

$$F^{\Omega}(c_1^{\Omega}(L_1), c_1^{\Omega}(L_2)) = c_1^{\Omega}(L_1 \otimes L_2).$$

and let

( ) $$\theta : L \longrightarrow \Omega(pt)$$

be the ~~corresponding~~ map defined by $F^{\Omega} \in \underline{L}(\Omega(pt))$. Let $\xi(X) \in \Omega(pt)[[X]]$ be the element $\mathcal{C}(F^{\Omega})$ of $\underline{N}(\Omega(pt))$ so that the group law $\xi(F^{\Omega}(\xi^{-1}X, \xi^{-1}Y)) = (\xi * F^{\Omega})(X, Y)$ is typical.

Put

$$\xi(X) = \sum a_i X^i \qquad a_i \in \Omega(pt) \qquad a_0 = 1$$
$$\bar{\xi}(X) = \sum a_i X^{i+1}$$

and let

$$\hat{\xi} : \Omega \longrightarrow \Omega$$

be the stable ring homomorphism given by

$$\hat{\xi}(f_* 1) = f_*\left(\tilde{\xi}(\nu_f)\right) \qquad \tilde{\xi}(L) = \sum a_i \left(c_1^\Omega(L)\right)^i .$$

<u>Proposition 1</u>: $\quad \hat{\xi} \circ \hat{\xi} = \hat{\xi}$

~~Proof: Let F_0 be the universal group law over L so that $\theta F_0 = F^\Omega$, and let $\bar{\xi}_0 \in \underline{N}(L)$ be $c(F_0)$ so that $\bar{\xi}_0 \cdot F_0 = i\pi F_0$. Then $\theta \bar{\xi}_0 = \bar{\xi}$. Now~~

<u>Proof</u>:

$$\left(\hat{\xi} \circ \hat{\xi}\right)\left(c_1^\Omega(L)\right) = \hat{\xi}\left(\bar{\xi}(c_1(L))\right)$$

$$= \sum_i \left(\hat{\xi}\, a_i\right) \cdot \left(\hat{\xi}\, c_1^\Omega(L)\right)^{i+1}$$

$$= \sum_i \left(\hat{\xi}\, a_i\right) \left[\bar{\xi}\left(c_1^\Omega(L)\right)\right]^{i+1}$$

Therefore $\hat{\xi} \circ \hat{\xi} = \hat{h}$ where $h \in \Omega(pt)[[X]]$ is

$$\bar{h}(X) = \sum_i \left(\hat{\xi}\, a_i\right) \left[\bar{\xi}(X)\right]^{i+1} .$$

~~Let $\xi_0(X) = \sum_{i \geq 0} a_{0i} X^{i+1}$, $a_{0i} \in L$ so that $\theta a_{0i} = a_i$~~

(To calculate $\hat{\xi}(a_i)$ we use that the a_i come from L:)

Let F_0 be the universal group law over L and let $\bar{\xi}_0 = c(F_0) \in \underline{N}(L)$. Then $\bar{\xi}_0 * F_0$ is typical, $\theta F_0 = F^\Omega$, and $\theta \bar{\xi}_0 = \bar{\xi}$. Note that

$$\hat{\xi}\, c_1^\Omega(L_1 \otimes L_2) = \hat{\xi}\, F^\Omega(c_1^\Omega(L_1), c_1^\Omega(L_2))$$
$$= (\hat{\xi} F^\Omega)(\hat{\xi}\, c_1^\Omega(L_1), \hat{\xi}\, c_1^\Omega(L_2))$$

where $\hat{\xi} F^\Omega$ means $\hat{\xi}$ applied to the coeffs of F^Ω or

$$\bar{\xi}\,(F^\Omega(c_1^\Omega L_1, c_1^\Omega L_2)) = (\hat{\xi} F^\Omega)(\bar{\xi}(c_1^\Omega L_1), \bar{\xi}(c_1^\Omega L_2))$$

for all L_1, L_2 or

$$\bar{\xi}\,(F^\Omega(X, Y)) = (\hat{\xi} F^\Omega)(\bar{\xi} X, \bar{\xi} Y)$$

or finally

$$(\hat{\xi} F^\Omega)(X,Y) = \bar{\xi}\,(F^\Omega(\bar{\xi}^{-1} X, \bar{\xi}^{-1} Y)) = (\bar{\xi} * F^\Omega)(X, Y).$$

Thus the diagram

$$\begin{tikzcd} F_0 \arrow[r, mapsto] \arrow[d, mapsto] & F^\Omega \arrow[d, mapsto] \\ \bar{\xi}_0 \cdot F_0 \arrow[r, mapsto] & (\bar{\xi} \cdot F^\Omega) \end{tikzcd}$$

$$\begin{tikzcd} L \arrow[r, "\theta"] \arrow[d, "\iota\pi"'] & \Omega(pt) \arrow[d, "\hat{\xi}"] \\ L \arrow[r, "\theta"] & \Omega(pt) \end{tikzcd}$$

commutes,

so

$$\bar{h}(X) = (\hat{\xi}\,\bar{\xi})(\bar{\xi} X) = (\theta\, \iota\pi\, \bar{\xi}_0)(\bar{\xi}_0 X).$$

But $\iota\pi\, \bar{\xi}_0$ = the power series X, since $c(\underset{\bar{\xi}_0 \cdot F_0}{F}) = e$ if $\underset{\bar{\xi}_0 \cdot F_0}{F}$ is typical.

$\therefore \ \bar{h}(X) = \bar{\xi}(X)$, so $\hat{\xi}\circ\hat{\xi} = \hat{h} = \hat{\xi}$. QED.

Let BP = image of $\hat{\xi}$

$$\Omega \xrightarrow{\pi} BP \xhookrightarrow{i} \Omega \qquad\qquad \hat{\xi} = i\pi$$

and define for any map $f: X \to Y$ proper oriented

$$f_*^{BP}(\pi x) = \pi f_* x$$

This is well defined because $\pi x = 0 \Rightarrow \hat{\xi} x = 0$
$\Rightarrow \pi f_* x = \pi \hat{\xi} f_* x = \pi\, f_*(\tilde{\xi}(\nu_f)\cdot \hat{\xi} x) = 0$. Then BP is a cohomology theory with product and Gysin for complex bundles; it satisfies the axioms since it's a quotient of Ω.

The map i is not compatible with Gysin. Instead if $x = \pi g_* 1 \in BP(X)$, where $g: Z \to X$ is prop-or., then

$$i f_*(x) = \hat{\xi}((fg)_* 1) = f_* g_* \ \tilde{\xi}(\nu_{fg}) \qquad\qquad \nu_{fg} = \nu_g + g^*\nu_f$$

$$= f_* g_*\left[g^*\tilde{\xi}(\nu_f)\cdot\tilde{\xi}(\nu_g)\right]$$

$$= f_*\left[\tilde{\xi}(\nu_f)\cdot\hat{\xi}(g_* 1)\right]$$

$$= f_*\left[\tilde{\xi}(\nu_f)\, i x\right]$$

Thus we have the following Riemann-Roch type result

$$\boxed{i(f_* x) = f_*\left(\tilde{\xi}(\nu_f)\cdot i x\right)}$$

Note that we have the following commutative diagram (see page 5)

$$\begin{tikzcd}[column sep=large]
\overset{F_0}{L} \arrow[r, "\pi"] \arrow[d, "\theta"] & \overset{F_t}{LT} \arrow[r, "i"] \arrow[d, "\theta'"] & \overset{\bar{\xi}_0 \cdot F_0}{L} \arrow[d, "\theta"] \\
\overset{F}{\Omega(pt)} \arrow[r, "\pi"] \arrow[rr, bend right, "\bar{\xi}"'] & \overset{\pi F}{BP(pt)} \arrow[r, "i"] & \overset{\bar{\xi} \cdot F}{\Omega(pt)}
\end{tikzcd}$$

where F_t is the universal typical group law (and θ' is induced by θ). It follows that there are ~~maps~~ ring homomorphisms

$$\Phi : LT \otimes_L \Omega(X) \longrightarrow BP(X)$$

$$\Psi : L \otimes_{LT} BP(X) \longrightarrow \Omega(X)$$

given by

$$\Phi(u \otimes x) = \theta' u \cdot \pi x$$

$$\Psi(v \otimes x) = \theta v \cdot i x$$

where Φ is compatible with f^* and f_* any f, defined to be the LT linear extension of f^*_Ω and f^Ω_*, (resp. where Ψ is compatible with f^*, but with f_* only if $\nu_f = 0$.)

<u>Proposition 2:</u> $\Phi : LT \otimes_L \Omega(X) \longrightarrow BP(X)$ is an isomorphism for all X.

<u>Proof:</u> We will show that the composition Φ':

$$BP \xrightarrow{i} \Omega \xrightarrow{1 \otimes id} LT \otimes_L \Omega$$

is an inverse to Φ. $\Phi \Phi' = id$ is clear; ~~as for other one~~

and to show that $\Phi' \Phi = id$ it is enough since $LT \otimes_L \Omega = \Omega / I \Omega$, $I = \operatorname{Ker}\{\pi : L \to LT\}$ to show that $\Phi' \Phi (f_* 1) - 1 \otimes f_* 1 = 1 \otimes \hat{\xi}(f_* 1) - 1 \otimes f_* 1 = 0$ or that

$$f_*\left(\tilde{\xi}(\nu_f) - 1\right) \in I \cdot \Omega(X)$$

if $f: Z \to X$ is prop-or. But recall that $\pi \tilde{\xi}_0(X) = X$, hence $\tilde{\xi}(L) = \sum a_i \, c_1(L)^i$ with $a_i \in I\Omega(pt)$ for $i \geq 1$. Thus $\tilde{\xi}(L) - 1 \in I \cdot \Omega$ hence $\tilde{\xi}(\nu_f) - 1 \in I \cdot \Omega$ by splitting principle. QED.

Our next project is to ~~define~~ define the correct f_* on $L \otimes_{LT} BP(X)$ so that Φ is compatible with Gysin. Set $Q(X) = L \otimes_{LT} BP(X)$ and define for $f: X \to Y$ prop-or a map

$$f_*^{BP} : Q(X) \longrightarrow Q(Y)$$

to be the L-linear extension of $f_*^{BP} : BP(X) \longrightarrow BP(Y)$. We search now for a new Gysin homomorphism for Q to be denoted by f_*^Q such that

$$\Phi f_*^Q = f_*^\Omega \Phi .$$

$\cancel{\Phi f_*^Q(v \otimes x) = f_*^\Omega(\Phi v \cdot i x)}$ ~~$\Phi v \in \Omega(pt)$~~

We set $$f_*^Q z = f_*^{BP}(\chi(\nu_f)\cdot z) \quad \text{where}$$

$$\chi(L) = \sum b_i\, c_1^{BP}(L)^i \qquad b_i \in Q(pt),\ b_0 = 1.$$

is to be determined.
(This is valid because $Q = L \otimes_{LT} BP$ satisfies the splitting principle since BP does; it doesn't need that Q is half exact which would require us to know at this stage that L is flat over LT.).

$$\begin{aligned}
\Psi(f_*^Q z) &= \Psi f_*^{BP}(\chi(\nu_f)\cdot z) \\
&= f_*^{\Omega}\left[\tilde{\xi}(\nu_f)\cdot \Psi(\chi(\nu_f)\cdot z)\right] \qquad (RR\text{-page } 6) \\
&= f_*^{\Omega}(\Psi z)
\end{aligned}$$

provided that

$$\tilde{\xi}(\nu_f)\,\Psi[\chi(\nu_f)] = 1$$

or that

$$\boxed{\tilde{\xi}(L)\,\Psi[\chi(L)] = 1}$$

for all line bundles L. Recall that

$$\begin{aligned}
\Psi\{c_1^{BP}(L)\} &= c_1^{\Omega}(L)\,\tilde{\xi}(L) \qquad (RR\text{-page } 6) \\
&= \bar{\xi}(c_1^{\Omega}(L))
\end{aligned}$$

Thus we want $$\bar{\xi}(X)\,(\Psi\chi)(\bar{\xi}(X)) = 1$$

i.e. $$(\Psi\cdot\bar{\chi})(\bar{\xi}(X)) = X,$$

where $\bar{\chi}(X) = X\chi(X)$ and $\Phi\chi$ denotes the power series $\sum \Phi(b_i)X^i$. Recall that $\bar{\xi}(X) = (\theta\bar{\xi}_0)(X)$ where $\xi_0 \in L[[X]]$ so we can take $\bar{\chi} = (\bar{\xi}_0)^{-1} \in L[[X]] \to (L\otimes_{LT} BP_{(p)})[[X]]$. So χ exists and we have constructed f_*^Q.

We now have a transformation of cohomology theories with products preserving the Gysin homomorphism

$$\Phi : Q \longrightarrow \Omega,$$

so by the universal property of Ω, Φ has a unique section Φ' compatible with f^*, f_*. To show $\Phi'\Phi = id_Q$.

$$\begin{array}{ccccc} Q & \xrightarrow{\Phi} & \Omega & \xrightarrow{\Phi'} & Q \\ \| & & & & \| \\ L\otimes_{LT} BP & & & & L\otimes_{LT} BP \end{array}$$

$$\Phi'(f_* 1) = f_*^{BP}(\chi(\nu_f))$$

$$\Phi(v\otimes x) = \theta v \cdot ix$$

$$\therefore \quad \Phi'\Phi(1\otimes x) = \Phi'(\iota x)$$

$$x = \pi g_* 1$$
$$\iota x = \hat{\xi} g_* 1$$

$$= \Phi'(\hat{\xi} g_* 1)$$

$$= \Phi'(g_* \tilde{\xi}(\nu_g))$$

$$= g_*^Q\{\Phi'\tilde{\xi}(\nu_g)\}$$

Now have to calculate the char. class $E \longmapsto \Phi'\tilde{\xi}(E)$

$$\Phi'\tilde{\xi}(L) = \Phi'[\tilde{\xi}(c_1^\Omega(L))] = (\Phi'\tilde{\xi})(\Phi' c_1^\Omega(L))$$

To show $\Psi'\Psi(1\otimes x) = 1\otimes x$ $\quad x\in BP(X)$

we calculate the power series associated to the ring operation $f_* 1 \longmapsto \Psi'\Psi(1\otimes \pi f_* 1)$.

$$\Psi'\Psi(1\otimes \pi c_1^{\Omega}(L)) = \Psi'\left[\hat{\xi}\, c_1^{\Omega}(L)\right]$$

$$= \Psi'\left[\bar{\xi}(c_1^{\Omega}(L))\right] = (\Psi'\bar{\xi})(\Psi' c_1^{\Omega} L)$$

But

$$\Psi' c_1^{\Omega}(L) = \iota_Q^* \iota_*^Q 1 = \iota_{BP}^* \iota_*^{BP} \tilde{\chi}(L)$$

$$= c_1^{BP}(L)\, \chi(c_1^{BP}(L)) = \bar{\chi}(c_1^{BP}(L)).$$

Thus

$$\Psi'\Psi(1\otimes \pi c_1^{\Omega}(L)) = (\Psi'\bar{\xi})(\bar{\chi}(c_1^{BP}(L)))$$

so I want to know that

$$(\Psi'\bar{\xi})(\bar{\chi}(X)) \overset{?}{=} X.$$

Here's how the argument goes. First

$$\Psi' c_1^{\Omega}(L) = \underset{\sim}{\bar{\chi}}(c_1^{BP}(L))$$ set $L_\xi = L_1 \otimes L_2$

$$\Longrightarrow \quad (\Psi' F^{\Omega})(\bar{\chi}(X), \bar{\chi}(Y)) = \bar{\chi}\{F^{BP}(X,Y)\}$$

$$\Longrightarrow \quad \boxed{\Psi' F^{\Omega} = \bar{\chi}_* F^{BP}}$$

This tells me that the map

$$L \xrightarrow{\theta} \Omega(pt) \xrightarrow{\Psi'} Q(pt) = L \otimes_{LT} BP(pt)$$

sends the canonical law F_0 to $\bar{\chi} * F^{BP}$. ~~hence~~ ~~sends $\bar\xi$ to~~ Secondly the diagram

$$\begin{tikzcd}
F_t \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & \bar{\xi}_0 \cdot F_0 \arrow[ddd, mapsto] \\
& LT \arrow[r, "i"] \arrow[d, "\theta"] & L \arrow[d, "\theta"] & \\
& BP(pt) \arrow[r] & \Omega(pt) & \\
\pi F^{\Omega} = F^{BP} \arrow[rrr, mapsto] & & & \bar{\xi} \cdot F^{\Omega}
\end{tikzcd}$$

i.e. in Q we have the identity

$$(\bar{\xi}_0 \cdot F_0) = F^{BP}.$$

(here is where $\otimes_{LT}$ is used). Now recall that

$$\bar{\chi} = \bar{\xi}_0^{-1} \qquad \text{in } Q[[X]]$$

hence $F_0 = \bar{\chi} \cdot F^{BP}$ which finally enables us to conclude that $\Phi'\Theta(v) = v$ for $v \in L$. Therefore

$$\Phi'\bar{\xi} = \Phi'\theta\bar{\xi}_0 = \bar{\xi}_0 = \bar{\chi}^{-1}$$

proving the ? in page 11 and completing the proof of

<u>Proposition 3:</u>

$$\boxed{\begin{gathered} \Phi : L \otimes_{LT} BP(X) \xrightarrow{\sim} \Omega(X). \\ \Phi(f_*^{BP} x) = f_*^{\Omega}\left[\tilde{\bar{\xi}}(\nu_f) \cdot \Phi x\right] \end{gathered}}$$

Lemma: Let R be a ~~ring~~ $\mathbb{Z}_{(p)}$-algebra let F be a formal group law over R, let $a \in R^*$, and let ~~[illegible]~~ $\bar{a} * F$ be the group law $(\bar{a} * F)(X, Y) = a F(a^{-1}X, a^{-1}Y)$. Let $c(F)$ be the power series constructed by Cartier such that $c(F) \cdot F$ is typical. Then

$$\boxed{c(\bar{a} * F) = \bar{a} * c(F).}$$

Proof: We review the definition of $c(F)$. $\exists$ a formal group G over R endowed with a coordinate

$$D \xrightarrow{\gamma_0} G$$

such that

$$F(x, y) = \gamma_0^{-1}(\gamma_0 x +_G \gamma_0 y)$$

Let

$$\gamma = \cancel{\quad} \operatorname{Cart}(\gamma_0) = \prod_{\substack{q \text{ prime} \\ q \neq p}} \left(1 - \frac{1}{q} V_q F_q\right) \gamma_0$$

be the Cartier projection of γ. Then the group law

$$F'(a, b) = \gamma^{-1}(\gamma a +_G \gamma b)$$

is typical and we have

$$F' = c(F) * F \qquad \text{where}$$

$$\boxed{c(F) = \gamma^{-1} \circ \gamma_0}$$

Now we have

$$(\bar{a} * F)(x, y) = a F(a^{-1}x, a^{-1}y)$$
$$= a \gamma_0^{-1}(\gamma_0 a^{-1}x +_0 \gamma_0 a^{-1}y)$$
$$= \gamma_{0a}^{-1}(\gamma_{0a}\, x +_0 \gamma_{0a}\, y)$$

where $\gamma_{0a}(x) = \gamma_0(a^{-1}x)$

or $\gamma_{0a} = [a^{-1}]\cdot\gamma_0$

$$\gamma_a = \mathrm{Cart}\,(\gamma_{0a}) = \prod_{q \neq p} \left(1 - \frac{1}{q} V_q F_q\right)\left([a^{-1}]\cdot\gamma_0\right)$$

But from Cartier's paper we have

$$\begin{cases} F_n[c] = [c^n] F_n \\ V_n[c^n] = [c] V_n \\ [c](\gamma + \gamma') = [c]\gamma + [c]\gamma' \end{cases}$$

so that

$$\gamma_a = [a^{-1}]\cdot \mathrm{Cart}\,\gamma_0 = [a^{-1}]\cdot\gamma$$

Hence

$c(\bar{a} * F)(x) =$

$$(\gamma_a^{-1}\gamma_{0a})(x) = a\,\gamma^{-1}(\gamma_0(a^{-1}x)) = (\bar{a} * c(F))(x)$$

QED

Suppose now that R is torsion-free, e.g p is a non-zero divisor and extend the base to $R[\frac{1}{p}] = L$. Then we have ~~an isomorphism~~ unique group ~~homomorphism~~ hom. log

$$\begin{tikzcd} D \arrow[r, "\gamma_0"] \arrow[rr, bend right, "l"'] & G \arrow[r, "\log"] & D \end{tikzcd}$$

such that $l(X) = X +$ higher terms. Then

$$\begin{aligned} l(F(x,y)) &= l\gamma_0^{-1}(\gamma_0 x + \gamma_0 y) \\ &= \log(\gamma_0 x + \gamma_0 y) = \log \gamma_0 x + \log \gamma_0 y \\ &= l(x) + l(y) \end{aligned}$$

Also as log is a homomorphism it commutes with Cartier operator so

$$\log \gamma = \prod_{\substack{q \text{ prime} \\ q \neq p}} \left(1 - \frac{1}{q} V_q F_q\right) \underbrace{\log \gamma_0}_{l =}$$

but
$$\sum_{\zeta^q = 1} \zeta^k = \begin{cases} 0 & \text{if } k \nmid q \\ q & \text{if } k \mid q \end{cases}$$

so
$$\frac{1}{q}(V_q F_q)(X^k) = \frac{1}{q} \sum_{\zeta^q = 1} (\zeta X)^k = \begin{cases} X^k & \text{if } k \mid q \\ 0 & \text{otherwise} \end{cases}$$

Thus if
$$l(X) = \sum_{n \geq 1} a_{n-1} \frac{X^n}{n} \qquad a_0 = 1$$

$$l'(X) = \log \gamma(X) = \sum_{a \geq 0} a_{p^a - 1} \frac{X^{p^a}}{p^a} \qquad \text{new logarithm}$$

and
$$\boxed{c(F) = (l')^{-1} \circ l} \qquad \boxed{l' \circ c(F) = l}$$

Effect of homotheties on cobordisms:

Here is a strengthened form of the Thom isomorphism:

$\check\beta \leftarrow \beta$ $\qquad$ $\alpha \mapsto \hat\alpha$

$$\mathrm{Hom}^{\otimes}(K,F) = \{\alpha : K \longrightarrow F^* \quad \text{abelian gp. hom. compatible with } f^*\}$$

$$\cong$$

$$\mathrm{Hom}^{\otimes}(\Omega,F)' = \{\beta : \Omega \longrightarrow F \quad \text{ring homs comp with } f^* \text{ such that } \check\beta(1) \in F(pt)^*\}$$

$1 =$ trivial line bundle over pt

$$\cong$$

$$\Big\{\varphi(X) = \sum_{i \geq 0} a_i X^i \;\Big|\; a_i \in F(pt),\ a_0 \in F(pt)^*\Big\}$$

where $\quad \check\beta(E) = \iota_*^{-1}\beta\,\iota_* 1 \qquad i : X \longrightarrow E$ zero section

$$\hat\alpha(f_* 1) = f_*(\alpha(\nu_f))$$

(The point is that $\check\beta(E+F) = \check\beta(E)\,\check\beta(F)$ but if $\check\beta(1)$ is not invertible I won't get a map $K \longrightarrow F^*$.)

Let R be a ring and let $a \in R^*$. Then taking $\varphi(X) = a \in \Omega(pt) \otimes R$, I get a homomorphism

$$\begin{cases} \hat a : \Omega \longrightarrow \Omega \otimes R \\ \hat a(f_* x) = f_*(a^{\dim \nu(f)} \cdot \hat a x) \\ \tilde a(E) = a^{\dim E} \\ \hat a(f_* 1) = a^{\deg(f_* 1)} \cdot f_* 1 \end{cases}$$

This action of $\mathbb{G}_m$ on Ω is clearly that given by the standard grading of Ω. Clearly

$$\boxed{\hat{a}\circ\hat{b} = \widehat{(ab)}}$$

<u>Proposition 4</u>: If $\hat{\xi}$ is the idempotent endomorphism of Ω localized at p defined by the Cartier projector, then

$$\boxed{\hat{\xi}\circ\hat{a} = \hat{a}\circ\hat{\xi}}$$

<u>Proof:</u> $\hat{\xi}\circ\hat{a} = \hat{h}$ where

$$\bar{h}(X) = \hat{\xi}\bar{a}\left(\bar{\xi}(X)\right) = \hat{\xi}\bar{\bar{a}}\cdot\bar{\xi}(X) = a\cdot\bar{\xi}(X)$$

since $\hat{\xi}$ acts trivially on R. $\hat{a}\circ\hat{\xi} = \hat{h}_1$ where

$$\bar{h}_1(X) = (\hat{a}\cdot\bar{\xi})(\bar{a}(X)) = (\hat{a}\bar{\xi})(aX)$$

To calculate $\hat{a}\bar{\xi}$ we use that $\bar{\xi}$ comes from L.

$$\begin{tikzcd}
F_0^{\prime,\bar{\xi}_0} \arrow[rrr, mapsto] \arrow[ddd] & & & F^{\Omega,\bar{\xi}} \arrow[ddd] \\
& L \arrow[r, "\theta"] \arrow[d, "\hat{a}"] & \Omega(pt)\otimes R \arrow[d, "\hat{a}"] & \\
& L \arrow[r, "\theta"] & \Omega(pt)\otimes R & \\
\bar{a}_* F_0^{\prime,?} \arrow[rrr, mapsto] & & & \hat{a}F^{\Omega,\hat{a}\bar{\xi}} = \bar{a}_* F^{\Omega}
\end{tikzcd}$$

where we have denoted by $\hat{a} : L \to L$ the $\underset{\text{unique}}{}$ map such that $\hat{a}F_0 = \bar{a} * F_0$. By Lemma

$$\hat{a}\bar{\xi}_0 = c(\bar{a} * F_0) = \bar{a} * \bar{\xi}_0$$

hence $? = \bar{a} * \bar{\xi}$ and so

$$\hat{a}\bar{\xi} = \theta(\bar{a} * \bar{\xi}_0) = \bar{a} * \bar{\xi}$$

Hence

$$\bar{h}_1(X) = (\bar{a} * \bar{\xi})(aX) = a\,\bar{\xi}(a^{-1}aX) = a\,\bar{\xi}(X)$$

so $h = h_1$. **QED.**

<u>Corollary</u>: BP has a natural grading and the maps $\Omega \xrightarrow{\pi} BP \xrightarrow{i} \Omega$ are compatible with the grading.

February 13, 1969:

The universal prop of ΩT
equivalence of cat schemes $\Rightarrow$ same spectral sequence for $\Omega_{(p)}$ + ΩT

Some examples of BP-theories.

Proposition 5: (Universal property). If Q is a cohomology theory with values in $\mathbb{Z}_{(p)}$-algebras with Gysin homomorphism, splitting principle, etc. then

$$\mathrm{Hom}'^{\otimes}(BP, Q) \cong \left\{ \varphi(X) = \sum_{i \geq 0} a_i X^i \;\middle|\; \begin{array}{l} a_i \in Q(pt) \\ a_0 \in Q(pt)^* \end{array} \right.$$

such that $\bar{\varphi} * F^Q$ is typical $\}$.

Proof: Hom$'$ denotes those transformations $\beta : BP \longrightarrow Q$ such that $\check{\beta}$(trivial line bundle) $\in Q(pt)^*$. By prop. 2 β is the same as $\hat{\varphi} : \Omega \longrightarrow Q$ carrying F^Ω into a typical law. But we calculate

$$\boxed{\hat{\varphi} F^\Omega = \bar{\varphi} * F^Q}$$

QED.

Example 1: Complex cobordism over $\mathbb{Q} = \mathbb{Z}_{(0)}$. Here one takes $p = 1$ and the only typical law is the additive group G_a. Thus $LT = \mathbb{Q}$ and we have the decomposition

$$L_{\mathbb{Q}} \otimes_{\mathbb{Q}} BP = \Omega_{\mathbb{Q}}$$

$L_{\mathbb{Q}} = \mathbb{Q}[a_1, a_2, \cdots]$ where the logarithm on $L_{\mathbb{Q}}$ is

$$\ell(X) = \sum_{i \geq 0} a_i \frac{X^{i+1}}{i+1}$$

and $\theta a_i = P_i$. Of course by Thom's results it follows that

(*) $$BP(pt) = \mathbb{Q}$$

and hence $$BP(X) = H^*(X, \mathbb{Q})$$

It would be nice to have a direct proof of (*) not using homotopy theory. The best I can do so far is to use K-theory and periodicity as follows: Let an almost complex manifold X be given and embed it $X \xhookrightarrow{i} S^{2n} = (\mathbb{C}^n)^+$. and let $p: S^{2n} \longrightarrow pt.$, $f = pi$. Grothendieck tells us that there is a characteristic class u_k in $\Omega_{\mathbb{Q}}$ (k = codim i) such that $u_k(i_! 1) = i_* 1$. This tells us that $i_* 1$ is in the Chern subring of S^{2n}. But for BP-theory one should be able to calculate the Chern ring of $BP(S^{2n})$ and show p_* carries it into dimensions ≥ 0 of $BP(pt.)$.

Note that as we have a decomposition

$$L_{\mathbb{Q}} \otimes BP \simeq \Omega_{\mathbb{Q}}$$

one has a decomposition of spectra

$$L_{\mathbb{Q}} \otimes BP \simeq MU_{\mathbb{Q}}$$

Taking $H_*(\cdot, \mathbb{Q})$ one finds that $H_*(BP) = \mathbb{Q}$, whence $BP(pt) = \pi_*(BP) = \mathbb{Q}$ by homotopy theory.

<u>Example 2:</u> Complex cobordism over $\mathbb{Z}_{(p)}$.

$$L \otimes_{LT} BP \simeq \Omega \qquad (\text{over } \mathbb{Z}_{(p)})$$

By Cartier LT is a poly ring with generators in degrees $p^a - 1$ and by Lazard L has generators in each degree. Hence $\Omega = Q \otimes BP$ where Q has generators in degrees $\neq p^a - 1$. Apply homology to isomorphism of ring spectra

$$\underset{\sim}{Q} \otimes BP \simeq MU$$

$$Q \otimes H_*(BP) \simeq H_*(MU) \simeq H_*(BU) \qquad H_*(\ ;\mathbb{Z}_{(p)})$$

and one sees that $H_*(BP)$ is a poly ring $\overbrace{\text{with}}^{\text{over } \mathbb{Z}_{(p)}}$ generators in degrees $p^a - 1$. As $H_*(MU)^{\mathbb{F}_p} \twoheadrightarrow H_*(BP)^{\mathbb{F}_p} \twoheadrightarrow (\mathcal{A}/(\beta))' \hookrightarrow \mathcal{A}'$ (this says that $\mathcal{A}/(\beta) \hookrightarrow H^*(MU)$), it follows that $H_*(BP)^{\mathbb{F}_p} \overset{\sim}{=} (\mathcal{A}/(\beta))'$ so BP is the Brown-Peterson spectrum.

<u>Universal property of BP.</u>

$$\mathrm{Homst}(BP, Q) = \text{power series } \varphi(X) = \textstyle\sum a_i X^i \quad a_0 = 1 \quad a_i \in Q(pt)$$
such that $\varphi * F^Q$ is typical

$$\pi_*(BP) = LT = \text{poly. ring over } \mathbb{Z}_{(p)} \text{ with generators of degrees } p^a - 1.$$

$H^*(BP, \mathbb{F}_p)$ free module rank 1 over $\mathcal{A}/(\beta)$.

<u>Ring of (stable) operations in BP</u>: If R is a $BP(pt) = TL$ algebra, then

$$\mathrm{Hom}_{st}^{\otimes}(BP,\ R \otimes_{TL} BP) = \Big\{ \bar{\varphi}(X) = \sum a_i X^{i+1} \;\Big|\; a_i \in R \quad a_0 = 1$$

such that $\bar{\varphi} * F$ is typical, where F is the group law over R given by the (structural) map $TL \longrightarrow R$. $\Big\}$

It seems to be impossible to understand this properly without category schemes. Thus introduce two category schemes:

$\mathfrak{F}$:
$$(\mathrm{Ob}\,\mathfrak{F})(R) = \underline{F}(R) \qquad \text{formal group laws over } R$$

$$\mathrm{Hom}_{\mathfrak{F}(R)}(F_1, F_2) = \left\{ \begin{array}{l} \text{power series } \bar{\varphi}(X) = \sum r_i X^{i+1} \\ r_0 = 1 \text{ such that } \bar{\varphi} * F_1 = F_2 \end{array} \right\}$$

$\mathfrak{F}$ is therefore the category scheme given by the group $\underline{N}$ acting on $\underline{F}$.

$\mathfrak{F}\mathfrak{T}$:
$$(\mathrm{Ob}\,\mathfrak{F}\mathfrak{T})(R) = \underline{FT}(R) \qquad \text{typical formal group laws over } R$$

$$\mathrm{Hom}_{(\mathfrak{F}\mathfrak{T})(R)}(F_1, F_2) = \{ \text{power series } \bar{\varphi} \ni \bar{\varphi} * F_1 = F_2 \}$$

$\mathfrak{F}\mathfrak{T}$ is a full subcategory scheme of the category scheme $\mathfrak{F}$. Moreover by Cartier the inclusion functor

$$\mathfrak{F}\mathfrak{T} \hookrightarrow \mathfrak{F}$$

is an equivalence of category schemes.

<u>Corollary</u>: The Novikov-Adams spectral sequences ~~[illegible]~~ for complex bordism and for BP theory coincide when localized at p.

(Note there is such a spectral sequence since $BP_*(BP)$ is flat over $BP_*(pt)$ and since a convergent MU resolution is good for BP. $BP_*(BP)$ is a direct summand of $BP_*(MU)$ $= BP_*[b_1, b_2, \cdots]$.)

<u>Question</u>: Is there an analogue of the Novikov-Landweber algebra for BP? More precisely we have $\underline{N}$ acting on $\underline{F}$ and subscheme $\underline{FT}$ meeting every ~~[illegible]~~ orbit. ~~[illegible]~~ ~~[illegible]~~ Does there exist a subgroup $\underline{NT}$ of $\underline{N}$ so that $(\underline{FT}, \underline{NT}) \longrightarrow (\underline{F}, \underline{N})$ is an equivalence of category schemes?

Yes

Want to write a paper on cobordism theory and formal groups.

Results:

~~universal property of cobordism theory~~

~~twisting a theory by a characteristic class~~

(i) $\Omega(pt)$ + $\mathcal{N}(pt)$ as universal rings.

(ii) formula for $f_*: \Omega(PE) \longrightarrow \Omega(X)$ as residue

(iii) decompositions

$$\begin{cases} \Omega^*_{(p)} \simeq BP^* \otimes_{LT} L \\ \mathcal{N}^* \simeq H^* \otimes \mathcal{N}(pt) \\ \Omega^*_{\mathbb{Q}} \simeq H^* \otimes \Omega(pt)_{\mathbb{Q}} \end{cases}$$

(iv) Universal descriptions of

$H^*(X, \mathbb{Z}_2)$

BP^*

K on the category of manifolds

~~Question: to set up Ω need λ structure on v.b.~~

~~to set up products we need $\otimes$ structure on v.b.~~

~~What corresponds to the λ structure on bundles?~~

(v) Operations in BP theory

Outline of paper

1. A univ. prop. of cob.
2. Chern classes in cobordism
3. Residues and the Gysin homomorphism for a projective bundle
4. $\Omega(pt)$ = the Lazard ring.
5. A universal property of K-theory + Conner Floyd thm.
6. Proof of the R-R thm. after Grothendieck

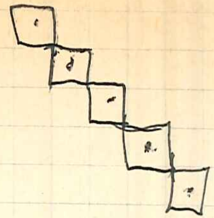
New results.

Decomposition theorem + operations

$\Omega\Gamma$ + its structure

Blowups.

1. Review of formal groups



$\boxed{\text{defn}}$: ... inverse series and iterates F, n_F, $(-1)_F$, $I(X)$

classification in char. 0.

invariant differential $\omega_F(Z)$

Lazard ring + universal law. $L \otimes_{\mathbb{Z}} \mathbb{Q} \xleftarrow{\sim} \mathbb{Q}[p_1, p_2, \cdots]$

2. Review of cobordism theory.

(1) Universal property of cobordism theory

(2) Complex cobordism and Chern classes.

$$\begin{cases} c_1(L) = i^* i_* 1 \\ U(\mathbb{P}E) \cong U(X)[Z]/\cdots \quad \text{relation.} \\ F^u(c_1^u L, c_1^u L') = c_1^u(L \otimes L'). \end{cases}$$

Theorem: $h: L \xrightarrow{\cong} U(pt)$

$F_{univ} \longmapsto F^u$

Proof: (1) Myshenko's formula

$\Downarrow$

$$h(p_i) = \mathbb{P}_i$$

(2) Thom thm. $\Rightarrow$ $h: L \otimes \mathbb{Q} \xrightarrow{\sim} U^*(pt) \otimes \mathbb{Q}$

Lazard $\Rightarrow$ L torsion-free $\Rightarrow$ $L \to U^*(pt)$ injective.

Known that $U^*(pt)$ is generated by $\mathbb{P}_n$ and by non-singular hypersurfaces Z of degree 1,1. in $\mathbb{P}^n \times \mathbb{P}^m$

$$\begin{array}{ccc} \mathbb{P}^n \times \mathbb{P}^m & \xrightarrow{s} & \mathcal{O}(1) \boxtimes \mathcal{O}(1) \\ \uparrow i & & \uparrow \\ Z & \xrightarrow{i} & \mathbb{P}^n \times \mathbb{P}^m \end{array}$$

$$\begin{array}{ccc} Z & \xrightarrow{i} & \mathbb{P}^m \times \mathbb{P}^n \\ & \searrow^{f} & \downarrow \pi \\ & & pt \end{array}$$

$$[Z] = f_* 1 = \pi_* i_* 1 = \pi_* c_1(\mathcal{O}(1) \boxtimes \mathcal{O}(1))$$
$$= \pi_* F(x, y) = \pi_* \textstyle\sum' a_{ij} x^i y^j = \cdots$$

3. Residues and Gysin hom. for projective bundle.

Residues - definition etc.

Formula for Gysin.

~~can assume $E = L_1 + \dots + L_n$ over~~

(i) check that residue is defined

(ii) can assume $E = L_1 + \dots + L_n$ ~~$L_i = p$~~ over ~~[illegible]~~ $(\mathbb{C}P^N)^n$ L_i = inverse image of $\mathcal{O}(1)$ on ith factor.

then use induction.

4. Application

(i) Myshenko formula

(ii) cobordism class of a blowup.

(iii) geometrical Chern classes.

Outline:

1. Review of Ω and the formal group law.
2. Residues and Gysin hom. for a proj. bundle.
3. Applications
   1.) Mycenko's formula
   2.) Group laws compatible with blowing up.
   3.) Geometric versus actual Chern classes.

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4. Review of Operations + (schemes category.
   The basic isomorphism in abstract form!
5. Operations in Ω. reduction in case of char. 0.
6. Typical laws; reduction of Ω to ΩT.
7. Laws of height ∞; reduction of η to $H(X, \mathbb{Z}_2)$.
   application to Milnor's thm.
8. Laws of height 1; relation of Ω to K.
   variations on the Stong-Hattori theorem.
