obselete, June 11, 1969

Decomposition theorem $\quad \Omega = L \otimes_{LT} BP$, where $BP = LT \otimes_L \Omega$.

In the following we work over $\mathbb{Z}_{(p)}$. L Lazard ring, F_u canonical group law. If F is a group law over a $\mathbb{Z}_{(p)}$-algebra R, let $c(F)$ be the Cartier coordinate change so that $c(F)*F$ is typical. Let LT be the Lazard ring for typical laws, F_t the canonical typical law. Then we have maps

$$\begin{array}{ccccc} L & \xrightarrow{\pi} & LT & \xrightarrow{i} & L \\ F_u & \longmapsto & F_t & \longmapsto & c(F_u)*F_u \end{array}$$

such that $\qquad \pi i = id_{LT} \quad$ since

$$\pi\left(c(F_u)*F_u\right) = c(F_t)*F_t = F_t.$$

Let $BP(X) = LT \otimes_L \Omega(X)$. Then ~~...~~ for any standard theory Γ (splitting principle) with F_Γ typical we have a unique morphism $BP \longrightarrow \Gamma$. If F_Γ not nec. typical, set

$$\bar{\xi}_\Gamma = c(F_\Gamma)$$

and introduce a new Gysin homomorphism on Γ by

$$f_*^{\Gamma!}x = f_*^{\Gamma}\left(\tilde{\xi}_F(\nu_f)\,x\right).$$

Then

$$c_1^{\Gamma!}(L) = \bar{\xi}_\Gamma(c_1^\Gamma(L))$$

so

$$F_{\Gamma!} = \bar{\xi}_\Gamma * F_\Gamma = c(F_\Gamma)*F_\Gamma \quad \text{is typical}$$

and so $\exists$ a unique $\overbrace{\text{natural ring}}$ homomorphism $\alpha : BP \longrightarrow \Gamma!$ with

$$\alpha\left(f_*^{BP} x\right) = f_*^{\Gamma}\left(\tilde{\xi}_{\Gamma}(\nu_f) \cdot \alpha(x)\right).$$

Taking $\Gamma = \Omega$ we have a map

$$i : BP \longrightarrow \Omega$$

$$i\left(f_*^{BP} x\right) = f_*^{\Omega}\left(\tilde{\xi}_{\Omega}(\nu_f)\, \alpha(x)\right).$$

So we have a diagram

$$\star \qquad \begin{tikzcd} \overset{F_u}{L} \arrow[r, "\pi"] \arrow[d, "\theta"] & \overset{F_t}{LT} \arrow[r, "i"] \arrow[d, "\theta"] & \overset{c(F_u) * F_u}{L} \arrow[d, "\theta"] \\ \underset{F_{\Omega}}{\Omega(?)} \arrow[r, "\pi"] & \underset{F_{BP}}{BP(?)} \arrow[r, "i"] & \underset{c(\Gamma_{\Omega}) * F_{\Omega} = F_{\Omega!}}{\Omega!(?)} \end{tikzcd}$$

which commutes in virtue of where the group laws go. The first square is cartesian by definition; also

$$i\pi\, f_*^{\Omega} x = f_*^{\Omega}\left(\tilde{\xi}_{\Omega}(\nu_f)\, i\pi x\right)$$

Thus $\boxed{i\pi = \hat{\xi}_{\Omega}}$. Moreover

$$\pi i \pi\, f_*^{\Omega} x = f_*^{\Omega}\left(\pi \tilde{\xi}_{\Omega}(\nu_f) \cdot \pi i \pi x\right)$$

and

$$\pi\, \tilde{\xi}_{\Omega}(L) = \pi\left(\xi_{\Omega}\left(c_1^{\Omega}(L)\right)\right) = \left(\pi \xi_{\Omega}\right)\left(c_1^{BP} L\right) = 1$$

since

$$\pi \bar{\xi}_{\Omega} = \pi\, c(F_{\Omega}) = \pi \theta\, c(F_u) = \theta \pi\, c(F_u) = \text{the power series } (\theta 1) X. \implies \pi \xi_{\Omega} = 1.$$

Therefore $\pi i \pi = \hat{1} = \pi$ so as π is surjective

$$\boxed{\pi i = id_{BP}}$$

so BP is a retract of Ω and therefore ~~comes~~ from a spectrum.

Also $\hat{\xi}_\Omega$ is an idempotent endomorphism of Ω.

~~From the diagram we~~ see the second square is cartesian so we obtain an isomorphism of theories

$$\boxed{\Phi : L \otimes_{LT} BP(X) \xrightarrow{\cong} \Omega I(X)}$$

Φf_*^{BP}

Let

$$Q! = L \otimes_{LT} BP$$

with $f_*^{Q!}$ the L linear extension of f_*^{BP}. ~~Let~~ Q be the theory corresponding to Q! Thus

$$f_*^{Q} x = f_*^{Q!}(\tilde{\chi}_{Q!}(\nu_f)x)$$

where $\tilde{\chi}_{Q!}(L) = \chi_{Q!}(c_1^{Q!}(L))$

& $\bar{\chi}_{Q!} = (\bar{\xi}_Q)^{-1}$

Let
$$Q(X) = L \otimes_{LT} BP(X)$$

so that there is a cocartesian square

$$\begin{tikzcd} LT \arrow[r, "i"] \arrow[d, "\theta"] & L \arrow[d, "in_1"] \\ BP(X) \arrow[r, "in_2"] & Q(X) \end{tikzcd}$$

and a map $I : Q(X) \longrightarrow \Omega(X)$ given by $I in_1 = \theta$, $I in_2 = i$.
$Q(X)$ is a contravariant functor. Introduce the Gysin homomorphism $f_*^{\#}$ as the L-linear extension of f_*^{BP} ie.

$$\begin{cases} f_*^{\#}\, in_2 = in_2 f_*^{BP} \\ f_*^{\#}\, in_1 = in_1 \end{cases}$$

Now let
$$\bar{\chi} = in_1\, c(F_u)^{-1} \in Q[[X]]$$
and define
$$f_*^{Q}(x) = f_*^{\#}\left(\tilde{\chi}(\nu_f) x\right).$$

Lemma: (i) $F_Q = in_1 F_u$

(ii) $f_*^{Q!} = f_*^{\#}$

(iii) $I f_*^{Q} = f_*^{\Omega} I$.

Proof: (i)
$$c_1^{Q}(L) = \iota^* \iota_*^{Q} 1 = \iota^* \iota_*^{\#}(\tilde{\chi}(L)) = c_1^{\#}(L)\, \chi(c_1^{\#} L)$$
$$= \bar{\chi}\, c_1^{\#}(L) = \bar{\chi}\left(in_2\, c_1^{BP} L\right)$$

Therefore

$$F_Q = \bar{\chi} * in_2 F_{BP}$$
$$= \bar{\chi} * in_2 \theta F_t$$
$$= \bar{\chi} * in_1 (c(F_u) * F_u)$$
$$= in_1 c(F_u)^{-1} * (in_1 c(F_u) * in_1 F_u)$$
$$= in_1 F_u .$$

(ii). $\bar{\xi}_Q = c(F_Q) = in_1 c(F_u) = \bar{\chi}^{-1}$

$$f_*^{Q!} x \overset{df}{=} f_*^{Q}(\tilde{\xi}_Q(\nu_f)\cdot x) = f_*^{\#}(\tilde{\chi}(\nu_f)\tilde{\xi}_Q(\nu_f)x)$$

But

$$L \longmapsto \tilde{\chi}(L)\,\tilde{\xi}_Q(L) = \chi(c_1^{\#}L)\cdot \xi_Q(c_1^{Q}L)$$
$$= \chi(c_1^{\#}L)\,\xi_Q(\bar{\chi}\, c_1^{\#}L).$$

is the operation associated to

$$\chi(X)\,\xi_Q(\bar{\chi}(X)) = \frac{\bar{\xi}_Q(\bar{\chi}(X))}{X} = 1 .$$

$$\therefore\ f_*^{Q!} = f_*^{\#}$$

(iii) We know that $I: Q! \to \Omega$ is a morphism of theories hence also $I: Q \longrightarrow \Omega$ must be.

$$\Big(I f_*^{Q} x = I f_*^{\#}(\tilde{\chi}(\nu_f)x) = f_*^{\Omega!}(I(\tilde{\chi}(\nu_f))\cdot Ix)$$
$$= f_*^{\Omega}(\tilde{\xi}_\Omega(\nu_f)\cdot I(\tilde{\chi}(\nu_f))\cdot Ix) \Big)$$

$I\tilde{\xi}_Q(\nu_f)\tilde{\chi}(\nu_f)$

Claim I: $Q \xrightarrow{\sim} \Omega$. In fact given Γ one gets a diagram

$$\begin{tikzcd} LT \arrow[r, "i"] \arrow[d, "\theta"] & L \arrow[d, "\theta"] \\ BP \arrow[r, "\imath"'] & \Gamma! \end{tikzcd}$$

hence a ! map of theories $Q! \longrightarrow \Gamma!$ over L, hence a ! map $Q \longrightarrow \Gamma$.

June 12, 1969

Operations in cohomology theories and operations in ΩT.

§1. Calculation of $\mathrm{Aut}^{\otimes}$ for Ω and ΩT.

Let Ω be complex cobordism theory regarded as a contravariant functor from manifolds to graded commutative (in the sense of topology) rings. All rings shall be of this type from now on. Consider the functor $C = \mathrm{Aut}^{\otimes}\Omega$ from rings to groupoids ~~categories~~ defined by

$$\mathrm{Ob}\ C(R) = \mathrm{Hom}\,(\Omega(pt), R)$$

$$\underset{C(R)}{\mathrm{Hom}}\,(u, v) = \mathrm{Isom}^{\otimes}_{R}(\Omega_u, \Omega_v),$$

where if $u: \Omega(pt) \longrightarrow R$ is a morphism then Ω_u denotes the contravariant functor from manifolds to R-algebras given by

$$\Omega_u(X) = R_u \underset{\Omega(pt)}{\otimes} \Omega(X)$$

and where $\mathrm{Isom}^{\otimes}_{R}$ denotes the set of isomorphisms of functors compatible with R-algebra structures.

Let L be the functor from rings to sets given by

$$L(R) = \left\{ F(X,Y) = \sum_{k,\ell \geq 0} a_{k\ell} X^k Y^\ell \;\middle|\; \begin{array}{l} F \text{ formal (comm.) group law} \\ a_{k\ell} \in R_{2k+2\ell-2} \end{array} \right\}$$

and let G be the functor from rings to groups given by

$$G(R) = \left\{ \varphi(X) = \sum_{n \geq 0} r_n X^{n+1} \;\middle|\; r_n \in R_{2n},\ r_0 \in R^* \right\}$$

with group structure given by composition of power series.
Introduce the functor $\mathcal{L}$ from rings to ~~category~~ groupoids given by

$$\mathrm{Ob}\ \mathcal{L}(R) = L(R)$$

$$\mathrm{Hom}_{\mathcal{L}(R)}(F, F') = \{ \varphi \in G(R) \mid \varphi * F = F' \}$$

with evident composition. Here

$$(\varphi * F)(X, Y) = \varphi\left(F(\varphi^{-1}X, \varphi^{-1}Y)\right).$$

so that

$$\psi * (\varphi * F) = (\psi\varphi) * F .$$

Let us make $\mathcal{L}$ act on Ω as follows. Let F_{univ} over A be a universal law so that

$$\mathrm{Hom}(A, R) \simeq L(R)$$
$$u \longmapsto u(F_{\mathrm{univ}}),$$

and let $c : A \longrightarrow \Omega(pt)$ be the map corresponding to F^{Ω}. Then Ω is a functor from manifolds to A-algebras. If $F \in L(R)$, let

$$\Omega_F = R_u \otimes_A \Omega$$

where $u : A \longrightarrow R$ is the map sending F_{univ} to F. It is easily seen that Ω_F is the initial object of the category of

Chern theories over R with group law F. Let F' be another law over R and let $\varphi \in G(R)$ be an invertible power series. ~~Theres~~ There is a unique multiplicative characteristic class

$$\tilde{\varphi} : K \longrightarrow \Omega_{F'}$$

given on line bundles by

$$\tilde{\varphi}(L) = \frac{\varphi(c_1 L)}{c_1 L}.$$

Twisting the Gysin homomorphism of $\Omega_{F'}$ by means of $\tilde{\varphi}$ one obtains a new Chern theory $\Omega_{F'}^{\varphi}$ with

$$c_1^{\varphi}(L) = \varphi(c_1 L)$$

and hence with group law $\varphi * F'$ since

$$c_1^{\varphi}(L \otimes L') = \varphi(F'(c_1 L, c_1 L')) = (\varphi * F')(c_1^{\varphi} L, c_1^{\varphi} L').$$

Thus by the universal property of Ω_F, where $F = \varphi * F'$, we have a unique natural map

$$\hat{\varphi} : \Omega_F \longrightarrow \Omega_{F'}$$

such that $$\hat{\varphi}(f_* x) = f_*(\hat{\varphi} x \cdot \tilde{\varphi}(\nu_f)).$$

Therefore we obtain an action of $\mathcal{L}$ on Ω by associating to a law F over R, the Chern theory Ω_F, and to the morphism

from F to F' in $\mathcal{L}(R)$ given by $\varphi \in G(R)$ with $\varphi * F = F'$ the morphism $\widehat{\varphi^{-1}}$ from Ω_F to $\Omega_{F'}$.
(It is straightforward to check that $\widehat{\psi^{-1}}\,\widehat{\varphi^{-1}} = \widehat{(\psi\varphi)^{-1}}$
and hence that $F \mapsto \Omega_F$ is a functor.)

Define a morphism of functors

(A) $$\underline{\mathrm{Aut}}^{\otimes}\,\Omega \longrightarrow \mathcal{L}$$

by associating to a homomorphism $\Omega(pt) \longrightarrow R$ the law induced from F^{Ω} and to an isomorphism

$$\theta : \Omega_u \xrightarrow{\sim} \Omega_v$$

the series $\varphi(X)^{-1} \in R[[X]]$ such that

$$\theta(c_1^u L) = \varphi(c_1^v L) = \sum_{n \geq 0} r_n (c_1^v L)^{n+1}$$

To see that r_0 is invertible note that if $i : pt \to \mathbb{P}^1$ is the inclusion of a point, then $i_* 1 = c_1(\mathcal{O}(1))$ and there is a map of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \Omega_u(pt) \arrow[r, "i_*"] \arrow[d, "\cdot r_0"] & \Omega_u(\mathbb{P}^1) \arrow[r, "i^*"] \arrow[d, "\theta", "\cong"'] & \Omega_u(pt) \arrow[r] \arrow[d, "\theta = id_R", "\cong"'] & 0 \\
0 \arrow[r] & \Omega_u(pt) \arrow[r, "i_*"] & \Omega_v(\mathbb{P}^1) \arrow[r, "i^*"] & \Omega_u(pt) \arrow[r] & 0
\end{tikzcd}$$

which shows that multiplication by r_0 is an isomorphism. Moreover

$$\theta(c_1^u(L\otimes L')) = \theta(F_u(c_1^u L, c_1^u L')) = F_u(\varphi(c_1^v L), \varphi(c_1^v L'))$$

$$= \varphi(c_1^v(L\otimes L')) = \varphi(F_v(c_1^v L, c_1^v L'))$$

so $\varphi^{-1} * F_u = F_v$. It is clear that (A) is a functor for a fixed ring R.

<u>Theorem 1:</u> (A) <u>is an isomorphism of functors from rings to groupoids.</u> In particular <u>$\mathrm{Aut}^{\otimes}\Omega$ is representable.</u>

<u>Proof</u>. (A) is an isomorphism on objects since $A \xrightarrow{\sim} \Omega(pt)$; this is the theorem that F^{Ω} is a universal law. Next, by means of the action of $\mathcal{L}$ on Ω ~~just~~ constructed above we have ~~obtained~~ a map right inverse to the map

$$\mathrm{Isom}^{\otimes}_R(\Omega_F, \Omega_{F'}) \overset{\widehat{\varphi} \dashleftarrow \varphi}{\longrightarrow} \mathrm{Hom}_{\mathcal{L}(R)}(F, F')$$

induced by (A), so this map is surjective. Finally we must show that if

$$\theta : \Omega_F \xrightarrow{\cong} \Omega_{F'}$$

gives rise to φ by the formula

$$\theta(c_1^F L) = \varphi(c_1^{F'} L)$$

then in fact $\theta = \widehat{\varphi}$. But this follows from Riemann-Roch. Indeed endow $\Omega_{F'}$ with the twisted Gysin homomorphism such that

Θ is compatible with first Chern classes; by RR Θ is compatible with Gysin, hence $\Theta = \hat{\varphi}$ by the universal property of Ω_F. q.e.d.

Let p be a fixed prime and consider the theory $\Omega_{(p)}$. (We consider only $\mathbb{Z}_{(p)}$-algebras in the following.) Let $\mathcal{L}T \subset \mathcal{L}$ be the "full subcategory" which is the functor associating to a ring R the full subcategory of $\mathcal{L}(R)$ consisting of the typical laws. As $\mathcal{L}T \longrightarrow \mathcal{L}$ is an equivalence ~~it is possible to~~ and as $\mathcal{L}$ acts on $\Omega_{(p)}$, we know $\Omega_{(p)}$ can be decomposed into simpler theories. We propose now to make this explicit.

Let F_{univ} over $A_{(p)}$ be a universal law for laws over $\mathbb{Z}_{(p)}$-algebras and let F_{typ} over AT be a universal typical law. There is a canonical surjection

$$f : A_{(p)} \longrightarrow AT$$

sending F_{univ} to F_{typ}. Let

$$\Omega T = AT \otimes_{A_{(p)}} \Omega_{(p)} = AT \otimes_A \Omega$$

It is clear that ΩT is a universal Chern theory (over $\mathbb{Z}_{(p)}$) with typical group law.

By a theorem of Cartier every law over a $\mathbb{Z}_{(p)}$-algebra R is isomorphic to a typical law. Thus element in $G(PT)$ with

$$\varphi_{can} * s(F_{typ}) = t(F_{typ}).$$

Putting this together we have the following isos.

$$\begin{array}{ccc} AT & & PT \\ \| & & \| \\ \mathbb{Z}_{(p)}[q_1, q_2, \ldots] & & \mathbb{Z}_p[t_1, t_2, \ldots, q_1, q_2, \ldots] \end{array}$$

with $t\, q_i = q_i$. Now I need formulas for ε, s, Δ. Clearly $\varepsilon(t_i) = 0$, $\varepsilon(q_i) = q_i$. To calculate s we need to know how to find

$$s(q_i) = q_i\left(\varphi_{can} * t(F_{typ})\right)$$

It doesn't seem possible to find closed formulas for these but by working over $\mathbb{Q}$, which is legitimate since AT and PT are torsion-free we can use the logarithm to get simpler parameters than the q_i. So for any typical F let $r_i(F)$ be defined by

$$\ell_F \gamma = \sum_{i \geq 0} V^i\left[\frac{r_i(F)}{p^i}\right]\gamma \qquad r_0 = 1 \quad \deg r_i = p^i - 1.$$

Applying logarithm to both sides of $*$ on page 10 we find

$$\ell F\gamma = F \ell\gamma = \sum V^{i-1}\left[\frac{r_i(F)}{p^{i-1}}\right]\gamma.$$

$$\| $$

$$\sum V^{i-1}[q_i(F)]\, \ell\gamma = \sum V^{i-1}[q_i(F)]\, V^j\left[\frac{r_j(F)}{p^j}\right]\gamma$$

Thus

$$\sum_{i\geq 1} r_{i+1}(F)\,\frac{X^{p^i}}{p^i} = \sum_{m\geq 0}{}' \, p^m q_{m+1}(F)^{p^j}\,\frac{r_j(F)}{p^{j+m}}\,X^{p^{m+j}}$$

$$\boxed{r_{i+1}(F) = \sum_{m+j=i} p^m q_{m+1}(F)^{p^j}\, r_j(F)}$$

This gives a recursion formula for q_i and r_i since ~~modulo earlier things one has~~ $r_{i+1} = p^i q_{i+1} +$ earlier q's $+\, r$

So now we have

$$\mathbb{Z}_{(p)}[q_1, q_2, \dots] \longleftrightarrow \mathbb{Z}_{(p)}[r_1, r_2, \dots]$$

and this is an isomorphism over $\mathbb{Q}$. Recall that

$$\varphi_{can} = \sum_{i\geq 0}^{t(F_{typ})} t_i X^{p^i}$$

And $s(F_{typ}) = \varphi_{can}^{-1} * t(F_{typ})$ so taking logarithms we have

$$\sum_{i\geq 0} s(r_i)\,\frac{X^{p^i}}{p^i} = \sum_{i\geq 0} t(r_i)\,\frac{X^{p^i}}{p^i} \circ \varphi_{can}$$

$$= \sum_{j\geq 0}\,\sum_{i\geq 0} p^j t(r_i)\,\frac{1}{p^{i+j}}\left(t_j X^{p^j}\right)^{p^i}$$

hence

$$\boxed{s(r_i) = \sum_{j+h=i} p^j \, t(r_h) \, t_j^h}$$

June 12, 1969.

Some basic geometry

Let V be a vector space over $\mathbb{R}$ of dimension n and let V^+ be the 1-point compactification of V. V^+ is a n-sphere. I claim that after blowing up ∞ one obtains $\mathbb{P}(V \oplus 1)$. To see this identify $\mathbb{P}(V+1)$ with the projective completion of V, thus $\mathbb{P}(V+1) = V \amalg \mathbb{P}V$ set theoretically. But $\widetilde{V^+} = V \amalg \mathbb{P}(\text{normal space to } V^+ \text{ at } \infty) = V \amalg \mathbb{P}V$. Now check that manifold structures are the same. $\widetilde{V^+} - \text{origin} = \text{pairs } (\ell, x)$ where ℓ is a line in V passing thru 0 and where $x \in \ell - (0)$ possibly with $x = \infty$. This is clearly the same as $\mathbb{P}(V+1)$ − the line $\mathbb{R}\cdot(0,1)$. The blowup map is

$$f : \mathbb{P}(V+1) \longrightarrow V^+$$

$$f \begin{cases} \mathbb{R}(x,1) = x \\ \mathbb{R}(x,0) = \infty \end{cases}$$

(Near $\mathbb{P}V$ a line $\ell \subset V+1$ same as a line $\bar{\ell}$ in V + an element of the dual of $\bar{\ell}$. This checks with the normal bundle of $\mathbb{P}V$ in $\mathbb{P}(V+L)$ as being $\mathcal{O}(1) \otimes L$.

Arbeitstagung notes, June 18, 1969

Wall conversation:

Here is the inductive step in the Novikov argument. Recall we are trying to prove that if M is a C^∞ (PL-) manifold homeomorphic to $K \times \mathbb{R}^n$, K a topological manifold with $\pi_1(K)$ abelian and $\dim K \geq 5$, then M is C^∞ isom. to $N \times \mathbb{R}^n$. The case $n=1$ is handled by Siebenmann and h-cobordism. Notice that it suffices to find a smooth submanifold $N \subset M$ with trivial normal bundle and such that $N \to M$ is a heq; In effect put a boundary on M and use that $\bar{M}$ − Int tub. nbd. of N is an s-cobordism.

So take anchor ring $T^{n-1} \times \mathbb{R} \subset \mathbb{R}^n$

$$\begin{tikzcd} M \arrow[r, "\text{homeo}"'] & K \times \mathbb{R}^n \\ C^\infty\ \cup_{\text{open}} & \cup \\ \mathcal{U} \arrow[r, "\text{homeo}"'] & K \times T^{n-1} \times \mathbb{R} \end{tikzcd}$$

By induction $\mathcal{U} \underset{C^\infty}{\cong} M_1 \times \mathbb{R}$ where $M_1 \to K \times T^{n-1}$ is a heq. Now take covering with respect to the last factor $\tilde{M}_1 \to K \times T^{n-1} \times \mathbb{R}$ whence by induction $\tilde{M}_1 \underset{C^\infty}{\cong} M_2 \times \mathbb{R}$. It is possible to put $M_2 \hookrightarrow M_1$ with trivial normal bundle and again $M_2 \to K \times T^{n-1}$ is a heq. So done

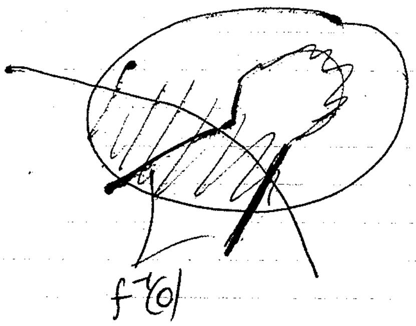
---

Key part of Kirby-Siebenmann is that in dimensions ≥ 5 a topological submanifold (locally flat) of codim 1 of a smooth manifold may be isotoped to a smooth submanifold. Assuming relative version of this I show now how to get topological transversality.

Theorem. Let $X \subset E$ be a microbundle cordim

and let $f: Y \longrightarrow E$ be a map where Y is a top. manifold. Then f may be moved by an isotopy until it is transversal to X.

Proof: By obvious induction we may suppose E trivial over X and that f is transversal in a neighborhood of the boundary. Now by successive projecting to $\mathbb{R}$ can assume that $X < E$ is $0 \longrightarrow \mathbb{R}$. Now use coordinate charts of Y.



shedded region indicates region of transversality

Now in the chart we can arrange coordinates so that $f^{-1}(0)$ is smooth and the make map smooth inside without affecting the transversality already obtained. So done.

Tate lecture:

Basic recall

Prop: F ~~[illegible]~~ alg. closed $\Longrightarrow$ $K_2(F)$ $\mathbb{Q}$ vector space

Proof. $K_2(F) = F^* \otimes F^* / R$ $\quad$ R gen. by $r_a = a \otimes (1-a)$ $\quad a \neq 0, 1$.

F^* divisible $\Rightarrow$ $F^* \otimes F^*$ uniquely divisible. $\therefore$ Enough to show R divisible. ~~By~~ Given $a \in F$ $\quad a \neq 0, 1$ write

$$T^n - a = \prod_{i=1}^{n} (T - a_i)$$

Then

$$a \otimes (1-a) = \sum_i a_i^n \otimes (1 - a_i) = n \sum_{i=1}^{n} a_i \otimes (1 - a_i)$$

~~Prop: $K_2(F) \longrightarrow K_2(\bar{F})$ induces an isomorphism $\otimes \mathbb{Q}$.~~

Need the formula that if $F \longrightarrow E$ finite extension of degree n then $\exists$ $N : K_2(E) \longrightarrow K_2(F)$ $\ni$

$$N\{x, y\}_E = \{x, N_{E/F}\, y\}_F \qquad \text{if } x \in F,\ y \in E.$$

From this one finds that if $a, b \in F$ then $N\{a,b\}_E = n\{a,b\}_F$ so that the kernel of $K_2(F) \to K_2(E)$ is killed by n, hence

Cor: $\operatorname{Ker}\{K_2(F) \longrightarrow K_2(\bar{F})\}$ = torsion subgroup of $K_2(F)$.

Prop: F quasi-alg. closed $\Longrightarrow$ $K_2(F)$ ~~[illegible]~~ divisible

Proof: One know that q. alg. cl $\Rightarrow$ (forms of degree < no. of vbls have $\neq 0$ roots,

$\Rightarrow N_{E/F}$ surjective $\Longrightarrow$ $\{a, b\}_F = \{a, N_{F(a^{1/m})/F}\, x\}_F$

$$= N\{a, x\}_{F(a^{1/m})} = m\, N\{a^{1/m}, x\}_{F(a^{1/m})}.$$

<u>Cor</u>: F finite field $\Longrightarrow K_2(F) = 0$ (Steinberg)

Also $K_2(\overline{\mathbb{F}}_p) = 0$.

Arbeitstagung, June 13, 1969

Kuiper talks on Kirby's solution of the annulus conjecture.

first part - definition of stable homeomorphism of $\mathbb{R}^n$ and why the notion is local. Why "every homeo. of $\mathbb{R}^n$ stable $\Rightarrow$ annulus conjecture (~~[illegible]~~ any two embeddings of $S^{n-1}\times S^0$ into S^n are isomorphic). Stable manifolds and the notion of a stable homeomorphism of two stable manifolds.

second part:

Lemma 1. (Connell) If $h: \mathbb{R}^n \longrightarrow \mathbb{R}^n \ni |h(x)-x| < K$ then h is stable.

Proof. Embed $\mathbb{R}^n$ as interior of D^n in $\mathbb{R}^n$. h extends to closure and can be completed to a homeomorphism ~~[illegible]~~ which is id off the disk. The latter is somewhere the identity, hence h is stable.

Lemma 2: (Kirby) Any homeomorphism of T^n is stable.

Proof. Translations any isomorphisms of $T^n = \mathbb{R}^n/\mathbb{Z}^n$ coming from unimodular transformations of $\mathbb{R}^n$ are stable. One may thus compose h with such a stable thing and suppose $h(0)=0$ and that h acts trivially on $\pi_1(T^n, 0)$. It follows that $\tilde{h}$ on $\tilde{T}=1$ leaves $\mathbb{Z}^n$ ~~[illegible]~~ fixed hence is of bounded displacement and so is stable by lemma 1. On the other hand by ~~[illegible]~~ the local nature of stability h is stable.

And now the key. Suppose given $h: \mathbb{R}^n \to \mathbb{R}^n$ a homeomorphism. One knows that T^n − small D^n can be immersed in $\mathbb{R}^n$. Thus one gets a diagram (commutative)

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "h"] & \mathbb{R}^n \\
T-D \arrow[u, "\text{immers.}"'] \arrow[r, "h_1"] & V \arrow[u, "\text{immers.}"']
\end{tikzcd}$$

where h_1 is a homeo. and the vertical arrows are PL "etale". The point is that h_1 can be embedded in a diagram

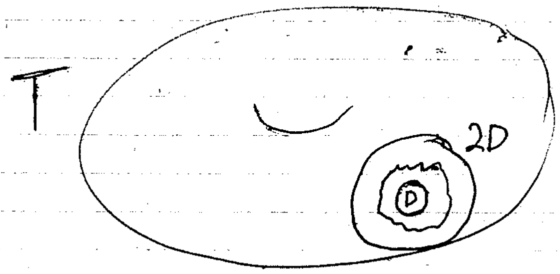
$$\star \qquad \begin{tikzcd}
T-D \arrow[r, "h_1", "\text{homeo}"'] \arrow[d] & V \arrow[d, "\text{PL open embedding}"] \\
T \arrow[r, "h_2", "\text{homeo}"'] & \tau
\end{tikzcd}$$

where τ is a PL-manifold, necessarily homeomorphic to T. By Wall one knows that a finite covering of τ is PL isomorphic to T. Hence one gets

$$\begin{tikzcd}
T \arrow[r, "h_2"] & \tau \\
\hat{T} \arrow[u] \arrow[r, "h_3"] & \hat{\tau} \arrow[u] \arrow[l, bend left, dashed, "g"]
\end{tikzcd} \qquad \leftarrow \text{finite covering}$$

Now g PL isom., so g stable; gh_3 stable by lemma 2 $\Longrightarrow$ h_3 stable and hence that h is stable by the local nature of stability.

<u>Proof of the point:</u> The situation is that we are given a homeomorphism $h_1: T - D \longrightarrow V$ where V is a PL (in fact C^∞) manifold. V is an open C^∞ manifold with an end ~~[illegible]~~ admitting S^{n-1} as a topological boundary. By Siebenmann ~~[illegible]~~ and Smale (seem to need $n \geq 6$) we can put a smooth S^{n-1} on. Let $\bar{V}$ be the completed manifold and let consider the picture



where the squiggly S^{n-1} is the inverse image under h_1 of a parallel S^{n-1}_ε to $\partial\bar{V}$. By Schonflies (Mazur-Brown) the inside of the squiggly S^{n-1} is homeomorphic to D^n. ~~[illegible]~~ Thus we get a homeomorphism

$$T \longrightarrow \bar{V}_{\geq\varepsilon} \underset{S^{n-1}_\varepsilon}{\cup} D^n$$

which is h_1 on (closed) outside of squiggly S^n and which is the homeomorphism of the (closed) inside with D^n. This does what we want (at least if we shrink h_1 to $T - 2D$)

Is ~~Probably~~ use of the Smale - Siebenmann ~~[illegible]~~ unnecessary? Need to know that a cont. embedding $S^{n-1} \times I \longrightarrow R^n$ has a smooth S^{n-1} in the middle.

~~A~~ Review of Griffiths lecture on ~~[illegible]~~ algebraic cycles. Griff

Problem is to describe algebraic part of $H_*(V, \mathbb{Z})$.
Critical case: dim $V = 2n$ and the primitive part of $H_{2n}(V)$ (over $\mathbb{Q}$), that is, the part perpendicular to $H_{2n}(S)$ where S is a generic hyperplane.

Lefschetz method — a kind of Morse theory used already by Poincaré for curves on surfaces. To simplify assume V fibers over $\mathbb{P}^1$ with non-degenerate critical points. Griffiths ~~[illegible]~~ ~~[illegible]~~ wants to construct (primitive algebraic) cycles on V by the Poincaré method. First he constructs a generalized Jacobian $J \xrightarrow{\pi} \mathbb{P}^1$ which ~~is~~ a complex bundle of complex abelian lie groups, (not necessarily abelian varieties), which are tori at the good fibers ~~[illegible]~~ and a product of a torus and $\mathbb{G}_m$ at the the bad ones. He then shows that a primitive cycle on V defines a section of π. It turns out that sections of π correspond ~~[illegible]~~ essentially to ~~[illegible]~~ data satisfying Hodge conditions. Next he tries to take a point in the generic ~~[illegible]~~ fiber of π and lift it back to a cycle on the corresponding fiber of f (Jacobi inversion theorem). Unfortunately ~~[illegible]~~ except in special cases (curves on surfaces, low degree hypersurfa[ces]) the group of Jacobi-invertible point of J of the generic fiber is a countable subgroup hence can't do this. ~~[illegible]~~ In some of the special cases one can obtain a continuous part of J and thus construct algebraic cycles and thu[s] verify Hodge conjecture in some non-trivial codim >1 cases.

Since the ~~section~~ point of J_λ corresponding to a class in S_λ the fiber over λ algebraically equivalent to zero vanishes, Griffiths can start with a non-trivial primitive algebraic class in V. and its intersection with S_{gen} is then homologous to zero but no multiple is alg. equivalent to 0. He calls this the "discreteness" of Jacobi inversion in higher codimension.

Comments of Adams lectures on my stuff.

new ideas

(i) The hypersurfaces H_{ij} of degree 1,1 in $\mathbb{P}^i \times \mathbb{P}^j$ as i and j run over integers ≥ 0 form a system of multiplicative generators.

Proof using my theorem: let $F(X,Y) = \sum_{i,j\geq 0} a_{ij} X^i Y^j$ be the formal group law of cobordism theory. Then

$$[H_{mn}] = \pi_* \, c_1(\mathcal{O}(1) \boxtimes \mathcal{O}(1))$$

$$= \pi_* \, F(z_1, z_2) \qquad\qquad \pi_*(z^i z^j) = P_{m-i} P_{n-j}$$

$$= \sum_{i,j\geq 0} a_{ij} P_{m-i} P_{n-j}$$

hence $\quad [H_{mn}] \equiv$

$$\sum_{\substack{j\geq 0 \\ j<m}} a_{mj} P_{n-j} + \sum_{\substack{i\geq 0 \\ i<n}} a_{in} P_{m-i} + a_{mn}$$

modulo decomposables. $\qquad$ decomposable unless $m+j-1=0$ ie $m=1$

Thus $[H_{mn}] \equiv a_{mn}$ modulo decomposable unless $m=1$ or $n=1$. As $H_{1n} = \mathbb{P}_{n-1}$, $H_{m1} = \mathbb{P}_{m-1}$ one sees that the H_{mn} generate the Lazard ring.

(ii) Use of Lazard's theory to simplify proof that $\pi_*(MU)$ is a polynomial ring as well as a simplification of Lazard's theorem that L is a polynomial ring.

~~[illegible]~~

To prove L is a polynomial ring with one generator of every (even) degree, it suffices to show that the map

$$L \xrightarrow{\varphi} \mathbb{Z}[b_1, b_2, \cdots] = B$$

$$F \longmapsto \sum_{n\geq 0} b_n X^{n+1} * (X+Y)$$

induces a monomorphism

$$Q(L) \longrightarrow Q(B)$$

In fact ~~[illegible]~~ Adams shows this map can be identified ~~[illegible]~~ in dim $2n$ with

$$a_n\mathbb{Z} \longrightarrow \mathbb{Z}$$

where $a_n = 1$ if n not of form $p^i - 1$, p a prime, and ~~[illegible]~~ if $n = p^i - 1$, then $c_n = p$. To prove this last fact it suffices to consider laws over rings of the form $\underset{0}{\mathbb{Z}} + 0 + 0 + 0 + \underset{2n}{G} + 0 + 0$. where G is an abelian group and when they can be reduced to additive form. (This is just Fröhlich's generalization to abelian groups.)

To prove my theorem and Milnor's we need the following consequence of the Adams spectral sequence for $\pi_*(MU)$

$$Q_{2n}(\pi_*(MU)) \otimes \mathbb{Z}/p = \mathbb{Z}/p \qquad \text{for all primes } p \text{ and } n \geq 1.$$

~~[illegible]~~ It follows that

$$Q_{2n}\pi_*(MU) = \mathbb{Z}.$$

He shows that

$$\begin{tikzcd} L \arrow[r] \arrow[d, "\varphi"'] & \pi_*(MU) \arrow[d, "h"] \\ B = \mathbb{Z}[b_1, b_2, \ldots] \arrow[r, "\cong"] & H_*(MU) \end{tikzcd}$$

commutes and considers the induced diagram

$$\begin{tikzcd} Q_{2n}(L) \arrow[r] \arrow[d, hook] & Q_{2n}(\pi_*(MU)) \arrow[d, hook] \\ Q_{2n}(B) \arrow[r, "\cong"] & Q_{2n}(H_*(MU)) \end{tikzcd}.$$

The index of the first inclusion in c_n hence that of the second $| c_n$. As one knows that the index of the second is at ~~most~~ least c_n by characteristic class considerations, one has that the upper arrow is an isom, hence $L \to \pi_*(MU)$ is surjective and so done.

June 19, 1969

Notes on Wall's Lecture on Kirby–Siebenman

The key theorem is the following which works for Top/O and Top/PL, but which I state for Top/O.

Theorem: Let M be a compact top. manifold of dim ≥ 5 (≥ 6 if $\partial M \neq \phi$). Let $\mathcal{S}_{Top/O}(M)$ be the set of isom. classes of C^∞ structures on M modulo isotopy. Then

$$\mathcal{S}_{Top/O}(M) \longrightarrow \{\text{homotopy classes of liftings } \begin{array}{ccc} & & BO \\ & \nearrow & \downarrow \\ M & \xrightarrow{\tau} & BTop \end{array}$$

is bijective.

This results from the following

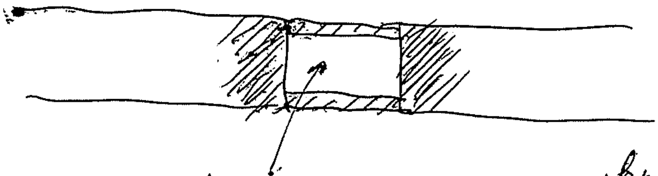
Key theorem: If M is a topological manifold (dim $\gg 5$) and let there be given a smooth structure on $M \times \mathbb{R}$. Then there is is a smooth structure on M giving up to isotopy the structure on $M \times \mathbb{R}$.

Probably another way of stating this is as follows. Suppose that V is a smooth manifold of dim ≥ 6 and that M is a locally flat topological submanifold of codimension 1. Suppose that M is a smooth submanifold in a neighborhood of every point x of $F \cap M$, where F is a closed subset of V. Then there is an arbitrarily small topological isotopy of V keeping F pointwise fixed and carrying M into a smooth submanifold of V.

The construction of g from h.

$$
\begin{tikzcd}
D^k \times T^n \arrow[r, "g", "\text{homeo}"'] & W^{n+k} \\
D^k \times (T^n - D^n) \arrow[u, phantom, sloped, "\subset", "C^\infty\ \text{open}"'] \arrow[r, Rightarrow] \arrow[d, "\mathrm{id} \times \alpha"] & D^k \times (T^n - D^n) \arrow[u, phantom, sloped, "\subset", "C^\infty\ \text{open}"'] \arrow[d, "C^\infty"] \\
D^k \times \mathbb{R}^n \arrow[r, "h"] & V^{k+n}
\end{tikzcd}
$$

↑ inherits PL str. from immersion



boundary is a smooth homotopy sphere bounds parallelizable manifold with signature 0. In PL case use Shonflies.

$$
\begin{tikzcd}
 & D^{k+n} \arrow[rr, "\text{exists since}"] & & D^{k+n} \arrow[ddddll, bend left, "f"] \\
D^k \times D^n \arrow[ur] \arrow[ddr] & D^k \times \mathbb{R}^n \arrow[u, "\text{comp.}"] \arrow[d, "\text{covering}"] \arrow[rr, "\tilde{g}"] & & D^k \times \mathbb{R}^n \arrow[u, "\text{comp}"] \arrow[d] \\
 & D^k \times T^n \arrow[r] & W^{k+n} \arrow[r, "\text{PL}"] & D^k \times T^n \\
 & & & \\
 & D^k \times \mathbb{R}^n \arrow[r, "h"] & V^{k+n} &
\end{tikzcd}
$$

exists if α chosen carefully

Choose f so that it's an embedding.

The geometry gives a $P(h)$ problem on $D^{k-1} \times \mathbb{R}^{n}$; by hypothesis it is solvable if $\times \mathbb{R}$. Thus $Q(g) \times S^1$ solvable so by s-cobordism $Q(g)$ has a solution. ∎

smooth PL top Poincaré sys.

~~Thm~~: $\mathcal{S}_{B/A}(X)$ = A structures on a B-manifold

$\mathcal{T}_{B/A}(X)$ = liftings
$$\begin{tikzcd} & BA \arrow[d] \\ X \arrow[r] \arrow[ur, dashed] & BB \end{tikzcd}$$

$\boxed{\text{Theorem:}}$ $\mathcal{S}_{PL/O} \xrightarrow{\cong} \mathcal{T}_{PL/O}$

$\mathcal{S}_{Top/PL}(X) \longrightarrow \mathcal{T}_{Top/PL}(X)$ bijective $\begin{cases} \dim X \geq 5 \\ \dim X \geq 6 \text{ al} \end{cases}$

false in dim 3.

$L_{n+1}(\pi_1 X) \to \mathcal{S}_{G/PL}(X) \longrightarrow \mathcal{T}_{G/PL}(X) \longrightarrow L(\pi_1 X)$ $\left[\dim X \geq 5 \text{ ∂i} \right.$

Same results for G/Top

Step 1: a tangential structure on X determines a geometrical A-structure on $X \times \mathbb{R}^n$ n large

Step 2: Geom. structure on $X \times \mathbb{R}$ induces one on X "Product theorem". (fails for G/?)

$P(h)$ denotes property: $h: D^k \times \mathbb{R}^n \longrightarrow V^{k+n}$
is a homeo of PL-manifold, PL near bdry
want isotopy compact support of h
to a homeo which is PL on $D^k \times D^n_{\text{small}}$

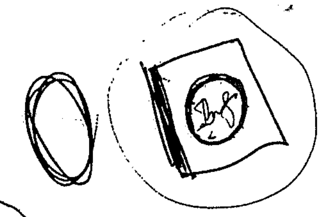
Choose PL immersion $\alpha: T^n - D^n \longrightarrow \mathbb{R}^n$

$\alpha \times 1: D^k \times (T^n - D^n) \longrightarrow D^k \times \mathbb{R}^n$

$Q(g)$: $g: D^k \times T^n \longrightarrow W^{k+n}$ ~~same as above~~
a homeo of PL-manifold want a homotopy
relative bdry to a PL-homeo

Prop 1: Given h can construct g such that
$Q(g)$ soluble $\Longleftrightarrow$ $P(h)$ soluble.

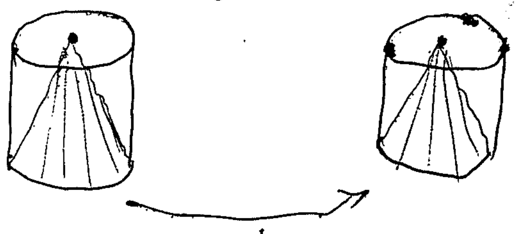
$$
\begin{tikzcd}
 & D^k \times \mathbb{R}^n \arrow[r, "h"] & W^{n+k} & & \\
D^k \times B^n_\varepsilon \arrow[ur] \arrow[ddddr] & D^k \times (T^n - D^n) \arrow[r, "h'"] \arrow[u, "id \times \alpha"] & D^k \times (T^n - D^n)^{\sim} \arrow[u] & & \\
 & D^k \times T^n \arrow[u, phantom, "\cap"] \arrow[r, "g"] & W^{n+k} \arrow[u, phantom, "\cap"] \arrow[r, "\text{diffeo}"] & D^k \times T^n & \\
 & D^k \times \mathbb{R}^n \arrow[u] \arrow[d, "\text{compactify}"] \arrow[rr, "\tilde{g}", "\text{bdd. displacement hence } g' \exists."'] & & D^k \times \mathbb{R}^n \arrow[u] \arrow[d] & \\
 & D^{k+n} \arrow[rr, "g'"] & & D^{k+n} \arrow[uuuul, bend right] &
\end{tikzcd}
$$



$\sim$ to id rel. bdry

put D^{k+n} back in V^{n+k}

g' ~~[illegible]~~ homeo. homotopic to identity rel. bdry hence by Alexander g' isotopic to identity



h

Assume g homotopic rel. bdry to a smooth ~~[illegible]~~ a imm.

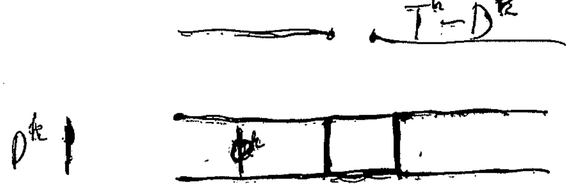
$P(h)$: $h : D^k \times \mathbb{R}^n \longrightarrow V^{k+n}$ $\overbrace{\text{homeo.} \ \exists}^{\text{PL near bdry}}$ isotopy of compact support to a homeo. which is smooth on $D^k \times B^n_\varepsilon$

$Q(g)$: $g : D^k \times T^n \longrightarrow W^{k+n}$ is a diffeo. homotopic mod bdry to a diffeo.

The construction of g from h.

$$\begin{tikzcd}
D^k \times \mathbb{R}^n \arrow[r, "\text{homeo}", "h"'] & V^{k+n} \\
D^k \times (T^n - D^n) \arrow[u, "id \times \text{Kirby immersion } \alpha"] \arrow[r, "\text{homeo}"] \arrow[d, hook] & V' \arrow[u, "\text{smooth immersion}"'] \arrow[d, "\text{put in } D^k \times D^n \text{ in}"] \\
D^k \times T^n \arrow[r, "g"'] & W^{k+n}
\end{tikzcd}$$

Recall that h diffeo. near $S^k \times \mathbb{R}^n$!



boundary of a parallelizable

June 18, 1969

Atiyah – Hirzebruch approach to the J-homomorphism:

Hirz. index formula
Atiyah's version of Adams J

$$\begin{tikzcd} Y \arrow[r, "j"] \arrow[dr, bend right, "f"'] & E \arrow[d, shift left, "\pi"] \\ & X \arrow[u, shift left, dashed, "i"] \end{tikzcd}$$

Let E be a complex bundle over a smooth manifold X of dimension $\gg \dim X$. Assume E fiber homotopically trivial so that there exists a submanifold Y of degree 1 over X with trivial normal bundle in E. Then $\exists\ a \in 1 + \bar{K}(X)$ with

$$j_! 1 = i_! a \qquad i \text{ zero section}$$

where these elements are in $\tilde{K}(X^E)$. Then

$$j_* 1 = ch\, j_! 1 = ch\, i_! a = i_*(ch a \cdot Todd(-E))$$

so applying π_* we have

$$1 = ch a \cdot Todd(-E) \qquad \text{or}$$

$$\boxed{ch a = Todd\ E \quad \text{with } a \in 1 + \bar{K}(X)}$$

Thus one gets a map

$$\begin{tikzcd} K(X) \arrow[r, "Todd"] \arrow[d] & 1 + \overline{H^{ev}}(X, \mathbb{Q}) \arrow[d] \\ J(X) \arrow[r, "e"] & {[1 + \overline{H^{ev}}(X, \mathbb{Q})] / ch(1 + \bar{K}(X))} \end{tikzcd}$$

so in particular for a sphere $X = S^{2n}$ we find that

$$(Todd\ E)_n = (\text{Bernoulli type coefficient})\, c_n E + \text{decomposable}.$$

and so e takes its values

$$J(S^{2n}) \longrightarrow \mathbb{Z} \cdot \frac{1}{k_n} / \mathbb{Z}$$

$\therefore\ (-1)^{n-1} s_n = n a_n$ so

$$\sum_{n\geq 1} (-1)^{n-1} s_{n-1} z^n = z \sum_{n\geq 1} n a_n z^{n-1} = z \frac{d}{dz} \log T(z)$$

$$= z \frac{T'(z)}{T(z)}$$

<u>Application</u>: Suppose

$$T(x) = \frac{x}{1-e^{-x}} \qquad \log T(x) = \log x + \log(1-e^{-x})$$

$$\frac{T'(x)}{T(x)} = \frac{1}{x} + \frac{-e^{-x}}{1-e^{-x}}$$

$$\sum_{n\geq 1} (-1)^{n-1} s_n x^n = 1 - \frac{xe^{-x}}{1-e^{-x}}$$

$$= 1 - \frac{x}{e^x - 1}$$

$\therefore\ s_n =$ coefficient of x^n in $\dfrac{x}{e^x-1}$

or $\boxed{(-1)^n s_n = \text{coeff of } x^n \text{ in } \dfrac{x}{1-e^{-x}}}$ $\quad n \geq 1$

Thus coefficient of $c_n E$ in $(\text{Todd}\, E)_n$ is by above $\overset{\text{approx}}{=} \dfrac{B_n}{n!}$

But $ch_n E = \dfrac{(-1)^{n-1} c_n(E)}{(n-1)!}$ + decomposable.

$$\therefore\ \mathbb{Z}\frac{1}{k_n}\Big/\mathbb{Z} = \mathbb{Z} \cdot \frac{B_n}{n!} \cdot (n-1)! \Big/ \mathbb{Z}$$

so $\quad k_n =$ denominator of $\dfrac{B_n}{n}$ when expresses in lowest terms.
