June 3, 1969

New proof of the decomposition theorem.

1. Let $\mathcal{A}$ be a category with a internal product operation $\otimes$, supposed to be unitary, associative, and commutati
and let

$$h : \mathcal{A} \longrightarrow \mathrm{Mod}\, A$$

be a tensor functor, i.e. there is given a natural transf.

$$\boxtimes : \quad hX \otimes hY \longrightarrow h(X \otimes Y)$$

compatible with the unit, the associativity and the commutativity isomorphism in Mod A.

Let P be a quasi-bigebra over A which acts on h in a fashion compatible with products. This means that there is given a natural transformation

$$\Delta : hX \longrightarrow P \otimes hX$$

such that (i) hX is a P-comodule

(ii)
$$\begin{tikzcd}
{}_1hX \otimes_1 hY \arrow[r, "\Delta\otimes\Delta"] \arrow[d, "\boxtimes"] & ({}_1P_2 \otimes {}_2hX) \otimes ({}_1P_2 \otimes {}_2hY) \arrow[r, "id\otimes T\otimes id"] & ({}_1P_2 \otimes {}_1P_2) \otimes {}_2hX \otimes \arrow[d, "\mu_P \otimes i"] \\
h(X\otimes Y) \arrow[r, "\Delta"] & {}_1P_2 \otimes {}_2h(X\otimes Y) & {}_1P_2 \otimes {}_2h(X)\otimes{}_2 \arrow[l, "id\otimes\boxtimes"']
\end{tikzcd}$$

is commutative for all X, Y.

A better way of expressing this is to say that we are given a tensor functor $h' : \mathcal{A} \longrightarrow \mathrm{Com}\, P$ such that

$$\begin{tikzcd} & \mathrm{Com}\,P \arrow[d, "\text{forget}"] \\ \mathcal{A} \arrow[ur, "h'"] \arrow[r, "h"'] & \mathrm{Mod}\,A \end{tikzcd}$$

is commutative.

Given a ring homomorphism $u : A \longrightarrow R$ we get a ⊗-functor

$$h_u : \mathcal{A} \longrightarrow \mathrm{Mod}\, R$$
$$X \longmapsto R_{[u]} \otimes_A hX .$$

Also given $\theta : P \longrightarrow R$ a ring homomorphism with $\theta s = u$ and $\theta t = v$ we obtain a transf. of ⊗-func

$$\bar{\theta} : h_u \longrightarrow h_v$$

defined to be the composition

$$R_{[u]} \otimes_A h(X) \xrightarrow{id \otimes \Delta} R_{[u]} \otimes_A P \otimes_A hX \xrightarrow{(id,\theta)\otimes id} R_{[v]} \otimes_A hX .$$

Here (id, θ) is the ring homomorphism from $R_{[u]} \otimes_A P$ to $R_{[v]}$ given by id on R and θ on P. (In formulas $(id,\theta)(r \otimes p) = r\theta p$.
$ru(a) \cdot \theta p = r \cdot u(a)\theta p = r \cdot \theta(s(a)p)$)

Notice that if θ, as a morphism in the category $(A(R), P(R))$, is an isomorphism, then $\bar{\theta}$ is an isomorphism. In particular if (A,P) is an affine groupoid, then $\bar{\theta}$ is an isomorphism

2. Suppose that

$$(2.1) \qquad (A, P) \longrightarrow (A', P')$$

is a morphism of affine categories; ~~corresponding to~~ it gives rise to a functor

$$(2.2) \qquad (A(R), P(R)) \longleftarrow (A'(R), P'(R))$$

in the opposite direction for each ring R. Then there is a commutative square of $\otimes$-functors

$$\begin{array}{ccc} \mathrm{Com}(P) & \xrightarrow{A'\otimes_A ?} & \mathrm{Com}\, P' \\ \downarrow \scriptstyle{\mathrm{forget}} & & \downarrow \scriptstyle{\mathrm{forget}} \\ \mathrm{Mod}\, A & \xrightarrow{A'\otimes_A ?} & \mathrm{Mod}\, A' \end{array}$$

~~so~~ Consequently if $h : \mathcal{A} \longrightarrow \mathrm{Mod}\, A$ is a tensor functor on which P acts, then $A'\otimes_A h : \mathcal{A} \longrightarrow \mathrm{Mod}\, A'$ is a tensor functor on which P' acts.

We shall say that the morphism (2.1) is fully faithful (resp. an equivalence) if for each R the functor (2.2) is fully faithful (resp. an equivalence). Fully faithful means that

$$(2.3) \qquad P' \longleftarrow A' \otimes_A P \otimes_A A'$$

One sees easily that if P represents the functor $\underline{\mathrm{End}}^{\otimes} h$ then P' given by (2.3) represents the functor $\underline{\mathrm{End}}^{\otimes} h'$, where $h' = A' \otimes_A h$.

Lemma 2.4: Let $(f, f_1) : (A, P) \longrightarrow (A', P')$ be an equivalence. Then there exists a morphism in the other direction $(g, g_1) : (A', P') \longrightarrow (A, P)$ which is quasi-inverse to (f, f_1) in the sense that there are isomorphisms of the two composites with the identities.

Proof: We know that for any ring R, the functor

$$(f^*, f_1^*) : (A'(R), P'(R)) \longrightarrow (A(R), P(R))$$

is an equivalence of categories, and we must show that we can find a quasi-inverse which is independent of R. Take $R = A$; then $id_A \in A(A)$ is an object of $(A(A), P(A))$ hence is isomorphic to $f^*(g)$ where $g : A' \longrightarrow A$. The isomorphism is given by $\theta : P \to$ such that $\theta s = id_A$, $\theta t = gf$. ~~Now the composite~~

~~is isomorphic to~~ ~~for any $u \in A(R)$ $f^*(g^* u)$ is isomorphic via θ to u, hence as (f^*, f_1^*) is an equivalence there exists a morphism [illegible] in $P'(R)$ with target $g^* u$ which is carried by f_1^* into the given isomorphism of u to $f^* g^* u$.~~

Now g^* maps objects of $(A, P)(R)$ to the objects of $(A', P')(R)$ and after composition with f^* there is an isomorphism of $u \in A(R)$ with $f^* g^* u = ugf$ given by $P \xrightarrow{\theta} A \xrightarrow{u} R$. Thus we know g^* extends to a functor quasi-inverse to (f^*, f_1^*) in a canonical way which is functorial in R. Hence we obtain $g_1 : A' \to P$, etc.

Definition: Let (A,P) be an affine category and let $u' : A \to A'$ and $u'' : A \to A''$ be ring morphisms. Then to u' we have associated an affine category (A', P') and a fully faithful faithful functor $(A,P) \longrightarrow (A',P')$. Same for u''. We say that u' and u'' are equivalent if there exists an equivalence g

$$\begin{tikzcd} (A',P') \arrow[rr, "g"] & & (A'',P'') \\ & (A,P) \arrow[ul, "u'"] \arrow[ur, "u''"'] & \end{tikzcd}$$

such that $gu' \simeq u''$.

In terms of categories we have a diagram

$$\begin{tikzcd} \mathcal{C}' \arrow[dr, "F'"'] & & \mathcal{C}'' \arrow[dl, "F''"] \\ & \mathcal{C} & \end{tikzcd}$$

where F' and F'' are fully faithful. Then F' and F'' are equivalent if they have the same essential image.

In concrete terms it means that there exist maps

$$\begin{tikzcd} \mathcal{C}' \arrow[r, shift left, "G_1"] & \mathcal{C}'' \arrow[l, shift left, "G_2"] \end{tikzcd}$$

~~and isomorphisms of functors~~ $\cancel{F''G_1 \simeq F'}$ $\cancel{G_2G_1 \simeq id}$ $\cancel{F'G_2 \simeq F''}$ $\cancel{G_1G_2 \simeq id}$

together with isomorphisms

$$\theta' : F' \longrightarrow F''G_1$$
$$\theta'' ; F'' \longrightarrow F'G_2 .$$

From these one deduces an isomorphism

$$F' \longrightarrow F''G_1 \longrightarrow F'G_2G_1$$

hence as F' is fully faithful an isomorphism

$$id \longrightarrow G_2G_1 .$$

Similarly one gets an isomorphism

$$id \longrightarrow G_1G_2$$

and we checked these two isomorphisms are compatible.

Therefore $u' : A \longrightarrow A'$ and $u'' : A \longrightarrow A''$ are equivalent if $\exists$ diagram

$$\begin{tikzcd}
A' \arrow[r, "g_2"'] & A'' \arrow[r, "g_1"'] & A' \\
& A \arrow[ul, "u'"] \arrow[u, "u''"'] \arrow[ur, "u'"'] \arrow[from=1-1, to=2-2, phantom, "\Uparrow\theta'"] \arrow[from=1-3, to=2-2, phantom, "\Uparrow\theta''"]
\end{tikzcd}$$

where θ' and θ'' are isomorphisms. From such a ~~these~~ collecti
one get isomorphisms

$$A''_{[g_2]} \otimes_{A'} h_{u'} \cong A''_{[g_2u']} \otimes_A h \overset{\theta'}{\cong} h_{u''}$$

$$A'_{[g_1]} \otimes_{A''} h_{u''} \cong A'_{[g_1u'']} \otimes_A h \overset{\theta''}{\cong} h_{u'}$$

Examples: Recall that if $\theta : \Omega \longrightarrow Q$ is a multiplicative transformation, where Q is a Chern theory, such that

$$\theta(i_* 1) = \lambda \cdot i_* 1$$

with $\lambda \in Q(pt)^*$ where $i : pt \to \mathbb{P}^1$, then by RR there is a unique multiplicative transformation $\tilde{p} : K \longrightarrow Q^*$ such that $\theta = \hat{p} : \Omega \longrightarrow Q$. Here the notation is as follows: $\bar{p}$ denotes a power series in $(Q(pt)[[X]])^*$, $\tilde{p}$ is the unique multiplicative transf $K \longrightarrow Q^*$ $\ni$

$$\tilde{p}(L) = \bar{p}(c_1^Q(L))$$

~~...~~ $p(X) = X\bar{p}(X)$ and

$$\hat{p} : \Omega \longrightarrow Q$$

is the unique multiplicative transf. $\ni$

$$\hat{p}(f_* x) = f_*(\hat{p}x \cdot \tilde{p}(\nu_f)).$$

or equivalently by R-R the unique transf $\ni$

$$\hat{p}(c_1^\Omega(L)) = p(c_1^Q(L)).$$

Let A be the Lazard ring with universal law F_{univ} and let $A \longrightarrow \Omega(pt)$ be given by $F_{univ} \longmapsto F^\Omega$. Let (A,P) be the affine category associating to each ring its category of formal group laws:

$$\text{objects} = \text{formal group laws}/R \cong \mathrm{Hom}(A,R)$$

$$\mathrm{Hom}(F,F') = \left\{ \text{power series } p(X) = \sum_{n\geq 0} a_n X^{n+1},\ a_0 \in R^* \;\middle|\; p * F = F' \right\}$$

Composition is defined as follows: Given $u : F \longrightarrow F'$ and $v : F' \longrightarrow F''$, say $u = (F, p)$ $v = (F', q)$, then $v \circ u = (F, qp)$.

Make (A, P) act on Ω as follows. Given a formal group law F over R, let

$$\Omega_F = R {}_{[u]}\!\otimes_A \Omega$$

where $u : A \longrightarrow R$ sends $F_{univ} \longrightarrow F$. Given $p(x) \in R[[X]]^*$ let

$$\widehat{p^{-1}} : \Omega \longrightarrow R {}_{[v]}\!\otimes_A \Omega$$

be the transformation (multiplicative) such that

$$\widehat{p^{-1}}\, c_1(L) = p^{-1}(c_1(L)).$$

Then

$$\widehat{p^{-1}}\, F^{\Omega} = p^{-1} * F^{\Omega}$$

where $u : A \longrightarrow R$ sends F_{univ} to F. Note that

$$in_1 : R \xrightarrow{\ \sim\ } \Omega_F(pt),$$

hence given $g(x) \in R[[X]]^*$ there is a unique multiplicative transformation

$$\widehat{g^{\#}} : \Omega \longrightarrow \Omega_F$$

such that

$$\widehat{g^{\#}}\, f_* x = f_*\left(\widehat{g^{\#}} x \cdot \widetilde{g^{\#}}(\nu_f)\right)$$

where

$$\widetilde{g^{\#}} : K \longrightarrow \Omega_F^{\,*}$$

is the multiplicative characteristic class (genus) given by

$$\tilde{g}(L) = (in_1 \tilde{g})(in_2\, c_1^{\Omega} L)$$
$$\| \qquad\qquad$$
$$c_1^{\Omega_F}(L).$$

From now on we identify $\Omega_F(pt)$ with R via in_1.
Note that then

$$F^{(\Omega_F)} = F.$$

Note that

$$\hat{g}\, c_1^{\Omega}(L) = g(c_1^{\Omega_F} L)$$

hence

$$\hat{g} F^{\Omega} = g * F,$$

so if $v : A \longrightarrow R$ sends F_{univ} to $g*F$, then the diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "v"] & \Omega(pt) \arrow[d, "\hat{g}"] \\
R \arrow[r, "in_1"] & \Omega_F(pt)
\end{tikzcd}$$

is commutative, so in fact $\hat{g}$ induces a mult. transformation

$$\hat{g} : \Omega_{g*F} \longrightarrow \Omega_F$$

which is R linear and satisfies

$$\hat{g}\, c_1(L) = g(c_1(L)).$$

Therefore if (F, p) is a morphism in $(A(R), P(R))$ from F to $F' = p * F$, we have a multiplicative transformation

$$\widehat{p^{-1}} : \Omega_F \longrightarrow \Omega_{F'}$$

Given $p, q \in R[[X]]^*$.

$$\Omega_F \xrightarrow{\widehat{p^{-1}}} \Omega_{p*F} \xrightarrow{\widehat{q^{-1}}} \Omega_{q*p*F}.$$

$$\begin{aligned} \widehat{q^{-1}}\,\widehat{p^{-1}}\,(c_1 L) &= \widehat{q^{-1}}(p^{-1}(c_1(L))) \\ &= p^{-1}(q^{-1}(c_1 L)) \qquad \widehat{q^{-1}} \text{ is } R\text{-linear} \\ &= \widehat{(qp)^{-1}}\, c_1(L). \end{aligned}$$

Thus we have defined an action of (A, P) on Ω.

Example 1: Suppose $h = \Omega \otimes \mathbb{Z}_{(p)}$ and (A, P) is the ~~[illegible]~~ affine category associating to each $\mathbb{Z}_{(p)}$ algebra its category of formal group laws. Let (A', P') be the full subcategory ~~associating~~ to a $\mathbb{Z}_{(p)}$-algebra its full subcategory of typical group laws, and let $f : (A, P) \longrightarrow (A', P')$ be the inclusion functor. Then this functor is an equivalence of categories; indeed Cartier's recoordinatization defines a map

$$A' \xrightarrow{g} A$$

such that $gf = id_{A'}$ and such that fg is P-isomorphic to id_A. Hence if we define

$$\Omega' = A'_{[f]} \otimes_A \Omega_{(p)}$$

then there is an isomorphism

$$A \underset{A'}{\otimes} \Omega' \xrightarrow{\sim} \Omega_{(p)}.$$

This isomorphism is the composition

$$A\otimes_{A'}\Omega'_{\varphi} = A\otimes_{A'}(A'\otimes_A \Omega_{(p)}) \cong A\otimes_{\mathbb{Z}_{(p)}\otimes A}\Omega_{(p)} \cong A\otimes_A \Omega = \Omega_{(p)}$$

Example 2: Suppose $h = \Omega\otimes\mathbb{Q}$, $(A(R),P(R))$ = formal group laws over ~~R~~ R (an alg. over $\mathbb{Q}$), $(A'(R),P'(R))$ full subcategory consisting of the law $X+Y$. Then

$$\Omega' = A'\otimes_A \Omega_{\mathbb{Q}}$$

is a universal Chern theory with values in $\mathbb{Q}$ algebras with law $X+Y$. By R-R this theory is $X \mapsto H^*(X,\mathbb{Q})$. Thus

$$\Omega_{\mathbb{Q}} \cong A\otimes_{\mathbb{Q}} H^*(X,\mathbb{Q}).$$

(~~more detail required here. K theory relation with Conner-Floyd thm.~~)

Example 3: $h = \mathfrak{N}$ unoriented cobordism
$(A,P)(R)$ = category formal laws of height ∞ over the $\mathbb{F}_2$-alg. R
$(A',P')(R)$ = full subcategory consisting of the law $X+Y$. Then

$$\mathfrak{N}(X) \cong A^*\otimes_{\mathbb{F}_2} \mathfrak{N}'(X)$$

By formal group law theory is a polynomial ring over $\mathbb{F}_2$ with generators in dimensions $\neq 2^i-1$. Thus using Thom's theorem $\mathfrak{N}'(pt) = \mathbb{Z}_2$ so by uniqueness theorem in generalized cohomology theory $\mathfrak{N}'(X) = H^*(X, \mathbb{Z}/2\mathbb{Z})$.

(more detail. One sees that $F^{\mathfrak{N}}$ has a unique log of form

$$\ell(X) = \sum a_n X^{n+1} \text{ with } \begin{cases} a_{2^i-1} = 0 & i>0 \\ a_0 = 1 \end{cases}$$

and that $\quad \eta(pt) \simeq \mathbb{F}_2[a_i]$.)

We now wish to consider the case of laws which are locally isomorphic for the fpqc topology. ~~Thus in the situation on bottom of page 6, we are given group laws F' over A' and F'' on A'' and morphisms $A' \xrightarrow{g_2} A \xrightarrow{g_1} A'$ together with isomorphism $g_2(F') \simeq F$, $g_1(F) \simeq F'$. Now we wish to understand what happens in the case that~~

Recall the situation in the bottom of page 6 where we said that $u' : A \longrightarrow A'$ and $u'' : A \longrightarrow A''$ are equivalent if $\exists$

$$\begin{tikzcd} A' \arrow[r, "g_2"] & A'' \arrow[r, "g_1"] & A' \\ & A \arrow[ul, "u'"] \arrow[u, "u''"] \arrow[ur, "u'"'] & \end{tikzcd}$$

and in this case

$$\begin{cases} A''_{g_2} \otimes_{A'} h_{A'} \simeq h_{A''} \\ A'_{g_1} \otimes_{A''} h_{A''} \simeq h_{A'} \end{cases}$$

Now suppose that u', u'' are not equivalent but instead that after a faithfully flat map $A' \xrightarrow{\varphi} B'$ the map $A \xrightarrow{u'} A' \xrightarrow{\varphi} B$ is equivalent to $A \xrightarrow{u''} A''$. Then what we have is an exact sequence

$$h_{A'} \longrightarrow h_{B'} \rightrightarrows h_{B' \otimes_A B'}$$

and that $\quad \eta(pt) \simeq \mathbb{F}_2[a_i].)$

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& A \arrow[u, "u''"] \arrow[ur, "u'"'] \arrow[ul, phantom, "\Uparrow"] \arrow[ur, phantom, "\Uparrow"] &
\end{tikzcd}$$

and in this case

$$\begin{cases} A''_{g_2} \otimes_{A'} h_{A'} \simeq h_{A''} \\ A'_{g_1} \otimes_{A''} h_{A''} \simeq h_{A'} \end{cases}$$

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$$h_{A'} \longrightarrow h_{B'} \rightrightarrows h_{B' \otimes_A B'}$$

and ~~as~~ isomorphisms

$$h_{B'} \cong B' \otimes_{A''} h_{A''}$$

$$h_{A''} \cong A'' \otimes_{B'} h_{B'}$$

I now wish to apply these considerations to K-theory ~~[illegible]~~ and Ω. First the formal group picture:

June 5, 1969 – June 7.

Characteristic numbers { operations in K^* (summary p. Stong-Hatt. thm. proof page 19; $H_.(BU)$ page 9; completeness of Wu relations p21.

Let Q be a ~~[illegible]~~ generalized cohomology theory with products and with ~~[illegible]~~ Thom isomorphism for ~~[illegible]~~ complex vector bundles. Then there is a ring homomorphism (called normal characteristic numbers with values in Q)

$$\Omega_.(pt) \longrightarrow Q_.(MU) \simeq Q_.(BU)$$

which I would like to describe using formal group laws.

Adams description of $Q_.(BU)$: $\quad Q_.(BU(1)) \simeq \bigoplus_{i=0}^{\infty} Q_.(pt)\, b_i$, where $\{b_i\}$ is dual base to $\{c_1^i\}$. Then $Q_.(BU(1)) \longrightarrow Q_.(BU)$ carries the b_i into generators for $Q_.(BU)$ where $b_0 = 1$. Thus

$$Q_.(BU) = Q_.(pt)[b_1, b_2, \cdots] \qquad \deg b_i = 2i$$

My description:

$$\operatorname{Hom}_{Q(pt)\text{-alg}}(Q_.(BU), R) \cong \{\text{Mult. char classes } \bar{K} \longrightarrow Q^{\cdot} \otimes_{Q(pt)} R\}$$

$$\cong \{\text{Power series } \sum_{n\geq 0} a_n X^n,\ a_n \in R,\ a_0 = 1\}$$

$$\cong \operatorname{Hom}_{Q(pt)\text{-alg}}(Q(pt)[b_1, \cdots], R)$$

Classical description of $\Omega_.(pt) \xrightarrow{\Phi} Q_.(BU)$:

Given $f: M^n \longrightarrow pt$ compact almost complex manifold, let

$$s_\alpha(M^n) = f_* c_\alpha(\nu_f)$$

be the α-th characteristic number of M^n. Then

$$\Phi[M^n] = \sum b_*^\alpha s_\alpha(M^n)$$

<u>My description</u>: Let $c_{\underline{b}} : \overline{K}(\cdot) \longrightarrow Q(\cdot)[b_1, b_2, \ldots$ be the universal characteristic class given on line bundles by

$$c_{\underline{b}}(L) = \sum_{i=0}^{\infty} b_i\, c_1(L)^i \qquad b_0 = 1.$$

Then there is a unique multiplicative homomorphism

$$s_{\underline{b}} = \widehat{c_{\underline{b}}} : \Omega(\cdot) \longrightarrow Q(\cdot)[\underline{b}]$$

given by

$$\hat{c}_b(f_* x) = f_*(x \cdot c_{\underline{b}}(\nu_f))$$

and

$$s_{\underline{b}} : \Omega(pt) \longrightarrow Q(pt)[\underline{b}]$$

is the characteristic numbers map Φ.

One knows in general that if Q' is a Chern theory, ~~and~~ if $\varphi : \overline{K} \longrightarrow (Q')^*$ is a mult. char. class, and if $\hat{\varphi} : \Omega \longrightarrow Q'$ is the induced operation, then

$$\hat{\varphi}\, c_1^{\Omega}(L) = c_1^{Q'}(L) \cdot \overline{\varphi}(L) = p(c_1^{Q'} L)$$

where $p(X) = \sum_{n \geq 0} a_n X^{n+1} \in Q'(pt)[[X]] \ni \dfrac{p(c_1 L)}{c_1 L} = \varphi(L)$. Hence

$$\hat{\varphi}(F^{\Omega}) = p_* F^{Q'}$$

and therefore for

$$s_{\underline{b}} = \hat{c}_b : \Omega(pt) \longrightarrow Q(pt)[\underline{b}] = Q_\cdot(BU)$$

we have

$$\boxed{\begin{array}{l} s_{\underline{b}}(F^\Omega) = p_{\underline{b}} * F^Q \\ \text{where} \quad p_{\underline{b}}(X) = \sum_{i\geq 0} b_i X^{i+1} \qquad b_0 = 1 \end{array}}$$

Examples. 1.

$$\pi_*(MU) \longrightarrow H_*(MU) \simeq H_*(BU) \cong \mathbb{Z}[\underline{b}]$$

$$F^\Omega \longmapsto \Big(\sum_{n\geq 0} b_n X^{n+1}\Big) * (X+Y)$$

2.

$$\pi_*(MU) \longrightarrow K_\cdot(BU) \cong \mathbb{Z}[\beta^{-1}, \beta, b_1, b_2, \ldots\ldots]$$

$$F^\Omega \longmapsto \Big(\sum_{n\geq 0} b_n X^{n+1}\Big) * (X+Y-\beta XY)$$

3.

$$\pi_\cdot(MO) \longrightarrow H_\cdot(BO, \mathbb{F}_2) \cong \mathbb{F}_2[b_1, b_2, \ldots\ldots]$$

$$F^\eta \longmapsto \Big(\sum_{n\geq 0} b_n X^{n+1}\Big) * (X+Y)$$

$K^{\bullet}$ theory – operations.

Let $K^{\bullet}$ be the periodic generalized cohomology theory of Atiyah–Hirzebruch. Let us review how this is defined. If X is a compact space (resp. pointed space), let

$$K(X) = \text{groth group of cx. v.b. over } X$$

$$(\text{resp. } \tilde{K}^{*}(X) = \operatorname{Ker}\{K(X) \longrightarrow K(\text{basepoint})\}.)$$

By the periodicity theorem of Bott there is a canonical element $\beta \in \tilde{K}(S^2)$ such that

$$\beta\cdot : K(X) \longrightarrow \tilde{K}(S^2 \wedge X)$$

is an isomorphism. This said we define for a pointed space X

$$K^{2n}(X) = \tilde{K}(X)$$

$$K^{2n+1}(X) = \tilde{K}(S^1 \wedge X)$$

and we define the suspension isomorphism

$\cancel{K^{2n}(X) \longrightarrow K^{2n+1}(S^1 \wedge X)}$ ~~to be identity~~

$$\begin{array}{ccc} K^{2n+1}(X) & \longrightarrow & K^{2n+2}(S^1 \wedge X) \\ \| \text{ defn} & & \| \text{ defn} \\ \tilde{K}(S^1 \wedge X) & \xrightarrow{\text{id}} & \tilde{K}(S^1 \wedge X) \end{array}$$

$$\begin{tikzcd}
K^{2n}(X) \arrow[r] \arrow[d, equal] & K^{2n+1}(S^1\wedge X) \arrow[d, equal] \\
\tilde{K}(X) \arrow[r] \arrow[dr, "\cong", "\beta\cdot"'] & \tilde{K}(S^1\wedge S^1\wedge X) \arrow[d, "\cong"] \\
 & \tilde{K}(S^2\wedge X)
\end{tikzcd}$$

The associated spectrum BU. : ~~For a pointed connected~~ ~~space~~ For a compact space X, let $\bar{K}(X)$ be $\mathrm{Ker}[K(X) \to H^0(X,\mathbb{Z})]$. Then there is a canonical isomorphism

$$[X, BU] \cong \bar{K}(X),$$

hence if X is a pointed connected ~~space~~ finite complex, there is an isomorphism

$$\tilde{K}(X) \cong \bar{K}(X) = [X, BU]_\bullet .$$ ← basepoint preserving

Therefore for a pointed finite complex

$$K^{2n+1}(X) = [S^1\wedge X, BU]_\circ = [X, U]_\circ$$

$$K^{2n}(X) = [X, \mathbb{Z}\times BU]_\circ .$$

From the suspension isom we have

$$\begin{cases} [X, U]_\circ \xrightarrow{\sim} [S^1\wedge X, \mathbb{Z}\times BU]_\circ \\ [X, \mathbb{Z}\times BU]_\circ \xrightarrow{\sim} [S^1\wedge X, U]_\circ \end{cases}$$

Therefore we have homotopy equivalences

$$\begin{cases} U \xrightarrow{\simeq} \Omega(\mathbb{Z}\times BU) \\ \mathbb{Z}\times BU \xrightarrow{\simeq} \Omega U \end{cases}$$

and so we have an Ω-spectrum $\underset{\sim}{BU}$ given by

$$\begin{cases} BU_{2n} = \mathbb{Z}\times BU \\ BU_{2n+1} = U. \end{cases}$$

Now I wish to check the Atiyah trick condition, i.e. that K^0 as a functor on the suspension category $\mathcal{a}$ is ind-represent by Künneth complexes. So given X in $\mathcal{a}$ and $u\in K^0(X)$ we have $X=\Sigma^{-2n}Y$ with Y a finite pointed complex for some $n\geq 0$ and

$$K^0(X) \cong K^{2n}(Y) = [Y, \mathbb{Z}\times BU]_0.$$

Now $BU = \varinjlim \text{Grass}_{mn}$. $K^{\cdot}(\text{Grass}_{mn})$ is finitely generated free $K^{\cdot}(pt)=\mathbb{Z}[\beta,\beta^{-1}]$ module. Clearly u comes from a map $Y\longrightarrow [-r,r]\times G_{mm} = E$, for some r, m, where E is a Künneth complex. Thus u comes from a map $X=\Sigma^{-2n}Y \longrightarrow \Sigma^{-2n}E$ in $\mathcal{a}$ where $\Sigma^{-2n}E$ is of Künneth type.

Since this condition holds I know that the functor $R\longmapsto (\underset{\sim}{\text{End}}^{\otimes} K^{\cdot})(R)$ is represented by the flat affine groupoid

$$K_{\cdot}(\underset{\sim}{BU}) = \varinjlim_{n} K_{\cdot+2n}(\mathbb{Z}\times BU)$$

over $K_{\cdot}(pt)$.

To calculate the functor $\underline{\text{End}}^{\otimes} K^{\cdot}$ suppose given a graded $K^{\cdot}(pt)$ algebra R and a multiplicative stable transf

$$K^{\cdot} \xrightarrow{\gamma} R\otimes_{K^{\cdot}(pt)} K^{\cdot}$$

composing γ with the ~~the~~ Todd map

$$\Omega^{\cdot} \xrightarrow{\Phi} K^{\cdot}$$

we obtain a stable multiplicative transf.

$$\Omega^{\cdot} \xrightarrow{\gamma\Phi} R \otimes_{K^{\cdot}(pt)} K^{\cdot}$$

which, as we know, is given by a power series

$$\varphi(X) = \sum_{n\geq 0} a_n X^{n+1}, \quad a_0 = 1 \qquad a_n \in \left(R \otimes_{K^{\cdot}(pt)} K^{\cdot}(pt)\right)^{-2n} \cong R^{-2n}.$$

However φ cannot be arbitrary; one knows that

$$(\gamma\Phi)(c_1^{\Omega}(L)) = \varphi(c_1^{K}L)$$

and hence that

$$(\gamma\Phi)(F^{\Omega}) = \varphi * F^{K^{\cdot}}$$

$$\parallel$$

$$\gamma(F^{K^{\cdot}})$$

Now $F^{K^{\cdot}}(X,Y) = X+Y-\beta XY$, therefore

$$\varphi * F^{K} = X+Y-\gamma(\beta)XY.$$

Conversely by the Conner-Floyd theorem

$$\Omega^{\cdot}(X) \otimes_{\Omega^{\cdot}(pt)} K^{\cdot}(pt) \xrightarrow{\sim} K^{\cdot}(X)$$

we know that if $\varphi * F^{K} = X+Y-\tau XY$ for some $\tau \in R^{-2} \cap R^{*}$, then $\hat{\varphi}$ induces a transf from $K^{\cdot}$ to $K^{\cdot}_R$. So we conclude

Proposition: $\mathrm{Hom}_{\text{rings}}(K_\bullet(\underset{\sim}{BU}), R) \cong \{(\sigma, \tau, \varphi) \mid$

$\sigma, \tau \in R^* \cap R^{-2}$, $\varphi(X) = \sum_{n \geq 0} a_n X^{n+1}$, $a_n \in R^{-2n}$ $a_0 = 1$ and

$$X + Y - \tau XY = \varphi * (X + Y - \sigma XY).$$

interchange τ and σ. Should $\tau = t(\beta)$ $\sigma = s(\beta)$.

Using this we shall calculate the structure of $K_\bullet(\underset{\sim}{BU})$. First observe that above implies

$$\varphi(X) + \varphi(Y) - \tau\varphi(X)\varphi(Y) = \varphi(X + Y - \sigma XY)$$

$$X + a_1X^2 + Y + a_1Y^2 - \tau XY = X + Y - \sigma XY + a_1(X+Y)^2 \quad \text{mod deg}$$

$$\therefore \quad -\tau = -\sigma + 2a_1$$

or

$$\boxed{\tau = \sigma - 2a_1}$$

$$\cancel{(-\tau^{-1}X) * (X+Y+XY)}$$

$$(-\tau^{-1}X) * (X+Y+XY) = \varphi * ((-\sigma^{-1}X) * (X+Y+XY))$$

Thus

$$(-\tau^{-1}X) \circ \varphi \circ (-\sigma^{-1}X)$$

is an automorphism of the law $X+Y+XY$. But one knows that the latter are of the form

$$\sum_{n \geq 1} b_n X^n \qquad b_1 \in R^*$$

where there is a map $\bigoplus_{n \geq 0} \mathbb{Z}\binom{T}{n} \longrightarrow R$ such that $\binom{T}{n} \to b_n$

Therefore

$$\boxed{K_\bullet(\underset{\sim}{BU}) \cong \mathbb{Z}[T,T^{-1}] \otimes_{\mathbb{Z}[T]} \bigoplus_{n\geq 0} \mathbb{Z}\binom{T}{n} \otimes_{\mathbb{Z}} \mathbb{Z}[\sigma,\sigma^{-1}]}$$

where s, t from $K_\bullet(pt) \cong \mathbb{Z}[\beta,\beta^{-1}]$ to $K_\bullet(\underset{\sim}{BU})$ are given by

$$t_*(\beta) = \sigma$$
$$s_*(\beta) = T\sigma$$

~~(I think s, t are interchanged)~~

I can use the same method to calculate $H_\bullet(\underset{\sim}{BU})$. Thus a ring homomorphism from $H_\bullet(\underset{\sim}{BU})$ to R is the same thing as a multiplicative stable operation $K^\bullet \longrightarrow H^\bullet \otimes_{\mathbb{Z}} R$, that is, a unit σ in R^{-2} and a power series $\varphi(X) = \sum_{n\geq 0} a_n X^{n+1}$, $a_0 = 1$, $a_n \in R^{-2n}$ such that

$$X + Y - \sigma XY = \varphi * (X+Y)$$

or $$(-\sigma^{-1}X) * (X+Y+XY) = \varphi * (X+Y)$$

or $$X+Y+XY = \left((-\sigma X)\circ\varphi\right) * (X+Y)$$

In other words $1 + (-\sigma X)\circ\varphi$ is an exponential function. One knows these are of the form

$$\sum_{n\geq 0} b_n X^n \qquad b_n \in R^{-n}$$

where there is a ring hom

$$\bigoplus_{n\geq 0} \mathbb{Z}\frac{T^n}{n!} \longrightarrow R$$

sending $\frac{T^n}{n!} \longrightarrow b_n$. Therefore as $b_1 = -\sigma$

$$H_\cdot(\underset{\sim}{BU}) \cong \bigoplus_{n\geq 0} \mathbb{Z}\frac{\sigma^n}{n!} \otimes_{\mathbb{Z}[\sigma]} \mathbb{Z}[\sigma,\sigma^{-1}] \cong \mathbb{Q}[\sigma,\sigma^{-1}].$$

This agrees well with what Kan told me!

The calculation of $K_\cdot(\underset{\sim}{BU})$ may be simplified as follows. Let R be a graded $K_\cdot(pt)$ algebra where $\beta \mapsto \sigma \in R_2 \cap (R)^*$. A multiplicative stable operation

$$\gamma : K^\cdot \longrightarrow R \otimes_{K^\cdot(pt)} K^\cdot$$

is determined by its restriction to K^0; the latter is the additive extension of the restriction to line bundles

$$\gamma(L) = \sum_{n\geq 0} b_n (L-1)^n$$

where $b_n \in R_0$. As $\gamma(LL') = \gamma(L)\cdot\gamma(L')$, one knows that intuitively

$$b_n = \binom{b_1}{n} \qquad \text{ie.}$$

$$\gamma(L) = L^{b_1}$$

and precisely that there is a ring homomorphism

$$\bigoplus_{n\geq 0} \mathbb{Z}\binom{T}{n} \longrightarrow R$$

$$\binom{T}{n} \longmapsto b_n$$

We now calculate $\gamma(\beta)$. Recall stability of $\gamma \Rightarrow$ commutativity of

$$\begin{array}{ccccccc}
\beta & \leftarrow & K^{-2}(pt) & \xrightarrow{\gamma} & \left(R \otimes_{K(pt)} K^\cdot(pt)\right)^{-2} & \rightarrow \sigma \otimes 1 = & \sigma\tau^{-1} \otimes \beta \\
\downarrow & & \wr\downarrow & & S\downarrow & & \downarrow \\
1-H^{-1} & & K^0(S^2) & \xrightarrow{\gamma} & R \otimes_{K^\cdot(pt)} K^\cdot(S^2) & & \sigma\tau^{-1} \otimes (1-H^{-1})
\end{array}$$

~~where~~ where H is the Hopf bundle $\mathcal{O}(1)$ on $S^2 \simeq CP^1$. Now

$$\gamma(1-H^{-1}) = 1 - H^{-b_1} = b_1(1-H^{-1}) \qquad \text{since } (1-H^{-1})^2 = 0$$

Thus $\sigma\tau^{-1} = b_1$ ie $\sigma = \gamma(\beta) = b_1 \tau$.

So we conclude that any stable operation

$$\gamma : K^\bullet \longrightarrow R \underset{t\ K^\bullet(pt)}{\otimes} K^\bullet \qquad K^\bullet(pt) \xrightarrow{t} R, \quad \beta \longmapsto \tau$$

is of the form

$$\begin{cases} \gamma(\beta) = t\tau \\ \gamma(L) = L^t \overset{i.e.}{=} \sum_{n\geq 0} \binom{t}{n}(L-1)^n \end{cases}$$

Thus

$$K_\bullet(\underset{\sim}{BU}) \cong \underbrace{\left[\left(\bigoplus_{n\geq 0} \mathbb{Z}\binom{T}{n}\right) \otimes_{\mathbb{Z}[T]} \mathbb{Z}[T,T^{-1}]\right]}_{\Gamma} \otimes_{\mathbb{Z}} K_\bullet(pt)$$

where $\operatorname{Spec}\Gamma = \underset{\sim}{\mathrm{Aut}}\ \widehat{\mathbb{G}}_m$. Moreover the canonical homomorphism

$$K^\bullet(X) \xrightarrow{\Delta} K_\bullet(\underset{\sim}{BU}) \otimes_{K^\bullet(pt)} K^\bullet(X) \cong \Gamma \otimes_{\mathbb{Z}} K^\bullet(X)$$

is the unique additive extension given as line bundles by

$$\Delta(L) = \text{"}L^T\text{" i.e. } \sum_{n\geq 0} \binom{T}{n}(L-1)^n$$

Stong – Hattori theorem as formulated by Adams says that

$$0 \longrightarrow \pi_\cdot(MU) \longrightarrow K_\cdot(MU) \underset{\Delta}{\overset{id\otimes 1}{\rightrightarrows}} K_\cdot(MU) \otimes K_\cdot(\underset{\sim}{BU})$$

is exact. I want to make these maps explicit. Recall that

$$\operatorname{Hom}_{K_\cdot(pt)}\left(K_\cdot(MU), R\right) = \operatorname{Hom}\left(\Omega^\cdot, R \otimes_{K(pt)} K^\cdot\right)$$

$$\cong \left\{ \sum a_n X^{n+1} \;\middle|\; a_0 = 1,\ a_n \in R_n \right\}.$$

i.e. $\exists$ canon. map

$$\gamma : \Omega^\cdot(X) \longrightarrow K_\cdot(MU) \otimes_{K(pt)} K^\cdot(X)$$

~~there ∃ canonical map~~ given by

$$\gamma\left(c_1^\Omega(L)\right) = \sum_{n\geq 0} b_n\, c_1^K(L)^{n+1} \qquad \gamma(F^\Omega) = \left(\sum b_n X^{n+1}\right)_* F$$

where $K_\cdot(MU) = \mathbb{Z}[b_1, b_2, \ldots\ldots] \otimes_{\mathbb{Z}} K^\cdot(pt)$.

Also $\exists$ canonical map

$$\gamma_1 : K^\cdot(X) \longrightarrow K_\cdot(\underset{\sim}{BU}) \otimes_{K(pt)} K^\cdot(X)$$

given by

$$\gamma_1(L) = L^T \qquad \gamma_1(\beta) = T\beta$$

where

$$K_\cdot(\underset{\sim}{BU}) = \bigoplus_{n\geq 0} \mathbb{Z}\binom{T}{n}\,[T^{-1}] \otimes_{\mathbb{Z}} K^\cdot(pt)$$

Therefore ~~we obtain~~ combining γ and γ_1^* we obtain

$$\cancel{\Omega^*(X) \xrightarrow{\gamma} K_*(MU) \otimes_{K^*(pt)} K^*(X) \xrightarrow{id \otimes \gamma_1} K_*(MU) \otimes_{K^*(pt)} K_*(BU) \otimes_{K^*(pt)} K^*}$$

an operation

$$\Omega^*(X) \xrightarrow{\gamma} K_*(MU) \otimes_{K^*(pt)} K^*(X) \xrightarrow{id \otimes \gamma_1} K_*(MU) \otimes_{K^*(pt)} K^*(\underset{\sim}{BU}) \otimes_{K^*(pt)} K^*(X$$

hence by the universal property of γ there exists a unique ~~a map~~ ring homomorphism

$$K_*(MU) \xrightarrow{\mu} K_*(MU) \otimes_{K^*(pt)} K_*(\underset{\sim}{BU})$$

such that the two compositions below are equal

$$\Omega^*(X) \xrightarrow{\gamma} K_*(MU) \otimes_{K^*(pt)} K^*(X) \overset{\mu \otimes id}{\underset{id \otimes \gamma_1}{\rightrightarrows}} K_*(MU) \otimes_{K^*(pt)} K^*(\underset{\sim}{BU}) \otimes_{K^*(pt)} K^*($$

I now wish to calculate μ. Now

$$K_*(MU) = \mathbb{Z}[b_1, b_2, \cdots] \otimes \cancel{K(pt)}\, \mathbb{Z}[\beta, \beta^{-1}]$$

$$K_*(\underset{\sim}{BU}) = \bigoplus_{n \geq 0} \mathbb{Z}\binom{T}{n}[T^{-1}] \otimes \mathbb{Z}[\beta, \beta^{-1}]$$

as a left module β acts as $T\beta$

Thus

$$K_*(MU) \otimes_{K^*(pt)} K_*(\underset{\sim}{BU}) = \mathbb{Z}[b_1, \cdots] \otimes \bigoplus_{n \geq 0} \mathbb{Z}\binom{T}{n}[T^{-1}] \otimes \mathbb{Z}[\beta, \beta^{-1}]$$

Moreover

$$(id \otimes \gamma_1)\, \gamma\, c_1^{\Omega} L = (id \otimes \gamma_1) \sum b_n \left(c_1^{K^*} L\right)^{n+1}$$

$$= \sum b_n \left(\gamma_1 c_1^{K^*} L\right)^{n+1}$$

Now

$$\gamma_1 c_1^{K^*} L = \gamma_1\left(\beta^{-1}(1 - L^{-1})\right) = (T\beta)^{-1}(1 - L^{-T})$$

$$\cancel{T^{-1} \beta^{-1} \frac{T(1-L^{-1})}{} c_1^{K^*} L}$$

~~Thus~~ $\cancel{(id \otimes \gamma_1)\gamma(c_1^{K^*} L) = \sum b_n \left\{T^{-1} \beta^{-1} (L^{-1})\right\}^{n+1} (c_1^{K^*} L)^{n+1}}$

1

$$= -\beta^{-1}T^{-1}\left(L^{-T}-1\right) = -\beta^{-1}T^{-1}\sum_{n\geq 1}{}' \binom{T}{n}\left(L^{-1}-1\right)^{n}$$

$$= T^{-1}\sum_{n\geq 1}\binom{T}{n}(-\beta)^{n-1}\left(\frac{1-L^{-1}}{\beta}\right)^{n}$$

$$\boxed{\gamma_1 c_1^{K^\cdot} L = T^{-1}\sum_{n\geq 1}\binom{T}{n}(-\beta)^{n-1}\left(c_1^{K^\cdot}L\right)}$$

so

$$\left(id\otimes\gamma_1\right)\left(\gamma c_1^{\Omega} L\right) = \left(\sum_{n\geq 0}{}' b_n X^{n+1} \circ T^{-1}\sum_{n\geq 1}\binom{T}{n}(-\beta)^{n-1}X^{n}\right)\left(c_1^{K^\cdot}L\right).$$

Therefore

$$\boxed{\sum_{n\geq 0}(\mu\, b_n)X^{n+1} = \left(\sum_{n\geq 0}{}' b_n X^{n+1}\right)\circ\left(T^{-1}\sum_{n\geq 1}\binom{T}{n}(-\beta)^{n-1}X^{n}\right)}$$

These formulas simplify with the following change of notation. Then write

$$K_\cdot(MU) \simeq \mathbb{Z}\left[a_0^{-1}, a_0, a_1, \ldots\right] \qquad t: K^\cdot(pt)\longrightarrow K_\cdot(M\ldots$$

$$\beta \longmapsto a_0^{-1}$$

with

$$\gamma: \Omega^\cdot(X)\longrightarrow K_\cdot(MU)\otimes_{K^\cdot(pt)} K^\cdot(X) \qquad \text{given by}$$

$$\gamma\left(c_1^{\Omega}(L)\right) = \sum_{n\geq 0} a_n\left(1-L^{-1}\right)^{n+1}$$

Thus

$$a_n = b_n\beta^{-n-1} \in K^{\cdot}_{-2}(MU) \quad \text{and}$$

$$\boxed{\sum_{n\geq 0}(\mu a_n)X^{n+1} = \left(\sum_{n\geq 0}a_n X^{n+1}\right)\circ\left(\sum_{n\geq 1}\binom{T}{n}(-1)^{n-1}X^{n}\right)}$$

Here is a summary of our calculations:

Proposition: 1) There is an isomorphism

$$K_\bullet(MU) \simeq \mathbb{Z}[a_0^{-1}, a_0, a_1, \dots] \qquad a_i \in K_{-2}(MU)$$

such that the canonical maps

$$K^\bullet(pt) \xrightarrow{t} K_\bullet(MU) \xleftarrow{s} \Omega^\bullet(pt)$$

$$\gamma : \Omega^\bullet(X) \longrightarrow K_\bullet(MU) \otimes_{K^\bullet(pt)} K^\bullet(X)$$

are given by

$$t(\beta) = a_0^{-1}$$

$$s(F^\Omega) = \Big(\sum_{n\geq 0} a_n X^{n+1}\Big) * (X+Y-XY)$$

$$\gamma\big(c_1^\Omega(L)\big) = \sum_{n\geq 0} a_n (1-L^{-1})^{n+1}$$

2) There is an isomorphism

$$K_\bullet(\underset{\sim}{BU}) \simeq \bigoplus_{n\geq 0} \mathbb{Z}\binom{T}{n} \otimes_{\mathbb{Z}[T]} \mathbb{Z}[T, T^{-1}] \otimes_{\mathbb{Z}} \mathbb{Z}[\tau, \tau^{-1}]$$

$$T \in K_0(\underset{\sim}{BU}) \qquad \tau \in K_2(\underset{\sim}{BU})$$

such that the canonical maps

$$K^\bullet(pt) \xrightarrow{t} K_\bullet(\underset{\sim}{BU}) \xleftarrow{s} K^\bullet(pt)$$

$$\gamma_1 : K^\bullet(X) \longrightarrow K_\bullet(\underset{\sim}{BU}) \otimes_{K^\bullet(pt)} K(X)$$

are given by

$$t(\beta) = \tau$$
$$s(\beta) = T\tau$$
$$\gamma_1(L) = L^T$$

3) The unique ring homomorphism

$$\mu : K_\cdot(MU) \longrightarrow K_\cdot(MU) \otimes_{K^\cdot(pt)} K_\cdot(\underset{\sim}{BU})$$

such that the following two compositions are equal

$$(\star) \quad \Omega^\cdot(X) \xrightarrow{\gamma} K_\cdot(MU) \otimes_{K^\cdot(pt)} K^\cdot(X) \underset{id \otimes \gamma_1}{\overset{\mu \otimes id}{\rightrightarrows}} K_\cdot(MU) \otimes_{K^\cdot(pt)} K_\cdot(BU) \otimes_{K^\cdot(pt)} K^\cdot$$

is given by

$$\sum_{n \geq 0} (\mu a_n) X^{n+1} = \left(\sum_{n \geq 0} a_n X^{n+1} \right) \circ \left(\sum_{n \geq 1} \binom{T}{n} (-1)^{n-1} X^n \right)$$

4) The Stong-Hattori theorem implies that for a torsion-free finite complex X, the sequence $(\star)$ is exact. Equivalently ~~given~~ an algebraic function on invertible power series

$$P\left(\sum a_n X^{n+1} \right) = P(a_0, a_1, \cdots) \in \mathbb{Z}[a_0^{-1}, a_0, a_1, \cdots]$$

can be expressed in the form

$$P\left(\sum a_n X^{n+1} \right) = Q\left(\left(\sum a_n X^{n+1} \right) * (X + Y - XY) \right)$$

where Q is an algebraic function on formal group laws (i.e. $Q \in \Omega^\cdot(pt)$) if and only if

$$P\left(\sum a_n X^{n+1}\right) = P\left\{\left(\sum a_n X^{n+1}\right) \circ \left(\sum_{n\geq 1} \binom{T}{n} (-1)^{n-1} X^n\right)\right\}.$$

Remark 1: In terms of schemes we have that the diagram

$$\begin{pmatrix}\text{inv.}\\ \text{power series}\end{pmatrix} \times \underline{\operatorname{Aut}} \hat{\mathbb{G}}_m \rightrightarrows \begin{pmatrix}\text{invertible}\\ \text{power series}\end{pmatrix} \longrightarrow \begin{pmatrix}\text{formal gp}\\ \text{laws}\end{pmatrix}$$

$$f \longmapsto f_*(X+Y-XY)$$

$$f \times g \longmapsto f, \qquad f \times g \longmapsto fg$$

becomes exact after $\mathbb{G}_a$ is applied

Remark 2: According to Hattori, the map $\pi_*(MU) \longrightarrow K_*(MU)$ is injective onto a direct summand, i.e. the cokernel of the map is ~~[illegible]~~ a free $\mathbb{Z}$-module. It would be nice to know if the sequence (it isn't see below)

$$(**) \qquad 0 \longrightarrow \pi_*(MU) \longrightarrow K_*(MU) \rightrightarrows K_*(MU) \otimes_{K(pt)} K_*(BU) \rightrightarrows K_*(MU) \otimes_{K(pt)} K_*(BU) \otimes_{K(pt)} K_*(B$$

~~to~~ were exact as this would yield Hattori's result as well as the fact that it remains exact after tensoring over $\mathbb{Z}$ with any ring R (any submodule of a flat $\mathbb{Z}$-module is flat).

In any case the sequence (**) is of the Amitsur form

$$0 \to A \longrightarrow B \rightrightarrows B \otimes_A B \rightrightarrows B \otimes_A B \otimes_A B \cdots\cdots \qquad A = \pi_*(MU), \quad B = K_*(MU)$$

This may be seen by noting that ~~[illegible]~~ a map

$$\underbrace{B \otimes_A B \otimes_A B \cdots}_{n \text{ times}} \longrightarrow R$$

is the same as giving $\overset{\text{invertible}}{}$ power series $f_1(X), \dots, f_r(X)$ with coefficients in R such that

$$f_i(X) * (X+Y-XY) = f_j(X) * (X+Y-XY) \quad \forall i, j,$$

or equivalently $\overset{\text{invertible}}{}$ series $f_1, u_2, \dots, u_r$ (with $f_i = f_1 u_2 \cdots u_i$) where u_i stabilizes $X+Y-XY$ and hence u_i gives rise to a map $\Gamma \longrightarrow R$, where Spec $\Gamma = \mathrm{Aut}\, \hat{G}_m$. Thus

$$\underbrace{B \otimes_A \cdots \otimes_A B}_{r} \simeq B \otimes \underbrace{\Gamma \otimes \cdots \otimes \Gamma}_{r-1} \simeq B \otimes_{K(pt)} K_\cdot(BU) \otimes \cdots \otimes_{K(pt)} K_\cdot(BU$$

CLAIM (**) not exact in degree 1. ~~[illegible]~~

To see this simplify notation and write (**) as a complex

$$0 \longrightarrow A \longrightarrow Q^0 \longrightarrow Q^1 \longrightarrow Q^2 \longrightarrow \cdots$$

Let $B^i = \mathrm{Ker}\, Q^{i-1} \to Q^i$ so that we have exact sequences

$$0 \longrightarrow A \longrightarrow Q^0 \longrightarrow B^1 \longrightarrow 0 \qquad \text{(by Stong - Hattori)}$$

$$0 \longrightarrow B^1 \longrightarrow Q^1 \longrightarrow B^2 \longrightarrow 0 \qquad \text{by hypothesis}$$

Tensoring with $\mathbb{Z}/p\mathbb{Z}$ this remains exact since $B^* \subset Q^*$ is torsion free. Thus

$$0 \longrightarrow A \otimes \mathbb{Z}/p\mathbb{Z} \longrightarrow Q^0 \otimes \mathbb{Z}/p\mathbb{Z} \longrightarrow Q^1 \otimes \mathbb{Z}/p\mathbb{Z}$$

is exact. But in the case at hand we know that

$$0 \longrightarrow (\Omega(pt) \otimes \mathbb{Z}/p\mathbb{Z})[\mathbb{P}_{p-1}] \longrightarrow K_\cdot(MU) \otimes \mathbb{Z}/p\mathbb{Z} \Longrightarrow K_\cdot(MU) \otimes \Gamma \otimes \mathbb{Z}/p\mathbb{Z}$$

is exact. (This is because modulo p^n we know that

$\mathrm{Aut}\,\hat{\mathbb{G}}_m = \mathbb{Z}_p^*$ is a flat subgroup of the group G of invertible power series under composition and that $G/\mathbb{Z}_p^* \simeq$ the functor of laws of height 1. $\simeq$ $\mathrm{Spec}\,(\Omega(pt)\otimes\mathbb{Z}/p^n\mathbb{Z})[\mathbb{P}_{p-1}]$. Denote this last by $\mathcal{L}_{1,\mathbb{Z}/p^n\mathbb{Z}}$. Faithfully flat descent shows us that

$$\Gamma(\mathcal{L}_{1,\mathbb{Z}/p^n\mathbb{Z}}, \mathcal{O}_Z) = \Gamma(G,\mathcal{O}_G)^{\Gamma(\mathbb{Z}_p^*,\,\mathcal{O}) = \Gamma\otimes\mathbb{Z}/p^n\mathbb{Z}} \quad)$$

CLAIM for each n that

$$0 \longrightarrow \Omega(pt)[\mathbb{P}_{p-1}^{-1}]\otimes\mathbb{Z}/p^n\mathbb{Z} \longrightarrow K_\bullet(MU)/p^n \longrightarrow K_\bullet(MU)/p^n\otimes\Gamma \longrightarrow \cdots$$

is exact. This is just faithfully flat descent for the morphism $G_{\mathbb{Z}/p^n\mathbb{Z}} \to \mathcal{L}_{1,\mathbb{Z}/p^n\mathbb{Z}}$ which is a torsor for $(\mathbb{Z}_p)^*$.

<u>Proof of Stong-Hattori theorem</u>: Let $z\in K_\bullet(MU)$ be a primitive element. Then by rational considerations there exists an integer $n\neq 0$ such that nz comes from $\Omega(pt)$. On the other hand I know that $z \in \varprojlim_n \Omega(pt)[\mathbb{P}_{p-1}^{-1}]\otimes\mathbb{Z}/p^n\mathbb{Z}$ which means

$$z = \sum_{g\leq N} (\mathbb{P}_{p-1})^{g}\, \omega_g$$

where the ω_g are polynomials in the other generators (with coeffs in $\mathbb{Z}_p$) tending to zero in the p-topology. For such a thing to be of the form ω/n with $\omega\in\Omega(pt)$ it must be that $\omega_g = 0$ for $g<0$. Thus $z \in \Omega(pt)\otimes\mathbb{Z}_p$ for all p so $z\in\Omega(pt)$.

Hattori's result that $\Omega(pt)$ is a direct summand of $K_\bullet(MU)$

follows easily. In effect as $K_\cdot(MU) = \mathbb{Z}[b_0, b_0^{-1}] \otimes \mathbb{Z}[b_1, \dots]$ is free in each dimension it suffices to show that if nz is in $\Omega_\cdot(pt)$ so is z. But if nz is primitive so is z as $K_\cdot(MU) \otimes \Gamma$ is torsion-free. (This last observation is what Adams must have meant by Hattori's proof showing that $\Omega_\cdot(pt) = P\, K_\cdot(MU)$.)

<u>Remark 3:</u> From the fact that (*) on page 16 is exact, one deduces that it is exact for any torsion free finite complex. Indeed after suspension to kill π_1 one can suppose that X is minimal and use skeleton induction. Thus for a torsion-free finite complex X, $\Omega^*(X)$ can be calculated algebraically from $K(X)$ with its Adams operations; $\Omega^*(X)$ is the invariant elements of $K_\cdot(MU) \otimes_{K(pt)} K(X)$.

The general picture about char. nos. and Wu relations:

Given a Chern theory Q with formal group law F^Q, let Γ_Q be the coordinate ring of $\underline{\mathrm{Aut}}\, F^Q$. Then we have maps (here $Q_\cdot(MU), \gamma$ have the universal property of representing stable ring homomorphisms from Ω to Q.)

$$(**)\qquad \Omega(X) \xrightarrow{\ \gamma\ } Q_\cdot(MU) \underset{Q(pt)}{\otimes} Q(X) \underset{id\otimes\gamma}{\overset{\mu\otimes id}{\rightrightarrows}} Q_\cdot(MU) \underset{Q(pt)}{\otimes} \Gamma_Q \underset{Q(pt)}{\otimes} Q(X)$$

where μ is the unique ring homomorphism such that these two compositions are equal. Identifying $Q_\cdot(MU) \simeq Q(pt)[b_1, \cdots]$ we have that

$$\begin{cases} \gamma = \left(\sum b_n X^{n+1}\right)^\wedge \\ \sum (\mu b_n) X^{n+1} = \left(\sum b_n X^{n+1}\right) \circ \left(\sum t_n X^{n+1}\right) \end{cases}$$

where $\sum t_n X^{n+1}$, $t_n \in \Gamma_Q$ is the generic auto of F_Q.

~~The characteristic nos~~

γ is essentially the characteristic numbers map (with values in Q) since
$$Q_\cdot(MU) \simeq \mathrm{Hom}^{cont}_{Q(pt)}(Q(MU), Q(pt)) \simeq \mathrm{Hom}^{cont}_{Q(pt)}(Q(BU), Q(pt)) \simeq \mathrm{Hom}^*(K, Q^*).$$
The ~~[illegible]~~ condition
$$(\mu\otimes id)(z) = (id \otimes \gamma)(z)$$
on an element $z \in Q_\cdot(MU) \otimes_{Q(pt)} Q(X)$ in the image of γ is called the Wu relations. To say that the Wu relations are complete means that $(**)$ is exact in the middle.

Example: Take $Q = H_s^\bullet(X, \mathbb{Z}/p)$ denoted $H_s^\bullet(X)$ in the following. Then for $X = pt$ the maps of $(**)$ correspond to maps of schemes

$$G_{1,/p} \times \mathrm{Aut}_{1,/p} \hat{G}_a \rightrightarrows G_{1/p} \xrightarrow{\gamma^t} \mathcal{L}_{/p}$$

[subscript denotes mod ... 1 denotes in leading term

One knows that the image of γ^t is $\mathcal{L}_{/p,\infty}$ = laws of height ∞ mod p. Thus

$$\Omega(pt)_{/p} \xrightarrow{\gamma} H_\bullet(MU)$$

kills the coefficients of $[p]_{F_\Omega}(X)$. Now we know that

$$G_{1/p} / \mathrm{Aut}_{1/p}^* G_a \simeq \mathcal{L}_{/p,\infty}$$

and in fact that there is a section due to Cartier which gives

$$G_{1,/p} \simeq \mathcal{L}_{/p,\infty} \times \mathrm{Aut}_{1/p} G_a$$

From this one ~~sees~~ sees that

$$\Omega(pt)_{/p} \longrightarrow H_\bullet(MU) \rightrightarrows H_\bullet(MU) \otimes (\mathcal{a}/\beta)^\vee$$

is exact in the middle, hence the Wu relations are complete.

Stong–Hattori says Wu relations complete for $K^\bullet$.

June 11, 1969

The cobordism class of a blowup

Let

$$\begin{tikzcd} \tilde{Y} \arrow[r, "j"] \arrow[d, "g"'] & \tilde{X} \arrow[d, "f"] \\ Y \arrow[r, "i"] & X \end{tikzcd} \qquad \begin{array}{l} \tilde{Y} = \mathbb{P}E \quad E = \nu_i \\ \dim E = r \end{array}$$

be a standard blowup diagram. The problem is to calculate $f_* 1$ in a Chern theory with group law F.

Theorem 1:

$$f_* 1 = 1 + i_*\left[\operatorname{res} \frac{\omega(Z)}{I(Z) \prod_{\nu=1}^{r} F(Z, x_\nu)}\right].$$

where $\quad c_t(E) = \prod_{\nu=1}^{r} (1 + t x_\nu).$

Proof: May assume the Chern theory involved is Ω.

Let $k : U \longrightarrow X$ be the complement of U. As f is an isomorphism off U, $f_* 1 - 1$ has a canonical trivialization over U and so defines an element of $\Omega_Y(X)$. As

$$i_* : \Omega(Y) \xrightarrow{\cong} \Omega_Y(X)$$

there is a unique u with $\quad i_* u = f_* 1 - 1 \quad$ as elements of $\Omega_Y(X)$. ~~By excision we may assume that X is E with i the zero section, and hence that j is the zero section of the line bundle $\mathcal{O}(-1)$ on $\mathbb{P}E$.~~

The situation is induced by a map $\varphi : X \longrightarrow MU(r)$ transversal to $BU(r)$ with Y the inverse image of $BU(r)$ under φ. We can therefore suppose that $Y = BU(r)$ and X is the canonical bundle E over $BU(r)$ with i the zero section. Moreover we can then identify j with the zero section of $\mathcal{O}(-1)$ over $\mathbb{P}E$. Finally by splitting principle can assume $Y = BU(1)^r$.

In this case $c_r E$ is a non-zero divisor, and so $\iota^* : \Omega_Y(X) \longrightarrow \Omega_*(Y)$ is injective. It suffices therefore to prove that

$$\iota^*(f_* 1 - 1) = c_r(E) \operatorname{res} \frac{\omega(Z)}{I(Z) \prod_{\nu=1}^{r} F(Z, x_\nu)}$$

Now to calculate the former use the diagram

$$\begin{tikzcd}
\mathbb{P}E \arrow[r, "j"] \arrow[d, "g"] & \mathcal{O}(-1) \arrow[r, "k"] \arrow[d, "f"] & g^*E \arrow[dl, "h"] \\
Y \arrow[r, "i"] & E &
\end{tikzcd}$$

where k is the natural inclusion in the canonical sequence

$$0 \longrightarrow \mathcal{O}(-1) \xrightarrow{k} g^*E \longrightarrow F \longrightarrow 0$$

over $\mathbb{P}E$. Then

$$\begin{aligned}
\iota^* f_* 1 = \iota^* h_* k_* 1 &= g_* j^* k^* k_* 1 \\
&= g_* j^* c_{r-1}(\nu_k) = g_* c_{r-1}(j^* \nu_k) \\
&= g_*(c_{r-1} F)
\end{aligned}$$

So the theorem 1 will follow from

<u>Theorem 2:</u> If F is the canonical quotient bundle on PE, then

$$g_*(c_{n-1}F) = 1 + c_n(E) \cdot \operatorname{res} \frac{\omega(Z)}{I(Z) \prod_{\nu=1}^{n} F(Z, x_\nu)}$$

<u>Proof:</u> We have with $\xi = c_1(\mathcal{O}(1))$, $\eta = c_1(\mathcal{O}(-1)) = I(\xi)$

$$c_t(F) = \frac{c_t(E)}{c_t(\mathcal{O}(-1))} = \frac{c_t(E)}{1+t\eta} = c_t(E)(1 - t\eta + t^2\eta^2 - \cdots)$$

so

$$c_{n-1}(F) = c_{n-1}(E) + (-\eta)c_{n-2}(E) + \cdots + (-\eta)^{n-1}$$

Let

$$\alpha(Z) = c_{n-1}(E) + (-I(Z))c_{n-2}(E) + \cdots + (-I(Z))^{n-1}$$

so that

$$\alpha(\xi) = c_{n-1}(F).$$

Then

$$c_n(E) - I(Z)\alpha(Z) = c_n(E) + (-I(Z))c_{n-1}(E) + \cdots + (-I(Z))^n$$

$$= \prod_{\nu=1}^{n} (x_\nu - I(Z))$$

$$= \prod_{\nu=1}^{n} \left[F(x_\nu, Z)\left\{1 + I(Z)\, G(I(Z), F(x_\nu, Z))\right\}\right]$$

so

$$(*) \qquad \frac{c_n(E) - I(Z)\alpha(Z)}{\prod_{\nu=1}^{n} F(x_\nu, Z)} = \prod_{\nu=1}^{n}\left[1 + I(Z)\, G(I(Z), F(x_\nu, Z))\right]$$

$$\equiv 1 \mod (Z).$$

Therefore

$$g_*(c_{n-1}(F)) = g_*(\alpha(\xi)) = \operatorname{res} \frac{\alpha(Z)\,\omega(Z)}{\prod\limits_{1\le \nu\le n} F(Z,x_\nu)}$$

$$= \operatorname{res} \frac{I(Z)\alpha(Z)\,\omega(Z)}{I(Z)\prod\limits_{\nu} F(Z,x_\nu)}$$

$$= c_n(E)\operatorname{res}\left(\frac{\omega(Z)}{I(Z)\prod\limits_{\nu} F(Z,x_\nu)}\right) + \operatorname{res}\left[\frac{c_n(E)-I(Z)\alpha(Z)}{\prod\limits_{\nu} F(X_\nu,Z)}\cdot\frac{dZ}{-I(Z)}\right]$$

since $I(Z)$ has a simple zero at $Z=0$ and $I'(0)=-1$ the last term using $(*)$ is seen to be 1, proving theorem 2.

I now wish to classify those group laws F for which a Chern theory having the law F has $f_*1=1$ for any blowup. By universal considerations this means that the residue term in theorem 1 is always zero.

Theorem 3: Let F be a formal group law over a ring R such that

$$(**)\qquad \operatorname{res}\frac{\omega(Z)}{I(Z)\prod\limits_{\nu=1}^{n} F(Z,x_\nu)} = 0 \qquad n\ge 1$$

for all nilpotent elements $x_1,\dots,x_n$ $n\ge1$ in any R-algebra. Then

$$F(X,Y) = X+Y+\beta XY$$

for some $\beta\in R$, Conversely any such F satisfies $(**)$.

Proof: We consider the converse statement first. If ~~F(x,y)=~~

$$F(X,Y) = X + Y + \beta XY$$

then

$$IX = \frac{-X}{1+\beta X}$$

and

$$\omega(X) = \frac{dX}{F_2(X,0)} = \frac{dX}{1+\beta X}$$

so

$$\operatorname{res} \frac{\omega(Z)}{I(Z)\prod_{\nu=1}^{n} F(Z,x_\nu)} = \operatorname{res} \frac{dZ}{-Z\left(\prod_\nu (Z + x_\nu + \beta Z x_\nu)\right)}$$

$$\frac{1}{\prod_{\nu=1}^{n}(1+\beta x_\nu)} \operatorname{res} \frac{dZ}{(-Z)\prod_\nu \left(Z + \frac{x_\nu}{1+\beta x_\nu}\right)}$$

which is zero for $n \geq 1$ in virtue of the following

Lemma: Let $f(Z), g(Z)$ be polynomials over R with g monic and degree $f <$ degree $g = n$. If a is the $(n-1)$th coefficient of f, then

$$\operatorname{res} \frac{f(Z)\,dZ}{g(Z)} = a$$

Proof of lemma: The easiest way to see this is to use that the sum of the residues is zero and that the residue at ∞ is $-a$.

~~But to be precise one first replaces R by an extension in~~

~~which $g(Z) = \prod(Z - x_\nu)$ and one uses induction on [illegible]~~

~~One can assume also~~ To be precise use induction on the degree of g and check the case $n=1$ from the definition of residue. Next one can easily reduce to the case where $R = k[x_1, \dots, x_n]$ is a polynomial ring and

$$g(Z) = \prod_{\nu=1}^{n} (Z - x_\nu) .$$

As $x_i - x_j$ is a non-zero-divisor in R it follows that we can embed R in the ring $R[(x_i-x_j)^{-1}]_{i \neq j}$ and so suppose that the $x_i - x_j$ are invertibles. Now given f we have

$$f(Z) = f_1(Z)(Z - x_n) + f(x_n)$$

so by induction are reduced to showing ~~that~~ to the case where $f = 1$. But we have by the division algorit

$$\prod (x_n - x_j) = \prod_{j<n} (Z - x_j) + g(Z)(Z - x_n) \qquad \text{degree } g \leq n-2$$

where the first is a unit and so are done by induction. This proves the lemma and the converse part of thm 3.

Now suppose that F is a law satisfying $(**)$. Taking all the $x_\nu = 0$ we see that

$$\operatorname{res} \frac{\omega(Z)}{I(Z)\, Z^r} = 0 \qquad \text{all } r \geq 1 .$$

Now $\dfrac{\omega(Z)}{I(Z)} = (-1 + a_1 Z + a_2 Z^2 + \dots) \dfrac{dZ}{Z}$ as $\omega(Z)$ is regular and $I(Z) = Z(-1 + \text{higher terms})$. Thus

$$\frac{\omega(Z)}{I(Z)} = -\frac{dZ}{Z}$$

and we are given that

$$\operatorname{res} \frac{dZ}{Z \prod_{\nu=1}^{r} F(Z, x_\nu)} = 0 \qquad r \geq 1.$$

Let x be a nilpotent element in an R-algebra. ~~and~~
Then

$$(Z - x) = F(Z, Ix)\big(1 + x\, G(x, F(Z, Ix))\big)$$

so taking $x_1 = I(x)$ and all other $x_\nu = 0$ we find

$$\operatorname{res} \frac{1 + x\, G(x, F(Z, Ix))}{Z - x} \frac{dZ}{Z^r} = 0 \qquad r \geq 1$$

But we can write by the division algorithm

$$1 + x\, G(x, F(Z, Ix)) = q(Z)(Z - x) + \underbrace{\big(1 + x\, G(x, F(x, Ix))\big)}_{\parallel \atop 1 + x\, G(x, 0)}$$

and this residue is

$$\underbrace{\operatorname{res} \frac{[1 + x\, G(x, 0)]\, dZ}{(Z - x) Z^r}}_{\overset{\parallel}{0} \text{ by lemma}} + \operatorname{res} \left[\frac{q(Z)\, dZ}{Z^r} \right] = 0 \qquad \forall r \geq 1$$

As $q(Z)$ is regular we must have $q(Z) = 0$ so

$$(Z - x) = F(Z, Ix)\big(1 + x\, G(x, 0)\big)$$

Thus as x was an arbitrary nilpotent element of any R-algebra

$$F(Z, IX) = \frac{Z - X}{1 + X\, G(X, 0)}.$$

Therefore

$$F(X,Y) = X + Y + XY\,G(X,Y)$$

is linear as a function of X, so $G(X,Y) = G(0,Y)$; by symmetry $G(X,Y) = G(X,0)$, so G is a constant β and theorem 3 is proved.

(Remark: See earlier paper (May 30) for indication that

$$\operatorname{res} \frac{\omega(Z)}{I(Z)\prod F(Z,x_\nu)} = \operatorname{res} \frac{\omega(Z)}{I(Z)Z^n} + \text{terms of degree} \geq n \text{ in } x_1, \dots, x_n.)$$

This appears below.

We observed before that in the stable range $i : Y \to X$ $2\overset{\mathbb{C}x.}{\operatorname{codim}} > \dim Y$,* then the cobordism class $f_* 1$, where $f : \tilde{X} \to X$ depends only on $i_* 1$. In fact it seems that

$$f_* 1 - 1 = a \cdot i_* 1$$

where $a \in \Omega^{-n}(pt)$ is the class where i is ~~a~~ $pt \to \mathbb{C}^n$, i.e.

$$a = \operatorname{res}\left\{\frac{\omega(Z)}{I(Z)Z^n}\right\}$$

* This is because by basic naturality the element $f_* 1$ depends only on the map $X \longrightarrow MU(n)$ represented by the embedding $Y \hookrightarrow X$.

Proof: The element $f_* 1 - 1$ of $\Omega^0_Y(X)$ is induced by a map $k: X \longrightarrow MU(r)$ transversal to $BU(r)$. ~~thus~~ As $\dim X \leq 4r-1$, kX may be deformed into the $4r-1$ skeleton of $MU(r)$, hence $k^*(c^\alpha \cdot \text{Thom class}) = 0$ if $|\alpha| \geq r$. Now

$$f_* 1 - 1 = \iota_* \left\{ \operatorname{res} - \frac{\omega(Z)}{Z \prod_{\nu=1}^{r} F(x_\nu, IZ)} \right\}$$

$$= \iota_* P(c_1, c_2, \cdots, c_r)$$

where $P = \sum \gamma_\alpha c^\alpha$ is a power series. In the stable range we know that the answer depends only on $\iota_* 1$. ~~thus~~ Moreover by the above $\iota_* c^\alpha = 0$ if $|\alpha| \geq r$. Thus one sees that $\gamma_\alpha = 0$ for $0 < |\alpha| < r$, so

$$f_* 1 - 1 = \iota_*(\gamma_0) = \gamma_0 \cdot \iota_* 1. \qquad ((\iota_* 1)^2 = 0).$$

To get γ_0 it suffices to take the case where $Y \to X$ is $pt \longrightarrow \mathbb{C}^h$.

Corollary of the above proof seems to be ~~is~~ that

$$\operatorname{res} \frac{\omega(Z)}{Z \prod_{\nu=1}^{r} F(x_\nu, IZ)} = \operatorname{res} \frac{\omega(Z)}{Z (IZ)^r} + \text{monomials of degree} \geq r \text{ in the } x_i$$

since $\iota_* : \Omega(BU) \xrightarrow{\sim} \Omega(MU)$.