July 1, 1969

Warning: This is incorrect since Ω_G for G abelian doesn't satisfy proj. bundle thm. see

The Conner-Floyd theorem in equivariant cobordism theory.

Theorem: $\Omega_G(X) \underset{\Omega_G(pt)}{\otimes} K_G(pt) \xrightarrow{\cong} K_G(X)$.

The proof is involved and consists of several steps.

1) The map $\Omega_G(X) \longrightarrow K_G(X)$ is surjective (see page 8).

2) It is enough to prove the theorem when $X = G_n(V)$ the Grassmannian of n-dimensional quotients of a G-module V: In effect supposethat $\alpha \in \Omega_G(X)$ goes to zero in $K_G(X)$ where X is a G-manifold. Let $\alpha = \sum \alpha_i$ with α_i homogeneous and represent α_i via a map

$$W_i^+ \wedge X \longrightarrow E_{n_i}(V_i)^+$$

where $E_n(V)$ is the canonical quotient bundle over $G_n(V)$. Suspending further we can take all $W_i = W$ and we can suppose all $V_i = V$. In fact there is a diagram

$$\begin{tikzcd}
 & Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{tr. cart}"] & \coprod G_{n_i}(V) \arrow[d] \\
X \arrow[r, "i"] & X \wedge W^+ \arrow[r, "\gamma"] & \bigvee_i E_{n_i}(V)^+
\end{tikzcd}$$

where $\alpha = \iota^* \gamma^*$ (Thom class). Let Y be the wedge and let C be the cone on γ. Then we have a diagram

$$\begin{tikzcd}
Q(C) \arrow[r, "u\otimes 1"] \arrow[d, "\text{surj}"', "\text{by 1)}"] & Q(Y) \arrow[r, "\alpha\otimes 1"] \arrow[d, "\cong"] & Q(X) \arrow[d] & {} \arrow[d, mapsto] \\
K_G(C) \arrow[r] & K_G(Y) \arrow[r] & K_G(X) & 0 \quad \leftarrow \text{exact}
\end{tikzcd}$$

where $Q(?) = \Omega_G(?) \otimes_{\Omega_G(pt)} K_G(pt)$. The rows comes from the long exact sequences for Ω_G and K_G; u is the Thom class. Note that the middle vertical arrow is an isomorphism, since it is a direct sum of the maps of the thm. for $X = E_n(V)^+$, hence by the Thom isomorphism for $X = G_n(V)$. (By hyp. these are isos.) Diagram chasing shows $\alpha \otimes 1 = 0$.

3. It suffices to prove the projective bundle theorem for Q: Now that if the projective bundle thm. holds that the thms holds for X iff it holds for $\mathbb{P}E$, where E is a bundle over X of constant dimension (In effect this comes to saying that $A^n \to B^n$ is an isomorphism iff $A \to B$ is). So starting with $G_n(V)$ we can pass through successive projective bundles up to the flag space of V and then down to a point.

4. Let V be a faithful representation of G and let Y be the flag space of V. I claim the projective bundle theorem is true (for Ω_G hence also Q) provided X is over Y. In effect there are spectral sequences

$$H^p(X/G,\ x \longmapsto \Omega_G^q(Gx)) \Longrightarrow \Omega_G^{p+q}(X)$$

~~and similarly one for PE~~

$$H^p(X/G,\ x \longmapsto \Omega_G^q(\mathbb{P}E|Gx)) \Longrightarrow \Omega_G^{p+q}(\mathbb{P}E)$$

which reduce us to checking the theorem over each orbit Gx. But as X is over Y, the isotropy group of x is contained in that of Y which is abelian. Hence the isotropy group at x is abelian so $E \restriction Gx$ is a sum of G-line bundles and one knows the result holds.

Finally we note that if $q : Y \longrightarrow pt$ then there is an element $\eta \in \Omega_G(pt)$ such that $q_* : \Omega_G(Y) \longrightarrow \Omega_G(pt)$ becomes surjective after inverting η and such that $\xi \mapsto 1$ in $K_G(pt)$. Thus working with the theory $\Omega_G(?)[\eta^{-1}] = Q'(?)$, we have an element $\xi \in Q'(Y)$ with $q_* \xi = 1$ and hence we can descend the projective bundle theorem for E_Y over $X \times Y$ to E over X for the theory Q' and hence also for Q which is a base extension of Q' as $\eta \in \Omega_G(pt)$. This finishes the proof.

Remark: Chern classes exist in the theory $\Omega_G(?)[\eta^{-1}]$.

Added July 14

How to modify equivariant cobordism Ω_G so as to get a universal theory having the projective bundle theorem. Let Y be the flag manifold of a faithful representation of G and set

$$\Omega'_G(X) = \operatorname{Ker}\left\{\Omega_G(X\times Y) \rightrightarrows \Omega_G(X\times Y\times Y)\right\}$$

<u>Claim</u>: $\Omega_G(X) \xrightarrow{\sim} \Omega'_G(X)$ if all isotropy groups of X are abelian.

and in fact the sequence

$$(*) \qquad \Omega_G(X) \longrightarrow \Omega_G(X\times Y) \rightrightarrows \Omega_G(X\times Y\times Y) \;\substack{\rightarrow\\\rightarrow\\\rightarrow}\; \cdots$$

is exact.

<u>Proof</u>: Over G-spaces with abelian isotropy groups, the projective bundle theorem holds, hence the sequence $(*)$ is an Amitsur sequence of a faithfully flat map $\Omega_G(X) \to \Omega_G(X\times Y)$ and hence is exact.

Notice that there is an element $\xi \in \Omega_G(Y)$ which has the property that $(pr_1)_*\xi$ is a unit $pr_1 : X\times Y \longrightarrow X$ whenever X has abelian isotropy groups. Thus for general X $(pr'_1)_*\xi$ is a unit for $(pr'_1)_* : \Omega'_G(X\times Y) \to \Omega'_G(X)$. This means we can descend the projective bundle theorem for Ω'_G over Y to the theorem for Ω'_G in general. Thus Ω'_G has proj. bundle theorem. Finally if Q is an equivariant theory with projective bundle theorem, then from

$$\begin{tikzcd}
\Omega'_G(X) \arrow[r] \arrow[d] & \Omega_G(X\times Y) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \Omega_G(X\times Y\times Y) \arrow[d] \\
Q(X) \arrow[r] & Q(X\times Y) \arrow[r, shift left] \arrow[r, shift right] & Q(X\times Y\times Y) \quad \text{exact}
\end{tikzcd}$$

we deduce a map $\Omega'_G(X) \to Q(X)$, obviously unique. $\therefore$ Ω'_G universal.

How to calculate $\Omega_G(pt)$ $G = \mathbb{Z}/p\mathbb{Z}$ p a prime. (method of tom Dieck, possibly goes back to Conner-Floyd). Start with Gysin sequence for a representation V of G with $V^G = (0)$:

$$\Omega^{q}_{\mathbb{Z}/p}(X) \xrightarrow{e(V)} \Omega^{q+2n}_{\mathbb{Z}_p}(X) \longrightarrow \Omega^{q+2n}_{\mathbb{Z}/p}(X \times S(V)) \xrightarrow{\delta} \Omega^{q+1}_{\mathbb{Z}/p}(X)$$

Assume that X is a compact complex-oriented G-manifold. An element of the third group, since G acts freely on $S(V)$ is represented by a proper G-map with equivariant complex-orientation

$$\begin{tikzcd}
Z \arrow[r, "or"] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & X \times S(V) \arrow[r, "or"] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & S(V) \arrow[d] \\
Z/G \arrow[r, "or"] & (X \times S(V))/G \arrow[r, "or"] & S(V)/G
\end{tikzcd}$$

(In general the category of G-manifolds Z complex-oriented over a free G-manifold X is equivalent to the category of manifolds complex-oriented over X/G.) Now $S(V)/G \to pt$ is oriented so one gets that Z/G is oriented and hence we have Poincaré duality

$$\Omega^{q+2n}_{\mathbb{Z}/p}(X \times S(V)) \cong \Omega^{q+2n}((X \times S(V))/G)$$

$$\cong \Omega_{-q-2n+\dim X+2n-1}((X \times S(V))/G)$$

$$\cong \Omega_{-q+\dim X-1}((X \times S(V))/G).$$

So taking the limit as V runs over the reps with $V^0 = 0$ we get a long exact sequence

$$\cdots\ \Omega^{q}_{\mathbb{Z}/p}(X) \longrightarrow (S^{-1}\Omega)^{q}_{\mathbb{Z}/p}(X^{G}) \longrightarrow \Omega_{\dim X-(q+1)}(X_G) \longrightarrow \Omega^{q+1}_{\mathbb{Z}/p}(X)\ \cdots$$

since $S(V)$ as V gets larger + larger approaches E_G and

$$X_G = E_G \times_G X .$$

In particular for $X = pt$ we get

$$\cdots\ \Omega^{q}_{\mathbb{Z}_p}(pt) \longrightarrow (S^{-1}\Omega)^{q}_{\mathbb{Z}_p}(pt) \longrightarrow \underbrace{\Omega_{-q-1}\big(\overset{B\mathbb{Z}_p}{\cancel{pt}}\big)}_{\uparrow} \longrightarrow \Omega^{q+1}_{\mathbb{Z}_p}(pt) \longrightarrow \cdots$$

bordism classes of
free oriented complact
$\mathbb{Z}/p$ mainfolds.

July 4, 1969

Adams operations in Ω:

Define $\Psi^k : \Omega \longrightarrow \Omega \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{k}]$ to be the multiplicative operation such that

$$\Psi^k c_1^\Omega(L) = c_1^\Omega(L^{\otimes k})$$

Then ~~under the canonical map~~ the diagram

$$\begin{tikzcd}
\Omega \arrow[r] \arrow[d, "\Psi^k"'] & K \arrow[d, "\psi^k"] \\
\Omega \otimes \mathbb{Z}[\frac{1}{k}] \arrow[r] & K \otimes \mathbb{Z}[\frac{1}{k}]
\end{tikzcd}$$

is commutative. Unfortunately Ψ^k is not stable ~~so~~ as

$$c_1^\Omega(L^{\otimes k}) = k\, c_1^\Omega(L) + \cdots\cdots$$

To remedy this let

$$\varepsilon^k : \Omega \longrightarrow \Omega \otimes \mathbb{Z}[\tfrac{1}{k}]$$

be the multiplicative operation such that

$$\varepsilon^k\left(c_1^\Omega(L)\right) = \frac{1}{k} c_1^\Omega(L^{\otimes k}).$$

Then ε^k is stable.

<u>Proposition</u>: $\begin{cases} \varepsilon^k \circ \varepsilon^\ell = \varepsilon^{k\ell} \\ \varepsilon^k P_n = k^n P_n \end{cases}$

<u>Proof</u>: Look at $L \longmapsto \varepsilon^k \{\ell(c_1(L))\}$. It transforms ~~Sums in~~ tensor products of line bundles into sums and is a power series in $c_1(L)$ with leading term $c_1(L)$. Then

$$\boxed{\varepsilon^k\big(\ell(c_1(L))\big) = \ell(c_1(L))}$$

But

$$\varepsilon^k \ell(c_1(L)) = \sum_{n\geq 1} \varepsilon^k(P_{n-1}) \frac{\left(\frac{1}{k} c_1(L^{\otimes k})\right)^n}{n}$$

$$= \sum_{n\geq 1} \frac{\varepsilon^k P_{n-1}}{k^n} \frac{c_1(L^{\otimes k})^n}{n}$$

$$\ell(c_1(L)) = \frac{1}{k}\ell(c_1(L^{\otimes k})) = \sum_{n\geq 1} \frac{P_{n-1}}{k} \frac{c_1(L^{\otimes k})^n}{n}$$

Thus if $c_1(L^{\otimes k}) = \varphi_k(c_1(L))$ one has that

$$\sum_{n\geq 1} \frac{\varepsilon^k P_{n-1}}{k^n} \frac{\varphi_k(X)^n}{n} = \sum_{n\geq 1} \frac{P_{n-1}}{k} \frac{\varphi_k(X)^n}{n}$$

But over $\Omega(pt)[\frac{1}{k}]$ $\varphi_k(X)$ is invertible, hence

$$\varepsilon^k P_{n-1} = k^{n-1} .$$

For the second assertion we note this formula implies

~~$(\varepsilon^k \circ \varepsilon^j)(\ell) = \varepsilon^{kj}(\ell)$~~ $(\varepsilon^j \circ \varepsilon^k)(\ell) = \varepsilon^{jk}(\ell)$

Thus

~~[illegible]~~

$$(\varepsilon^{j}\circ\varepsilon^{k})\,\ell(c_1 L) = \varepsilon^{j}\big((\varepsilon^{k}\ell)(\varepsilon^{k}c_1 L)\big) = (\varepsilon^{j}\varepsilon^{k}\ell)(\varepsilon^{j}\varepsilon^{k}c_1 L)$$

$$= (\varepsilon^{jk}\ell)(\varepsilon^{j}\varepsilon^{k}c_1 L)$$

$$\ell(c_1 L) = (\varepsilon^{jk}\ell)(\varepsilon^{jk}c_1 L)$$

and so $\varepsilon^{jk}c_1 L = \varepsilon^{j}\varepsilon^{k}c_1 L$ which proves that $\varepsilon^{j}\varepsilon^{k} = \varepsilon^{jk}$.

Remarks: 1. The diagram

$$\begin{tikzcd}
\Omega \arrow[r] \arrow[d, "\varepsilon^k"] & K \arrow[d, "\frac{\psi^k}{k}"] \\
\Omega[\frac{1}{k}] \arrow[r] & K[\frac{1}{k}]
\end{tikzcd}$$

commuted.

2. Let $\Gamma = \bigoplus_{n\geq 0}\mathbb{Z}\binom{T}{n} \underset{\mathbb{Z}[T]}{\otimes} \mathbb{Z}[T,T^{-1}]$, and define

$$\varepsilon^{T} : \Omega \longrightarrow \Gamma \underset{\mathbb{Z}}{\otimes} \Omega$$

~~[illegible]~~ to be the multiplicative operation with

$$\varepsilon^{T}\big(c_1^{\Omega}(L)\big) = \text{"}\frac{1}{T}c_1(L^{T})\text{"} \quad \text{, that is,}$$

$$= \frac{1}{T}\sum_{n\geq 0}\binom{T}{n}c_1\big((L-1)^{n}\big)$$

I claim that ε^{T} is a stable operation and that the proposition on page 2 generalizes, i.e.

$$\begin{cases} \varepsilon^{T} \cdot \varepsilon^{T'} = \varepsilon^{TT'} & \text{as operations } \Omega \to \Gamma \otimes \Gamma \otimes \Omega . \\ \varepsilon^{T}(P_n) = T^{n} P_n . \end{cases}$$

Note that it suffices to check this after tensoring with $\mathbb{Q}$ as Γ, $\Omega(pt)$ are torsion-free. But

$$\Gamma \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \simeq \mathbb{Q}[T, T^{-1}]$$

and

$$\begin{tikzcd} \mathbb{Q}[T, T^{-1}] \arrow[r, hook] & \prod_{k \geq 1} \mathbb{Q} \\ T \arrow[r, mapsto] & (k) \end{tikzcd}$$

(any polynomial function is determined by its values on the positive integers). Thus to check the formulas in general it is sufficient to do so for each integer $k > 0$, which we've already done.

3. For η one has $\Gamma \underset{\mathbb{Z}}{\otimes} \mathbb{F}_2 = \underset{(\text{sets})}{\text{Homcont}} (\mathbb{Z}_2^*, \mathbb{F}_2)$.

Moreover $\varphi_2(X) = 0$. Thus the operation ε^{t} where t is a 2-adic ~~number~~ unit is always the identity. ~~if t ≡ 1 (mod 2) and the projection operator~~

Lemma: Let R be a ring and let $\varphi(X) = \sum_{n\geq 0} r_n X^n \in R[[X]]$ satisfy

$$\varphi(X+Y+XY) = \varphi(X)\,\varphi(Y)$$
$$\varphi(0) = 1$$

Let $\Gamma_0 = \bigoplus_{n\geq 0} \mathbb{Z}\binom{T}{n}$ be the subring of $\mathbb{Q}[T]$ consisting of all polynomials $P(T)$ such that $P(\mathbb{Z}) \subset \mathbb{Z}$. Then there is a unique ring homomorphism $u : \Gamma_0 \longrightarrow R$ sending $\binom{T}{n}$ to r_n.

Proof: The uniqueness of u is clear, as the $\binom{T}{n}$ are a base for Γ over $\mathbb{Z}$. There are integers a_{mni} such that

$$\binom{T}{m}\binom{T}{n} = \sum_{0\leq i\leq m+n} a_{mni}\binom{T}{i}$$

coming from the fact that the LHS is an integral valued polynomial. It suffices to show that

$$r_m r_n \overset{?}{=} \sum_i a_{mni}\, r_i \qquad \text{all } m,n, \qquad \left(\begin{matrix}\text{and that}\\ r_0 = 1\end{matrix}\right)$$

to conclude u is a ring homomorphism. That is it suffices to show that

$$\sum r_m X^m \cdot \sum r_n Y^n \overset{?}{=} \sum_i a_{mni}\, X^m Y^n r_i = \sum_i r_i\Big(\sum_{m,n} a_{mni} X^m Y^n\Big)$$

First we make the calculation in $\Gamma_0 \otimes \mathbb{Q}$

$$\sum_{m,n\geq 0}\binom{T}{m}\binom{T}{n}X^m Y^n = \sum\binom{T}{m}X^m \cdot \sum\binom{T}{n}Y^n$$
$$= e^{T\log(1+X)}\cdot e^{T\log(1+Y)}$$
$$= e^{T[\log(1+X)+\log(1+Y)]} = e^{T\log(1+X+Y+XY)}$$

$$= \sum_i \binom{T}{i} (X+Y+XY)^i$$

Thus we see that

$$\sum_{m,n}{}' a_{mni} X^m Y^n = (X+Y+XY)^i$$

and therefore what we have to prove is that

$$\varphi(X) . \varphi(Y) = \varphi(X+Y+XY)$$

which is our hypothesis.

The identity $\sum \binom{T}{m} X^m = e^{T \log (1+X)}$

in $\mathbb{Q}[T][[X]]$ follows from the fact that it holds ~~[illegible]~~ after taking T to be any positive integer.

July 5, 1969.

On K-theory characteristic numbers

§1. Formal groups of height 1

Let p be a fixed prime number. If A is an abelian group let $A_{/p^n} = A/p^nA$. Let L be the Lazard ring with universal law F_{univ}. Let $P_i \in L$ be the elements given by

$$\omega_{univ}(Z) = \sum_{n\geq 0} P_n Z^n dZ \qquad P_0 = 1$$

$$l_{univ}(Z) = \sum_{n \geq 0} P_n \frac{Z^{n+1}}{n+1}$$

where ω_{univ} and l_{univ} are the invariant differential and logarithm of F_{univ}.

We identify a scheme with the functor it represents from rings to sets. Let $\mathcal{L}$ be the Lazard scheme associating to R its set of formal group laws

$$\mathcal{L}(R) = \mathrm{Hom}_{(rings)}(L, R)$$

and let $\mathcal{G}$ be the affine group scheme

$$\mathcal{G}(R) = \left\{ \text{power series } \sum_{n\geq 0} r_n X^{n+1},\ r_n \in R,\ r_0 \in R^* \right\}$$
under composition.

$$= \mathrm{Hom}_{(rings)}\left(\mathbb{Z}[a_0^{-1}, a_0, a_1, \cdots\cdots], R \right)$$

Then $\mathcal{G}$ acts on $\mathcal{L}$ by

$$\begin{aligned} \mathcal{G}(R) \times \mathcal{L}(R) &\longrightarrow \mathcal{L}(R) \\ (\varphi\ ,\ F) &\longmapsto \varphi * F \end{aligned}$$

where $(\varphi * F)(X,Y) = \varphi\left(F(\varphi^{-1}X, \varphi^{-1}Y)\right).$

Of central importance for us will be the "orbit" of the $\overbrace{\text{law}}^{\text{multiplicative}}$ $X+Y+XY$ over $\mathbb{Z}$. Let $\mathcal{H} \subset \mathcal{G}$ be the stabilizer of this law, that is

$$\begin{aligned} \mathcal{H}(R) &= \{ \varphi \in \mathcal{G}(R) \mid \varphi(X+Y+XY) = \overbrace{\varphi(X)\varphi(Y)}^{\varphi(X)+\varphi(Y)+} \}. \\ &= \operatorname{Hom}_{(\text{rings})}(\Gamma, R) \end{aligned}$$

where

$$\Gamma = \bigoplus_{n \geq 0} \mathbb{Z}\binom{T}{n} \otimes_{\mathbb{Z}[T]} \mathbb{Z}[T, T^{-1}]$$

$$\varphi_{\text{univ}}(X) = \sum_{n \geq 1} \binom{T}{n} X^n.$$

~~Then we have a semi-simplicial [illegible] affine scheme~~

$$\cancel{\cdots \;\substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow}\; \mathcal{G} \times \mathcal{H} \times \mathcal{H} \;\substack{\rightarrow\\\rightarrow\\\rightarrow}\; \mathcal{G} \times \mathcal{H} \rightrightarrows \mathcal{G}}$$

~~where faces and degeneracies are given by~~

$$\cancel{d_i(g_0, g_1, \ldots, g_q) = \begin{cases} (g_1, \ldots, g_q) & i = 0 \\ (g_0, \ldots, g_{i-1}g_i, \ldots, g_q) & 0 < i < \end{cases}}$$

Then our problem is to understand how exact or non-exact the diagram of schemes

$$(1) \qquad \mathcal{G} \times \mathcal{H} \underset{\text{mult}}{\overset{pr_1}{\rightrightarrows}} \mathcal{G} \xrightarrow[\text{acting on } X+Y+XY]{} \mathcal{L}$$

is. In terms of coordinate rings these maps are

$$(2) \qquad L \longrightarrow \mathbb{Z}[a_0^{-1}, a_0, a_1, \ldots] \rightrightarrows \mathbb{Z}[a_0^{-1}, a_0, a_1, \ldots] \otimes \Gamma$$

$$F_{univ} \longmapsto \underbrace{\left(\sum a_n X^{n+1}\right)}_{\psi_{univ}} * (X+Y+XY)$$

$$\psi_{univ} \longmapsto \psi_{univ} \otimes 1$$

$$\longmapsto \psi_{univ} \otimes \circ \psi_{univ}$$

Now suppose that $p^{\nu}R=0$ for a fixed ν and let $p_F(X)$ be the pth iterate for the group law F over R. One knows that modulo p, $p_F(X)$ is a power series in X^{p^h} for some h, possibly ∞

$$p_F(X) \equiv a_{p^h} X^{p^h} + \cdots\cdots \pmod{pR}.$$

We say that F is of height $\geq h$, and if a_{p^h} is a unit in R, then we say that F is of height h. Example:

$$F(X,Y) = X+Y+XY = (1+X)(1+Y)-1$$

$$p_F(X) = (1+X)^p - 1 \equiv X^p \pmod{p}$$

is of height 1. The law F_{univ} over $\mathbb{Z}_{(p)}$ is isomorphic via Cartier to the law F' with logarithm

$$\ell_{F'}(X) = \sum_{a \geq 0} P_{p^a-1} \frac{X^{p^a}}{p^a}$$

and for this law

$$p_{F'}(X) = \ell_{F'}^{-1}(p \cdot \ell_{F'}(X)).$$

Now working mod (degree $p+1$) we have

$$\ell_{F'}(X) \equiv X + P_{p-1}\frac{X^p}{p}$$

$$\ell_{F'}^{-1}(X) \equiv X - P_{p-1}\frac{X^p}{p}$$

so

$$\begin{aligned} p_{F'}(X) &\equiv \left(pX + P_{p-1}X^p\right) - P_{p-1}\frac{(pX)^p}{p} \\ &\equiv pX + P_{p-1}(1-p^{p-1})X^p \\ &\equiv P_{p-1}X^p \quad \operatorname{mod}(p, \deg p+1) \end{aligned}$$

Therefore if F is a law over R satisfying $p^\nu R=0$, then F is a law of height 1 iff P_{p-1} goes into a unit under the canonical map $L \to R$.

Let $\mathcal{L}_{/p^\nu} = \mathcal{L} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Z}_{/p^\nu}$ and let $\mathcal{L}^1_{/p^\nu}$ be the open subset where P_{p-1} is invertible. Then clearly

$$\mathcal{L}^1_{/p^\nu}(R) = \left\{ \begin{array}{l} \text{laws over } R \text{ of height } 1 \text{ where} \\ R \text{ is a } \mathbb{Z}/p^\nu \text{ algebra.} \end{array} \right.$$

and we have a diagram with maps as (1) above

$$\mathcal{G}_{/p^\nu} \times \mathcal{H}_{/p^\nu} \rightrightarrows \mathcal{G}_{/p^\nu} \longrightarrow \mathcal{L}^1_{/p^\nu} . \tag{3}$$

The corresponding maps of coordinate rings are

$$(4) \qquad \begin{tikzcd} L_{/p^\nu}[P_{p-1}^{-1}] \arrow[r] & \mathbb{Z}_{/p^\nu}[a_0^{-1}, a_0, a_1, \dots] \arrow[r, shift left] \arrow[r, shift right] & \mathbb{Z}_{/p^\nu}[a_0^{-1}, a_0, a_1, \dots] \otimes_{\mathbb{Z}} \Gamma \end{tikzcd}$$

$$F_{univ} \longmapsto \psi_{univ} * (X+Y+XY)$$

$$\psi_{univ} \begin{array}{l} \longmapsto \psi_{univ} \otimes 1 \\ \longmapsto (\psi_{univ} \otimes 1) \circ (1 \otimes \psi_{univ}) \end{array}$$

<u>Theorem:</u> The diagram (3) of schemes is exact, and allows us to identify $\mathcal{L}'_{/p^\nu}$ with the homogeneous space $G_{/p^\nu}/H_{/p^\nu}$ in the sense of fpqc sheaves. The morphism $G_{/p^\nu} \longrightarrow \mathcal{L}'_{/p^\nu}$ is faithfully flat. The diagram (4) is left exact and the arrows of (4) are faithfully flat.

<u>Corollary:</u> If F is a law of height 1 over a ring R such that $p^\nu R = 0$, then there exists a faithfully flat extension $R \to R'$ such that over R', F becomes isomorphic to $X+Y+XY$.

The corollary follows from the theorem since one may take R' to fit in a cocartesian square of rings:

$$\begin{tikzcd} L_{/p^\nu}[P_{p-1}^{-1}] \arrow[r] \arrow[d] & \mathbb{Z}_{/p^\nu}[a_0^{-1}, a_0, a_1, \dots] \arrow[d] \\ R \arrow[r] & R' \end{tikzcd}$$

We show how the corollary implies the theorem. Consider (3) as a diagram of sheaves for the fpqc topology and let $\mathcal{G}_{/p^\nu}/\mathcal{H}_{/p^\nu}$ be the quotient sheaf. The map of sheaves

$$\mathcal{G}_{/p^\nu}/\mathcal{H}_{/p^\nu} \longrightarrow \mathcal{L}'_{/p^\nu}$$

is clearly injective by the definition of $\mathcal{H}$. By the corollary it is surjective locally and hence is an isomorphism. Thus (3) is exact as a ~~sequence~~ diagram of fpqc sheaves, hence also as a diagram of schemes. To see that $\mathcal{G}_{/p^\nu} \xrightarrow{\pi} \mathcal{L}^1_{/p^\nu}$ is faithfully flat, note that by the corollary ∃ a faithfully flat extension $X \longrightarrow \mathcal{L}'_{/p^\nu}$ such that one has that the pull back of π is

$$\begin{tikzcd}
Y = \mathcal{G}_{/p^\nu} \times_{\mathcal{L}_{/p^\nu}} X \arrow[r, "\pi'"] \arrow[d] & X \arrow[d] \arrow[dl, dashed] \\
\mathcal{G}_{/p^\nu} \arrow[r, "\pi"] & \mathcal{L}^1_{/p^\nu}
\end{tikzcd}$$

isomorphic to $pr_1 : X \times \mathcal{H}_{/p^\nu} \longrightarrow X$. As $\mathcal{H}_{/p^\nu}$ is the group schemes associated to the profinite group $\mathbb{Z}_p^*$ over $\mathbb{Z}_{/p^\nu}$, it follows that $\mathcal{H}_{/p^\nu} \longrightarrow \operatorname{Spec} \mathbb{Z}_{/p^\nu}$ is faithfully flat, hence also pr_1 is. Thus by descent π is faithfully flat.

Note that once (3) is known to be exact, the exactness of (4) follows by taking maps in $\mathbb{G}_a$.

We now prove the corollary using essentially a technique of Lazard's to handle the case over R/pR and then Lubin–Tate for the passage from R/pR to $R/p^\nu R$. Recall that Lazard proved that any two group laws of the same height over a separably closed field are isomorphic. Actually his proof shows that if

F and F' are laws of the same height h over a ring A of characteristic p, then one can construct ~~successive~~ a sequence of finite ~~[illegible]~~ étale extensions of A

( ) $$A \subset A_1 \subset A_2 \subset \cdots\cdots$$

where
$$A_1 \cong A[X]/(X^{(p^h-1)} - \alpha) \qquad \alpha \text{ a unit in } A$$

and for $n > 1$
$$A_n \cong A_{n-1}[X]/(X^{p^h} - X - \beta_n) \qquad \beta_n \in A_{n-1},$$

such that over $A_\infty = \varinjlim A_n$, the laws F and F' are isomorphic. Instead of entering into the details of Lazard's proof I shall ~~[illegible]~~ give instead the following modification. Let
$$p_F(X) = g(X^{p^h}) \qquad \text{where}$$
$$g(X) = \alpha X \quad \text{mod degree } 2 \quad \text{with} \quad \alpha \in A^*.$$

By Fröhlich there is a ~~successive~~ sequence of finite étale extensions of the form ( ) and a invertible series $u(X)$ with coefficients in A_∞ such that
$$(u \circ p_F \circ u^{-1})(X) = X^{p^h}.$$

It follows that $X \longmapsto X^{p^h}$ is an endomorphism of the law $u * F$ and consequently the coefficients of $u * F$ ~~[illegible] coefficients in~~ ~~comes from a law F_0 over $\mathbb{F}_{p^h}$ via a homomorphism~~ satisfy the equation

( ) $$\lambda^{p^h} = \lambda.$$

Let $\varphi(X)$ be the Cartier change of coordinates such that $\varphi * (u * F)$ is a typical law. Note that the coefficients of $\varphi(X)$ satisfy ( ) hence

$$P_{\varphi*(u*F)}(X) = \varphi(\varphi^{-1}(X)^{p^h}) = \varphi(\varphi^{-1}(X^{p^h})) = X^{p^h}.$$

Therefore over $A_\infty^{\bar{F}}$ the law F is isomorphic to a typical law F_0 with $P_{F_0}(X) = X^{p^h}$. By Cartier's theory this law F_0 is unique and is defined over $\mathbb{F}_p$. In fact Cartier's coordinates are defined by

$$P_{F_0}(X) = \sum_{n \geq 1}{}^{F_0} b_n X^{p^n}$$

hence all $b_n = 0$ except $b_h = 1$. We have therefore shown that F becomes isomorphic to F_0 after a sequence of finite etale extensions; similarly F′ becomes isomorphic to F_0 after another sequence.

Suppose now that the height $h = 1$ and that F and F′ are two laws of height 1 over a ring R such that $p^\nu R = 0$. Then after a sequence of finite etale extensions of $A = R/pR$ we obtain a ring A′ over which F and F′ are isomorphic. Now by the theorem on the topological invariance of the fundamental group, there is a finite etale extension $R \longrightarrow R_n$ reducing modulo p to $A \longrightarrow A_n$, and hence passing to the limit there is a faithfully flat ind-etale extension $R \longrightarrow R'$ ~~such that~~ reducing modulo p to $A \longrightarrow A'$. Let u be a invertible series over R′ such that $u*F \equiv F'$ mod pR'. Then the laws $u*F$ and F′ are both liftings to R′ of the same law $\bar{F}$ over R'/pR'. By <u>Lubin-Tate</u> such liftings up to isomorphism are parameterized by h-1 parameters where h is the height of $\bar{F}$. If $h = 1$ any two liftings are therefore isomorphic. Hence F and F′

Given a representation of G

$$\rho : G \longrightarrow U(n)$$

I know how to associate an element $[\rho]$ of $\tilde{K}^{1}(G) = \tilde{K}^{0}(\Sigma G)$

I want to associate an element of

$$\Omega^{-1}(G) \xrightarrow{\simeq} \Omega^{0}(\Sigma G)$$

which maps down onto the element $[\rho]$ if possible.

$$\begin{tikzcd}
X \arrow[r] & U \\
\Sigma X \arrow[r] & BU \\
\Omega^{0}(\Sigma X) & \Omega^{0}(BU) \arrow[l] \\
u & 1 \arrow[l, mapsto] \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r] & U \arrow[r, "\simeq"] & \Omega(BU)
\end{tikzcd}$$

$$\begin{tikzcd}
\Sigma U \arrow[r] & BU \\
\boxed{K^{0}(\Sigma U)} & K^{0}(BU) \ni \mathrm{id} \arrow[l]
\end{tikzcd}$$

thus you want maybe a way of lifting a rep $\rho : G \longrightarrow U$ into a cob. element $\alpha(\rho)$ such that the elements $\alpha(\rho_i)$ goe under $\Omega(G) \longrightarrow B \otimes_{A} K$, into generators for the latter

$$\begin{tikzcd}
\Sigma U(n) \arrow[r] & BU(n) \\
\Omega^{0}(\Sigma U(n)) \arrow[d] & \Omega^{0}(BU(n)) \arrow[l] \arrow[d] \\
K^{0}(\Sigma U(n)) & K^{0}(BU(n)) \arrow[l]
\end{tikzcd}$$

how does ψ^{k} act on the element of $K^{0}(\Sigma U(n))$ corresponding to the fund. representation

by consideration of the character one sees that

K theory characteristic numbers

$\Omega(X)$ even unitary bordism of X

$K(X)$ as usual.

Operations $\Omega \longrightarrow K$: $\exists$ a ring R with a left $\Omega(pt)$ algebra structure and a right $K(pt)$ module structure and a universal ~~[illegible]~~ operation

$$\theta : \Omega \longrightarrow R \otimes_{\mathbb{Z}} K(X).$$

~~In fact each such operation is given by a power~~ such that for the Hopf bundle $\mathcal{O}(1)$ on $\mathbb{C}P^1$ we have

$$\theta\left(c_1^{\Omega}\mathcal{O}(1)\right) = \text{unit} \cdot c_1^{K}(\mathcal{O}(1)).$$

Such an operation is given by a power series

$$\theta\left(c_1^{\Omega}(L)\right) = \sum_{n \geq 0} a_n \, c_1(L)^{n+1}$$

where $a_0 \in R^*$, $a_n \in R$.

Thus the universal operation is

$$\Omega(X) \longrightarrow \mathbb{Z}[a_0^{-1}, a_0, a_1, \ldots] \otimes_{\mathbb{Z}} K(X)$$

$$c_1(L) \longmapsto \sum_{i \geq 0} a_i \, c_1^K(L)^{i+1}$$

Think thru thm. on K-theory and Ω

assertion: Let k be an integer ≥ 1. let

$$\Omega(X) \underset{\Omega(pt)}{\otimes} \Omega(pt)\left[\frac{1}{P_{p-1}}\right] \underset{\mathbb{Z}}{\otimes} \mathbb{Z}/p^k = \Omega(X)[P_{p-1}^{-1}]/p^k$$

$$K(X) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}/p^k = K(X)/p^k$$

Then

$$\Omega(X)[P_{p-1}]/p^k \xrightarrow{\theta} \cancel{K(X)/p^k \otimes}$$

$$K_*(MU) \underset{\mathbb{Z}}{\otimes} K(X)/p^k \rightrightarrows K_*(MU) \otimes \Gamma \otimes K(X)/p^k$$

is exact and moreover the arrow θ is faithfully flat.

Better Γ/p^k acts freely on

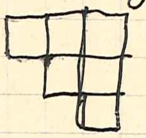
$$K_*(MU) \otimes_{\mathbb{Z}} K(X)/p^k$$

and the quotient is $\Omega(X)[P_{p-1}]/p^k$.

poor formulation! you want a good formulation and proof of the theorem!

(II) $B\otimes_A A\Omega(X) \xrightarrow{\sim} B\otimes_{\mathbb{Z}/p^n} K(X)/p^n$

Once you know that B is faithfully flat over A and that



$$A\Omega\cancel{(X)} \longrightarrow B\otimes K\cancel{(X)} \rightrightarrows B\otimes\Gamma\otimes K$$

when writing this up it will be necessary to carefully to present the descent theory separately from the rest.

Claim that once $\boxed{A\longrightarrow B \text{ known to be f.f.}}$ ~~and~~ with $B\otimes_A B \cong B\otimes_k\Gamma$, then the rest is formal.

$$\begin{tikzcd}
A\Omega \arrow[r] & B\otimes_k K \arrow[r, shift left] \arrow[r, shift right] & B\otimes_k\Gamma\otimes_k K \\
A \arrow[u] \arrow[r] & B \arrow[u] & A\Omega \arrow[dd] \\
 & A\Omega \arrow[ur, "\cong"] \arrow[d] & \\
 & B\otimes_A A\Omega \arrow[r, "\cong"] \arrow[d] & B\otimes_k K \arrow[d, shift left] \arrow[d, shift right] \\
 & B\otimes_A B\otimes_A A\Omega \arrow[r, "\cong"] & B\otimes_k\Gamma\otimes_k K
\end{tikzcd}$$

The natural map is an isom. because of the universal nature of both sides

this proves the exactness.

$$\Omega^{\cdot} \longrightarrow K_{\cdot}(MU) \otimes_{K^{\cdot}(pt)} K^{\cdot}(X) \rightrightarrows K_{\cdot}(MU) \otimes_{K^{\cdot}(pt)} K_{\cdot}(\underset{\sim}{BU}) \otimes_{K^{\cdot}(pt)} K^{\cdot}(X)$$

so in this way we can form the complex

$$K_{\cdot}(MU) \otimes \Gamma \otimes \cdots \otimes \Gamma \otimes K^{\cdot}(X)/p^n$$

and the assertion is that it is exact except in the first spot where it is

$$\Omega^{\cdot}(X) \cancel{\otimes} /p^n [P_{p-1}^{-1}].$$

How to prove this. Establish an isomorphism

~~$\Omega(pt)/p^n[P_{p-1}^{-1}] \otimes_{\Omega(pt)} \Omega^{\cdot}(X) \cong K(MU)/p \otimes K(X)$~~

set $A\Omega(X) = \Omega(X)/p^n[P_{p-1}^{-1}]$

$A = A\Omega(pt)$. universal ring over $\mathbb{Z}/p^n\mathbb{Z}$ with law of height 1

$$A \longrightarrow B = K_{\cdot}(MU)/p^n$$

(I) $A \rightarrow B$ faithfully flat and

$$B \otimes_A B \cong B \otimes_{\mathbb{Z}/p^n} \Gamma/p^n$$

| Galois situation

thus Spec $B \rightarrow$ Spec A is a principal fiber for Spec Γ/p^n.

geometric fact:

$$k \otimes_A A\Omega \xrightarrow{\sim} K$$ Conner-Floyd thm.

These are the ingredients, now for the proof of the following

Theorem: (i) $B \otimes_A A\Omega(X) \xrightarrow{\sim} B \otimes_k K(X)$

(ii) $A\Omega(X) \longrightarrow B \otimes_k K(X) \rightrightarrows B \otimes_k \Gamma \otimes_k K(X) \overset{\rightarrow}{\rightrightarrows}$ ~~exact~~ acyclic

(iii) $A\Omega(X) \longrightarrow B \otimes_k K(X)$ faithfully flat

Can you deduce Petrie?

$X = G$ compact s.c. Lie gp.

by Hodgkin

$K^{*}(X)$ ~~exterior~~ algebra over $K^{*}(pt)$ generators in $K^{1}(X)$

now how ~~do~~ the generators appear?

$$G \longrightarrow U = \Omega \mathbb{Z}$$

$$\Sigma G \longrightarrow BU$$

$$K^0(\Sigma G)$$
$$\|$$
$$K^{-1}(G)$$

how does $\frac{1}{k}\psi^k$ act on a suspension

$$\begin{tikzcd} K^0(\Sigma G) \arrow[d, "ch"] \\ H^{ev}(\Sigma G, \mathbb{Q}) \end{tikzcd}$$

RcA

$$\begin{tikzcd} K^{-1}(G) \arrow[r, "\sigma"] \arrow[d, "ch"] & K^0(\Sigma G) \arrow[d, "ch"] \\ H^{odd}(G) \arrow[r, "\sigma"] & H^{ev}(\Sigma G) \end{tikzcd}$$

July 6, 1969 – July 14, 1969

Real K-theory and cobordism theory

Let $G = \mathbb{Z}/2\mathbb{Z}$ act on a space X. Atiyah considers complex vector bundles over X with a compatible G-action which is semi-linear with respect to the conjugation – of $\mathbb{C}$. These things he calls real bundles on the G-space X. When G acts trivially on X this terminology is justified. $KR(X)$ is the associated K-group.

There is a corresponding cobordism theory $\Omega R(X)$ defined as the universal cohomology theory on G-manifolds endowed with Gysin homomorphism for proper maps endowed with a "real" structure on the stable normal bundle. (I shall only work with even degree Gysin morphisms for the moment)

Chern classes for real bundles in cohomology: Clearly

$$\operatorname{Pic}R(X) = H^1(X, G; \mathcal{O}_X^*)$$

where $\mathcal{O}_X$ denotes complex valued functions and G acts on $\mathcal{O}_X$ via conjugation. Usual sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 0$$

shows that

$$H^1(X, G; \mathcal{O}_X^*) \xrightarrow{\sim} H^2(X, G; \mathbb{Z}(1))$$

where $\mathbb{Z}(1)$ denotes the integers where G acts via sign representation This defines the first Chern class. Higher Chern classes are defined as usual and

$$c_q(E) \in H^{2q}(X, G; \mathbb{Z}(q)).$$

If E is a real bundle on a G-space X, then $\Gamma(X,E)$ has a conjugation on it and so $\Gamma(X,E) = \mathbb{C}\otimes_{\mathbb{R}}\Gamma(X,E)^G$. This shows that any bundle is a direct summand of a bundle of the form f^*n where $n = \mathbb{C}^n$ over pt. Also any n-dimensional real bundle is induced by a G-map

$$X \longrightarrow \mathrm{Grass}_n(\mathbb{C}^N)$$

for N sufficiently large, where G acts on the right by conjugation. Observe that the fixed set is $\mathrm{Grass}_n(\mathbb{R}^N)$. Also

$$\begin{aligned}\widetilde{KR}(X) &= \text{real virtual bundles of dim } 0\\ &= [X, BU]_G\\ KR(X) &= [X, \mathbb{Z}\times BU]_G\end{aligned}$$ G acts trivially on the $\mathbb{Z}$.

In general if Q is a Chern theory in this real setup then

$$\mathrm{Hom}_{(sets)}(\widetilde{KR}, Q) = Q(pt)[c_1, c_2, \text{---}].$$

Take $Q(X) = H^*(X, G; \mathbb{Z}(-)) \oplus H^*(X, G; \mathbb{Z}(+))$ which is a Chern theory. Now

$$\begin{aligned}Q(pt) = H^*(pt, G; \mathbb{Z}^{sg}) &\oplus H^*(pt, G, \mathbb{Z})\\ \cong \quad &\quad\ \ \|\\ H^*(\mathbb{R}P^\infty, \mathbb{Z})\cdot\eta &\oplus H^*(\mathbb{R}P^\infty, \mathbb{Z})\end{aligned}$$

where η is the generator of $H^1(pt, G; Z^{sg}) = \dfrac{\text{Ker } 1+\sigma \text{ on } \mathbb{Z}^{sg}}{\text{Im } 1-\sigma \text{ on } \mathbb{Z}^{sg}} = \mathbb{Z}/2$
Now $\eta^2 =$ generator of $H^2(\mathbb{R}P^\infty; \mathbb{Z})$ as one sees by reducing modulo 2 and using that $H^*(\mathbb{R}P^\infty, \mathbb{Z}/2) = \mathbb{Z}/2[\eta]$. Thus

$$Q(pt) = \mathbb{Z} \oplus \eta\,\mathbb{Z}/2[\eta] = \mathbb{Z}[\eta]/(2\eta)$$

and

$$Q(BU_n) = \mathbb{Z}[\eta, c_1, c_2, \dots, c_n]/(2\eta) .$$

As the real bundles over a points are trivial it follows that the Chern aspect for real bundles over a point is the same as for complex bundles and Ω. Thus

$$\Omega R(BU_1) = \Omega R(pt)[c_1]$$

$$c_1(L \otimes L') = F(c_1 L, c_1 L')$$

for some law over $\Omega R(pt)$. This law determines the behavior of Chern classes just as for Ω.

Special features of ΩR: The fixed point submanifold functor $X \longmapsto X_{\mathbb{R}}$ is compatible with normal bundles so gives a morphism of theories

$$\Omega R(X) \longrightarrow \eta(X_{\mathbb{R}})$$

compatible with Gysin morphism and hence compatible with Chern classes. Forgetting G-action gives a morphism

$$\Omega R(X) \longrightarrow \Omega(X)$$

compatible with Gysen. This shows that $\Omega R(pt)$ is an augmented $\Omega(pt)$-algebra. ~~On the other [illegible]~~

~~defines a morphism~~

$$\cancel{KO(X) \longrightarrow KR(X\times X).}$$

~~In effect G acts on $E\times E$ over $X\times X$ and we can endow $E\times E$ with a complex structure by defining~~

$$\cancel{i(e,e) = (e,-e)}$$

$$\cancel{i(e,-e) = (-e,-e)}$$

~~?~~

Suppose that Q is a cohomology theory on G-manifolds with Thom isomorphism for real vector bundles. Following Atiyah let $\mathbb{R}^{p,q}$ be $\mathbb{R}^{p+q}$ where G acts antipodally on the first p factors. Let $\Sigma^{p,q} = \mathbb{R}^{p,q} \cup \{\infty\}$ and set

$$Q^{p-q}(X) = \tilde{Q}(X \wedge \Sigma^{p,q})$$

Observe this is well-defined since there is the Thom isomorphism

$$Q(X) \xrightarrow{\cong} Q(X \wedge \Sigma^{1,1}).$$

More precisely $\bigoplus_{p,q\geq 0} \tilde{Q}(X\wedge \Sigma^{p,q})$ is a graded ring and one divides out by the relation $b-1$ where $b \in \tilde{Q}(\Sigma^{1,1})$ is the Thom class.

Atiyah studies the case where $Q = KR$ in which case $KR^*(X) = KO^*(X)$, the period 8 real K-theory when X has trivial G-action. I shall now briefly review Atiyah's proof of periodicity. He considers the theory $X \longmapsto KR^*(X\times S^{p,0})$ where $S^{p,q}$ (resp. $B^{p,q}$) denotes the unit sphere (resp. unit ball) in $\mathbb{R}^{p,q}$ and shows using the fields $\mathbb{R}, \mathbb{C}, \mathbb{H}$ that is of period $2p$ for $p=1,2,4$. This is because there is then a G-isomorphism

$$(X\times S^{p,0})\times(B^{p,q}, S^{p,q}) \longrightarrow (X\times S^{p,0})(B^{0,p+q}, S^{0,p+q})$$

given by the multiplication

$$\begin{array}{ccc} S^{p,0} \times \mathbb{R}^{p,0} & \longrightarrow & \mathbb{R}^{0,p} \\ (\xi\ ,\ v) & \longmapsto & \xi v\ , \end{array}$$

and hence

$$(*) \qquad KR^{p-q}(X \times S^{p,0}) \cong KR^{-p-q}(X \times S^{p,0}).$$

Next he considers the Gysin sequence for the sphere bundle $S^{p,0} \longrightarrow pt.$ which is the long exact sequence for the pair $(X \times \underline{B}^{p,0}, X \times \underline{S}^{p,0})$:

$$\begin{tikzcd} {} \arrow[r] & \widetilde{KR}^q(X \wedge \Sigma^{p,0}) \arrow[r] \arrow[d, phantom, "\wr"] & KR^q(X \times B^{p,0}) \arrow[r] \arrow[d, phantom, "\wr"] & KR^q(X \times S^{p,0}) \arrow[r, "\delta"] & {} \\ & KR^{q+p}(X) \arrow[r, "\cup \xi"] & KR^q(X) \arrow[r, "\pi^*"] & KR^q(X \times S^{p,0}) & \end{tikzcd}$$

where $\xi \in KR^{-p}(pt) = \widetilde{KR}(\Sigma^{0,p})$ is the image of 1 under

$$KR(pt) \xrightarrow{\cong} \widetilde{KR}(\Sigma^{p,p}) \xrightarrow{\text{rest.}} \widetilde{KR}(\Sigma^{0,p})$$

and hence is the cup product of the same restriction for $p = 1$ which is easily seen to be the restriction

$$\begin{array}{ccccc} KR(pt) & \xrightarrow{\text{Gysin}} & \widetilde{KR}(\mathbb{P}^1(\mathbb{C})) & \longrightarrow & KR(\mathbb{P}^1(\mathbb{R})) \\ 1 & & c_1(\mathcal{O}(1)) & \longmapsto & c_1(\mathcal{O}(1)) = \eta \end{array}$$

Atiyah shows using Clifford algebras that

$$\eta^3 = 0$$

so that there are exact sequences for $p \geq 3$

$$0 \longrightarrow KR^q(X) \longrightarrow KR^q(X \times S^{p,0}) \longrightarrow KR^{q+p+1}(X) \longrightarrow 0.$$

Taking $p=4$ and the element ξ of $KR^{-8}(S^{4,0})$ corresponding to 1 under (*) we get a diagram

$$\begin{tikzcd}
0 \arrow[r] & KR^{q}(X) \arrow[r] \arrow[d, "\cup\tau"] & KR^{q}(X\times S^{4,0}) \arrow[r] \arrow[d, "\cup\xi", "\cong"'] & KR^{q+5}(X) \arrow[r] \arrow[d, "\cup\tau"] & 0 \\
0 \arrow[r] & KR^{q-8}(X) \arrow[r] & KR^{q-8}(X\times S^{4,0}) \arrow[r] & KR^{q-3}(X) \arrow[r] & 0
\end{tikzcd}$$

where τ is some element in $KR^{-8}(pt)$ deduced from ξ. The identification of τ uses Clifford algebra theory. The diagram shows that $\cup\tau$ is both injective + surjective so is an isomorphism.

The above arguments of Atiyah use peculiar properties of KR-theory only where Clifford algebras are used. The rest of the arguments should hold in general, e.g. when $p=1,2,4$ one should have the basic isomorphism

$$Q^{q}(X\times S^{p,0}) \cong Q^{q-2p}(X\times S^{p,0})$$

More generally suppose that we are given a bilinear mapping

$$\varphi : \mathbb{R}^{p}\times\mathbb{R}^{n} \longrightarrow \mathbb{R}^{n}$$

such that $\forall x\in\mathbb{R}^{p}-0$ $\varphi(x,?)$ is an isomorphism. Then φ induces an isomorphism compatible with G

$$(X\times S^{p,0})\wedge\Sigma^{n,0} \cong (X\times S^{p,0})\wedge\Sigma^{0,n}$$

so we obtain

$$Q(X\times S^{p,0}) \cong Q^{-2n}(X\times S^{p,0}).$$

Similarly there is always the Gysin sequence

$$\cdots\longrightarrow Q^{q+p}(X)\xrightarrow{\cup\eta^{p}} Q^{q}(X)\longrightarrow Q^{q}(X\times S^{p,0})\xrightarrow{\delta} Q^{q+p+1}(X)-$$

where η (up to sign) is the element of $Q^{-1}(pt) = Q(\Sigma^{0,1}) = Q(\mathbb{P}^1(\mathbb{R}))$ represented by the restriction of $c_1(\mathcal{O}(1))$ in $Q(\mathbb{P}^1(\mathbb{C}))$.

Basic calculation: Let $Q(X) = \bigoplus_{q\geq 0} H^{2q}(X, G; \mathbb{Z}(q))$

which satisfies the projective bundle theorem for real bundles and hence has a Thom isomorphism for real bundles. To calculate $Q^*(pt)$.

$$Q^{0}(pt) = \bigoplus_{q\geq 0} H^{4q}(pt, G; \mathbb{Z}) \oplus \underset{\underset{\displaystyle 0}{\shortparallel}}{\bigoplus_{q\geq 0} H^{4q+2}(pt, G; \mathbb{Z}^{sg})}$$

$$\boxed{Q^{0}(pt) = \mathbb{Z}[\eta^{4}]/(2\eta^{4})}$$

Here η is the generator of $H^{1}(pt, G; \mathbb{Z}^{sg})$.

$$Q^{-n}(pt) = \tilde{Q}(\Sigma^{0,n})$$

Note that there is a spectral sequence

$$E_2^{p,q} = H^{p}_{gp}(G, \tilde{H}^{q}(\Sigma^{0,n}, \mathbb{Z}(\pm 1))) \Longrightarrow \tilde{H}^{p+q}(\Sigma^{0,n}; G; \mathbb{Z}(\pm 1))$$

which degenerates as only $E_2^{*,n} \neq 0$. ~~Thus we find~~

~~so~~

~~$\tilde{H}^{p}(\Sigma^{0,n}; G; \mathbb{Z}(k)) = H^{p-n}_{gp}(G, \mathbb{Z}(k)\,\sigma^{n})$~~

~~$Q^{-n}(pt) = \tilde{Q}(\Sigma^{0,n}) = \bigoplus_{q\geq 0} \tilde{H}^{2q}(\Sigma^{0,n}, G; \mathbb{Z}(q))$~~

$$Q^{-1}(pt) = \tilde{Q}(\Sigma^{0,1})$$

$$= \bigoplus_{q\geq 0} \tilde{H}^{4q}(\Sigma^{0,1}, G; \mathbb{Z}) + \bigoplus_{q\geq 0} \tilde{H}^{4q+2}(\Sigma^{0,1}, G, \mathbb{Z}^{sg})$$

$$\begin{tikzcd}[row sep=small]
\bigoplus_{q\geq 0} \tilde{H}^{4q}(\Sigma^{0,1}, G; \mathbb{Z}) \arrow[d, phantom, "\wr"] & \bigoplus_{q\geq 0} \tilde{H}^{4q+2}(\Sigma^{0,1}, G, \mathbb{Z}^{sg}) \arrow[d, phantom, "\wr"] & \text{as a } H^{4*}(pt, G, \mathbb{Z}) \text{ module} \arrow[l] \\
H^{4q-1}(pt, G, \mathbb{Z}) \arrow[d, phantom, "\shortparallel"] & \bigoplus_{q\geq 0} \tilde{H}^{4q+1}(pt, G, \mathbb{Z}^{sg}) & \\
0 & = \mathbb{Z}[\eta^4]/(2\eta^4) \cdot \eta\sigma &
\end{tikzcd}$$

$$\boxed{Q^{-1}(pt) = \mathbb{Z}[\eta^4]/(2\eta^4) \cdot \eta\sigma = Q^0(pt) \cdot \eta\sigma}$$

$$Q^1(pt) = \tilde{Q}(\Sigma^{1,0}) = \bigoplus_{q\geq 0} \tilde{H}^{4q}(\Sigma^{1,0}, G; \mathbb{Z}) \oplus \bigoplus_{q\geq 0} \tilde{H}^{4q+2}(\Sigma^{1,0}, G; \mathbb{Z}^{sg})$$

$$\begin{tikzcd}[row sep=small]
\bigoplus_{q\geq 0} \tilde{H}^{4q}(\Sigma^{1,0}, G; \mathbb{Z}) \arrow[d, phantom, "\wr\ \text{as a } H^{4*}(pt, G; \mathbb{Z}) \text{ module}"] & \bigoplus_{q\geq 0} \tilde{H}^{4q+2}(\Sigma^{1,0}, G; \mathbb{Z}^{sg}) \arrow[d, phantom, "\wr"] \\
\bigoplus_{q\geq 1} \tilde{H}^{4q-1}(pt, G; \tilde{H}^1(\Sigma^{1,0}; \mathbb{Z})) \arrow[d, phantom, "\wr"] & \oplus \bigoplus_{q\geq 0} H^{4q+1}(pt, G, \mathbb{Z}) \arrow[d, phantom, "\shortparallel"] \\
 & 0
\end{tikzcd}$$

$$\tilde{H}^1(\Sigma^{1,0}; \mathbb{Z}) = \mathbb{Z}^{sg}$$

$$\boxed{Q^{-1}(pt) = Q^0(pt) \cdot \eta^3\tau}$$

Here we think of σ as the generator of $\tilde{H}^1(\Sigma^{1,0}, G; \mathbb{Z})$ and $\eta \in H^1(pt, G, \mathbb{Z}^{sg})$ and $\eta\tau$ as the product of these elements in $\tilde{H}^2(\Sigma^{1,0}, G; \mathbb{Z}^{sg})$. Similarly τ denotes the generator of $\tilde{H}^1(\Sigma^{0,1}, G, \mathbb{Z}^{sg})$, $\eta^3 \in H^3(pt, G, \mathbb{Z}^{sg})$ and $\eta^3\tau \in \tilde{H}^4(\Sigma^{0,1}, G; \mathbb{Z})$. Somehow σ (resp. τ) is the true Thom class for $\Sigma^{0,1}$ (resp. $\Sigma^{1,0}$) and yet it doesn't belong to $\tilde{Q}(\Sigma^{0,1})$ only $\eta\sigma$ does. Admit the following calculations similarly done

$$Q^1(pt) = Q^0(pt)\cdot \eta^3\tau \qquad Q^{-1}(pt) = Q^0(pt)\cdot \eta\sigma$$

$$Q^2(pt) = Q^0(pt)\cdot \eta^2\tau^2 \qquad Q^{-2}(pt) = Q^0(pt)\, \eta^2\sigma^2$$

$$Q^3(pt) = Q^0(pt)\cdot \eta\tau^3 \qquad Q^{-3}(pt) = Q^0(pt)\, \eta^3\sigma^3$$

$$Q^4(pt) = Q^0(pt)\cdot \tau^4 \qquad Q^{-4}(pt) = Q^0(pt)\, \sigma^4$$

$$\sigma := \tau^{-1}$$

Thus $Q^*(X)$ is periodic with period 4. In fact $Q^*(pt)$ is the $Q^0(pt) = \mathbb{Z}[\eta^4]/(2\eta^4)$ subalgebra of $Q^0(pt)[\sigma, \sigma^{-1}, \eta]$ generated by $\eta\sigma$ and σ^4. The associated $\mathbb{Z}/4\mathbb{Z}$ graded theory is

$$Q^0(X) + Q^{-1}(X) + Q^{-2}(X) + Q^{-3}(X) = \bigoplus_{\substack{q\geq 0\\ \varepsilon=\pm 1}} H^q(X, G; \mathbb{Z}(\varepsilon))$$

In effect

$$Q^{-1}(X) = \tilde{Q}(\Sigma^{0,1}\wedge X) = \bigoplus_{q\geq 0} \tilde{H}^{2q}(\Sigma^{0,1}\wedge X, G, \mathbb{Z}(q))$$

$$= \bigoplus_{q\geq 1} H^{2q-1}(X, G; \mathbb{Z}(q))$$

I like the picture

$\varepsilon=+1$:	Q^0		η^2	Q^{-1}	Q^0	
$\varepsilon=-1$:		ηQ^{-1}	Q^0	η^3		

$q \longrightarrow$

Question: We know both twisted cohomology and KR when extended to a graded theory having Thom isomorphism for $\Sigma^{p,0}$ become periodic. Is the same true for the universal theory ΩR?

Spectral representation for ΩR: Let $X \longmapsto \Omega R(X)$ be the universal cohomology theory on G-manifolds endowed with Gysin homomorphism for proper maps f endowed with a real structure on ν_f. This means that after factoring f into $X \xhookrightarrow{i} Y \times \mathbb{C}^n \xrightarrow{pr_1} Y$, where n is very large, that the normal bundle of i has a complex structure for which G acts semi-linearly. Now by familiar arguments

$$\Omega R(X) \cong \varinjlim_n C(X \times \mathbb{C}^n)$$

where $C(X)$ denotes bordism classes of real-oriented proper maps $Z \to X$. Observe that $\Omega R(X) = \bigoplus_{j \in \mathbb{Z}} \Omega R^j(X)$, grading by codimension.

Proposition: $$\Omega R^j(X) = \varinjlim_n \left[X \wedge \Sigma^{n-j}, MU_n \right]_G .$$

Proof: We show that the right side has the correct universal property. Denote the RHS by $\overline{\Omega R}^j(X)$ and let $\overline{\Omega R}(X) = \bigoplus_j \overline{\Omega R}^j(X)$. Then $\overline{\Omega R}$ is evidently a contravariant functor satisfying homotopy axiom. If E is a real bundle of dim. n over X, then the classifying map

$$X \longrightarrow BU_n$$

$$E \longrightarrow MU_n$$

$$X^E \longrightarrow MU_n$$

gives a Thom class $U \in \overline{\Omega R}^n(X^E)$. The Thom isomorphism

$$\begin{tikzcd}
\overline{\Omega R}^j(X) \arrow[r, "\sim"] \arrow[d, equal] & \overline{\Omega R}^{j+n}(X^E) \arrow[d, equal] \\
\varinjlim_n [X \wedge \Sigma^{n-j, n-j}, MU_n]_G \arrow[d, equal] & \varinjlim_n [X^E \wedge \Sigma^{n,n}, MU_{j+n+\dim E}]_G \\
\varinjlim_{E'} [X^{E'}, MU_{j+\dim E'}]_G &
\end{tikzcd}$$

is clear from cofinality arguments. Thus $\overline{\Omega R}$ has a Thom isomorphism and so is a cohomology theory with Gysin for proper-oriented maps. Thus there is a map $\Omega R \longrightarrow \overline{\Omega R}$. Conversely given an element $\alpha \in [X \wedge \Sigma^{n-j,n-j}, MU_n]_G$; it can be represented by a map $f : X \times \Sigma^{n-j,n-j} \longrightarrow$ smooth off the inverse image of the base point of MU_n

$E_{n,N}$
$G_{n,N}$

$$\begin{tikzcd}
 & G_{n,N} \arrow[d] \\
U \arrow[r, "f"] & MU_n \\
X \times \mathbb{C}^{n-j} \arrow[u, phantom, "\cap \text{open}"] &
\end{tikzcd}$$

where f^{-1} compact is proper over X. This may be embedded in the diagram

$$\begin{tikzcd}
 & Z \arrow[r, "\psi"] \arrow[d] \arrow[dr, phantom, "\text{cart.}"] & G_{n,N} \arrow[d] \\
U \arrow[r] \arrow[d] & U \times \mathbb{C}^N \arrow[r, "\text{smooth}"'] \arrow[d] & EU_n \\
X \arrow[r, "i"] & X \times \mathbb{C}^N &
\end{tikzcd}$$

φ

where Z is proper over $X \times \mathbb{C}^N$ and oriented of codimension j. Thus $i^* \varphi_* \psi^* 1$ gives an element of $\Omega R^j(X)$, which one must show depends only on the original element $\alpha \in \widetilde{\Omega R}^j(X)$. UGH. But this is clearly the same ugliness one encounters in proving the Thom homotopy formula for cobordism when there is transversality.

Note that the notation ΩR^j is not consistent with Ω^j. Here we have

$$\begin{aligned} \Omega R^{j,q}(X) &= \widetilde{\Omega R}^j(X \wedge \Sigma^{q,0}) \\ &= \varinjlim_n \left[X \wedge \Sigma^{n-j-q,\, n-j}, MU_n\right]_G \end{aligned}$$

It seems desirable to <u>change notation</u> and put

$$\Omega R^{p,q}(X) = \varinjlim_n \left[X \wedge \Sigma^{n-p,\, n-q}, MU_n\right]_G$$

Then there ~~is~~ are canonical maps:

$$\boxed{\begin{array}{l} \Omega R^{p,q}(X) \longrightarrow KR^{q-p}(X) \\ \Omega R^{p,q}(X) \longrightarrow \Omega^{p+q}(X) \\ \Omega R^{p,q}(X) \longrightarrow \eta^q(X_{\mathbb{R}}) \end{array}}$$

Let $\eta \in \Omega R^{1,0}(pt) = \Omega R^{1,1}(\Sigma^{0,1})$ be represented by the map $\Sigma^{0,1} = \mathbb{P}^1(\mathbb{R}) \hookrightarrow \mathbb{R}^1(\mathbb{C}) \hookrightarrow MU_1$ or equivalently (up to sign) by the Chern class of $\mathcal{O}(1)$ on $\mathbb{P}^1(\mathbb{R})$. Then I make the following conjectures:

1.) <u>Conner-Floyd thm:</u> $$KR^*(pt) \otimes_{\Omega R^{**}(pt)} \Omega R^{**}(X) \xrightarrow{\sim} KR^*(X).$$

2.) <u>"Complexification" exact sequence</u> (after Atiyah)

$$\cdots \xrightarrow{\delta} \Omega R^{p-1,q}(X) \xrightarrow{\eta} \Omega R^{p,q}(X) \longrightarrow \Omega^{p+q}(X) \xrightarrow{\delta} \Omega R^{p-1,q+1}(X) \longrightarrow \cdots$$

3.) <u>Restriction to fixpt. set isomorphism:</u> (after tom Dieck)

$$\varinjlim \left\{ \to \Omega R^{p,q}(X) \xrightarrow{\eta} \Omega^{p+1,q}(X) \xrightarrow{\eta} \right\} \cong \eta^q(X_{\mathbb{R}})$$

<u>Proof of 2.):</u> This is the Gysin sequence for $S^{1,0} \longrightarrow pt$

$$\begin{tikzcd}
\tilde{\Omega R}^{p,q}(X \wedge \Sigma^{1,0}) \arrow[r] \arrow[d, equal] & \Omega R^{p,q}(X \times B^{1,0}) \arrow[r] \arrow[d, equal] & \Omega R^{p,q}(X \times S^{1,0}) \arrow[r, "\delta"] \arrow[d, equal] & \tilde{\Omega R}^{p,q+1}(X \wedge \Sigma^{1,0}) \arrow[d, equal] \\
\Omega R^{p-1,q}(X) \arrow[r, "\cup\eta"] & \Omega R^{p,q}(X) \arrow[r] & \Omega R^{p,q}(X \times S^{1,0}) \arrow[r, "\delta"] & \Omega R^{p-1,q+1}(X)
\end{tikzcd}$$

$$\Omega R^{p,q}(X \times S^{1,0}) = \varinjlim_n \left[(X \times S^{1,0}) \wedge \Sigma^{n-p,n-q}, MU_n \right]_G$$

$$= \varinjlim_n \left[X \wedge \Sigma^{n-p,n-q}, MU_n \right] = \Omega^{p+q}(X).$$

<u>Proof of 1.):</u> This is done by the method of Conner-Floyd. First note that existence of first Chern class shows that

$$\Omega R^{11}(X) \longrightarrow KR^0(X)$$

is surjective, whence by suspension that $\Omega^{1,q+1}(X) \longrightarrow KR^{1+q}(X)$ is surjective for $q \geq 0$ and $\Omega^{p+1,1}(X) \longrightarrow KR^{-p}(X)$ is surjective for $p \geq 0$. Thus the map is surjective. The next step is to take $\alpha \in \operatorname{Ker}\{\Omega R^{**}(X) \to KR^*(X)\}$ and write $\alpha = \sum \alpha_{pq}$ where α_{pq} is represented by a map $X \wedge \Sigma^{n-p,n-q} \longrightarrow MU_n$, n suff. large and independent of p,q. Then one gets that α is represented by a map γ

$$X \wedge \Sigma^{n,n} \xrightarrow{\gamma} \bigvee_{0 \le p,q \le n} EU^{+}_{n,N} \wedge \Sigma^{p,q}$$

where $EU_{n,N}$ is the canonical bundle over the Grassmannian $G_{n,N}$. Then the GF argument reduces us to proving the theorem for $EU^{+}_{n,N} \wedge \Sigma^{p,q}$, and hence for $G_{n,N}$ by Thom isomorphism. Here it follows from the projective bundle theorem for $\Omega R^{**}(X)$ and $KR^{*}(X)$ and the fact that the theorem is true over a point.

<u>Proof of 3)</u>: Consider the localized theory

$$F^{**}(X) = \Omega R^{**}(X)[\eta^{-1}]$$

and the map

$$F^{pq}(X) \longrightarrow \eta^{q}(X_{R}) \otimes \eta^{p}$$

$$F^{**}(X) \longrightarrow \eta^{*}(X_{R}) \otimes \mathbb{Z}_2[\eta, \eta^{-1}]$$

Then F^{**} is a coh. theory on G-manifolds endowed with Gysin morphism for proper stably-real maps. Moreover F^{**} is the universal such theory with the element $\eta \in F^{10}(pt)$ invertible.

~~[illegible] has the same property.~~

Moreover we have a restriction isomorphism

$$(*) \qquad F^{**}(X) \xrightarrow{\simeq} F^{**}(X_R).$$

To see this consider the morphism of spectral sequences

$$\begin{tikzcd}
E_2^{pq} = H^p(X/G, x \mapsto F^{*q}(Gx)) \arrow[r, Rightarrow] \arrow[d] & F^{*,p+q}(X) \arrow[d] \\
E_2^{pq} \quad H^p(X_{\mathbb{R}}, x \mapsto F^{*q}(x)) \arrow[r, Rightarrow] & F^{*,p+q}(X_{\mathbb{R}})
\end{tikzcd}$$

and note that if $x \notin X_{\mathbb{R}}$, then $\eta = 0$ on Gx since the bundle $\mathcal{O}(1)$ on $\mathbb{P}^1(\mathbb{R})$ gets a section when lifted to Gx. Thus $F^{**}(Gx) = 0$ so the sheaf on X/G has support on $X_{\mathbb{R}}$ and the map on E_2 terms is an isomorphism.

~~Now we are going to show that $X \mapsto \mathbb{Z}/2[\eta, \eta^{-1}] \otimes_{\mathbb{Z}} \eta^*(X)$ is a universal cohomology theory on G-manifolds with Gysin homomorphism for proper stably-real oriented maps and such that η is invertible.~~

Here's a better proof of the restriction to fixpoint isomorphism (*). ~~Start~~ with long exact sequence

$$\longrightarrow F(X, X_{\mathbb{R}}) \longrightarrow F(X) \longrightarrow F(X_{\mathbb{R}}) \longrightarrow \quad .$$

It suffices to show that $\eta^k = 0$ on $F(X, X_{\mathbb{R}})$. But $F(X, X_{\mathbb{R}})$ is an $F(X - X_{\mathbb{R}})$ module so we show $\eta^k = 0$ in $X - X_{\mathbb{R}}$. $X - X_{\mathbb{R}}$ is G-free, hence there is an equivariant map $X - X_{\mathbb{R}} \longrightarrow S^{n,0}$ for some n. So (we are) reduced to proving $\eta^n = 0$ in $S^{n,0}$ which comes from the Gysin type sequence

$$F(pt) \xrightarrow[\cong]{\eta^n} F(pt) \longrightarrow F(S^{n,0}) \longrightarrow \cdots\cdots$$

So $F(S^{n,0}) = 0$.

Consider the theory $Q^{**}(X) = \mathbb{Z}_{/2}[\eta,\eta^{-1}] \otimes \eta^*(X_{\mathbb{R}})$ with $Q^{pq}(X) = \eta^p \otimes \eta^q(X_{\mathbb{R}})$ on the category of G-manifolds. If E is a real bundle of cx dim. n over a G-space X, then $(\mathbb{P}E)_{\mathbb{R}} = \mathbb{P}(E_{\mathbb{R}})$; hence $X \longmapsto \eta^*(X_{\mathbb{R}})$ satisfies the projective bundle theorem. This means that we can define Chern classes for real bundles

$$c_i^Q(E) = \eta^i \otimes c_i^\eta(E_{\mathbb{R}})$$

and that we have a Thom isomorphism

$$\tilde{Q}^{**}(X) \xrightarrow{\sim} Q^{*+n,*+n}(X^E)$$

permitting us to define Gysin morphism for real oriented proper maps of G-manifolds. It's also clear that we have Thom isomorphisms

$$\begin{tikzcd} Q^{pq}(X) \arrow[r, "\cup\eta"] \arrow[d, equal] & \tilde{Q}^{p+1,q}(X \wedge \Sigma^{1,0}) \arrow[d, equal] \\ \eta^p \otimes \eta^q(X) \arrow[r, "\eta\otimes id"] & \eta^{p+1} \otimes \eta^q(X_R) \end{tikzcd}$$

and

$$\begin{tikzcd} Q^{pq}(X) \arrow[r, "\cup\varepsilon"] \arrow[d, equal] & \tilde{Q}^{p,q+1}(X \wedge \Sigma^{0,1}) \arrow[d, equal] \\ \eta^p \otimes \eta^q(X) \arrow[r, "id\otimes susp"] & \eta^p \otimes \tilde{\eta}^{q+1}(X \wedge \Sigma^{0,1}) \end{tikzcd}$$

where $\varepsilon \in \tilde{\eta}^1(\Sigma^{0,1})$ is the canonical generator. Therefore Q^{**} has Gysin morphism for proper stably-real-oriented maps.

Define a map

$$\Phi \; Q^{**}(X) \longrightarrow F^{**}(X)$$

by requiring it to be the $\mathbb{Z}/2[\eta, \eta^{-1}]$ linear extension of the map

$$\alpha : \eta^{q}(X)_{\mathbb{R}} \longrightarrow F^{0q}(X)$$

defined as follows. Consider $F^{0*}(X)$ as a cohomology functor on manifolds with trivial G-action. If E is an ordinary orthogonal bundle over X of dim n, then its complexification $\tilde{E}_{\mathbb{C}}$ is a real bundle àla Atiyah and there is a Thom isomorphism

$$\begin{tikzcd} F^{0q}(X) \arrow[r] & \tilde{F}^{\eta, n+q}(X^{E_{\mathbb{C}}}) \arrow[r, "\sim"] & \tilde{F}^{\eta, n+q}(X^{E}) \arrow[d, "\wr", swap] \arrow[d, "\eta^{-n}"] & \text{rest. to fixpts.} \\ & & \tilde{F}^{0, n+q}(X^{E}) & \end{tikzcd}$$

and hence F^{0*} has a Gysin morphism for proper maps of manifolds This gives a canonical map

$$\eta^{q}(X) \longrightarrow F^{0,q}(X)$$

for X a trivial G-manifold. This map is compatible with products and Gysin homomorphism and in virtue of the fixpoint isomorphism defines α above. Note that the map

$$\Phi : Q^{**}(X) \longrightarrow F^{**}(X)$$

is uniquely determined by the condition that it is natural multiplicative and compatible with Thom isomorphism. Hence it must be inverse to the natural map in the opposite direction.

Prop: (Milnor). The natural map

$$\rho : \Omega^*(X) \longrightarrow \eta^*(X)$$

for $X = pt$, has for image the subring of squares of $\eta^*(pt)$.

Proof: The map is compatible with Gysin homomorphism and hence for a complex line bundle L

$$\rho\left(c_1^{\Omega}(L)\right) = c_2^{\eta}(L).$$

We are going to compute what happens to the formal group laws. Let $L_i = pr_i^* \mathcal{O}(1)$ on $X = \mathbb{R}P^\infty \times \mathbb{R}P^\infty$ and let $x_i = c_1^{\eta}(L_i)$. Then in $\eta^*(X) = \eta^*(pt)[[x_1, x_2]]$ we have

$$\begin{aligned}
(\rho F^{\Omega})(x_1^2, x_2^2) &= (\rho F^{\Omega})\left(c_2^{\eta}(L_1 \oplus iL_1), c_2^{\eta}(L_2 \oplus iL_2)\right) \\
&= \rho\left(F^{\Omega}\left(c_1^{\Omega}(L_{1,\mathbb{C}}), c_1^{\Omega}(L_{2,\mathbb{C}})\right)\right) \\
&= \rho\, c_1^{\Omega}\left(L_{1,\mathbb{C}} \otimes_{\mathbb{C}} L_{2,\mathbb{C}}\right) \\
&= \rho\, c_1^{\Omega}\left((L_1 \otimes_{\mathbb{R}} L_2)_{\mathbb{C}}\right) \\
&= c_2^{\eta}\left(\quad " \quad\right) \\
&= c_1^{\eta}\left(L_1 \otimes_{\mathbb{R}} L_2\right)^2 \\
&= F^{\eta}\left(c_1^{\eta} L_1, c_1^{\eta} L_2\right)^2 \\
&= \left(\mathrm{frob}(F^{\eta})\right)(x_1^2, x_2^2)
\end{aligned}$$

where frob denotes squaring operation. Thus we have a commutative diagram

$$\begin{tikzcd}
F^{\Omega} = F_{univ} \arrow[r, mapsto] & F^{\mathfrak{N}} \\
\Omega(pt) \arrow[r, two heads] \arrow[dr, "f"'] & \mathfrak{N}(pt) \arrow[d, "frob"] \\
& \mathfrak{N}(pt)
\end{tikzcd}$$

proving the <u>proposition</u>.

~~Maybe a better method of proving 3), p.13 is to define a map in the opposite direction. Thus if X is G-trivial~~

~~$[X \wedge \Sigma^{0,n-k}, MO_n] \xrightarrow{\sim} [X \wedge \Sigma^{0,n-k}, MU_n]_G$~~

~~so taking inductive limits over n we get maps~~

~~$\mathfrak{N}^k(X) \longrightarrow \varinjlim_n [X \wedge \Sigma^{0,n-k}, MU_n]_G \downarrow \varinjlim_n$~~

<u>Proposition</u>: $2\eta = 0$ where $\eta = c_1(\mathcal{O}(1)) \in \Omega^{1,1}(\mathbb{R}P^1) \simeq \Omega^{1,0}(pt)$.

<u>Proof</u>: η is represented by the map

$$\Sigma^{01} = \mathbb{R}P^1 \longrightarrow \mathbb{R}P^\infty = MO_1 \subset MU_1$$

The addition of an element is carried out using the cogroup structure of $\Sigma^{0,1}$, hence adding it to itself is the map

$$\Sigma^{1,0} \xrightarrow{\deg 2} \Sigma^{1,0} \longrightarrow MO_1 \subset MU_1 .$$

As this element $\Sigma^{1,0} \xrightarrow{\deg 2} \mathbb{R}P^1 \subset \mathbb{R}P^2$ is homotopic to zero the result follows.

Returning to the exact sequence 2) we see that

Corollary 1: $\Omega^{p,q}(pt)$ is a finitely generated abelian group all of whose torsion is of order 2.

Corollary 2:

$$0 \longrightarrow \Omega R^{p,q}(X)[\tfrac{1}{2}] \longrightarrow \Omega^{p+q}(X)[\tfrac{1}{2}] \longrightarrow \Omega R^{p-1,q+1}(X)[\tfrac{1}{2}] \longrightarrow 0$$

We already know by consideration of the formal group laws that $\Omega R^{p,p}(pt) \longrightarrow \Omega^{2p}(pt)$ is surjective and makes $\bigoplus_p \Omega^{p,p}(pt)$ an augmented $\Omega^*(pt)$ module. Thus

$$\Omega R^{p-1,p+1}(pt)[\tfrac{1}{2}] = 0$$

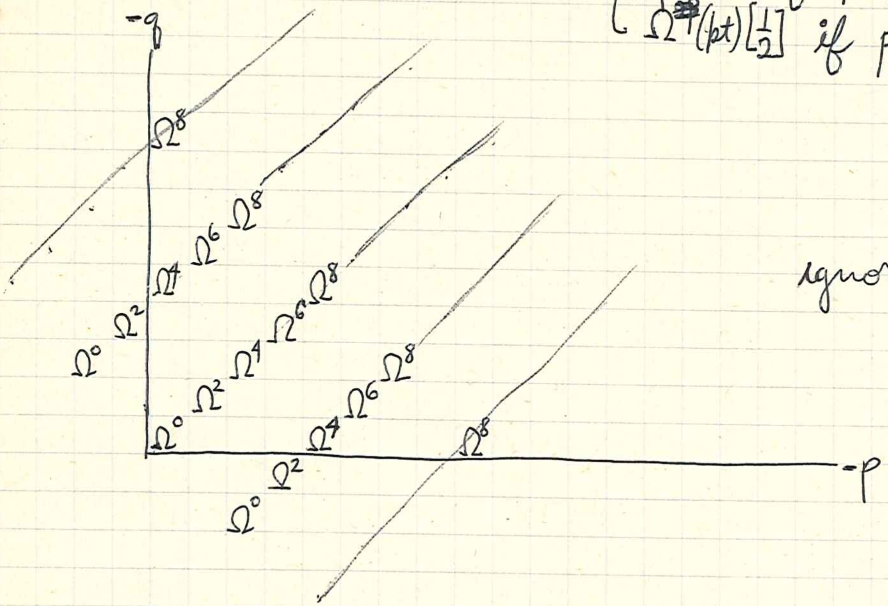
$$\Omega_R^{p-2,p+2}(pt)[\tfrac{1}{2}] \xleftarrow{\sim} \Omega^{2p}(pt)[\tfrac{1}{2}]$$

Note that $\Omega R^{p,q}(pt)[\frac{1}{2}]$ is a free $Z[\frac{1}{2}]$-module of finite rank since it is a submodule of $\Omega^{p+q}(pt)[\frac{1}{2}]$. Thus when we consider the sequence

$$0 \longrightarrow \Omega_R^{p-2,p+2}(pt)[\tfrac{1}{2}] \longrightarrow \Omega^{2p}(pt) \longrightarrow \Omega R^{p-3,p+3}(pt)[\tfrac{1}{2}] \longrightarrow 0$$

the first map must be an isomorphism. Conclude that

$$\Omega R^{p,q}(pt)[\tfrac{1}{2}] = \begin{cases} 0 & \text{if } p+q \text{ odd} \\ 0 & \text{if } p+q \text{ even and } p-q \equiv 2 \ (4) \\ \Omega^{p+q}(pt)[\tfrac{1}{2}] & \text{if } p+q \text{ evens and } p-q \equiv 0 \ (4) \end{cases}$$



ignoring 2 torsion

Observe that this agrees with Conner Floyd and $KR^{q}(pt)$ which we know modulo 2-torsion is $\mathbb{Z}, 0, 0, 0$ periodic.

Another of tom Dieck's tricks is to pass to the limit in the Gysin sequences

$$\cdots \Omega R^{p,q}(X) \xrightarrow{\eta^k} \Omega R^{p+k,q}(X) \longrightarrow \Omega R^{p+k,q}(S^{k,0} \times X) \xrightarrow{\delta} \Omega R^{p,q+1}(X)$$

Before doing this note that as G acts freely on $S^{k,0} \times X$ there is no transversality problem and so an element of $\Omega^{p,q}(S^{k,0} \times X)$ is represented by a proper map $Z \to S^{k,0} \times X$ with stably-real normal orientation of codimension $+p,+q$. Hence if $p+q \overset{(k-1,+}{>} \dim X$, then $\dim Z = \overset{-1}{k} + \dim X - p - q < 0$ and so $\Omega R^{p,q}(S^{k,0} \times X) = 0$. Thus

$$\Omega R^{\overset{-1}{p,q}}(X) \xrightarrow[\cong]{\eta} \Omega R^{p,q}(X)$$

provided $p+q > \dim X$.

Now we wish to define a commutative square

$$\begin{tikzcd}
{} \arrow[r] & \Omega R^{p,q}(X) \arrow[r, "\eta^k"] \arrow[d, "id"] & \Omega R^{p+k,q}(X) \arrow[r] \arrow[d, "\eta^{l-k}"] & \Omega^{p+k,q}(S^{k,0} \times X) \arrow[r, "\delta"] \arrow[d, "\alpha"] & \Omega R^{p,q+1}(X) \arrow[r] \arrow[d, "id"] & {} \\
{} \arrow[r] & \Omega R^{p,q}(X) \arrow[r, "\eta^l"] & \Omega R^{p+l,q}(X) \arrow[r] & \Omega^{p+l,q}(S^{l,0} \times X) \arrow[r, "\delta"] & \Omega R^{p,q+1}(X) \arrow[r] & {}
\end{tikzcd}$$

where α is the Gysin morphism for the embedding $S^{k,0} \to S^{l,0}$ which is [illegible] in a [illegible] neighborhood of $S^{k,0}$ of the form $S^{k,0}$ [illegible] $\longrightarrow S^{k,0} \times \mathbb{R}^{l-k,0}$. Now pass to limit over k and we obtain a long exact sequence

$$\boxed{\cdots \longrightarrow \Omega R^{p,q}(X) \longrightarrow \eta^q(X_R) \longrightarrow \varinjlim_k \Omega R^{p+k,q}(S^{k,0} \times X) \longrightarrow \Omega R^{p,q+1}(X) \longrightarrow}$$

By the argument above about transversality for G-free manifolds we find that

<u>Corollary:</u> $\Omega R^{p,q}(X) \xrightarrow{\cong} \eta^q(X_{\mathbb{R}})$ if $p+q > \dim X$.

In Landweber's Bulletin announcement, he proposes to study $\Omega R^{p,q}(X)$ by using the spectral sequence furnished by the exact sequences

$$\begin{tikzcd}
{} \arrow[r, squiggly] & \Omega R^{p-1,q}(X) \arrow[d, squiggly] & & \Omega R^{p-2,q+1}(X) \arrow[r] \arrow[d] & \Omega^{p+q-1}(X) \\
& \Omega R^{p,q}(X) \arrow[r, squiggly] & \Omega^{p+q}(X) \arrow[r, squiggly] & \Omega R^{p-1,q+1}(X) \arrow[r] \arrow[d, squiggly] & \Omega^{p+q}(X) \\
& & & {} \arrow[r, squiggly] & {}
\end{tikzcd}$$

Thus for this spectral sequence we have by our preceding arguments that one of the two maps

$$\Omega R^{p,q}(X) \longrightarrow \Omega^{p+q}(X) \longrightarrow \Omega R^{p-1,q+1}(X)$$

~~X torsion-free~~ X = pt (possibly X torsion-free) is an isomorphism modulo 2-torsion. So here is a spectral sequence such that the E_0 is torsion-free and such that the differentials (after the first) are all 2-torsion. ~~It follows that all differentials are 0.~~ It is pretty clear that the composition

$$\Omega^{p+q}(X) \longrightarrow \Omega R^{p-1,q+1}(X) \longrightarrow \Omega^{p+q}(X)$$

can be interpreted as taking a proper complex-oriented map $Z \longrightarrow X$, regarding the corresponding proper stably-real-oriented map $Z \times S^{1,0} \longrightarrow X \times S^{1,0} \longrightarrow X$ grace a stably-real-orientation of $S^{1,0}$ depending on p, q and then forgetting the real structure to obtain a proper complex-oriented map $Z \times S^{1,0} \longrightarrow X$. Thus this composition is either zero or multiplication by 2 depending on p, q. Without giving a careful proof here is what this should be I think

<u>Claim</u>: $\Omega^{p+q}(X) \xrightarrow{\delta} \Omega R^{p,q}(X) \xrightarrow[G]{\text{forget}} \Omega^{p+q}(X)$ is 0 if $p+q$ is odd or if $p+q$ is even and $p-q \equiv 2$ (4). If $p+q$ even and $p-q \equiv 0$ (4) it is multiplication by 2.

This is almost certainly clear for X = pt and on the other hand should be independent of X.

Thus the spectral sequence for $X = pt$ looks as follows

$$\begin{tikzcd}
{} \arrow[r, "\times 2"] & \Omega^{2p-2} \arrow[r] & {} \arrow[r] \arrow[d] & \Omega^{2p-2} \arrow[r] \arrow[rr, bend left, "\times 2"] & \Omega^{p-3,p+1} \arrow[r] \arrow[d] & \Omega^{2p-2} & \\
& \cancel{\Omega^{2p-1}} \arrow[r, "0"] & \Omega^{p-1,p} \arrow[r, "0"] \arrow[d, hook] & \cancel{\Omega^{2p-1}} \arrow[r, "0"] & \Omega^{p-2,p+1} \arrow[r, "0"] \arrow[d, hook] & \cancel{\Omega^{2p-1}} & \\
{} \arrow[r] & \Omega^{2p} \arrow[r] \arrow[rr, bend left, "2\times"] & \Omega^{p,p} \arrow[r] \arrow[d] & \Omega^{2p} \arrow[r] & \Omega^{p-1,p+1} \arrow[r] \arrow[d] & \Omega^{2p} \arrow[r, "2\times"] & {} \\
& \cancel{\Omega^{2p+1}} \arrow[r, "0"] & \Omega^{p+1,p} \arrow[r, "0"] & \cancel{\Omega^{2p+1}} \arrow[r, "0"] & \Omega^{p,p+1} \arrow[r, "0"] & \cancel{\Omega^{2p+1}} &
\end{tikzcd}$$

It is tempting to ask whether $\Omega^{p,q} \longrightarrow \Omega^{p+q}$ is always onto but this would imply by Conner Floyd that $KR^*(pt) \longrightarrow K^*(pt)$ is onto, which is false for $* \equiv 2,4,6\ (8)$. But one might conjecture ~~that~~ since

$$\widetilde{KR}(\Sigma^{p,q}) \longrightarrow \tilde{K}(\Sigma^{p,q})$$

is surjective when $p-q \equiv 0\ (8)$ that

$$\bigoplus_r \Omega R^{r,r}(\Sigma^{p,q}) \longrightarrow \bigoplus_r \Omega^{2r}(\Sigma^{p,q})$$

is surjective, i.e. that

<u>Conjecture</u>: $\Omega R^{p,q}(pt) \longrightarrow \Omega^{p+q}(pt)$ surjective for $p-q \equiv 0\ (8)$.

(This would follow from your conjecture that for a torsion-free X $\Omega^*(X)$ is the Chern ring of $K^*(X)$, however this last conjecture is false.)

~~Conjecture: $\Omega R^{p,q}(pt) \longrightarrow \Omega^{p+q}(pt)$~~

<u>Conjecture</u>: $\Omega R^{p,q}(pt) \twoheadrightarrow 2\cdot\Omega^{p+q}(pt) \hookrightarrow \Omega^{p+q}(pt)$ if $p-q \equiv 4\ (8)$.

Landweber's results are more precise than this. He shows that the second conjecture is true by noting that the image of $\Omega^{p,q}(pt)$ in $\Omega^{p+q}(pt)$, $p-q \equiv 4\ (8)$ contains $2\,\Omega^{p+q}(pt)$ and on the other hand is contained therein, since $\widetilde{KR}(\Sigma^{p,q}) = 2\,\tilde{K}(\Sigma^{p,q})$ and the Stong - Hattori theorem.

Your reason for conjecturing the first is wrong:

For Σ^{2n} it is false that $\tilde{\Omega}(\Sigma^{2n})$ is generated by Chern classes of elements in $\tilde{K}(\Sigma^{2n})$. If so it would follow that $\tilde{H}^{2n}(\Sigma^{n};\mathbb{Z})$ is generated by Chern classes which isn't so since c_n always a multiple of $(n-1)!$ by Bott.

How to calculate the multiplicative transformation

complexification: $KO^*(pt) \longrightarrow K^*(pt)$

First note we have

$$\begin{tikzcd}
KO(X) \arrow[r, "\otimes_{\mathbb{R}}\mathbb{C}"] \arrow[rr, bend right, "\times 2"'] & K(X) \arrow[r, "\text{rest.}"] \arrow[rr, bend left, "\psi^{-1}+id"] & KO(X) \arrow[r, "\otimes_{\mathbb{R}}\mathbb{C}"] & K(X)
\end{tikzcd}$$

from which follows that

$$\begin{tikzcd}
\widetilde{KO}(S^{4j}) \arrow[r] & \tilde{K}(S^{4j}) \arrow[r] & KO(S^{4j}) \\
\mathbb{Z} \arrow[r, "\alpha"] & \mathbb{Z} \arrow[r, "\beta"] & \mathbb{Z}
\end{tikzcd}$$

$\beta\alpha = 2.$

Now

	0	1	2	3	4	5	6	7	8	
$KO^*(pt)$:	$\mathbb{Z}$	$\mathbb{Z}_2\eta$	$\mathbb{Z}_2\eta^2$	0	$\mathbb{Z}\tau$	0	0	0	$\mathbb{Z}\sigma$	. .
$K^*(pt)$:	$\mathbb{Z}$	0	$\mathbb{Z}\beta$	0	$\mathbb{Z}\beta^2$	0	$\mathbb{Z}\beta^3$	0	$\mathbb{Z}\beta^4$	

where σ and β are units. It follows that $\sigma \longmapsto \pm\beta^4$ and that $\tau \longmapsto \pm 2\beta^2$ ~~then c is not a unit~~ otherwise the periodicity of KO^* would be 4. Thus $\tau^2 = \pm 4\sigma$. Observe that the restriction map is $1 \longmapsto 2$, $\beta^2 \longmapsto \pm\tau$, $\beta^4 \longmapsto \pm 2\sigma$ since $\beta\alpha = 2$. It is likely that $\beta \longmapsto \eta^2$.

Final notes: Landweber doesn't systematically ~~[illegible]~~ use the theories $\Omega R^{p,q}(S^{k,0}\times X)$ with their ~~[illegible]~~ periodicities. It would appear that knowledge about these theories such as KR-theory characteristic numbers would yield lots of information about the spectral sequence.

Recall that the periodicities come from maps as follows

$$\begin{tikzcd}
& \Omega R^{p,q}(X\times S^{k,0}) \cong \Omega^{p+m,q-m}(X\times S^{k,0}) \\
& \exists \text{ a } G \text{ equiv. elliptic map } \varphi : S^{k,0}\times \mathbb{R}^{m,0} \longrightarrow \mathbb{R}^{0,m} \arrow[u, Rightarrow] \\
& \exists \text{ an elliptic bilinear map } \mathbb{R}^{k}\otimes\mathbb{R}^{m} \longrightarrow \mathbb{R}^{m} \arrow[u, Rightarrow] \\
& \exists \text{ a } *\text{-homomorphism } C(k-1) \longrightarrow \mathrm{End}(\mathbb{R}^{m}) \arrow[u, Rightarrow] \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \\
2^{k-2} \mid m & \exists\ k-1 \text{ independent vector fields on } S^{m-1}
\end{tikzcd}$$

Here elliptic means that $\varphi(\xi, ?)$ is non-singular for $\xi \in \mathbb{R}^k - 0$. $C(k-1)$ is the Clifford algebra with generators e_i $1 \le i \le k-1$ anticommuting and $\ni$ $e_i^2 = -1$. a $*$-homomorphism $C(k-1) \longrightarrow \mathrm{End}(\mathbb{R}^m)$ means that the e_i act skew-adjointly on $\mathbb{R}^m$. Thus

$$(e_i v, v) = 0$$

$$\|e_i v\| = \|v\|^2$$

$$(e_i v, e_j v) = 0 \qquad i \neq j$$

giving $k-1$ orthogonal vector fields over S^{m-1}. Also if $e_0 = 1$ we have

$$\Big\| \sum_{i=0}^{k-1} \lambda_i e_i \, v \Big\|^2 = \Big(\sum_{i=0}^{k-1} \lambda_i^2 \Big) \cdot \|v\|^2$$

hence an elliptic pairing $\mathbb{R}^k \otimes \mathbb{R}^m \longrightarrow \mathbb{R}^m$.

July 21, 1969

More on an action~~s~~ of compact group G on a manifold X.

Recall that there is an open dense set of principal orbit types. To each x consider conjugacy class of the stabilizer G_x or the isomorphism class of the orbit of x; this is the orbit type of x. The principal orbit types are those ones consisting of $x \ni G_x$ has trivial isotropy representation. This set is open and dense.

In general the set of points of a given orbit type is a submanifold. To see this take the local representation around the orbit $G \times_{G_x} V_x$, V_x = normal space to orbit; then the same orbit type as x is $G \times_{G_x} (V_x^{G_x})$ otherwise the stabilizer gets smaller. One gets a stratification of X by the submanifolds of given orbit type.

If G is a finite group of odd order, then any non-trivial irreducible complex representation is real irreducible and the complexification of any non-trivial real irreducible representation splits into two complex irreducible representations. Equivalent if V is irred. over $\mathbb{R}$ then $\mathrm{Hom}_G(V,V) \cong \mathbb{C}$. To see this we show that if W is non-trivial complex irred. then $W \not\cong W^*$. Suppose on the contrary that $W \cong W^* \Longrightarrow (W \otimes W)^G$ is 1-dimensional $\Rightarrow$ either $S_2 W$ or $\Lambda_2 W$ has an invariant $\neq 0$ but not both $\Longrightarrow \int \chi(g^2)\,dg = \langle 1, \psi^2 W \rangle = \pm 1$, where χ = char. of W. But G of odd order $\Longrightarrow \sum_{g \in G} \chi(g^2) = \sum \chi(g) = 0$.

(G odd order)

Consequently if V is a real repn not containing the trivial repn then dim V is even. Claim that the stratas of X are all of even codimension. This is because the stratum through x is

$G\times_{G_x}(V_x)^{G_x} \subset G\times_{G_x} V$ = nbd. of Gx. and $(V_x)^{G_x}$ is of even codimension in V_x but what we've just shown.

Suppose G of odd order with no non-trivial characters. Then if X is a non-trivial G-manifold, it follows that the fixed point submanifold X^G is of even codimension ≥ 4

If G is finite odd order and acts faithfully on X, connected then G acts freely on the open set of principal orbit type. This is because the stabilizer $x \mapsto H_x$ is a locally constant function of x, hence constant as the principal orbit set is connected. Thus H_x is a normal subgp. fixing all of X by density and this contradicts the faithfulness of the representation of G on X.

Let U be the principal set on which the stabilizers are all reduced to the identity and let $F = X - U$. Would like to know if F is a divisor with normal crossings? So if $x \in F$ has stabilizer H and if the isotropy representation is V, then the stratum through x is V^H. If H acts without fixed points on the unit sphere of V/V^H, then this strata is purely of codimension dim V/V^H in F. It is therefore important to know that the non-trivial parts of the isotropy representations don't act freely on the unit sphere unless the representation is of complex dimension 1.

Suppose X is a G-manifold with a single orbit type G/H. Then the map

$$G \times X^H \longrightarrow X$$

is surjective. If N is the normalizer of H in G, then N acts on X^H. If $x, y \in X^H$ and $gx = y \Longrightarrow gH_xg^{-1} = H_y$. As there is a single orbit type, we have $H_x = H_y = H$, so $g \in N$. Thus

$$G \times_N X^H \xrightarrow{\sim} X$$

We would like to classify complex G bundles on X, or what is the same N bundles on X^H. Note that N/H acts freely on X^H, since $nx = x \Longrightarrow n \in H$. Thus we have the following problem.

(*) Suppose H is a normal subgroup of G and that G/H acts freely on X. Determine G-bundles on X.

special cases: 1) $X = G/H \times Y$. Then a G-bundle over X is the same as a bundle over Y endowed with an H-action. Thus

$$K_G(G/H \times Y) = K_H(Y) \underset{\uparrow}{=} K(Y) \otimes R(H)$$

holds since bundles are complex.

2) G acts freely on X. Then

$$K_G(X) = K(X/G)$$

Try to combine these

$$X \longrightarrow X/G = Y.$$

$$R(G)\otimes K(Y) = K_G(Y) \longrightarrow K_G(X)$$

A bundle over Y is the same as a bundle over X on which H acts trivially. Let W be an irreducible representation of H and let E be a G-bundle over X. Let E^W be the subbundle of E whose fibers are isomorphic to $\overset{\text{sums of}}{\text{copies}}$ of W. E^W is not stable under G necessarily. Let $g \in G$ and $x \in X$ and $\varphi : W \longrightarrow E_x$ be an H-map. Then

$$g : E_x \longrightarrow E_{gx}$$

$$g(he) = (ghg^{-1})(ge).$$

so $\psi = g\circ\varphi : W \longrightarrow E_{gx}$ satisfies $\psi(hw) = ghg^{-1}\psi(w)$. Inother words if we denote by W^g the h-module $h * w = g^{-1}hg$ then $\psi(h*w) = h\cdot\psi(w)$. Thus g carries $E^W \subset E^{W^g}$. The conclusion is that $\bigoplus_{W\in\mathcal{O}} E^W$ $\mathcal{O}$ an orbit in $\check{H}$ under G is a G-bundle.

~~[illegible] acts trivially~~

~~Suppose for simplicity that G acts trivially on $\check{H}$. [illegible]~~

~~Suppose that $E^W = E$. Then~~

$$\cancel{x \longmapsto \operatorname{Hom}_H(W, E_x)}$$

~~is what kind of animal. It is a bundle over X. Now~~

The moral is that if G/H acts freely on X, then the only relations ones has relate

$$K(X/G) \text{ and } K_H(pt) \quad \text{to} \quad K_G(X)$$

For example cohomologically we have

$$H^*_G(X) = H^*(P_G \times_G X)$$

$$= H^*(B_H \times_{G/H} X)$$

and there is a fibration

$$B_H \longrightarrow B_H \times_{G/H} X \longrightarrow X/G$$

giving a spectral sequence

$$E_2^{pq} = H^p(X/G, H^q_H(pt)) \Longrightarrow H^{p+q}_G(X).$$

Similarly there is Segal's spectral sequence

$$E_2^{pq} = H^p(X/G, K^q_H(pt)) \Longrightarrow K^{p+q}_G(X).$$

The lack of Chern classes in Ω_G, G non-abelian is even more striking for η_G which fails to satisfy the projective bundle theorem unless all irred. real reps of G are 1-dimensional, i.e. unless $G = (\mathbb{Z}/2\mathbb{Z})^k$.

July 24, 1969

On the universal nature of cobordism theories.

Outline

1. The category $\underline{St}(X)$ of stable bundles over a manifold X.

$$\operatorname{Ob} \underline{St}(X) = \text{pairs } (E,F) \text{ of } C^\infty \text{ v.b.} /X$$

$$\operatorname{Hom}_{\underline{St}(X)}\big((E,F),(E',F')\big) = \varinjlim_{G\in I} \pi_0 \underline{\operatorname{Isom}}(E+F'+G,\ E'+F+G)$$

where I is the category of vector bundles with injections for maps.

~~the Picard category structure of St(X)~~

$$\pi\ \underline{\operatorname{Hom}}(X, \mathbb{Z}\times BO) \underset{\text{equiv.}}{\simeq} \underline{St}(X)$$

the Picard category structure of $\underline{St}(X)$.

universal property of $\underline{St}(X)$.

2. $\nu_f \in \operatorname{Ob} \underline{St}(X)$.

the isomorphisms $\qquad \nu_{f'} \simeq g'^* \nu_f$

$$\nu_{gf} \simeq f^* \nu_g + \nu_f$$

compatibility between these isoms.

3. Φ orientations of stable bundles $\qquad \Phi : A \longrightarrow \mathbb{Z}\times BO$

the auto. "-1" of a stable bundle

$\underline{\Phi}$-orientations of manifolds

~~composite~~ pull back of orientations

~~orientations~~ orientations and homotopies

orienting compositions when A has H-space structure

$\left\{\begin{array}{l}\text{associativity}\\ \text{Commutativity}\end{array}\right.$

④ The spectrum $M\Phi$ and Ω_Φ

$\Omega_\Phi(X)$ = bordism classes of $Z \longrightarrow X$ proper + Φ-oriented

<u>Thom's thm:</u> $\Omega_\Phi(X) \cong \{X, M\Phi\}$.

~~[illegible]~~

universal property of Ω_Φ as a functor to sets
ab. groups

description of the inverse and sum.

Gysin morphism with Φ preserves H-structure

universal property of Ω_Φ as a functor with Gysin

---

product structure, associativity, unity,
commutativity.

Universal property of Ω_Φ as a functor to rings with Gysin.

<u>Questions:</u> What are spectra? Can they be constructed by an inversion procedure? Can you make anything of the fact that the true algebraic category of stable bundles is finer than the one $\pi(\underline{\mathrm{Hom}}(X, Z \times BO))$?

<u>First question</u>: What is a stable bundle over a manifold?

Let A be a paracompact topological space. We are going to define the category of stable vector bundles over A, denoted $\underline{St}(A)$. For objects take pairs (E, F) where E and F are bundles over A, and define

$$\mathrm{Hom}_{\underline{St}(A)}\big((E,F),(E',F')\big) = \varinjlim_{I} \pi_0\, \underline{\mathrm{Isom}}(E \oplus F' \oplus G, E' \oplus F \oplus G)$$

where the direct limit is taken over the category I whose objects are the vector bundles G. over A and with $\mathrm{Hom}_I(G, G') = \pi_0 \,\mathrm{Inj}(G, G')$. We ~~k~~now check this makes good sense. Clearly I is a category. Given an injection $G' \longrightarrow G$ choose a complement $G'' \longrightarrow G$. Than we get a map using $\oplus G''$

$$(*) \qquad \mathrm{Isom}(E \oplus F' \oplus G', E' \oplus F \oplus G') \longrightarrow \mathrm{Isom}(E \oplus F \oplus G, E \oplus F' \oplus G).$$

The set of complements is principally homogeneous under $\mathrm{Hom}(G'', G')$ hence any two splittings are homotopic and so the map induced by (*)

$$\pi_0\big(\mathrm{Isom}(E \oplus F' \oplus G', E' \oplus F \oplus G')\big) \longrightarrow \pi_0\big(\mathrm{Isom}(E \oplus F' \oplus G, E' \oplus F \oplus G)\big)$$

is independent of the choice of the complement. Moreover one sees this is a functor. ~~Finally by a standard argument I have~~ It is also clear that that this map depends only on the homotopy class of $G' \longrightarrow G$ as an injection since a small motion of G' remains complementary to G''. Thus the dired limit above is well-defined and moreover I is <u>clearly filtering</u>.

If $f : B \to A$ is a map of paracompact spaces, then we have

$$f^* : \underline{St}(A) \longrightarrow \underline{St}(B)$$

making $\underline{St}$ a fibered category over the category of spaces!

Suppose A is a finite dimensional CW complex, i.e. a manifold. Then the category I has as a cofinal subcategory the bundles n, $n \geq 0$. One sees that

$$\begin{aligned} \pi_0 \{ \underline{St}(A) \} &\xrightarrow{\sim} KO(A) \\ \bigstar \qquad \pi_1 \{ \underline{St}(A) \} &\xrightarrow{\sim} KO^{-1}(A) \end{aligned}$$

Let $BO = \varinjlim_{mn} G_{mn}$ be the infinite Grassmannian. Then over G_{mn} there is a ~~stable~~ canonical bundle E_{mn} of dim n and so we get a functor

$$\pi \{ \underline{Hom}(A, G_{mn}) \} \longrightarrow \underline{St}(A)$$

$$f \longmapsto f^*(E_{mn} - n)$$

This gives us a map

$$\pi \{ \underline{Hom}(A, \mathbb{Z} \times BO) \} \xrightarrow{\cong} \pi(\mathbb{Z}^A) \times \varinjlim_{mn} \pi(A, G_{mn})$$

$$\longrightarrow \underline{St}(A)$$

which is an equivalence of categories, because of $\bigstar$. This equivalence enables one to "identify" a stable bundle over A with a map $A \longrightarrow \mathbb{Z} \times BO$, ~~isomorphic and homotopic~~ isomorphic stable bundles with ~~isomorphic~~ homotopic maps, and homotopic isomorphisms with homotopic homotopies.

If $f: X \longrightarrow Y$ is a map of manifolds, then we define its stable normal bundle to be

$$\nu_f = f^*\tau_Y - \tau_X \in \mathrm{Ob}\ \underline{\mathrm{St}}(X).$$

Then we have the following sorites

(i) Given $X \xrightarrow{f} Y \xrightarrow{g} Z$

there is a canonical isomorphism

$$\nu_f + f^*\nu_g \simeq \nu_{gf}$$

(In fact this is the map

$$\begin{array}{ccc}(f^*\tau_Y - \tau_X) + f^*(g^*\tau_Z - \tau_Y) & \cong & (gf)^*\tau_Z - \tau_X \\ \| \text{ defn} & & \\ (f^*\tau_Y + f^*g^*\tau_Z) - (\tau_X + f^*\tau_Y) & \nearrow\!\!\!\!\!/ & \end{array}$$

this is the basic isomorphism $(E-F) \cong (E+G) - (F+G)$.

Question should I write

$$f^*\nu_g + \nu_f \cong \nu_{gf}$$

~~$f^*_*(f^* x \cdot y) \cong x f_* y$.~~

$$f_! x = f_*(x \cdot \varphi(\nu_f))$$

$$g_! f_! x = g_*(f_*(x\, \varphi(\nu_f)) \cdot \varphi(\nu_g)) = g_* f_* (x\, \varphi(\nu_f + f^*\nu_g)).$$

(ii) Given a transversal cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, "h"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

there is a canonical isomorphism

$$g'^* \nu_f \cong \nu_{f'}$$

(In fact one has ~~an exact sequence~~ a bicartesian square

$$\begin{tikzcd} \tau_{X'} \arrow[r] \arrow[d] & g'^* \tau_X \arrow[d] \\ f'^* \tau_{Y'} \arrow[r] & \cancel{g^* f^*}\, h^* \tau_Y \end{tikzcd}$$

so that $\exists$ exact sequence

$$0 \longrightarrow \tau_{X'} \longrightarrow f'^* \tau_{Y'} \oplus g'^* \tau_X \longrightarrow h^* \tau_Y \longrightarrow 0$$

hence an isomorphism

$$\tau_{X'} + h^* \tau_Y \cong f'^* \tau_{Y'} + g'^* \tau_X$$

and so an isomorphism

$$f'^* \tau_{Y'} - \tau_{X'} \cong \cancel{g}\, h^* \tau_Y - g'^* \tau_X$$

$$= g'^* (f^* \tau_Y - \tau_X).$$

)

Suppose now that $\Phi : A \longrightarrow \mathbb{Z}\times BO$ is a fibration. Then it defines a functor

$$\Phi_* : \pi(A^X) \longrightarrow \pi((\mathbb{Z}\times BO)^X) \text{ ~~[illegible]~~}$$

for any manifold X, which is a fibration of categories. ~~and [illegible] compatible with the [illegible]~~ If I agree to identify ~~as before I shall define a ξ-structure on~~ stable bundles with maps from X to $\mathbb{Z}\times BO$, then a ~~[illegible]~~ ~~[illegible]~~ Φ-structure on a stable bundle ξ will be by definition ~~an object α of the category $\pi(A^X)$ together with [illegible] of $\Phi\alpha$ with ξ in $\pi((\mathbb{Z}\times BO)^X)$~~ an isomorphism class of the fiber of Φ_* over ξ.

So instead of using the map Φ suppose that we are given a ~~[illegible]~~ morphism

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "\Phi"] \arrow[dr] & & \underline{St} \arrow[dl] \\ & \underline{Man} & \end{tikzcd}$$

where $\mathcal{E}$ is a fibred category over Man with groupoids for fibres. For the moment suppose that Φ is fibrant and then we can define ~~[illegible]~~ a Φ structure on ξ over X to be an isomorphism class of the fiber of Φ over ξ.

Basic assumption is that if $f : X \longrightarrow Y$ is a heq in Man, then $f^* : \mathcal{E}_Y \longrightarrow \mathcal{E}_X$ is an equivalence of groupoids. Note that this implies that if $f, g : X \longrightarrow Y$ are homotopic then $f^*, g^*: \mathcal{E}_Y \to \mathcal{E}_X$ are canonically isomorphic. In effect it is enough to do for $\iota_0, \iota_1 : X \to X \times \mathbb{R}$. Then we have

$$\begin{tikzcd}
\mathcal{E}_X \arrow[r, "pr_1^*"] \arrow[rr, bend left=60, "id"] \arrow[rr, bend right=60, "id"'] & \mathcal{E}_{X \times \mathbb{R}} \arrow[r, shift left, "\iota_0^*"] \arrow[r, shift right, "\iota_1^*"'] \arrow[u, phantom, "\Uparrow"] \arrow[d, phantom, "\Downarrow"] & \mathcal{E}_X \\
\end{tikzcd}$$

Thus a canonical isom $\iota_0^* pr_1^* \cong \iota_1^* pr_1^*$, and since pr_1^* is an equivalence, a canonical isomorphism $\iota_0^* \cong \iota_1^*$.

How to associate a spectrum to $\Phi : \mathcal{E} \longrightarrow \underline{St}$, Φ assumed to be fibrant and having groupoids for fibers. Also we assume that if $f: X \to Y$ is a heq, then $f^*: \mathcal{E}_Y \to \mathcal{E}_X$ is an equivalence of groupoids. You consider the category of ~~compact~~ pairs (X, ξ) where $\xi \in \mathcal{E}_X$ and where a morphism $(X, \xi) \longrightarrow (Y, \eta)$ is a pair consisting of a map $f: X \to Y$ and an isomorphism $\xi \simeq f^* \eta$ in $\mathcal{E}_X$. To a pair (X, ξ) we associate the Thom spectrum $X^{\Phi(\xi)}$ which is (hopefully) a definite object of the stable homotopy category so that the functor

$$\{?, X^{\Phi(\xi)}\}$$

is well-defined and depends only on $\Phi(\xi)$. Then you take the limit over the category of pairs

$$\Omega_{\Phi}(?) = \varinjlim_{(X,\xi)} \{?, X^{\Phi(\xi)}\}.$$

The next thing to note is that if the map $(X,\xi) \longrightarrow (Y,\eta)$ is a homotopy equivalence, then $X^{\Phi(\xi)} \longrightarrow Y^{\Phi(\eta)}$ is a homotopy equivalence of spectra. ~~Maybe~~ One ought to <u>suppose</u> that the category of pairs is filtering up to homotopy so that one has a good inductive limit

So with all these definitions it should be true that we have the Thom isomorphism

$$\Omega_{\Phi}(X) = \left\{ \begin{array}{l} \text{bordism classes of maps } Z \longrightarrow X \text{ which} \\ \text{are proper and } \Phi\text{-oriented.} \end{array} \right.$$

<u>Universal property</u> of Ω_{Φ}: It's a $\overbrace{\text{contravariant}}^{\text{universal}}$ functor on <u>Man</u> endowed with ~~maps for proper Φ-oriented maps~~ an element $1 \in \Omega_{\Phi}(pt)$ and a ~~[illegible]~~ fundamental class $f_* 1_X \in \Omega_{\Phi}(Y)$ for any map $f : X \longrightarrow Y$ proper + Φ-oriented, subject to the homotopy + transversality axioms.

Next suppose that $\Phi : \mathcal{E} \longrightarrow \underline{St}$ is a morphism ~~of fibers~~ compatible with Picard category structures on the fibers. In other words we suppose that for each space X

$$\mathcal{E}_X \longrightarrow \underline{St}_X$$

is a morphism of Picard categories. Then ~~[illegible]~~

Given ~~[illegible]~~ maps of ~~[illegible]~~ manifolds

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

if ν_f and ν_g are oriented, say $\Phi(\lambda_f) = \nu_f$, $\Phi(\lambda_g) = \nu_g$, ~~then an exact~~ then we obtain ~~an~~ Φ-orientation of ν_{gf} by setting

$$\lambda_{gf} = \lambda_f + f^* \lambda_g$$

since then $\underset{\substack{\uparrow \\ \Phi \text{ morphism} \\ \text{of Picard}}}{\Phi(\lambda_{gf}) \cong} \nu_f + f^* \nu_g \underset{\substack{\uparrow \\ \text{canonical} \\ \text{isomorphism}}}{\cong} \nu_{gf}$.

In this case we can define a Gysin morphism

$$f_* : \Omega_\Phi(X) \longrightarrow \Omega_\Phi(Y)$$

~~By~~ for an oriented proper map $f : X \longrightarrow Y$ by

$$f_* [Z \xrightarrow{g} X] = [Z \xrightarrow{fg} Y]$$

where fg is oriented using that of f & of g. ~~Check the basic axioms:~~

$$\begin{tikzcd} & Z \arrow[d, "h"] \\ X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

~~move g trans to fh~~

~~it follows that g' trans to h~~

~~If you make a small motion g_t of g then you get a family X'_t~~ ?

~~so~~ $$g^* f_* h_* 1 = g_t^* f_* h_* 1 = (f'_t)_* (g'_t)^* h_* 1 = (f'_t)_* (h''_t)_* (g''_t)^* 1$$

Here is how to handle the signs

So you start with $\Phi : A \longrightarrow \mathbb{Z} \times BO$, just a map at first, and it gives you an abelian group

$$\Omega_\Phi(X) = \{X, M_\Phi\}. = \text{bordism classes of proper maps } Z \to X \text{ with } \Phi\text{-orientation}$$

Next you suppose that Φ is a map of H-spaces so that if ξ, η are stable bundles over X with Φ orientation then $\xi \oplus \eta$ is Φ-oriented in a natural way. This means grace the isomorphism

$$\nu_{gf} \cong f^* \nu_g + \nu_f$$

that gf is oriented once g and f are. Note that A need not be a homotopy commutative H-space, nor if it is, need Φ respect the commutativity isomorphism. Next make the following calculation: suppose given a trans cartesian square

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"'] & Y
\end{tikzcd}$$

and let $h = gf' = fg'$. ~~Given orientations of f, g we get orientations of f', g' and hence two orientations of h.~~] isom

1) $\nu_{gf'} \cong f'^* \nu_g + \nu_{f'} \cong f'^* \nu_g + g'^* \nu_f$

2) $\nu_{fg'} \cong g'^* \nu_f + \nu_{g'} \cong g'^* \nu_f + f'^* \nu_g$

suppose given orientations λ_f, λ_g of f, g, ~~then~~ there are two possibilities for orienting h namely

$$f'^*\lambda_g + g'^*\lambda_f$$

$$g'^*\lambda_f + f'^*\lambda_g$$

which might be unrelated to each other. Suppose now given a commutativity homotopy for A, then ~~the two orientations for h are homotopic and~~ we ~~have an isom~~ get a isomorphism

$$f'^*\lambda_g + g'^*\lambda_f \cong g'^*\lambda_f + f'^*\lambda_g$$

Applying Φ to this isomorphism we get together with 1) and 2) above an automorphism of ν_h. Two cases to consider

a) Φ compatible with comm. isom. Then this auto. of ν_h is the identity.

b) Φ sign-compatible with comm. isom. e.g. given $\xi, \eta : X \longrightarrow A$ then

$$\begin{tikzcd}
\Phi(\xi+\eta) \arrow[r, phantom, "\overset{\Phi(\text{comm. isom of } A)}{\cong}"] \arrow[d, phantom, "\wr\|\ \Phi\text{ an H-map}"] & \Phi(\eta+\xi) \arrow[d, phantom, "\wr\|\ \Phi\text{ an H-map}"] \\
\Phi(\xi)+\Phi(\eta) \arrow[r, phantom, "\underset{\text{comm iso. of } Z\times BO}{\cong}"] & \Phi(\eta)+\Phi(\xi)
\end{tikzcd}$$

commutes with sign $(-1)^{pq}$ $\quad p = \dim \xi$, $q = \dim \eta$

In this case the auto. of ν_h is $(-1)^{(\dim f)(\dim g)}$.

Now consider basic products in Ω_{Φ}. By definition

$$f_* 1 \cdot g_* 1 \overset{\text{defn}}{=\!=} h_* 1 \quad \text{where } h \text{ is oriented by}$$

$$\lambda_h (= \lambda_{fg'}) = g'^* \lambda_f + f'^* \lambda_g .$$

(If this definition is correct, then the projection formula should follow from it: Given $x : Z \longrightarrow X$ $\quad y : W \longrightarrow Y$ assume y transversal to f and form diagram

$$\begin{tikzcd}
Z' \arrow[r, "y''"] \arrow[d] & Z \arrow[d, "x"] \\
X' \arrow[r, "y'"'] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
W \arrow[r, "y"] & Y
\end{tikzcd}$$

(assume x transversal to y')

Then y' represents $f^* y$ and $x \cdot f^* y = x_* y''_* 1$. Thus $f_*(x \cdot f^* y) = f_* x_* y''_* 1$. But $f_* x \cdot y = (fx)_* y''_* 1$ so these two are equal.)

Next I calculate $x \boxtimes y$. Let x be represented by $f : Z \longrightarrow X$ and y by $g : Z \longrightarrow W$, then we have that $x \boxtimes y = pr_1^* x \cdot pr_2^* y$ where $pr_1^* x$ is represented by $f \times id_Y : Z \times Y \longrightarrow X \times Y$ with orientation induced from x. Similarly $pr_2^* y$ is represented by $id_X \times g$. Since we have the cartesian transversal square

$$\begin{tikzcd}
Z\times W \arrow[r, "id_Z\times g"] \arrow[d, "f\times id_Y"'] & Z\times Y \arrow[d, "f\times id_Y"] \\
X\times W \arrow[r, "id_X\times g"] & X\times Y
\end{tikzcd}$$

it follows that $pr_1^* x \cdot pr_2^* y$ is represented by $(f\times id_Y)_*(id_Z\times g)_*\, 1$. In other words: $f_*1 \boxtimes g_*1 = (f\times g)_*\, 1$ where

$f\times g$ *is oriented as* $(f\times id)(id\times g)$.

Properties of the product:

I. Suppose A is associative and that $\Phi : A \longrightarrow Z\times BO$ is compatible with the associativity isomorphisms of both spaces. Then we get that the multiplication is associative. In effect we have to check that $(x\boxtimes y)\boxtimes z = x\boxtimes(y\boxtimes z)$, but representing these ~~as~~ by map f, g, h, we have

$$(x\boxtimes y)\boxtimes z = \left((f\times g)\times h\right)_*\, 1$$

$$\cancel{= \left((f\times g)\times id_Z\right)_* (id_{X\times Y}\times h)_*\, 1}$$

where $(f\times g)\times h$ is oriented as $\left[(f\times id)(g\times id)\times id\right](id\times h)$

i.e. with $\left(pr_1^*\lambda_f + pr_2^*\lambda_g\right) + pr_3^*\lambda_h$. Similarly

$$x\boxtimes(y\boxtimes z) = \left(f\times(g\times h)\right)_*\, 1$$

there $f\times(g\times h)$ is endowed with the orientation

$pr_1^*\lambda_f + (pr_2^*\lambda_g + pr_2^*\lambda_h)$. These two orientations are the same by the associativity of **A** and the compatibility.

I′. (Unitarity). We suppose that Φ is compatible with units in the H-spaces, ie, that $\Phi(0) \cong 0$ and that

$$\begin{tikzcd}
\Phi(0+\xi) \arrow[r, phantom, "\cong"] \arrow[d, phantom, sloped, "\cong"] & \Phi(\xi) \arrow[d, phantom, sloped, "\cong"] \\
\Phi(0)+\Phi(\xi) \arrow[r, phantom, "\cong"] & 0+\Phi(\xi)
\end{tikzcd}$$

commutes, ~~this means that~~ and also in the other direction. This has the effect of making the composite orientation on $f \circ id$ and $id \circ f$ the same as that of f. Hence 1 is a unit for the multiplication.

II. (Commutativity). (Case a): Assume that **A** endowed with a commutativity isomorphism and that Φ preserves this. Then as we saw for the diagram

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, "h"] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"'] & Y
\end{tikzcd}$$

the two orientations $g'^*\lambda_f + f'^*\lambda_g$ and $f'^*\lambda_g + g'^*\lambda_f$ are the same and hence

$$f_*1 \cdot g_*1 = g_*1 \cdot f_*1.$$

Case b: It is first necessary to understand what is the "-1" automorphism of a stable bundle. ~~If~~ the stable bundle is of the form $E-F$ it is the isomorphism

$$\begin{tikzcd} E-F \arrow[r, phantom, "\cong"] & (E+1)-(F+1) \arrow[d, "\cong"', "(id+(-id),\, id+id)"] \\ E-F \arrow[r, phantom, "\cong"] & (E+1)-(F+1) \end{tikzcd}$$

The important thing to check is that if f is Φ-oriented via an isom. $\theta_f : \Phi(\lambda_f) \simeq \nu_f$, then for the new orientation

$$-\theta_f : \quad \Phi(\lambda_f) \simeq \nu_f \overset{-1}{\simeq} \nu_f$$

if we denote this new oriented map by $-f$, we have

$$(-f)_* 1 = -f_* 1 \quad \text{in } \Omega_\Phi(X).$$

Now we suppose that Φ is sign-commutative. ~~Then~~ Recall we have

$$f_* 1 \cdot g_* 1 = h_* 1,$$

h oriented by the isomorphism

$$\nu_h \simeq \nu_{fg'} \cong g'^*\nu_f + \nu_{g'} \cong g'^*\nu_f + f'^*\nu_g$$

$$\cong g'^*\Phi(\lambda_f) + f'^*\Phi(\lambda_g) \cong \Phi(g'^*\lambda_f + f'^*\lambda_g)$$

and also we have

$$g_*1 \cdot f_*1 = h'_*1$$

where h' is the map h oriented by the isomorphism

$$\nu_h = \nu_{gf'} \cong f'^*\nu_g + \nu_{f'} \cong f'^*\nu_g + g'^*\nu_f$$

$$\cong f'^*\Phi(\lambda_g) + g'^*\Phi(\lambda_f) \cong \Phi(f'^*\lambda_g + g'^*\lambda_f).$$

Now the commutativity of A gives an isomorphism

$$\varphi : f'^*\lambda_g + g'^*\lambda_f \cong g'^*\lambda_f + f'^*\lambda_g$$

such that the diagram

$$\begin{tikzcd}
\Phi(f'^*\lambda_g + g'^*\lambda_f) \arrow[r, phantom, "\overset{\Phi(\varphi)}{\cong}"] \arrow[d, phantom, sloped, "\cong"] & \Phi(g'^*\lambda_f + f'^*\lambda_g) \arrow[d, phantom, sloped, "\cong"] \\
\nu_h \arrow[r, phantom, "\overset{(-1)^{\deg f \cdot \deg g}}{\cong}"] & \nu_h
\end{tikzcd}$$

is commutative.

Consequently we have the formula

$$\boxed{g_*1 \cdot f_*1 = (-1)^{\deg f \cdot \deg g} f_*1 \cdot g_*1}$$