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Cobordism modulo torsion in algebraic geometry.

Let $\underset{\sim}{\mathcal{A}}$ be the category of quasi-projective ~~varied~~ non-singular varieties over a field k. Let Q $: Y \mapsto Q(Y)$ $f \mapsto f^*$ be a contravariant functor from $\underset{\sim}{\mathcal{A}}$ to Rings endowed with a Gysin map f_* for proper maps f satisfying the following axioms:

I. $Y \longmapsto Q(Y)$ $f \longmapsto f_*$ is a covariant functor with values in $\underset{\sim}{Ab}$ from the sub category of $\underset{\sim}{\mathcal{A}}$ with same objects and with proper maps for morphisms.

II. (Gysin commutes with transversal base change).

III. (homotopy) $\quad pr_1^* : Q(Y) \xrightarrow{\ \sim\ } Q(Y \times \mathbb{A}^1)$

IV. (additivity) $\quad Q(Y_1 \amalg Y_2) \underset{\cong}{\xrightarrow{(in_1^*, in_2^*)}} Q(Y_1) \times Q(Y_2)$

V. (multiplicativity). If $f_i : X_i \longrightarrow Y_i$ are proper $i = 1, 2$ then

$$(f_1 \times f_2)_* (x_1 \boxtimes x_2) = (f_1)_* x_1 \boxtimes (f_2)_* x_2$$

for $x_i \in Q(X_i)$ where as usual

$$x_1 \boxtimes x_2 = pr_1^* x_1 \cdot pr_2^* x_2 \quad \text{in } Q(X_1 \times X_2).$$

VI. (splitting principle). If E is a vector bundle of dimension n over X, then $Q(PE)$ is a free $Q(X)$ module with basis $1, \xi, \ldots, \xi^{n-1}$ where $\xi = c_1(\mathcal{O}(1))$.

VII. (exactness). If $Y \overset{i}{\hookrightarrow} X$ is a closed embedding with complement $j: U \longrightarrow X$, then

$$Q(Y) \xrightarrow{i_*} Q(X) \xrightarrow{j^*} Q(U)$$

is exact.

VIII. If $i: Y \longrightarrow X$ is a closed embedding, then

$$i^* i_* y = (i^* i_* 1) y \qquad \text{for all } y \in Q(Y).$$

(need this only for a non-singular divisor).

Example 1: Take $Y \longmapsto K(Y)$ with Gysin map defined by Grothendieck:

$$f_*(F) = \sum (-1)^q R^q f_*(F).$$

Example 1': $Y \mapsto Q(Y) \otimes_{Q(pt)} R$ where R is flat over $Q(pt)$.

Example 2: Let Q be a theory satisfying I–VIII. We will show below that Q has a theory of Chern classes and in that particular to a power series $\varphi(X) = \sum_{n \geq 0} g_n X^n$, $g_0 \in Q(pt)^*$, $g_n \in Q(pt)$ there is a unique natural transformation

$$\tilde{\varphi}: K \longrightarrow Q^*$$

which is a homomorphism and is such that

$$\tilde{\varphi}(L) = \varphi(c_1^Q(L))$$

for all line bundles L. ($\tilde{\varphi}$ is called a multiplicative characteristic class). In fact

$$\tilde{\varphi}(E) = \operatorname{Norm}_{Q(\mathbb{P}E') \to Q(X)} \varphi({}^{Q}c_1 \mathcal{O}(1))$$

for any vector bundle E.

Given a multiplicative characteristic class we can define a new Gysin homomorphism by

$$f^\varphi_*(x) = f_*(\tilde{\varphi}(\nu_f)\, x).$$

Proposition 1: The theory $Y \mapsto Q(Y)$, $f \mapsto f^*$, f proper $\mapsto f^\varphi_*$ satisfies axioms I – VIII.

Proof. I. Given $X \xrightarrow{f} Y \xrightarrow{g} Z$ all proper we have

$$(gf)^\varphi_*(x) = (gf)_*(\tilde\varphi(\nu_{gf})x) = g_* f_*(\tilde\varphi(\nu_f + f^*\nu_g)\, x)$$

$$= g_* f_*(f^*\tilde\varphi(\nu_g)\ \tilde\varphi(\nu_f)\, x)$$

$$= g_*(\tilde\varphi(\nu_g)\ f_*(\tilde\varphi(\nu_f)x)) = g^\varphi_*\, f^\varphi_*\, x.$$

We used the projection formula

Lemma: If $f: X \to Y$ proper, then

$$f_*(x f^* y) = f_* x \cdot y$$

Proof: Using II + V: The diagram

$$\begin{tikzcd} X \arrow[r, "(id, f)"] \arrow[d, "f"'] & X \times Y \arrow[d, "f \times id"] \\ Y \arrow[r, "\Delta_Y"] & Y \times Y \end{tikzcd}$$

is (transversal) cartesian so

$$f_*(x \cdot f^* y) = f_*(id, f)^*(x \boxtimes y) = \Delta_Y^*(f \times id)_*(x \boxtimes y)$$

$$= \Delta_Y^*(f_* x \boxtimes y) = f_* x \cdot y.$$

II. Given

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

cartesian we have

$$\begin{aligned} g^* f_*^{\varphi} x &= g^* f_*(\tilde\varphi(\nu_f)\, x) \\ &= f'_*(g')^*(\tilde\varphi(\nu_f)\, x) \\ &= f'_* \tilde\varphi(\underbrace{g'^*\nu_f}_{\nu_{f'}})\, g'^* x = (f')_*^{\varphi}\, g'^* x \end{aligned}$$

III, IV. Clear.

V.

$$(f_1 \times f_2)_*^{\varphi}(x_1 \boxtimes x_2) = (f_1 \times f_2)_*\left(\tilde\varphi(\nu_{f_1 \times f_2})\, x_1 \boxtimes x_2\right)$$

But $\tilde\varphi(\nu_{f_1\times f_2}) = \tilde\varphi(\nu_{f_1} \boxtimes \nu_{f_2}) = \tilde\varphi(\nu_{f_1}) \boxtimes \tilde\varphi(\nu_{f_2})$, so it's clear.

VI. We are given that $Q(\mathbb{P}E)$ is a free module over $Q(X)$ with basis $1, \xi, \ldots, \xi^{n-1}$; we want the same conclusion but with ξ replaced by $c_1^{Q^\varphi}(\mathcal{O}(1)) = \bar\varphi(\xi)$ where $\bar\varphi(X) = \sum' q_n X^{n+1}$. We need the following ~~lemma~~ variant of the Weierstrass preparation theorem.

<u>Lemma</u>: Let R be a ring let $\varphi(X) = \sum' q_n X^{n+1} \in R^*[[X]]$, $q_0 \in R^*$ and let a_i ($1 \le i \le n$) be nilpotent elements of R. Then

$$\sum_{i=0}^{n} a_i\,\psi(X)^i = u(X)\sum_{i=0}^{n} b_i X^{n-i} \qquad a_0 = b_0 = 1$$

where $u(X)$ is a unit in $R[X]$ and where $b_i \in \sum_{i=1}^{n} Ra_i$, $1 \le i \le n$.

<u>Proof</u>: Set $\mathfrak{m} = \sum_{i=1}^{n} Ra_i$; it is a nilpotent ideal modulo which $\sum_{i=0}^{n} a_i \psi(X)^{n-i} \equiv q_0^n X^n$. Thus the Weierstrass argument shows it is $u(X)$ times a distinguished polynomial. qed.

$$\text{So}\quad Q(\mathbb{P}E) \xleftarrow[\;\xi \,\leftarrow\, Z\;]{\cong} Q(Y)[[Z]]\Big/\Big(\sum_{i=0}^{n} a_i Z^{n-i}\Big) \qquad \text{where } a_i = \pm c_i(E')$$

$$\bar\varphi(\xi) \longleftarrow W, \quad \varphi(Z) \leftarrow W$$

$$= Q(Y)[[W]]\Big/\Big(\sum_{i=0}^{n} a_i (\bar\varphi^{-1} W)^{n-i}\Big)$$

$$\cong Q(Y)[[W]]\Big/\Big(\sum_{i=0}^{n} b_i W^{n-i}\Big)$$

by the lemma.

<u>VII</u>. Enough to observe that $\iota_*^{\varphi}(y) = \iota_*(\tilde\varphi(\nu_i)y)$ where $\tilde\varphi(\nu_i)$ is a unit in $Q(Y)$, hence ι_*^{φ} + ι_* have same image.

<u>VIII</u>.

$$\begin{aligned}
\iota^*\iota_*^{\varphi}(y) &= \iota^*(\iota_*\,\tilde\varphi(\nu_i)y) \\
&= (\iota^*\iota_* 1)(\tilde\varphi(\nu_i)y) \\
&= (\iota^*\iota_*\tilde\varphi(\nu_i))\,y \\
&= (\iota^*\iota_*^{\varphi} 1)\,y
\end{aligned}$$

✓ Prop 1 complete.

The theory Q, $f \mapsto f^*$, f proper $\mapsto f_*^{\varphi}$ will be called the result of <u>twisting</u> Q by the characteristic class φ and will be denoted Q^{φ}

Chern classes

Let Q, $f \mapsto f^*$, f proper $\mapsto f_*$ be a theory satisfying I–VIII. We shall check that we can construct Chern classes (with values in Q) by using Grothendieck's procedures.

~~Thus let E be a vector bundle of dimension n over Y, and let $\mathcal{O}(1)$ be the canonical line bundle on PE', the bundle of hyperplanes in E.~~

If L is a line bundle over Y, we define its first Chern class $c_1(L) \in Q(Y)$ by

$$c_1(L) = i^* i_* 1$$

where $i : Y \longrightarrow L$ is the zero section.

<u>Proposition 2:</u> Let $j : X \longrightarrow Y$ be a non-singular divisor in Y and let L be the associated line bundle whose sheaf of sections $\underline{L}$ is given by the exact sequence

$$0 \longrightarrow \underline{L}^{-1} \longrightarrow \mathcal{O}_Y \longrightarrow j_* \mathcal{O}_X \longrightarrow 0$$

Then $c_1(L) = j_* 1_X$.

<u>Proof:</u> The transpose of $\underline{L}^{-1} \longrightarrow \mathcal{O}_Y$ is a map $\mathcal{O}_Y \longrightarrow \underline{L}$ given by a section s of L which is transversal to the zero section i. Thus

$$\begin{array}{ccc} X & \xrightarrow{j} & Y \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle i} \\ Y & \xrightarrow{s} & L \end{array}$$

is transversal cartesian and so (II)

$$j_* 1 = s^* \iota_* 1$$

But s is homotopic to i via the map

$$\begin{aligned} Y \times \mathbb{A}^1 &\longrightarrow L \\ (y, t) &\longmapsto t \cdot s(y) \end{aligned}$$,

so $s^* = \iota^*$ (III). qed.

Now let E be a vector bundle over Y of dimension n, let $f: \mathbb{P}E^\vee \longrightarrow X$ be the projective bundle of hyperplanes in E and let $\mathcal{O}(1)$ be the canonical quotient line bundle on $\mathbb{P}E^\vee$. By VI, $f^*: Q(X) \longrightarrow Q(\mathbb{P}E^\vee)$ makes the latter a free module over $Q(X)$ with basis $1, \xi, \dots, \xi^{n-1}$ where $\xi = c_1(\mathcal{O}(1))$, hence we may define the Chern classes $c_i(E) \in Q(Y)$ as coefficients of the relation

$$\xi^n - \left(f^* c_1(E)\right) \xi^{n-1} + \cdots + (-1)^n \left(f^* c_n(E)\right) = 0 .$$

To check signs suppose $n=1$, then $f: \mathbb{P}E^\vee \xrightarrow{\sim} X$ and $\mathcal{O}(1) \simeq f^*E$; the relation is

$$\xi - f^* c_1(E) = 0 ,$$

and hence this definition agrees with the old.

For a vector bundle E over Y which may not be everywhere of the same dimension we set

$$\sum_{i=0}^{\infty} c_i(E) Z^i = \det_{Q(\mathbb{P}E^\vee)/Q(X)} (1 + Z\xi)$$

It is easy to see this agrees with the old definition when E is of dimension n. (Indeed the matrix of ξ with respect to $1, \xi, \dots, \xi^{n-1}$ is

$$\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & & & \\ & & & 0 & 1 \\ (-1)^{n-1}c_n & & & -c_2 & +c_1 \end{bmatrix}, \quad \text{so } \det(1-Z\xi) = \det \begin{bmatrix} 1 & -Z & & & \\ & 1 & -Z & & \\ & & & & \\ & & & 1 & -Z \\ (-1)^{n}Zc_n & \cdots & & +c_2 Z & -c_1 Z+1 \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & 0 & & & \\ & 1 & & 0 & \\ (-1)^{n}Zc_n & (-1)^{n}Z^2c_n+(-1)^{n-1}Zc_{n-1} & & (-1)^{n}Z^3c_n+(-1)^{n-1}Z^2c_{n-1}+(-1)^{n-2}Zc_{n-2} & \end{bmatrix}$$

$$= 1 - c_1 Z + c_2 Z^2 - \cdots\cdots .)$$

By the additivity axiom IV this definition means that $c_i(E)$ can also be defined by decomposing Y ~~into~~ (finite in number) pieces on which E is of constant dimension and then using the first definition.

The key thing to verify about these Chern classes is the Whitney sum formula:

Theorem 3: If $0 \to E' \to E \to E'' \to 0$ is an exact sequence of vector bundles over Y, then

$$c_Z(E) = c_Z(E')\, c_Z(E'') .$$

We will give two proofs of this; the second assumes a stronger property of Q than ~~seems~~ doesn't seem to follow from the other axioms but which I also seem to need to prove the residue formula (for f_*) in complete generality.

<u>First proof:</u> From the splitting principle one knows that f^* is injective where f is a projective bundle, hence if f is the associated flag bundle of a vector bundle. Using this, additivity and the functorality of Chern classes we may suppose E' and E'' possess filtrations with quotients L_i' and L_j''. By theorem 3' below

$$c_Z(E) = \prod_i c_Z(L_i') \prod_j c_Z(L_j'') = c_Z(E')\, c_Z(E'').$$

<u>Theorem 3':</u> If E has a filtration by subbundles

$$0 = E_n \subset \cdots \subset E_1 \subset E_0 = E$$

with $\quad E_i / E_{i+1} = L_{i+1}$ a line bundle, then

$$c_Z(E) = \prod_{i=1}^{n} \left(1 + Z c_1(L_i)\right)$$

<u>Proof:</u> We must show that

$$\prod_{i=1}^{n} \left(\xi - f^* c_1(L_i)\right) = 0.$$

~~Now we have a sequence of subvarieties each a divisor in the preceding~~ Now we have a (non-singular) divisor

$$\mathbb{P}(E_{n-1}^{\perp}) \overset{i}{\subset} \mathbb{P}(E^{\vee})$$

~~The associated line bundle~~ on the complement $j: U \to \mathbb{P}(E^{\vee})$ of which the bundle $\mathcal{O}(1)$ becomes isomorphic to L_n. Thus $j^*(\xi - c_1(L_n)) = 0$

so by the exactness axiom $\underline{VII}$

$$Q(\mathbb{P}(E/E_{n-1})^{\vee}) \xrightarrow{\iota_*} Q(\mathbb{P}E^{\vee}) \xrightarrow{j^*} Q(U)$$

there is an element u with $\iota_* u = \xi - c_1(L_n)$. By induction hypothesis

$$\iota^* \prod_{i<n} (\xi - c_1(L_i)) = 0$$

hence

$$\prod_{i=1}^{n} (\xi - c_1(L_i)) = \prod_{i<n} (\xi - c_1(L_i)) \cdot \iota_* u$$

$$= i_* \Big(\iota^* \prod_{i<n} (\xi - c_1(L_i)) \cdot u \Big) = 0. \qquad \text{qed.}$$

<u>Second proof</u>: We shall ~~use~~ assume the following strengthening of axiom $\underline{III}$:

$\underline{III}'$. If $f: X \longrightarrow Y$ is an affine bundle (i.e. a torsor for a vector bundle), then $f^*: Q(Y) \longrightarrow Q(X)$ is an isomorphism.

Now suppose given an exact sequence $0 \to E' \to E \to E'' \to 0$ and consider the embedding

$$\mathbb{P}(E''^{\vee}) \xrightarrow{i} \mathbb{P}(E^{\vee})$$

The complement $j: U \longrightarrow \mathbb{P}(E^{\vee})$ of i is the variety of hyperplanes of E not containing E'; to such a hyperplane H we obtain a hyperplane $H' = H \cap E'$ whence a projection $g: U \to \mathbb{P}(E'^{\vee})$

whose fiber over H' is the set of H with $H \cap E' = H'$, i.e. the set of splittings of the sequence

$$0 \longrightarrow E'/H' \rightarrow E/H' \longrightarrow E'' \longrightarrow 0.$$

Thus U is an affine bundle over $\mathbb{P}E'$ ~~associated to~~ with associated vector bundle $\underline{\mathrm{Hom}}_{\mathbb{P}E'}(E'', \mathcal{O}(1))$. Thus we have by exactness $\underline{\mathrm{VII}}$

$$\begin{array}{ccccc} Q(\mathbb{P}(E''^{\vee})) & \xrightarrow{i_*} & Q(\mathbb{P}E^{\vee}) & \xrightarrow{j^*} & Q(U) \\ & & & & {\scriptstyle g^*}\uparrow\wr \quad \text{by } \underline{\mathrm{III}}' \\ & & & & Q(\mathbb{P}E'^{\vee}) \end{array}$$

Note that $j^* \mathcal{O}_{\mathbb{P}(E^{\vee})}(1) \simeq g^* \mathcal{O}_{\mathbb{P}(E'^{\vee})}(1)$; in effect this is the isomorphism

$$E/H \simeq E'/_{H \cap E'} \qquad \text{where } H \not\supset E'.$$

We have an exact sequence

$$0 \longrightarrow Q(\mathbb{P}E''^{\vee}) \xrightarrow{i_*} Q(\mathbb{P}E^{\vee}) \xrightarrow[\text{ring hom}]{(g^*)^{-1} j^*} Q(\mathbb{P}E'^{\vee}) \longrightarrow 0$$

$$\xi \longmapsto \xi'$$

Indeed it's exact at the left since ξ' generates $Q(\mathbb{P}E'^{\vee})$; by consideration of ranks over $Q(Y)$ one sees that i_* is injective. This is an exact sequence of free finite type $Q(Y)[T]$-modules where T acts by multiplication by ξ'', ξ, ξ', resp. Thus

$$\det_{Q(\mathbb{P}E^{\vee})/Q(Y)}(1 + Z\xi) = \det_{Q(\mathbb{P}E'^{\vee})/Q(Y)}(1 + Z\xi') \det_{Q(\mathbb{P}E''^{\vee})/Q(Y)}(1 + Z\xi'')$$

i.e.

$$c_Z(E) = c_Z(E') \cdot c_Z(E'') \qquad \text{qed.}$$

Comments on $\underline{III}$: It's true for K-theory. In effect if $X \longrightarrow Y$ is an affine bundle with associated vector bundle E then X is the bundle of splittings of an exact sequence of vector bundles

$$0 \longrightarrow E \longrightarrow M \longrightarrow 1 \longrightarrow 0$$

and we have by Grothendieck's calculation an exact sequence

$$\begin{array}{ccccccc} K(\mathbb{P}E) & \xrightarrow{\iota_*} & K(\mathbb{P}M) & \xrightarrow{j^*} & K(X) & \longrightarrow & 0 \\ \wr\Vert & & \wr\Vert & & & & \\ K(Y)[Z']/(\varphi(Z')) & \longrightarrow & K(Y)[Z]/(Z\varphi(Z)) & & & & \end{array} \qquad \iota_*(Z')^i = Z^{i+1}$$

One sees that the cokernel of ι_* is isomorphic to $K(Y)$, and in fact that $K(Y) \longrightarrow K(\mathbb{P}M) \longrightarrow K(X)$ is an isomorphism

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<u>Corollary 4</u>: $\quad \mathrm{Hom}^+(K, Q^*) \cong \{ \sum q_n X^n \in Q(pt)[[X]] \mid q_0 \in Q(pt)^* \}$

<u>Proof</u>: Given $\varphi(X) \in (Q(pt)[[X]])^*$ and a vector bundle E over Y set

$$\tilde{\varphi}(E) = \mathrm{Norm}_{Q(\mathbb{P}E^\vee)/Q(Y)} \varphi(c_1(\mathcal{O}(1)))$$

have to show $c_1(L)$ nilpotent for any L before this makes sense, see p. 22

I claim that if

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

is exact, then

$$\tilde{\varphi}(E) = \tilde{\varphi}(E') \cdot \tilde{\varphi}(E'')$$

~~In effect we have the following~~ To prove this we ~~need~~ may suppose E' and E'' possess flags and we are reduced to showing that

$$\tilde{\varphi}(E) = \prod_{i=1}^{n} \tilde{\varphi}(L_i).$$

if E has a flag with quotients L_i $1 \le i \le n$. But in this case

$$Q(PE^\vee) = Q(Y)[Z] / \left(\prod_{i=1}^{n} (Z - c_1(L_i)) \right)$$

so it suffices to prove

Lemma 5: Let R be a ring and let $x_1, \dots, x_n$ (be nilpotent elements of R). Let $\varphi(X) \in R[[X]]$ and let ξ be the image of Z in $R[Z] / \left(\prod_{i=1}^{n} (Z - X_i) \right)$. Then ξ is nilpotent and

$$\operatorname{Norm}_{R[\xi]/R} \varphi(\xi) = \prod_{i=1}^{n} \varphi(x_i)$$

Proof: (see page 17) Since $\prod(\xi - X_i) = 0$, $\xi^n \equiv 0$ modulo the ideal generated by $x_1, \dots, x_n$ in $R[\xi]$; hence ξ is nilpotent and $\varphi(\xi)$ is defined. By base extension we may replace R by $k[[x_1, \dots, x_n]]$, $k = \mathbb{Z}$; the x_i are no longer nilpotent but they are topologically nilpotent and similarly for ξ = image of Z in $R[[Z]] / \left(\prod_{i=1}^{n} (Z - x_i) \right)$.

$$\begin{array}{ccccc} R[\xi] & \longrightarrow & R[\xi, (x_i - x_j)^{-1}]_{i \neq j} & \cong & \prod_{i=1}^{n} R[(x_i - x_j)^{-1}]_{i \neq j} \\ \uparrow & & \uparrow & & \\ R & \hookrightarrow & R[(x_i - x_j)^{-1}]_{i \neq j} & & \end{array}$$

$$\xi \longmapsto (x_i)$$

Call the larger ring R'. Then by ~~base extension~~ compatibility of norms with base extension

$$\mathrm{Norm}_{R[\xi]/R}\,\varphi(\xi) = \mathrm{Norm}_{R'[\xi]/R'}\,\varphi(\xi) = \prod_{i=1}^{n} \varphi(x_i). \quad \text{qed}$$

~~So~~ having proved our claim we have a natural homomorphism

$$\tilde{\varphi} : K \longrightarrow Q^*$$

and it's clearly the unique ~~transformation~~ natural homomorphism with

$$\tilde{\varphi}(L) = \varphi(c_1(L))$$

for all line bundles L. Thus we have constructed ~~a bijection~~ map

$$\sim : (Q(pt)[[X]])^* \longrightarrow \mathrm{Hom}^+(K, Q^*).$$

Conversely given $\theta : K \to Q^*$ we apply it to $\mathcal{O}(1)$ on $\mathbb{P}^n$ and find

$$\theta(\mathcal{O}(1)) \in Q(\mathbb{P}^n)^* \cong \{Q(pt)[[X]]/(X^{n+1})\}^*$$

so

$$\theta(\mathcal{O}(1)) = \sum_{k=0}^{n} q_k\, c_1(\mathcal{O}(1))^k$$

where $q_k \in Q(pt)$ are uniquely determined. Letting $n \to \infty$ we find $\exists!\ \varphi(X) \in (Q(pt)[[X]])^*$ such that

$$\theta(L) = \varphi(c_1(L))$$

for all line bundles which are generated by their sections. It's clear that $\theta \mapsto \varphi$ furnishes a right inverse to $\varphi \mapsto \tilde{\varphi}$

and hence $\sim$ is injective. It remains to show that ~~the formula holds for all line bundles~~ $\tilde{\varphi} = \Theta$, or since an additive transformation $K \longrightarrow Q^*$ is determined by its values on line bundles it's enough to show that $\Theta(L) = \varphi(c_1(L))$ for all line bundles L. But any line bundle is induced from $\mathcal{O}(-1) \boxtimes \mathcal{O}(1)$ on $\mathbb{P}^n \times \mathbb{P}^n$ for some n. So it suffices to show that $\tilde{\varphi} = \Theta$ on

$$K(\mathbb{P}^n \times \mathbb{P}^m) = \mathbb{Z}[T_1, T_2] \big/ \left((T_1 - 1)^{n+1}, (T_2 - 1)^{n+1}\right).$$

As Θ and $\tilde{\varphi} : K \longrightarrow Q^*$ are both homomorphisms and as they coincide on the elements $T_1{}^k T_2{}^\ell = c_1(\mathcal{O}(k) \boxtimes \mathcal{O}(\ell))$, $k, \ell \geq 0$ it follows that they are equal on $K(\mathbb{P}^n \times \mathbb{P}^n)$. qed.

Remarks: Lemma 5 should read more generally that

$$\det_{R[\xi]/R} \left(1 + Z\varphi(\xi)\right) = \prod_{i=1}^{n} \left(1 + Z\varphi(x_i)\right)$$

(same proof), so that

$$\mathrm{tr}_{R[\xi]/R}\, \varphi(\xi) = \sum_{i=1}^{n} \varphi(x_i)$$

By same proof as cor. 4 one has

Corollary 6: $$\mathrm{Hom}^+(K, Q) \cong Q(pt)[[X]]^*$$

$$\tilde{\varphi} \longleftarrow\!\shortmid\; \varphi$$

$$\tilde{\varphi}(E) = \mathrm{tr}_{Q(PE^\vee)/Q(Y)}\, \varphi(\xi) \qquad \xi = c_1(\mathcal{O}(1)).$$

Chern classes in K-theory: For a vector bundle E one checks that

$$i_* 1 = \text{class of } \mathcal{O}_X \text{ in } K(E)$$

$$i^* i_* 1 = \sum (-1)^q \operatorname{Tor}_q^{\mathcal{O}_E}(\mathcal{O}_X, \mathcal{O}_X)$$

$$= \sum (-1)^q \Lambda^q E^\vee = \lambda_{-1}(E^\vee)$$

Thus for a line bundle

$$c_1^K(L) = 1 - L^{-1}.$$

Proposition 7: If E is a vector bundle of dimension n over Y and if $i: Y \to E$ is the zero section, then

$$i^* i_* 1 = c_n(E)$$

Proof. Uses induction on n. By splitting principle we may assume that $\exists$ exact sequence

$$0 \to F \xrightarrow{u} E \to L \to 0$$

where L is a line bundle. Consider diagram

$$\begin{tikzcd} F \arrow[r, "u"] & E \arrow[d, shift right, "\pi"'] \\ & Y \arrow[ul, "j"] \arrow[u, shift right, "i"'] \end{tikzcd}$$

where i, j are zero-sections and where π is the projection. Then $E \to L$ gives a section of $\pi^* L$ transversal to zero with zero set F. Thus by prop. 2

$$u_* 1 = \pi^* c_1(L)$$

hence

$$\begin{aligned} \iota^* \iota_* 1 &= j^* u^* u_* j_* 1 \\ &= j^* [(u^* u_* 1)\, j_* 1] && \text{axiom VIII.} \\ &= \iota^* \pi^* c_1 L \cdot j^* j_* 1 \\ &= c_1(L) \cdot c_{n-1}(F) && \text{induction} \\ &= \underline{c_n(E)} && \text{Whitney sum fmla} \end{aligned}$$

Pf of Lemma 5 unnecessarily complicated. It suffices to observe that $R[\xi]$ has a composition series with quotients [illegible] $R[[Z]]/(Z-x_i)$.

Formal group law

Let Q be as before

Proposition 8: There is a unique commutative formal group law $F(X,Y)$ with coefficients in $Q(pt)$ such that

$$c_1(L \otimes L') = F(c_1 L, c_1 L')$$

for all line bundles L, L' over the same Y.

Proof: As

$$Q(\mathbb{P}^n \times \mathbb{P}^m) \cong Q(pt)[X,Y]/(X^{n+1}, Y^{m+1})$$
$$x = c_1(pr_1^* \mathcal{O}(1)) \longleftrightarrow X$$
$$y = c_1(pr_2^* \mathcal{O}(1)) \longleftrightarrow Y$$

~~then~~ we have

$$c_1(\mathcal{O}(1) \boxtimes \mathcal{O}(1)) = \sum_{\substack{0 \le k \le n \\ 0 \le l \le m}}' a_{kl}^{(n,m)} X^k Y^l$$

for uniquely determined elements $a_{kl} \in Q(pt)$. One sees that $a_{kl}^{(n,m)}$ is independent of ~~n~~ n, m for $n \ge k$ and $m \ge l$, hence there is a unique power series $F(X,Y) = \sum a_{kl} X^k Y^l \in Q(pt)[[X,Y]]$ such that

$$(*) \qquad c_1(L \otimes L') = F(c_1 L, c_1 L')$$

for all line bundles L, L' over the same Y which are generated by their sections.

Now on $\mathbb{P}^n \times \mathbb{P}^m \times \mathbb{P}^b$ we have three line bundle L_1, L_2, L_3 which are the inverse images of the canonical line bundles on each factor. Moreover

$$Q(\mathbb{P}^n \times \mathbb{P}^m \times \mathbb{P}^p) = Q(pt)[x_1, x_2, x_3] \quad \text{with relations} \quad x_1^n = x_2^m = x_3^p = 0$$

where $\qquad x_i = c_1 L_i$. As

$$F(F(x_1,x_2),x_3) = c_1(L_1 \otimes L_2 \otimes L_3) = F(x_1, F(x_2,x_3))$$

we see that

$$F(X, F(Y,Z)) = F(\ \ F(Y,Z))$$

modulo (X^n, Y^m, Z^p) for all n,m,p and hence identically. Similarly one has

$$F(X,Y) = F(Y,X) \qquad F(X,0) = X = F(0,X)$$

and therefore F is a $\overbrace{\text{formal}}^{\text{commutative}}$ group law.

It remains to show: that $(*)$ holds ~~identically~~ for line bundles not necessarily generated by sections. I claim it's enough to prove $(*)$ when L^{-1} ~~generated by sections~~ >0 and $L' > 0$ where >0 means generated by sections. In effect assume this; then if $L_1, L_2 > 0$ we have

$$c_1(L_1^{-1} \otimes L_2^{-1}) +_F c_1(L_2) = c_1(L_1^{-1})$$

$$\text{so} \quad c_1(L_2^{-1}) = -_F\, c_1(L_2) \qquad\qquad (\text{take } L_1 = 1)$$

$$\text{and} \quad c_1(L_1^{-1} \otimes L_2^{-1}) = c_1(L_1^{-1}) -_F c_1(L_2) = c_1(L_1^{-1}) +_F c_1(L_2^{-1}).$$

Next given arbitrary line bundles L, L' one has

$$L = M \otimes N^{-1} \qquad L' = M' \otimes (N')^{-1}$$

where $M, N, M', N' > 0$, so

$$c_1(L\otimes L') = c_1(M\otimes M' \otimes (N\otimes N')^{-1})$$
$$= c_1(M\otimes M') +_F c_1(N^{-1}\otimes N'^{-1})$$
$$= c_1(M) +_F c_1(M') +_F c_1(N^{-1}) +_F c_1(N'^{-1})$$
$$= c_1(M\otimes N^{-1}) +_F c_1(M\otimes N'^{-1}) = c_1(L) +_F c_1(L')$$

which proves (*) in the general case.

By natural we ~~have~~ only ~~to prove~~ ~~that~~

$$c_1(\mathcal{O}(-1)\boxtimes \mathcal{O}(1)) = F(c_1(\mathcal{O}(-1)\otimes 1, 1\otimes c_1\mathcal{O}(1))$$

on $\mathbb{P}^n\times\mathbb{P}^n$ for n fixed. ~~So~~ consider the two maps

given by

$$\mathbb{Z} \longrightarrow Q(\mathbb{P}^n\times\mathbb{P}^n)$$
$$k \longmapsto c_1(\mathcal{O}(k)\boxtimes\mathcal{O}(1))$$
$$\longmapsto F(c_1(\mathcal{O}(k))\otimes 1, 1\otimes c_1\mathcal{O}(1))$$

By naturality it suffices to prove that

$$c_1(T_1^{-1}\otimes T_2) = F(c_1T_1^{-1}, c_1T_2)$$

on $\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}$ where $T_i = pr_i^*\mathcal{O}(1)$ $i=1,2$. I claim that the ~~two~~ maps

$$\mathbb{Z} \longrightarrow Q(\mathbb{P}^n\times\mathbb{P}^m)$$
$$k \longmapsto c_1(T_1^k\otimes T_2)$$
$$\longmapsto F(c_1T_1^k, c_1T_2)$$

are of polynomial type in the sense of the following definition

<u>Definition</u>: If A is an abelian group, then a map $v: \mathbb{Z} \to A$ is said to be of polynomial type if it satisfies the following equivalent conditions

(i) $\exists\ a_i \in A \quad 0 \le i \le n$ such that

$$v(k) = \sum_{i=0}^{n} \binom{k}{i} a_i$$

(ii) $\exists$ an n such that

$$k \longmapsto \sum_{i=0}^{n} (-1)^i \binom{n}{i} v(k+i)$$

is identically zero.

From the ~~second~~ first condition one sees that if $v: \mathbb{Z} \to A$ and $w: \mathbb{Z} \to B$ are of polynomial type, then so is $v \otimes w$. Now in $K(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1})$ we have $(T_1 - 1)^n = 0$ so

$$\sum (-1)^i \binom{n}{i} T_1^{i+k} = 0 \quad \text{for all } k$$

Thus $k \longmapsto T_1^k$ in $K(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1})$ is a polynomial type function; hence the same is true after composing with c_1 and $c_1(\cdot T_2)$, which are homomorphisms $K(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}) \longrightarrow Q(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1})$. Next note that the image of c_1 on $K(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1})$ is a finitely generated group of nilpotent elements of $Q(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1})$ hence there is an $N \ni$ $(c_1 T_1^k)^N = 0$ for all k and so there is a polynomial $G(X) \in Q(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1})[X]$ such that

$$k \longmapsto F(c_1 T_1^k, c_1 T_2) = G(c_1 T^k)$$

Therefore both functions ~~—~~ at the bottom of page 20 are of polynomial type. But they coincide for $k \geq 0$ hence for all k. qed.

Remark: I needed to have an argument showing that $c_1(L)$ is nilpotent for any line bundle L way back in the proof of corollary 4. But any bundle is induced from $T_1^{-1} T_2$ on $\mathbb{P}^n \times \mathbb{P}^n$ (for some n) and

$$c_1 : K(\mathbb{P}^n \times \mathbb{P}^n) \longrightarrow Q(\mathbb{P}^n \times \mathbb{P}^n)$$

is an additive homomorphism by Whitney sum formula (Th 3). But $K(\mathbb{P}^n \times \mathbb{P}^n)$ has a $\mathbb{Z}$ basis $T_1^k T_2^\ell$ $\quad 0 \leq k, \ell \leq n$ and these have nilpotent c_1, hence $c_1(T_1^{-1} T_2)$ is nilpotent.

~~*Lemma*: (a corollary to theorem 3). Given $0 \to E' \to E \to E'' \to 0$ an exact sequence of vector bundles on Y, we have an exact sequence~~

$$0 \to Q(\mathbb{P}E''^{\vee}) \xrightarrow{i_*} Q(\mathbb{P}E^{\vee}) \xrightarrow{u} Q(\mathbb{P}E'^{\vee}) \to 0$$

~~where i is the inclusion $\mathbb{P}E''^{\vee} \to \mathbb{P}E^{\vee}$ and where u is the unique $Q(Y)$-algebra homomorphism sending $c_1(\mathcal{O}_{\mathbb{P}E^{\vee}}(1))$ to $c_1(\mathcal{O}_{\mathbb{P}E''^{\vee}}(1))$.~~

$$i_* 1 = c_{n'}(\mathcal{O}(1) \otimes f^* E'^{\vee})$$

$$= (\text{unit in } Q(\mathbb{P}E^{\vee}))\left(\xi^{n'} - f^*(c_1(E'))\xi^{n'-1} + \cdots + (-1)^{n'} f^*(c_n E')\right)$$

Proof of VI on page 4 is too complicated. We are given

$$Q(\mathbb{P}E) \cong Q(Y)[[Z]] / (Z^n - c_1(E)Z^{n-1} + \cdots)$$

where $c_i(E)$ are nilpotent $i>0$, hence $\mathfrak{m} = (c_1(E), \ldots, c_n(E))$ is a nilpotent ideal in $Q(Y)$. Now

$$Q(\mathbb{P}E)/\mathfrak{m}\,Q(\mathbb{P}E) \underset{\bar\varphi(Z)}{\overset{\xi \leftarrow Z}{\simeq}} (Q(Y)/\mathfrak{m})[[Z]]/(Z^n)$$

$$\bar\varphi(\xi) \leftarrow W$$

$$\simeq (Q(Y)/\mathfrak{m})[[W]]/((\bar\varphi^{-1}(W))^n)$$

But $\bar\varphi^{-1}(W) \equiv W \cdot$ mod higher terms so the ideal $((\bar\varphi^{-1}(W))^n)$ is (W^n). Thus $Q(\mathbb{P}E)/\mathfrak{m}\,Q(\mathbb{P}E)$ has a basis $1, \ldots, \bar\varphi(\xi)^{n-1}$ and as $\mathfrak{m}$ is nilpotent and $Q(\mathbb{P}E)$ is free over $Q(Y)$ it follows that $1, \ldots, \bar\varphi(\xi)^{n-1}$ is a basis for $Q(\mathbb{P}E)$ as a $Q(Y)$-module.

NOTE: We have used that $c_i(E)$ is nilpotent, a fact that we don't know until after Thm 3 + remark on p. 22

residue formula

Theorem 9: Let E be a vector bundle over Y which is the direct sum of line bundles $L_1, \ldots, L_n$. Then the Gysin homomorphism $f_*: Q(PE^\vee) \longrightarrow Q(Y)$ is given by

$$f_*(a(\xi)) = \operatorname{res} \frac{a(Z)\,\omega(Z)}{\prod_{i=1}^{n} F(Z, I c_1(L_i))}$$

$\xi = c_1(\mathcal{O}(1))$
$a(Z) \in Q(Y)[[Z]]$

where $\omega(Z) = \dfrac{dZ}{F_1(0,Z)}$ is the invariant differential form and I is the inverse for the group law.

Proof: We use induction on n. Suppose $n=1$; ~~[illegible]~~ then f^* and f_* are isomorphisms inverse to each other and $f^*L = \mathcal{O}(1)$. It suffices to ~~[illegible]~~ prove that

$$a(x) = \operatorname{res} \frac{a(Z)\,\omega(Z)}{F(Z, I c_1(L))}$$

Now in

$$F(X,Y) = X + Y + XY\,G(X,Y)$$

set $Y = F(Z, IX)$ and you obtain

$$Z - X = F(Z, IX)\{1 + X\,G(X, F(Z,IX))\}.$$

Set $x = c_1(L)$ so $\quad Z - x = F(Z, Ix)\underbrace{\{1 + x\,G(x, F(Z, Ix))\}}_{\text{unit since } x \text{ nilpotent}}$

$$\therefore \operatorname{res} \frac{a(Z)\omega(Z)}{F(Z, I_x)} = \operatorname{res} \frac{\left(1 + xG(x, F(Z, I_x))\right)^{a(Z)} \omega(Z)}{Z - x}$$

$$= \frac{1 + xG(x, 0)}{F_2(x, 0)}\, a(x) = \qquad a(x)$$

since $F_2(X,Y) = 1 + XG(X,Y) + XYG_2(X,Y)$.

Suppose now that $n > 1$ *and let* $F = L_1 + \cdots + L_{n-1}$, $L = L_n$. *Let*

$$\mathbb{P}\check{L} \xrightarrow{\ i\ } \mathbb{P}\check{E} \xleftarrow{\ f\ } \mathbb{P}\check{F}$$

be the canonical inclusions. Then $\mathbb{P}\check{L}$ is the subvariety of $\mathbb{P}\check{E}$ where the section of $\mathcal{O}(1) \otimes f^*\check{F}$ given by

$$\mathcal{O} \xrightarrow{\text{can}} \mathcal{O}(1) \otimes f^*\check{E} \longrightarrow \mathcal{O}(1) \otimes f^*\check{F}$$

vanishes. This section is easily seen to be transversal to the zero-section. (Indeed this section associates to hyperplane H in $E(y)$ the composition map

$$F(y) \longrightarrow E(y) \longrightarrow E(y)/H = \mathcal{O}(1)(H) \qquad f(H) = y$$

which is identified with an element of $(\mathcal{O}(1) \otimes f^*\check{F})(y)$. The second vanishes when $H = F(y)$ and the vertical tangent space to $\mathbb{P}\check{E}$ at this point is canonically isomorphic to $\operatorname{Hom}(F, \mathcal{O}(1))(H)$; this latter is applied isomorphic to the fiber of $F \otimes f^*\mathcal{O}(1)$ at (H) so the section is transversal). Therefore by combining prop 7 + proof of prop. 2 we have that

$$i_* 1 = c_{n-1}(\mathcal{O}(1) \otimes f^*\check{F}) \prod_{j=1}^{n-1} F(\xi, f^* I x_j) \qquad x_j = c_1 L_j$$

$$=$$

where the latter results from the Whitney sum formula and the formal group law ~~the~~ (prop 8). Similarly

$$j_* 1 = c_1(\mathcal{O}(1) \otimes f^* \check{L}) = F(\xi, f^* I x_n)$$

I claim that the residue formula holds if $a(Z) = b(Z) F(Z, I x_n)$, indeed

$$f_*\left(b(\xi) F(\xi, f^* I x_n)\right) = f_*\left(b(\xi) j_* 1\right)$$
$$= f_* j_* \left(b(\xi')\right) \qquad \xi' = c_1(\mathcal{O}(1)) \text{ on } \mathbb{P}\check{F}$$

$$\operatorname{res} \frac{b(Z) F(Z, I x_n)\, \omega(Z)}{\prod_{j=1}^{n} F(Z, I x_j)} = \operatorname{res} \frac{b(Z)\, \omega(Z)}{\prod_{j=1}^{n-1} F(Z, I x_j)}$$

and these are equal by induction hypothesis applied to $\mathbb{P}\check{F} \longrightarrow Y$

~~Similarly the residue formula holds if $a(Z)$ is a multiple~~

~~of $\prod_{j}^{n-1}$~~. Since

$$Z - x_n = F(Z, I x_n) \cdot (\text{unit in } Q(Y)[[Z]])$$

the residue formula is true if $a(Z)$ is a multiple of $Z - x_n$. Similarly one proves the formula is true if $a(Z)$ is a multiple of $\prod_{j=1}^{n-1}(Z - x_j)$. By division algorithm

$$\prod_{j<n}(Z - x_j) = g(Z)(Z - x_n) + \prod_{j<n}(x_n - x_j)$$

so the formula holds if $a(Z)$ is a multiple of $\prod_{j<n}(x_n - x_j)$. Thus

$$(*) \qquad \prod_{j<n}(x_n - x_j)\left\{ f_* \, a(\xi) - \operatorname{res} \frac{a(Z)\,\omega(Z)}{\prod_{j=1}^{n} F(Z, I x_j)} \right\} = 0$$

But now we can argue universally: It suffices to take $a(Z) = Z^i$, take $Y = (\mathbb{P}^N \times \mathbb{P}^N)^n$ with $L_i = \mathcal{O}(1) \boxtimes \mathcal{O}(1)$ on the ith factor. Passing to the inverse limit as $N \to \infty$ we obtain an equation

$$\prod_{j<n}(x_n - x_j) \cdot u = 0$$

in the power series ring $Q(pt)[[y_1, z_1, \dots, y_n, z_n]]$ where

$$x_i = F(y_i, I z_i) = (y_i - z_i)(1 + \text{higher terms})$$

Thus

$$x_n - x_j = (y_n - z_n) + (y_j - z_j) + \text{higher terms}$$

This is a non-zero divisor since it is so in the associated graded ring, hence $u = 0$ and so the term of $(*)$ in brackets is zero. qed.

Corollary 10: The invariant differential form for F is

$$\omega(Z) = \sum_{n \geq 0} P_n Z^n \, dZ$$

where $P_n \in Q(pt)$ denotes the element $[\mathbb{P}_n \to pt]_* 1$. Consequently the logarithm of F (defined over $Q(pt) \otimes \mathbb{Q}$) is

$$\ell(Z) = \sum_{n \geq 0} \frac{P_n}{n+1} Z^{n+1}$$

<u>Proof</u>: Apply th. 9 to $f: \mathbb{P}^n \longrightarrow pt$ one finds

$$P_n = f_* 1 = \operatorname{res} \frac{\omega(Z)}{Z^{n+1}} \quad ,$$

hence P_n is the coefficient of $Z^n dZ$ in $\omega(Z)$. The assertion about $\ell(Z)$ results <u>from</u>: $d\ell(Z) = \omega(Z)$.

<u>Theorem 9'</u>: Assume that Q satisfies III' (page 10). If E is a vector bundle over Y then the Gysin homomorphism $f_* : Q(\mathbb{P}\check{E}) \longrightarrow Q(Y)$ is given by

$$f_* \, a(\xi) = \operatorname{res} \left[\frac{a(Z)\, \omega(Z)}{\operatorname{Norm}_{Q(\mathbb{P}\check{E})[[Z]]/Q(Y)[[Z]]} F(Z, I\xi)} \right]$$

for all $a(Z) \in Q(Y)[[Z]]$.

<u>Proof</u>. To see the residue is defined one may by the additivity axiom suppose that E has constant rank n. Since

$$F(Z, I\xi) = (\text{unit})(Z - \xi)$$

$$\operatorname{Norm} F(Z, I\xi) = (\text{unit})(Z^n - c_1(E) Z^{n-1} + \cdots)$$

which shows that $Q(Y)[[Z]] / (\operatorname{Norm} F(Z, I\xi))$ is free over $Q(Y)$ of rank n. Thus the residue is defined.

To check the formula we may by the splitting principle and the fact that Gysin commutes with transversal base change suppose that E has a quotient line bundle

$$0 \longrightarrow F \longrightarrow E \longrightarrow L \longrightarrow 0.$$

Let $g: X^{\mathcal{E}} \longrightarrow Y$ be the bundle of splittings of this exact sequence. Then g is an affine bundle with associated vector bundle $\mathrm{Hom}(L, E)$. By III$'$ g^* is an isomorphism so we suppose that E splits off a line bundle, ie $E \simeq L \oplus F$. Continuing this we may suppose that E is a sum of line bundles whence it follows from th 9. qed.

May 16, 1969

Riemann–Roch

We state it in the following form:

Theorem 11: Let Q and Q' be two theories satisfying I–VIII and let $\gamma : Q \longrightarrow Q'$ be a natural ring homomorphism. If $\gamma : Q(\mathbb{P}^n) \longrightarrow Q'(\mathbb{P}^n)$ carries $c_1(\mathcal{O}(1))$ into $c_1'(\mathcal{O}(1))$ for all n, then γ commutes with Gysin homomorphisms.

To deduce the usual R-R theorem take $Q = K$ and $Q' = A \otimes \mathbb{Q}$ but where the Gysin homorphism of Q' is the twist of that of A by the inverse Todd class. Thus

(see example 2 page 2-5.

$$f'_*(x) = f^A_*\left(x \cdot \mathrm{Todd}(\nu_f)^{-1}\right)$$

where $\mathrm{Todd}(X) = \dfrac{X}{1-e^{-X}} \in \mathbb{Q}[[X]]$.

For γ we take the character

$$ch : K \longrightarrow A \otimes \mathbb{Q} .$$

Now

$$ch\ c_1^K(L) = ch(1 - L^{-1}) = 1 - e^{-c_1^A(L)}$$

and

$$c_1'(L) = c_1^A(L)\,\mathrm{Todd}(c_1^A(L))^{-1} = c_1^A(L)\,\frac{1-e^{-c_1^A(L)}}{c_1^A(L)}$$

so these two are equal. Thus by thm. 11

$$ch\ f^K_* x = f^A_*\left(ch\,x \cdot \mathrm{Todd}(\nu_f)^{-1}\right)$$

which is the Riemann-Roch theorem of Grothendieck.

<u>Proof</u>: We check all the steps of Grothendieck's proof. So given a proper map $f: X \to Y$ we factor it

$$\begin{tikzcd} X \arrow[r, hook, "x \mapsto (x,0)"] \arrow[dr, "f"'] & X \times \mathbb{P}^2 \arrow[r, hook] \arrow[d, "f pr_1"] & Y \times \mathbb{P}^n \arrow[dl, "pr_1"] \\ & Y & \end{tikzcd}$$

Thus we have to show γ is compatible with Gysin for

$$pr_1 : Y \times \mathbb{P}^n \longrightarrow Y$$

and for an embedding $i: X \longrightarrow Y$ with two properties:

(i) i can be factored $X \xrightarrow{\varepsilon_0} X \times \mathbb{P}^1 \xhookrightarrow{h} Y$ $\quad \varepsilon_0(x) = (x, 0)$. + h embedding

(ii) ν_i splits off a trivial bundle of rank 2 ie $\nu_i \cong 2 \oplus E''$.

<u>Case of a projection</u>: Let F and F' be the formal group laws of Q and Q'. Suppose (for the moment) that the homomorphism $\gamma: Q(pt) \to Q'(pt)$ takes F to F'. We must show that

$$\gamma (pr_1)_* (z) = (pr_1)'_* \gamma z$$

for $z \in Q(Y \times \mathbb{P}^n)$. Both $\gamma (pr_1)_*$ and $(pr_1)'_* \gamma$ are $Q(Y)$-homomorphisms so it suffices to take $z = pr_2^* \, c_1(\mathcal{O}(1))^j$ where $\mathcal{O}(1)$ is the canonical bundle on $\mathbb{P}^n$. As Gysin is compatible with

transversal
basechange it suffices to prove that

$$\gamma f_* c_1(\mathcal{O}(1))^i \overset{?}{=} f'_* \gamma c_1(\mathcal{O}(1))^i$$

where $f: \mathbb{P}^n \to pt$ is the canonical map.

By assumption $\gamma c_1(\mathcal{O}(1)) = c_1'(\mathcal{O}(1))$, so we must show that

$$\gamma f_* c_1(\mathcal{O}(1))^i = \gamma \operatorname{res}\left(\frac{Z^i \omega(Z)}{Z^{n+1}}\right) \quad \text{and}$$

$$f'_*(c_1'(\mathcal{O}(1)))^i = \operatorname{res}\left(\frac{Z^i \omega'(Z)}{Z^{n+1}}\right)$$

are equal. But this is clear since $\gamma F = F' \Rightarrow \gamma\omega = \omega'$.

~~Case of an embedding satisfying (i) + (ii). To conform to Grothendieck's notation we write $i: Y \to X$, and introduce the Cartesian (not transversal) square~~

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{j} & \tilde{X} \\ \downarrow g & & \downarrow f \\ Y & \xrightarrow{i} & X \end{array}$$

~~where $f: \tilde{X} \to X$ is the result of blowing up~~

~~It remains to show that $\gamma F = F'$. I claim that for any line bundle L over a variety Y that~~

$$\cancel{\gamma c_1(L) = c_1'(L)}$$

~~or more generally that~~

$$\begin{array}{ccc} K(Y) & \xrightarrow{c_2} & Q(Y)[[Z]]^* \\ & \searrow^{c_2'} & \downarrow \gamma \\ & & Q(Y)[[Z]]^* \end{array}$$

~~[illegible]~~

It remains to show that $\gamma F = F'$.

<u>Lemma</u>: If $y \in K(Y)$, then $\gamma \, c_j(y) = c_j'(y)$.

<u>Proof</u>: ~~Consider $Y \to Q'(Y)[[Z]]$~~

Fix an integer n and let

$$Y \longmapsto Q'(Y) \otimes_{\mathbb{Z}} \mathbb{Z}[Z]/(Z^{n+1})$$

be considered as a theory in the obvious way (example 1'). Then

$$K(Y) \rightrightarrows \left[Q'(Y)[Z]/(Z^{n+1})\right]^*$$

$$y \longmapsto \sum_{j=0}^{n} c_j'(y) Z^j$$

$$y \longmapsto \sum_{j=0}^{n} \gamma \, c_j(y) Z^j$$

are two natural homomorphisms coinciding on the canonical bundles on P^N for all N; by corollary 4 they are equal. qed.

From the lemma one sees that

$$\gamma \, c_1(L) = c_1'(L)$$

for all line bundles L, hence

$$c_1'(L \otimes L') = \gamma \, c_1(L \otimes L')$$

$$= \gamma \, F(c_1 L, c_1 L')$$

$$= (\gamma F)(c_1' L, c_1' L')$$

But the formal group law is unique by Prop 8 hence $\gamma F = F'$.

<u>Case of an embedding satisfying (i) + (ii).</u> To conform to Grothendieck's notation let the embedding ~~by~~ $i: Y \to X$. Let

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{j} & \tilde{X} \\ \downarrow g & & \downarrow f \\ Y & \xrightarrow{i} & X \end{array}$$

be the (non-transversal) cartesian square where $\tilde{X}$ is the result of blowing up X along Y. Then ~~$\tilde{Y}$~~

$$\tilde{Y} = \mathbb{P}(\nu_i)$$

and $\tilde{Y}$ is a divisor in $\tilde{X}$ associated to the bundle $\mathcal{O}_{\tilde{X}}(-1)$. By assumption i may be factored

$$Y \xrightarrow{\varepsilon_0} Y \times \mathbb{P}^1 \xrightarrow{h} X \qquad\qquad \varepsilon_t(y) = (y, t)$$

where h is an embedding. Thus one has a distinguished (non-vanishing) section of ν_i the tangent to the motion i_t at $t=0$. ~~Claim~~ Let $s: Y \to \tilde{Y}$ be the induced section of $\mathbb{P}\nu_i$.

<u>Lemma</u>: (i) $f'_* 1 \in Q'(X)^*$

(ii)
$$f^* i_* y = j_*(s_* 1 \cdot g^* y) \qquad \text{for } y \in Q(Y)$$
$$f'^* i'_* y' = j'_*(s'_* 1 \cdot g'^* y') \qquad \text{for } y' \in Q'(Y).$$

(iii) $s_* 1$ is divisible by $j^* j_* 1$ in $Q(\tilde{Y})$

(iv) $\gamma s_* 1 = s'_* 1$

(v) $\gamma j_* 1 = j'_* 1$.

Assuming these we will prove

$$\gamma\, \iota_* y = \iota'_* \gamma y \qquad \text{for } y \in Q(Y).$$

By (i) and the projection formula f'^* is injective so it suffices to prove that these become equal after f'^* is applied. Note that by (iii) $\exists u$ with

$$s_* 1 = j^* j_* 1 \cdot u$$

so

$$\begin{aligned} s_* 1 \cdot g^* y &= j^* j_* 1 \cdot u \cdot g^* y \\ &= j^*(j_*(u \cdot g^* y)) && \text{by axiom VIII} \\ &= j^* a \end{aligned}$$

where $a = j_*(u \cdot g^* y) \in Q(\tilde{X})$. Thus

$$\begin{aligned} f'^*(\gamma\, \iota_* y) &= \gamma f^* \iota_* y \\ &= \gamma\, j_*(s_* 1 \cdot g^* y) && \text{(ii)} \\ &= \gamma\, j_* j^* a \\ &= \gamma\, (j_* 1 \cdot a) \\ &= j'_* 1 \cdot \gamma a && \text{(v)} \end{aligned}$$

$$\begin{aligned} f'^*(\iota'_* \gamma y) &= j'_*(s'_* 1 \cdot g'^* \gamma y) && \text{(ii)} \\ &= j'_*(\gamma(s_* 1 \cdot g^* y)) \\ &= j'_*(\gamma\, j^* a) = j'_* j'^* \gamma a \\ &= j'_* 1 \cdot \gamma a \end{aligned}$$

and so we are done.

It remains to prove the lemma.

(i): Let U be the complement of Y in X. Then as f is an isomorphism over U

$$\begin{tikzcd} U \arrow[r] \arrow[d, "id"'] & \tilde{X} \arrow[d, "f"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

is transversal cartesian and $j^* f_* 1 = 1$. Thus $j^*(f_* 1 - 1) = 0$ so by exactness of

$$Q(Y) \xrightarrow{\iota_*} Q(X) \xrightarrow{j^*} Q(U)$$

there is an element $v \in Q(Y)$ with $\iota_* u = f_* 1 - 1$. But

$$(\iota_* u)^2 = \iota_*(u \cdot (\iota^* \iota_* u))$$

and $\iota^* \iota_* u = \iota_1^* \iota_* u$ (by homotopy axiom) where ι_1 is $Y \xrightarrow{y \mapsto (y,1)} Y \times \mathbb{P}^1 \hookrightarrow X$ and

$$\begin{tikzcd} \phi \arrow[r] \arrow[d] & Y \arrow[d, "\iota"] \\ Y \arrow[r, "\iota_1"] & X \end{tikzcd}$$

cartesian + transversal $\Rightarrow$ $\iota_1^* \iota_* x = 0$ $\quad (Q^*(\phi) = 0$ by additivity$)$

$\therefore$ $(\iota_* u)^2 = 0$ so

$$f_* 1 = 1 + \iota_* u$$

is a unit. Similarly for Q'.

(ii). We need the following

<u>Lemma:</u> If $a, b \in \mathbb{P}^1$ then $(\varepsilon_a)_* = (\varepsilon_b)_*$
where $\varepsilon_t : Y \longrightarrow Y \times \mathbb{P}^1$ is the map $y \longmapsto (y, t)$

<u>Proof:</u> $(\varepsilon_a)_*$ and $(\varepsilon_b)_*$ are $Q(Y)$ module homomorphisms so its enough to show $(\varepsilon_a)_* 1 = (\varepsilon_b)_* 1$. ~~and by base extension~~ However these are both ~~the classes of O(1)~~ equal to $1 \otimes c_1 \mathcal{O}(1)$, $\mathcal{O}(1)$ = canonical line bundle on $\mathbb{P}^1$.

The point is that ~~the possibility~~ there exists a lifting

$$\begin{tikzcd} & \tilde{X} \arrow[d] \\ Y \times \mathbb{P}^1 \arrow[ur, dashed, "\theta"] \arrow[r, hook, "h"] & X \end{tikzcd}$$

In effect if I is the ideal of Y in X, then $h^* I$ is the ideal of the embedding $Y \xrightarrow{\varepsilon_0} Y \times \mathbb{P}^1$ which is principal as this is a divisor; hence by the universal property of a blowup θ exists. Now from the way s was defined

$$\begin{tikzcd} Y \arrow[rr, "j"] & & \tilde{X} \arrow[d, "f"] \\ Y \arrow[u, "s"] \arrow[r, "\varepsilon_0"] & Y \times \mathbb{P}^1 \arrow[ur, "\theta"] \arrow[r, hook, "h"] & X \end{tikzcd}$$

commutes. For $t \neq 0$

$$\begin{tikzcd} Y \arrow[r, "\theta\varepsilon_t"] \arrow[d, "id"] & \tilde{X} \arrow[d, "f"] \\ Y \arrow[r, "h\varepsilon_t"] & X \end{tikzcd}$$

is cartensian since f is an isomorphism over $X - Y$. Thus

$$f^* \iota_* y = f^* h_*(\varepsilon_0)_* y$$

$$= f^* h_*(\varepsilon_1)_* y \qquad \text{by lemma}$$

$$= \theta_*(\varepsilon_1)_* y \qquad \text{by cartesian axiom}$$

$$= \theta_*(\varepsilon_0)_* y \qquad \text{by lemma}$$

$$= j_* s_* y$$

$$= j_* s_*(s^* g^* y) \qquad gs = id_Y$$

$$= j_*(s_* 1 \cdot g^* y) \qquad \text{proj. fmla}$$

Same argument works for Q'.

(iii). Here I assume known that the divisor $\tilde{Y}$ in $\tilde{X}$ has associated line bundle $\mathcal{O}_{\tilde{X}}(-1)$ and $j^* \mathcal{O}_{\tilde{X}}(-1) = \mathcal{O}_{\tilde{Y}}(-1)$. Thus

$$j^* j_* 1 = j^* c_1(\mathcal{O}_{\tilde{X}}(-1)) = c_1(\mathcal{O}_{\tilde{Y}}(-1))$$

Also s is defined by means of a trivial sub-line-bundle of ν_i which I will now denote E. Thus

$$E = 1 \oplus E' = 1 \oplus 1 \oplus E''$$

and $s : Y \longrightarrow \tilde{Y} = \mathbb{P}E$ is the line generated by 1. ~~It follows that~~ ~~It is where the section~~

~~vanishes, so~~

Let

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow g^*E \xrightarrow{\pi} F \longrightarrow 0$$

be the canonical exact sequence. Then we get a section

$$\mathcal{O} = g^*1 \longrightarrow g^*1 \oplus g^*E' = g^*E \xrightarrow{\pi} F$$

of F which vanishes ~~where~~ at those lines in $\mathbb{P}E$ where the line equals the 1 in the decomposition $E = 1 + E'$. Thus this section of F, which is evidentally transversal to zeros, vanishes on $s(Y)$ so

$$s_* 1 = c_{n-1}(F)$$

But

$$c_t(\mathcal{O}(-1)) \cdot c_t(F) = g^* c_t(E) = g^* c_t(E'')$$

so comparing coefficients of t^{n-1} we have

$$1 \cdot c_{n-1}(F) + c_1 \mathcal{O}(-1) \cdot c_{n-2}(F) = g^* c_{n-1}(E'') = 0$$

since $\dim E'' = n-2$. Thus $s_* 1 = c_{n-1}(F)$ is divisible by $c_1 \mathcal{O}(-1) = j^* j_* 1$ as claimed.

(iv), (v). We just saw that $s_* 1 = c_{n-1}(F)$, $j_* 1 = c_1(\mathcal{O}_{\tilde{X}}(-1))$ so the result follows from the fact that γ preserves Chern classes by the lemma on page 32.

The proof of R-R is now complete

Corollary 12. (universal property of K-theory). Let Q be a theory satisfying I – VIII whose group law is

$$F(X,Y) = X + Y - XY$$

Then there is a unique natural multiplicative transformation $\gamma : K \longrightarrow Q$ compatible with Gysin homomorphism.

Proof: ~~If it exists is the additive transf from K to Q given on line bundles by~~

$$\cancel{\gamma(c_1^K(L)) = c_1^Q(L)}$$

~~that~~

$$\cancel{\gamma(L) = c_1^Q(L^{-1})}$$

If γ exists then

$$c_1^Q(L) = \gamma\, c_1^K(L) = \gamma(1 - L^{-1}) = 1 - \gamma(L)^{-1}$$

so

$$\gamma(L) = \frac{1}{1 - c_1^Q(L)}$$

for all line bundles L. Thus γ if it exists is the unique additive transformation from K to Q given on line bundles by this formula; in terms of cor. 6 it is the additive transf. associated to the power series $\frac{1}{1-X}$. So γ is unique.

To show γ exists define it to be ~~the~~ this unique additive transformation. By R-R th 11 we must only prove γ is a ring homomorphism, i.e.

$$\gamma(E \otimes F) = \gamma(E)\,\gamma(F)$$

for ~~all~~ vector bundles E, F over the same Y. We may suppose

E and F possess flags and so by induction on dimension we reduce to proving that

$$\mathcal{T}(L \otimes L') = \mathcal{T}(L)\,\mathcal{T}(L')$$

for two line bundles L, L' over the same Y. But

$$\mathcal{T}(L\otimes L') = \frac{1}{1-c_1^Q(L\otimes L')} = \frac{1}{1-c_1^Q L - c_1^Q L' + c_1^Q L \cdot c_1^Q L'}$$

$$= \frac{1}{1-c_1^Q L}\cdot\frac{1}{1-c_1^Q L'} = \mathcal{T}(L)\,.\,\mathcal{T}(L')$$

qed.

<u>Remark:</u> Corollary 12 says that K is the initial object of the category of theories satisfying I–VIII with group law $X+Y-XY$. We define cobordism theory Ω to be the initial object (if it exists) of the category of all theories. In the next section we shall prove the existence of $\Omega \otimes_{\mathbb{Z}} \mathbb{Q}$, the initial object of the category of theories with values in $\mathbb{Q}$-algebras.

We now derive the Conner-Floyd theorem giving K as a functor of Ω assuming the latter exists. Let F_u be a formal group law over a ring L which is universal in the sense of Lazard, i.e. every law over R comes from a unique homomorphism $L \to R$. Then there is are canonical maps

$$L \longrightarrow \Omega(pt)$$
$$L \longrightarrow \mathbb{Z}$$

given by the group law of Q and the law $X+Y-XY$.

~~[illegible]~~

There is also a canonical ~~map~~ morphism of theories

$$\Omega \longrightarrow K$$

given by the universal property of Ω. The diagram

$$\begin{tikzcd} L \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\ \Omega(Y) \arrow[r] & K(Y) \end{tikzcd}$$

is commutative (for each Y) whence we obtain a ring homomorphism

$$(*) \qquad \Omega(Y) \otimes_L \mathbb{Z} \longrightarrow K(Y)$$

which in fact is compatible with Gysin homomorphisms, defined by base extension from that of Ω.

? $\underline{\text{Corollary 13}}$. $\overbrace{}^{\text{(Conner-Floyd theorem)}}$ The map $(*)$ is an isomorphism. ?

~~Proof. (It seems impossible to give a nice proof of this by the way things have been arranged before).~~

It seems at this point that I need some extra conditions. The argument I want to use runs as follows: Both sides are initial objects of the category of theories with law $X+Y-XY$. Unfortunately the left side doesn't satisfy the exactness ~~thm~~ axiom. Two remedies:

A. Replace exactness axiom by stronger axiom

VII'. If $i : Y \to X$ is a closed embedding with complement $j : U \to X$, then

$$Q(Y) \xrightarrow{i_*} Q(X) \xrightarrow{j^*} Q(U) \longrightarrow 0$$

is exact.

and work with theories satisfying this additional axiom.

B. Assume Ω is universal for theories satisfying all axioms except VII and that Ω itself satisfies VII. Then go through preceding and show that everything works without Q satisfying VII. For example since you have $\Phi : \Omega \to Q$ and $\Phi\, c_i^{\Omega}(E) = c_i^{Q}$ theorem 3 holds + Chern classes are nilpotent, there exists a unique formal group law + residue formula th 9 holds. Check that R-R holds (note that use of exactness to prove (i) p. 33 not necessary since this comes from Ω)

Existence of Ω_Q

In this section we shall work with theories Q such that $Q(Y)$ is an algebra over $\mathbb{Q}$. We recall that a formal group law F over Q possesses a logarithm series $\ell(X)$ ~~[illegible]~~ uniquely characterized by

$$\begin{cases} \ell(X) = X + \cdots \\ \ell(F(X,Y)) = \ell(X) + \ell(Y) \end{cases}$$

Moreover

$$d\ell(Z) = \omega(Z) = \frac{dZ}{F_2(0,Z)}$$

Thus for a theory Q we have

$$\ell(X) = \sum_{n\geq 0} P_n \frac{X^{n+1}}{n+1}$$

where $P_n = [\mathbb{P}_n \longrightarrow pt]_* 1 \in Q(pt)$.

Define the character

$$ch : K \longrightarrow Q$$

to be the unique additive transformation (Cor. 6) given on line bundles by

$$ch\, L = e^{\ell(c_1^Q(L))}$$

Then

$$ch(L_1 \otimes L_2) = e^{\ell(c_1^Q(L_1\otimes L_2))} = e^{\ell(F(c_1^Q L_1, c_1^Q L_2))} = e^{\ell(c_1^Q L_1) + \ell(c_1^Q L_2)}$$

$$= ch\, L_1 \cdot ch\, L_2 .$$

One deduces from this that ch is a ring homomorphism. For a line bundle L one has that

$$ch\, c_1^K(L) = ch(1-L^{-1}) = 1 - e^{-\ell\, c_1^Q(L)}$$

Therefore if $\psi(X) = \frac{1-e^{-\ell(X)}}{X} \in Q(pt)[[X]]^*$ we obtain a new Gysin homomorphism f'_* on Q given by

$$f'_*(x) = f_*(x \cdot \Psi(\nu_f))$$

and

$$c_1'(L) = \iota^* \iota'_*(1) = \iota^* \iota_* \Psi(c_1 L)$$

$$= c_1 L \cdot \psi(c_1 L) = 1 - e^{-\ell(c_1 L)} = ch\, c_1^K(L)$$

so by R-R, ch is compatible with Gysin homom. i.e.

$$\boxed{ch\, f_*^K x = f_*^Q\left(d x \cdot \Psi(\nu_f)\right)}$$

Now let L be a polynomial ring over $\mathbb{Q}$ with generators $p_1, p_2, \ldots$ and set

$$K_L(Y) = L \otimes_{\mathbb{Z}} K(Y)$$

~~it is a contravariant functor from Var to (rings)~~

As L is flat over $\mathbb{Z}$, K_L is a theory with Gysin homomorphism induced by that of K, i.e.

$$f_*^{K_L}(\ell \otimes x) = \ell \otimes f_*^{K}(x)$$

Let $\varphi(X) \in \left(L \otimes_{\mathbb{Z}} K(pt)\right)[[X]]^*$ be a power series with leading coefficient 1 to be determined later. We define a theory Γ by twisting K_L via the (multiplicative) characteristic class associated to φ. Thus

$$\Gamma(Y) = K_L(Y)$$

$$f_*^{\Gamma}(z) = f_*^{K_L}(z \cdot \tilde{\varphi}(\nu_f)).$$

Define a natural ring homomorphism

$$\gamma : L \otimes_{\mathbb{Z}} K(Y) \longrightarrow Q(Y)$$

by requiring γ to be ch on $K(Y)$ and

$$\gamma(\rho_i) = P_i \,.$$

I claim that φ may be chosen so that $\gamma : \Gamma \longrightarrow Q$ is a morphism of theories. As γ is a ring homomorphism it suffices (by R-R) to make γ compatible with Chern classes of line bundles, i.e. we want

$$\gamma\, c_1^{\Gamma}(L) = c_1^{Q}(L)$$

But

$$c_1^{\Gamma}(L) = \bar{\varphi}(c_1^{K}(L)) \qquad \text{so}$$

$$\gamma c_1^{\Gamma}(L) = (\gamma \bar{\varphi})(\gamma c_1^{K}(L)) = (\gamma\bar{\varphi})(ch\; c_1^{K}(L))$$

$$= (\gamma\bar{\varphi})\left(1 - e^{-\ell(c_1^Q(L))}\right)$$

Therefore we ~~…~~ have to show that we can choose φ so that $\gamma\bar{\varphi}$ is the inverse of $1 - e^{-\ell(X)}$. But

$$\ell(X) = \sum_{n\geq 0} P_n \frac{X^{n+1}}{n+1} = \gamma \sum p_n \frac{X^{n+1}}{n+1}$$ $P_0 = 1$ $p_0 = 1$

$\therefore$ We take

$$\bar{\varphi}(X) = \text{inverse of the series } 1 - e^{-\sum_{n\geq 0} p_n \frac{X^{n+1}}{n+1}}$$

$$\text{in } \quad L \otimes_{\mathbb{Z}} K(pt)^*[[X]] = \mathbb{Q}[p_1, p_2, \cdots][[X]].$$

Theorem 12: Given a theory Q ~~…~~ with values in $\mathbb{Q}$-algebras, there is a unique morphism of theories $\gamma : \Gamma \longrightarrow Q$.

We've just proved the existence of γ. To prove uniqueness one first notes that ~~…~~ γ must carry the ~~…~~ group law of Γ into that of Q. ~~…~~ But the Gysin homomorphism ~~…~~ of Γ has been rigged so that the logarithm is

$$\ell^\Gamma(X) = \sum p_n \frac{X^{n+1}}{n+1}$$

In effect

$$1 - e^{-\sum p_n \frac{(c_1^K L)^{n+1}}{n+1}} = c_1^K(L)$$

so

$$\sum p_n \frac{(c_1^\Gamma L)^{n+1}}{n+1} = \frac{1}{1 - c_1^K(L)}.$$

The ~~diff~~ right side is additive for tensor products implying that ~~…~~ $\sum p_n \frac{z^{n+1}}{n+1}$ is the logarithm of the group law of Γ. Therefore $\gamma \ell^\Gamma = \ell^Q$ so

$$\gamma(p_n) = P_n$$

Next note that

$$K \xrightarrow{1\otimes} \Gamma \xrightarrow{\gamma} Q$$

is an additive transformation, hence given by a power series u in $Q(pt)[[X]]$

$$\gamma(1 \otimes L) = u(c_1^Q(L)),$$

and as γ is a ring homomorphism we must have

$$u(0) = 1$$

$$u(F^Q(X,Y)) = u(X) \cdot u(Y)$$

Therefore

$$\log u(X) = \alpha \ell^Q(X). \qquad \alpha \in Q(pt)$$

so

$$(*) \qquad \gamma(1 \otimes L) = e^{\alpha \ell^Q(c_1^Q(L))}$$

for some $\alpha \in Q(pt)$. We show $\alpha = 1$ as follows. Note that

$$K \xrightarrow{1\otimes} \Gamma$$

satisfies a R-R theorem

$$1 \otimes f_*^K x = f_*^\Gamma\left((1 \otimes x)\, \tilde{\varphi}(\nu_f)\right).$$

~~If~~ As $\gamma : \Gamma \longrightarrow Q$ commutes with Gysin the same is true for $\gamma_0(1\otimes)$.

Thus $\gamma_0(1\otimes)$ commutes with Gysin for f with $\nu_f = 0$.

However the R-R for $(*)$ shows that one gets a factor α. More precisely if $L = \mathcal{O}(1)$ on $\mathbb{P}^1$ and $i: pt \longrightarrow \mathbb{P}^1$

$$\gamma(1\otimes(1 - L^{-1})) = 1 - e^{-\alpha \ell^{Q}(c_1^{Q}(L))} = \alpha c_1^{Q}(L)$$

$$\|$$

$$\gamma(1\otimes i_* 1) = i_*^{Q}(\gamma(1\otimes 1)) = i_* 1 = c_1^{Q}(L)$$

so $\alpha = 1$ as claimed. Therefore

$$\gamma(1\otimes x) = ch\, x$$

so the uniqueness of γ is proved and with it theorem 12!

<u>Remarks:</u> Thus we have proved the existence of $\Omega\otimes Q$ and ~~proved~~ also proved there is an isomorphism

$$\boxed{\Omega(Y)\otimes_{\mathbb{Z}} Q \cong Q[p_1, p_2, \cdots]\otimes_{\mathbb{Z}} K(Y)}$$

Moreover the cobordism ring tensored with Q is a polynomial ring generated by the projective spaces. $\Omega\otimes_{\mathbb{Z}} Q$ is a twist of K-theory base-extended by $\mathbb{Z} \longrightarrow Q[p_1, p_2, \cdots]$.

<u>Question:</u> Is $\Omega(Y)\otimes_{\mathbb{Z}} Q$ a universal Chern ring over Q for $K(Y)$?

(Probably although to prove so one would have to show how to define Gysin homomorphism on the Chern ring)

May 21, 1969

Affine categories and ~~[illegible]~~ operations in generalized cohomology theories.

$\boxed{\text{1. Cogebras.}}$

Definition: A- dicogebra.

$$\left(\begin{array}{l} \varepsilon : P \longrightarrow A \\ \Delta : P \longrightarrow P \otimes_A P. \end{array}\right.$$

$Com(P)$ left comodules.

$$M \longrightarrow P \otimes_A M.$$

Example: Two structures coincide

$\boxed{\text{2. Endomorphisms of a functor } h: \mathcal{A} \longrightarrow \text{Mod } A.}$

Assume $\exists$ right A-module $P \ni$

$$\mathrm{Hom}_{A^\circ}(P, M) \xrightarrow{\sim} \mathrm{Hom}_{\underline{\mathrm{Hom}}(\mathcal{A}^\circ, \mathcal{Ab})}(h, M \otimes h)$$

Then P is an A-dicogebra and h induces

$$h': \mathcal{A} \longrightarrow \mathrm{Com}\, P.$$

Prop: If $f: \mathrm{Com}\, P \longrightarrow \mathrm{Mod}\, A$ is the forgetful functor then $\mathrm{End}(f) = P$.

Proof: ~~[illegible]~~ ✓

~~The identity morphism of h gives rise to a A-bimodule ~~[illegible]~~ map~~

~~$$\varepsilon : P \longrightarrow A$$~~

~~and the ~~map~~ composition~~

~~$$h \longrightarrow P \otimes h$$~~

(3.) Sufficient conditions for P to exist

<u>Proposition</u>: Suppose

$$h(X) = \varinjlim \operatorname{Hom}(X, E_i)$$

where $h(E_i) \in P(A)$. Then $P = \varinjlim h(E_i)^\vee$.

(4.) ~~[illegible] products~~ ~~Dibigebras~~ Tensor products

A commutative

$P * P'$

$$\operatorname{Com}(P) \times \operatorname{Com}(P') \longrightarrow \operatorname{Com}(P * P')$$

~~[illegible]~~

(5.) Affine categories ~~+ groupoids.~~

define ⊗digebras over A. | represent functor
$\operatorname{Com}(P)$ = a tensor category. | antipode or inversion
ass. comm. unit. ⊗

<u>Prop</u>: $h : \mathcal{A} \longrightarrow \operatorname{Mod} A$ ⊗-functor, then $\underline{\operatorname{End}}^{\otimes} h$ is an affine category.

(6) Affine groupoids and existence of an antipode

~~[illegible]~~ Cohomology operations

$\mathcal{A}$ stable finite homot. cat

$h^*: \mathcal{A}^\circ \longrightarrow \text{grMod}\ h^*(pt)$

h gen. coh. theory with products

$$h^*(X) = \varinjlim_i \{X, E_i\}^* \quad \Big| \quad \text{where } h^*(E_i) \text{ proj. f.t.}$$

~~[illegible]~~

$$P = \varinjlim h^*(E_i)^\vee$$

moreover $\exists$ canonical map

$$h^*(X) \longrightarrow P \otimes_A h^*(X)$$

$$h^*(X) = \varinjlim_i \mathrm{Hom}^*(X, E_i)$$

$$\longrightarrow \varinjlim_i \mathrm{Hom}_{\mathrm{Modgr}(h^*pt)}(h^*E_i, h^*X)$$

$$\cong \varinjlim_i h^*_*(E_i)' \otimes_{h^*(pt)} h^*(X)$$

$$P \otimes_A h^*(X).$$

Conclusion: If F is a gr $h^*(pt)$ module, then

<u>Definition</u>: Given ~~[illegible], A, P as above~~ an affine category (A, P) we say that two morphisms $A \xrightarrow{u} R$ and $v: A \longrightarrow S$ are **equivalent** if the extended affine categories $(R, R_u \otimes_A P \otimes_A R_u)$ and $(S, S_v \otimes_A P \otimes_A S_v)$ are equivalent.

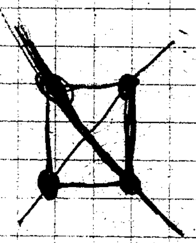
The reason for this definition is as follows. Let $h: \mathcal{A} \longrightarrow \mathrm{Com}\, P$ be a tensor functor. Then the extended functors h_u and h_v determine each other. In effect we are given

$$f: R \longrightarrow S$$

$$f_1: R \otimes P \otimes R \longrightarrow S \otimes P \otimes S$$

$$\begin{tikzcd} C_2 \arrow[r] \arrow[dr, "f.f."'] & C_3 \arrow[d, "f.f."] \\ & C_1 \end{tikzcd}$$

The basic question is it representa $\boxed{\text{no}}$



Lemma: P has unique ring structure $\ni$

$\gamma : h \longrightarrow P \otimes h$ compatible with products i.e.

$$\begin{tikzcd}
h(X) \otimes h(Y) \arrow[r, "\gamma \otimes \gamma"] \arrow[dd] & (P \otimes hX) \otimes (P \otimes hY) \arrow[d] & \\
& P \otimes P \otimes hX \otimes hY \arrow[dd] \arrow[dr] & \\
h(X \otimes Y) \arrow[dr, "\gamma"] & & P \otimes hX \otimes hY \arrow[dl, "\#"] \\
& P \otimes h(X \otimes Y) &
\end{tikzcd}$$

commutes. Moreover P is commutative + associative with unit. Finally if R is an A, A alg, then a map of A, A modules ${}_1P_2 \longrightarrow {}_1R_2$ is a ring homomorphism iff $h \longrightarrow R \otimes h$ is compatible with tensor product.

Proof: Uniqueness of ring structure. Suppose given $\mu : P * P \longrightarrow P \ni$ diag. commutes. Consider subcat gen. by E_i; then $hX \otimes hY \xrightarrow{\cong} h(X \otimes Y)$ isom so $\#$. isom so have a transf $hX \otimes hY \longrightarrow P \otimes hX \otimes hY$ which by above corresponds to the map

$$\mu_P : P * P \longrightarrow P$$

shows μ_P unique + how to define it. To show when thus defined, diagram above commutes for all X, Y. But we know it commutes if $X = E_i$ (resp $Y = E_j$) and any element of $\xi \in h(X)$ (resp $\eta \in h(Y)$) comes from $h(E_i)$ by maps $X \longrightarrow E_i$ (resp. ———).

Thus μ_P exists. To show P commutative we compare the maps

$$hX \otimes hY \longrightarrow h(X \otimes Y) \simeq h(Y \otimes X) \longrightarrow P \otimes h(Y \otimes X) \simeq P \otimes hY \otimes hX \simeq P \otimes hX \otimes hY$$

etc.

So the present attempt at proof runs as follows:

Given $\quad {}_1hX \otimes {}_1h'X' \longrightarrow {}_1F_2 \otimes {}_2hX \otimes {}_2h'X'$

fix X whence by the universal property of γ' it can be expressed uniquely as a composition

$$ {}_1hX \otimes {}_1h'X' \xrightarrow{id \otimes \gamma'} {}_1hX \otimes {}_1P'_2 \otimes {}_2h'X' \xrightarrow{\delta^{(X)} \otimes id} {}_1F_2 \otimes \cancel{P} {}_2hX \otimes {}_2h'X' $$

where

$$ \delta^{(X)} : {}_1hX \otimes {}_1P'_2 \longrightarrow {}_1F_2 \otimes {}_2hX $$

is a A-bimodule map, which is natural in X by ~~uniqueness~~ uniqueness. Thus by the universal property of γ^{3}, $\delta(X)$ ~~has~~ can be uniquely expressed as a composition

$$ \begin{array}{ccccc} {}_1hX \otimes {}_1P'_2 & \xrightarrow{\gamma \otimes id} & {}_1P_2 \otimes {}_2hX \otimes {}_1P'_2 & \xrightarrow[\sim]{id \otimes T} & {}_1P_2 \otimes {}_1P'_2 \otimes {}_2hX \\ & & & & \downarrow{\scriptstyle \mu \otimes id} \\ & & & & {}_1F_2 \otimes {}_2hX \end{array} $$

where $\mu : {}_1P_2 \otimes {}_1P'_2 \longrightarrow {}_1F_2$ is an A-bimodule map. Putting these together we find that θ may be uniquely expressed as the composition

$$ {}_1hX \otimes {}_1h'X' \xrightarrow{\gamma''} ({}_1P_2 \otimes {}_1P'_2) \otimes {}_2hX \otimes {}_2h'X \xrightarrow{\mu \otimes id \otimes id} {}_1F_2 \otimes {}_2hX \otimes {}_2h'X' $$

where ~~${}_1P_2 \otimes {}_1P'_2 \longrightarrow {}_1F_2$ is~~ γ'' is the composition

$$ {}_1hX \otimes {}_1h'X' \xrightarrow{\gamma \otimes \gamma'} ({}_1P_2 \otimes {}_2hX) \otimes ({}_1P'_2 \otimes {}_2h'X') \xrightarrow{id \otimes T \otimes id} ({}_1P_2 \otimes {}_1P'_2) \otimes ({}_2hX \otimes {}_2h'X') $$

Therefore $\mathrm{End}(h \otimes h')$ is representable by ${}_1P_2 \otimes {}_1P'_2$ as claimed ~~qed~~ q.e.

According ~~to~~ prop. it has a dicogebra structure which we ~~left~~ leave

Notes on G's notes of my stuff.

1. ~~Formal cats.~~ | the definition,
2. ~~Hom~~ Complexes of DR in general. | def.
3. Lie algebra assoc. to DR cx. |

~~Also attention to the first integral questions of generality~~

clean up

$\boxed{\text{operations}}$: $h : \mathcal{A}^{\circ} \longrightarrow \text{Mod } A$

assume that $\exists$ ind object $\{E_i \quad i \in I\}$ of $\mathcal{A}$ such that

$$h(X) = \varinjlim_i \operatorname{Hom}_{\mathcal{A}}(X, E_i)$$

work over a base X and establish

$$\varprojlim_E \Omega(P(E)) =$$

and that $h(E_i) \in P(\mathcal{A})$.

E

Lemma 1: For every A°-module F

$$\operatorname{Hom}_{\underline{Hom}(\mathcal{A}^{\circ}, ab)}(h, F \otimes h) = \operatorname{Hom}_{A^{\circ}}(P, F)$$

where $P = \varinjlim_i h(E_i)^{\vee}$ $\qquad h(E_i)^{\vee} = \operatorname{Hom}_A(h(E_i), A)$

Proof:

$$\operatorname{Hom}_{\underline{Hom}(\mathcal{A}^{\circ}, ab)}(h, F \otimes h) = \varprojlim_i \operatorname{Hom}(\operatorname{Hom}(?, E_i), F \otimes h)$$

$$= \varprojlim_i F \otimes h(E_i) = \varprojlim_i \operatorname{Hom}_{A^{\circ}}(h(E_i)^{\vee}, F)$$

$$= \operatorname{Hom}_{A^{\circ}}(P, F).$$

Question: Does $\exists$ a direct proof.

$$h : \mathcal{Q}^{\circ} \longrightarrow \mathrm{Modgr}(A) \qquad\qquad h(X) \underset{\substack{\cancel{h(*)} \\ A}}{\otimes} h(Y) \longrightarrow h(X \wedge Y)$$

suppose that

$$h^*(X) = \varinjlim_i \mathrm{Hom}^*(X, E_i)$$

for any A-module F ~~set~~

$$\mathrm{Hom}^*_{\mathbb{Z}}(h^*, F \otimes h^*) = \varprojlim_i F \otimes h^*(E_i) \qquad \text{Yoneda}$$

$$= \varprojlim_i \mathrm{Hom}_A(h^*(E_i)^{\vee}, F)$$

$$= \mathrm{Hom}_A(P, F)$$

where $P = \varinjlim_i \mathrm{Hom}_A(h^*(E_i), A)$

Structure of P: Call ~~[illegible]~~ above the right A-module structure

(i) left st.

$$\begin{tikzcd} h \arrow[r, "\gamma"] \arrow[d, "a"'] & P \otimes h \arrow[d, "\lambda_a \otimes id"] \\ h \arrow[r, "\gamma"] & P \otimes h \end{tikzcd}$$

Thus P is a bimodule. Check

$$\mathrm{Hom}_A(h, B \otimes h) = \mathrm{Hom}_{A,A}(P, B)$$

(ii) $h \xrightarrow[1 \otimes]{\cancel{id}} A \otimes h \Rightarrow \varepsilon : P \longrightarrow A$

(iii) $h \xrightarrow{\gamma} P \otimes h \xrightarrow{id \otimes \gamma} P \otimes P \otimes h \Longrightarrow \Delta : P \longrightarrow P \otimes_A P$

Proposition: Let A be a commutative ring and let

$$h : \mathcal{A} \longrightarrow \mathrm{Mod}\, A$$

$$h' : \mathcal{A}' \longrightarrow \mathrm{Mod}\, A$$

be functors such that End h (resp. End h') is represented by the dicogebra P (resp. P'). Let

$$h \otimes h' : \mathcal{A} \times \mathcal{A}' \longrightarrow \mathrm{Mod}\, A$$

be the functor $X, X' \longmapsto hX \otimes hX'$. Then End $(h \otimes h')$ is ~~also~~ represented by the dicogebra $P * P'$ defined as follows

$P * P'$ as an A-bimodule is the tensor product

$$({}_1P_2) \otimes_{A \otimes A} ({}_1P_2)$$

Thus $P * P'$ is generated by elts $p \otimes p'$ $\ni$

$$\begin{cases} ap \otimes p' = p \otimes ap' \\ pa \otimes p' = p \otimes p'a \end{cases}$$

and having the universal property associated with these identities

$$\begin{cases} \varepsilon_{P * P'}(p \otimes p') = \varepsilon_P(p)\, \varepsilon_{P'}(p') \\ \Delta_{P * P'} \qquad {}_1P_2 \otimes_{{}_1A^2} {}_1P'^{2} \longrightarrow \end{cases}$$

~~Thm~~ Make this affine category act on Ω as follows.

Given a group law F over a ring R, let

$$\Omega_F = R \underset{u}{\otimes}_A \Omega$$

where $u : A \longrightarrow R$ sends F_{univ} into F. Given another law F' and a power series $p(X) \ni p * F = F'$, one knows there exists a unique multiplicative transformation

$$\hat{p} : \Omega \longrightarrow \Omega_{F'}$$

such that $$\hat{p}\left(c_1^{\Omega} L\right) = (in_1 p)\left(in_2 c_1^{\Omega} L\right).$$

~~Here~~

$$R \xrightarrow{in_1} \Omega_F \xleftarrow{in_2} \Omega$$

are the inclusions. Moreover one notes that

$$\begin{tikzcd}
F_{univ} \arrow[r, mapsto] \arrow[drrr, mapsto, bend right] & F^{\Omega} \arrow[rr, mapsto] & & (in_1 p) * (in_2 F^{\Omega}) \\
A \arrow[r] \arrow[drr, bend right] & \Omega(pt) \arrow[r, "\hat{p}"] & \Omega_{F'}(pt) & \\
& & R \arrow[u, "in_1", "\wr"'] & F' \arrow[uu, mapsto]
\end{tikzcd}$$

commutes. Consequently $\hat{p}$ induces

$$\Omega_{F'} \longrightarrow \Omega$$

Let $\mathcal{A}$ = suspension category of finite complexes and let h^* be a generalized cohomology theory with products on the category of finite complexes. Let $A = h^*(pt)$. Denote by

$$h : \mathcal{A} \longrightarrow \mathrm{Modgr}(A)$$

the functor induced by $X \mapsto h(X)$. There is a natural transformation

$$h(X) \otimes_A h(Y) \longrightarrow h(X \wedge Y)$$

satisfying [illegible] some (rather obvious) ~~conditions of~~ associativity unit, and commutativity conditions.

Let $\mathcal{A}$ be the suspension category of finite CW complexes and let h be a generalized cohomology theory with products on the category of finite complexes. Let A be the graded (anti-)commutative ring $h^*(pt)$. Then h may ~~be~~ [illegible] viewed as [illegible] an additive functor

$$h : \mathcal{A} \longrightarrow \mathrm{Modgr}(A)$$

endowed with a natural transformation

$$(\quad) \qquad h(X) \otimes_A h(Y) \longrightarrow h(X \wedge Y)$$

[illegible] satisfying some rather obvious associativity, unit, and commutativity conditions.

If R is a commutative (anti-)commutative ring, ~~then~~ let [illegible] $(\mathrm{End}^{\otimes} h)(R)$ be the category whose objects are

~~ring homomorphisms $u: A \longrightarrow R$ and where a morphism from u to v is defined to be a natural transformation θ from h_u~~

If R is a (anti-commutative ~~[illegible]~~ graded) ring we define a category $(\mathrm{End}^{\otimes} h)(R)$ as follows. For objects we take the sets of morphisms $u: A \longrightarrow R$ of (anti-comm. graded) rings. Given such a u let

$$h_u : \mathcal{a} \longrightarrow \mathrm{Modgr}(R)$$

be the functor $X \longmapsto R_u \otimes_A h(X)$, where R_u denotes R ~~as an~~ ~~[illegible]~~ endowed with the A-algebra structure coming via u. Observe that ~~[illegible]~~ h_u is a tensor functor, ~~[illegible]~~ i.e. provided with a natural transformation

$$(*) \qquad h_u(X) \otimes_R h_u(Y) \longrightarrow h_u(X \wedge Y) .$$

~~[illegible]~~

$$\cancel{(R_u \otimes_A h(X)) \otimes_R (R_u \otimes_A h(Y)) \longrightarrow R \otimes (h X \otimes h Y) \longrightarrow R \otimes h(X \wedge Y)}.$$

We define a morphism in $(\mathrm{End}^{\otimes} h)(R)$ from u to v to be a stable R-linear natural transformation $\theta: h_u \longrightarrow h_v$ which is ~~[illegible]~~ compatible with the tensor ~~[illegible]~~ structure, i.e. the natural transformation $(*)$.

We wish ~~to~~ prove that the functor $\mathrm{End}^{\otimes} h$ from rings to categories is represented by a quasi-bigebra P over A. For this we need the following condition signalled by Adams in connection with ~~[illegible]~~ generalizations of the Adams spectral sequence [ ].

(**) There is a filtered inductive system E_i in $\mathcal{A}$ such that for all X

$$h(X) = \varinjlim_i \pi^*(X, E_i)$$

and such that for each i, $h(E_i)$ is a finitely generated projective A-module.

<u>Theorem:</u> Suppose h is a generalized cohomology theory with products satisfying (**). Then the functor $\mathrm{End}^{\otimes} h$ is represented by a quasi-bigebra P over A. Moreover if M is an A-module, then

$$\mathrm{Homset}^*_{\mathbb{Z}}(h, M \otimes_A h) \simeq \mathrm{Hom}^*_A(P, M)$$

~~§1. Endomorphisms of a tensor functor.~~

~~Suppose that A is commutative and that~~

~~$$h : \mathcal{A}^{o} \longrightarrow \mathrm{Modgr}(A)$$~~

~~is a functor. Suppose that~~

Proof: (In outline)

Yoneda's lemma

$$\mathrm{Homst}^{*}_{\mathbb{Z}}(h, M \otimes h) = \varinjlim_{i} M \otimes_{A} h(E_i)$$

$$= \varinjlim_{i} \mathrm{Hom}^{*}_{A}(h(E_i)^{\vee}, M)$$

$$= \mathrm{Hom}^{*}_{A}(P, M)$$

where

$$P = \varinjlim_{i} h(E_i)^{\vee}$$

Let this $\overset{\text{A-module}}{\text{~~structure~~}}$ of P be called the right one. To define the left one let $a \in A^{o}$. Then $\exists!\ a^{\#} \ni$

$$\begin{array}{ccc} hX & \xrightarrow{\ \gamma\ } & P \otimes hX \\ {\scriptstyle a\cdot}\downarrow & & \downarrow{\scriptstyle a.\otimes id} \\ hX & \xrightarrow{\ \gamma\ } & P \otimes hX \end{array}$$

Thus P becomes an A-bimodule. **Check that if M is an A-bimodule then**

$$\mathrm{Homst}^{*}_{A}(h, M \otimes h) = \mathrm{Hom}_{A,A}(P, M)$$

In effect $\Leftarrow$ clear and given $\theta : hX \to M \otimes hX$ left A-linear, let $P \xrightarrow{u} M$ be the right linear map $\ni$ $\theta = (u \otimes id)\gamma$. Then if $a \in A^{0}$ $(au \otimes id)\gamma x = a\theta x = \theta a x = (u \otimes id)\gamma a x = (ua \otimes id) \circ \gamma \Rightarrow u(ap) = a.u(p)$.

Define product structure on P as follows: Start with

$$_1h(X \wedge Y) \xrightarrow{\gamma} {}_1P_2 \otimes {}_2h(X \wedge Y).$$

Use $\boxed{A}$ Let $\mathcal{a}' \subset \mathcal{a}$ be the full subcategory containing the $\underbrace{E_i}_{\text{and suspensions}}$. Then

$$\mathrm{Homst}_{\mathbb{Z}}(h/\mathcal{a}', M \otimes h/\mathcal{a}') \cong \mathrm{Hom}_A(P, M)$$

(same argument)

$\boxed{B}$ **On this subcategory**

$$h(X \wedge Y) \xleftarrow{\sim} hX \otimes_A hY$$

so that we have

$$hX \otimes hY \longrightarrow$$

Claim: $\exists!\, \mu : {}_1P_2 \otimes {}_1P_2 \longrightarrow {}_1P_2 \ni$

$$\begin{tikzcd}
{}_1hX \otimes_1 hY \arrow[r, "\gamma \otimes \gamma"] \arrow[d] & ({}_1P_2 \otimes_2 hX) \otimes ({}_1P_2 \otimes_2 hX) \arrow[r, "\cong", "id \otimes T \otimes id"'] & ({}_1P_2 \otimes {}_1P_2) \otimes ({}_2hX \otimes_2 hY) \arrow[d, "\mu \otimes id"] \\
{}_1h(X \wedge Y) \arrow[r, "\gamma"] & {}_1P_2 \otimes_2 h(X \wedge Y) & {}_1P_2 \otimes ({}_2hX \otimes_2 hY) \arrow[l, "\square"']
\end{tikzcd}$$

To prove the claims we first work on the category $\mathcal{A}'$. Then the map $\square$ is an isom so we have

$$\theta : {}_1hX \otimes {}_1hY \longrightarrow {}_1P_2 \otimes {}_2hX \otimes hY$$

~~[illegible]~~ Fix X; by ~~[illegible]~~ universal property of γ θ comes from

$${}_1hX \xrightarrow{\otimes\, {}_1P_2} {}_1P_2 \otimes_2 hX$$

which also comes from

$$\mu : {}_1P_2 \otimes {}_1P_2 \longrightarrow {}_1P_2 .$$

~~[illegible]~~ It's clear that square // commutes if $X, Y \in \mathcal{A}'$ but holds in general since any element $x \in hX$ is induced by a map $X \longrightarrow E_i$.

So now I've shown μ exists. Next to show μ ring structure. ~~[illegible]~~ Associativity: Start with

$$hX \otimes hY \otimes hZ \longrightarrow$$

$$\cancel{hX \otimes hY \otimes hZ}$$

$$\cancel{h(X \wedge Y \wedge Z)}$$

May 21, 1969

Operations in generalized cohomology theories

§1. Cogebras.

Let A be a ring with unit but not necessarily commutative. Let $\mathcal{A}$ be a category and let

1.1 $$h : \mathcal{A}^{\circ} \longrightarrow \text{Mod}\, A$$

be a functor. For any right A-module M let

$$M \otimes h : \mathcal{A}^{\circ} \longrightarrow \mathcal{A}b$$

be the functor $X \longmapsto M \otimes_A hX$. ~~we say that End h is~~ Suppose that the functor from $\text{Mod}\, A^{\circ}$ to $\mathcal{A}b$

$$M \longmapsto \text{Hom}_{\text{Hom}(\mathcal{A}^{\circ}, \mathcal{A}b)}(h, M \otimes h)$$

is representable by a right A module P, i.e. there is a canonical morphism of functors

1.2 $$\gamma : h \longrightarrow P \otimes h$$

such that

1.3 $$\text{Hom}_{A^{\circ}}(P, M) \xrightarrow{\ \cong\ } \text{Hom}_{\underline{\text{Hom}}(\mathcal{A}^{\circ}, \mathcal{A}b)}(h, M \otimes h)$$

$$u \longmapsto (u \otimes id)\gamma$$

~~If $a \in A$, then multiplication by a gives a morphism~~

~~$$\lambda_a : h \longrightarrow h$$~~

~~and hence gives rise to an endomorphism $\lambda_a^{\#}$ of P as a right A-module. One sees that $\lambda_a^{\#} \circ \lambda_b^{\#} = \lambda_{ba}^{\#}$ so that~~

If $a \in A$, then multiplication by a defines a morphism of functors $\lambda_a : h \longrightarrow h$ with values in $\mathcal{A}b$, hence by the universal property of P, there is a unique endomorphism $\lambda_a^{\#}$ of P as an A°-module such that the diagram

$$\begin{array}{ccc} h & \xrightarrow{\ \gamma\ } & P \otimes h \\ \downarrow{\scriptstyle \lambda_a} & & \downarrow{\scriptstyle \lambda_a^{\#} \otimes \mathrm{id}} \\ h & \xrightarrow{\ \gamma\ } & P \otimes h \end{array}$$

is commutative. One sees that $\lambda_a^{\#} \circ \lambda_b^{\#} = \lambda_{ab}^{\#}$, so that $(a, p) \longmapsto \lambda_a^{\#}(p)$ defines the structure of a left A-module on P. Thus P is an A-bimodule in such a way that γ is a morphism of functors with values in Mod A. The following is clear:

<u>Proposition 1.4</u>: If F is an A-bimodule and $F \otimes h : \mathcal{A}^{\circ} \longrightarrow$ Mod A is the functor $X \longmapsto F \otimes_A hX$, then

$$\mathrm{Hom}_{\underline{\mathrm{Hom}}(\mathcal{A}^{\circ}, \mathrm{Mod}\, A)}(h, F \otimes h) = \mathrm{Hom}_{A, A^{\circ}}(P, F).$$

May 22, 1969:

Operations in generalized cohomology theories

§1. Dicogebras ~~[illegible]~~

Let $A = \bigoplus_{n\in\mathbb{Z}} A^n$ be a graded ring with unit but not necessarily commutative. Denote by Modgr A the category of graded (left) A-modules $M = \bigoplus_{n\in\mathbb{Z}} M^n$. It is a graded abelian category with

$$\mathrm{Hom}^{\delta}(M, N) = \text{homomorphisms } \varphi \text{ from } M \text{ to } N \text{ of degree } \delta$$

i.e. $\varphi = \{\varphi^n : M^n \longrightarrow N^{n+\delta}\} \ni$

$$\varphi(ax) = (-1)^{\delta(\deg a)} a \cdot \varphi(x).$$

~~[illegible]~~ By a graded A-bimodule M we mean a graded ~~abelian~~ ~~[illegible]~~ $A \otimes_{\mathbb{Z}} A^{0}$ bimodule. Thus M has a left and a right A structure and these commute. If M is ~~[illegible]~~ a graded A^{0} module and if N is a graded A module we can form the tensor product $M \otimes_A N$ which is a graded abelian group.

~~One calls~~ By a (graded) dicogebra over A one means a graded A-bimodule P endowed with morphisms of A-bimodules

$$\varepsilon : P \longrightarrow A$$

$$\Delta : P \longrightarrow P \otimes_A P$$

such that

$$(\varepsilon \otimes id)\Delta = 1 \otimes id$$
$$(id \otimes \varepsilon)\Delta = id \otimes 1$$
$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta .$$

~~[illegible]~~ When A is commutative (this means anti-commutative), then a cogebra over A is a dicogebra such that the left and right structures coincide.

A (left) ~~module of a~~ P-comodule is an A-module M endowed with $\Delta_M : M \longrightarrow P \otimes_A M$ $\ni$

$$(\varepsilon \otimes id)\Delta_M = 1 \otimes id$$
$$(\Delta_P \otimes id)\Delta_M = (id \otimes \Delta_M)\Delta_M$$

The ~~left comodules~~ P-comodules form an additive category which is abelian if P is ~~a~~ flat ~~as a~~ for the right A-module structure.

§2. Dibigebras.

Suppose from now ~~on~~ that A is commutative (in sense of topology). ~~Then~~ By a dibigebra over A ~~is~~ we mean a bicogebra P over A endowed with the structure of a commutative graded ring such that P is an algebra over A for both of its ~~the~~ left and right modules ~~structures~~ and such that

$$\varepsilon : P \longrightarrow A$$
$$\Delta : P \longrightarrow P \otimes_A P$$

are morphisms of $A \otimes A^{\circ}$ algebras.

(Better: Let A be commutative. Let P, Q be two dicogebras over A and let

$$P * Q = \cancel{P \otimes_{A\otimes A} Q} \qquad \underset{A \qquad\quad A}{P \otimes Q}$$

$P * Q$ is a dicogebra over A with

$$\begin{tikzcd}
P * Q \arrow[r, "\Delta_P * \Delta_Q"] \arrow[ddd, "\Delta_{P*Q}"] & (P \otimes_A P) \otimes (Q \otimes_A Q) \arrow[d, phantom, "\cong" rotate] \\
 & P \otimes Q \quad P \otimes Q \arrow[d] \\
 & \\
(P * Q) \otimes (P * Q) \arrow[r, equal] & (P \otimes Q) \otimes (P \otimes Q)
\end{tikzcd}$$

)

If R is a commutative ring then one obtains a category $C(R)$ as follows

$$\begin{cases} \operatorname{Ob} C(R) = \operatorname{Hom}(A, R) \\ \operatorname{Hom}_{C(R)}(u, v) = \operatorname{Hom}_{A,A\text{-algs.}}(P, {}_uR_v) \end{cases}$$

where composition

Let P be a dibigebra over A. Given a ring R we can define a category $C(R)$ having ~~with~~ as objects the ~~morphisms~~ ring homomorphisms $u: A \longrightarrow R$ as follows. The

Let P be a dibigebra over A. Given a ring R let $C(R)$ be the following category. The objects of $C(R)$ are the ring homomorphisms $u: A \longrightarrow R$. A morphism from u to v is a ring homomorphism $h: P \longrightarrow R$ such that $hs = u$ and $ht = v$. The identity morphism of u is the map $u\varepsilon: P \longrightarrow R$. Finally the composition of a morphism h in $C(R)$ from u to v and a morphism h' from v to w is defined to be the ~~map~~ composition

$$P \xrightarrow{\Delta} P \otimes P \xrightarrow{(h, h')} R$$

where

$$(h, h')(x \otimes y) = h(x) \cdot h'(y).$$

It is clear that $R \longmapsto C(R)$ is a covariant functor from (rings) to $\underset{\sim}{\mathrm{Cat}}$, the category of small categories. It is also clear that conversely a functor C from (rings) to

(Cat) which is representable is the same (up to isomorphism type) as a dibigebra P over A.

Let M and N be two P-comodules where P is an A-dibigebra. Then $M \otimes N$ ~~is also~~ has a natural structure as a P-comodule with $\Delta_{M\otimes N}$ defined as the composition

$$\begin{array}{ccc}
M \otimes N \xrightarrow{\Delta_M \otimes \Delta_N} ({}_1P_2 \otimes_2 M) \otimes ({}_1P_3 \otimes_3 N) & & p \otimes m \otimes q \otimes n \\
\downarrow & & \downarrow \\
{}_1P_2 \otimes {}_1P_2 \otimes_2 M \otimes_2 N & & p \otimes q \otimes m \otimes n \\
\downarrow & & \downarrow \\
P_2 \otimes_2 (M \otimes N) & & (pq) \otimes m \otimes n
\end{array}$$

Here the subscripts indicate the different A module structures. (?)

If $\mathcal{C}(R)$ is a groupoid for all R, then the inverse defines an antipode $i : P \longrightarrow P$ which is a ring homom reverses left and right A-algebra structures and which satisfies some other identities.

§3. Operations

Let A be a graded ring (not necessarily commutative), let $\mathcal{A}$ be a category and let

$$h : \mathcal{A}^\circ \longrightarrow \mathrm{Modgr}(A)$$

be a functor. We make the following hypothesis

($*$) There exists a right A-module P and a natural transformation ($\mathbb{Z}$-linear) $h \longrightarrow P \otimes h$ such that

$$\mathrm{Hom}_{A^\circ}(P, F) \xrightarrow{\sim} \mathrm{Hom}(h, F \otimes h)$$

for all A°-modules F.

Proposition 3.1: P has a natural dicogebra structure over A.

Proof: If $a \in A^n$, then mult. by a defines a map $h \longrightarrow h[n] = A[n] \otimes h$ hence $\exists !$ map $P \longrightarrow P[n] \ni$

$$\begin{array}{ccc} h & \longrightarrow & P \otimes h \\ \downarrow & & \downarrow \\ A[n] \otimes h & \longrightarrow & P[n] \otimes h \end{array}$$

commutes. Thus P inherits a left A-module structure. One sees that

(3.2) $$\mathrm{Hom}_{A,A}(P, F) \xrightarrow{\sim} \mathrm{Hom}_A(h, F \otimes h)$$

for all A-bimodules F.

The ~~identity~~ map $h \longrightarrow A \otimes h$ $u \longmapsto 1 \otimes u$ ~~gives~~
furnishes us with an A-bimodule map

$$\varepsilon : P \longrightarrow A$$

and the composition

$$h \xrightarrow{\gamma} P \otimes h \longrightarrow P \otimes P \otimes h$$

furnishes with an A-bimodule map

$$\Delta : P \longrightarrow P \otimes P.$$

It is straightforward to check that ε, Δ ~~endow~~ ~~constitute~~
a 'dicogebra ~~structure~~ on P. q.e.d.

§4. Multiplicative operations

~~Suppose that A is commutative and that~~

$$h : \mathcal{A} \longrightarrow Modgr(A) \qquad h' : \mathcal{A}$$

~~satisfies (*) of §3. We suppose that $\mathcal{A}$ is [illegible] given with a operation functor~~

$$\mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$$

$$X, Y \longmapsto X \otimes Y$$

~~and that there is given a natural transformation~~

$$\cancel{h(X \otimes Y)}$$

$$\delta : h(X) \otimes h(Y) \longrightarrow h(X \otimes Y)$$

~~Suppose (*) holds.~~

Suppose that A is commutative. Let

$$h : \mathcal{A} \longrightarrow Modgr(A)$$

$$h : \mathcal{A}' \longrightarrow Modgr(A)$$

be two functors satisfying (*) and let P, P' be the corresponding dia~~[illegible]~~gebras.

Proposition 4.1 The functor $h \otimes h' : \mathcal{A} \times \mathcal{A}' \longrightarrow Modgr(A)$ given by $X, Y \longmapsto hX \otimes h'Y$ satisfies (*) and the corresp. ~~di~~ dicógebra ~~is $P * P'$.~~ is $P * P'$ defined below —

~~Proof. Given [illegible] a natural transf~~

$$\cancel{hX \otimes h'X' \longrightarrow F \otimes (hX \otimes h'X')}$$

<u>Proof:</u> Let F, F' be an A-bimodules. In the following it will be necessary to work with tensor products where the two A-module structures have to be kept separate. ~~Thus~~ So we introduce a subscript notation to indicate the first and second, etc. ~~structures~~ and how they are correlated in a tensor product. For example ${}_1F_2 \otimes_2 F'_1$ is ~~the usual tensor, is and~~ is that tensor product ~~which is~~ such that $x, x' \longmapsto x \otimes x'$ satisfies:

$$\begin{cases} a.(x\otimes y) = (a.x)\otimes y = x \otimes (y.a) \\ (x \otimes y).a = \cancel{x\otimes(y.a} \; (x.a)\otimes y = (x \otimes a.y). \end{cases}$$

<u>Suppose</u> ~~to~~ given an A-~~bim~~ bimodule F and a natural transf

$$\Theta : {}_1hX \otimes {}_1h'X' \longrightarrow {}_1F_2 \otimes_2 hX \otimes_2 h'X'$$

~~Here~~ The subscript 2 on the ~~left~~ right indicates that we are taking the tensor product of F_2, hX, $h'X'$ as A-modules and Θ is required to be A-linear for the 1-structure. Fix X by the universal property of γ', Θ can be uniquely expressed as a composition

$$(\quad) \quad {}_1hX \otimes_1 h'X' \xrightarrow{id \otimes \gamma'} {}_1hX \otimes {}_1P'_2 \otimes_2 h'X' \xrightarrow{\delta(X)\otimes id} {}_1F_2 \otimes_2 hX \otimes_2 h'X'$$

where

$$\delta(X): \; {}_1hX \otimes {}_1P'_2 \longrightarrow {}_1F_2 \otimes {}_2hX$$

is linear for both "1" and "2" A-module ~~structures~~. By the uniqueness $\delta(X)$ is a natural transformation, hence by the universal property of γ, it may be expressed uniquely as a composition

$$(\;)\quad {}_1hX \otimes {}_1P'_2 \xrightarrow{\gamma\otimes id} {}_1P_2 \otimes {}_2hX \otimes {}_1P'_2 \xrightarrow{id\otimes T} {}_1P_2 \otimes {}_1P'_2 \otimes {}_2hX \xrightarrow{\mu\otimes id} {}_1F_2 \otimes {}_2hX$$

where $\mu: {}_1P_2 \otimes {}_1P'_2 \longrightarrow {}_1F_2$ is a morphism of A-bimodules. Here $T: M\otimes N \simeq N \otimes M$ denotes the interchange isomorphism. Putting ( ) and ( ) together, we find that Θ may be uniquely expressed as the composition

$${}_1hX \otimes {}_1hX' \xrightarrow{\gamma''} \left({}_1P_2 \otimes {}_1P'_2\right) \otimes {}_2hX \otimes {}_2hX' \xrightarrow{\mu\otimes id} {}_1F_2 \otimes {}_2hX \otimes {}_2hX'$$

where γ'' is the composition

$$(\;)\quad {}_1hX \otimes {}_1hX' \longrightarrow \left({}_1P_2 \otimes {}_2hX\right) \otimes \left({}_1P'_2 \otimes {}_2h'X'\right) \xrightarrow{id\otimes T\otimes id} \left({}_1P_2 \otimes {}_1P'_2\right) \otimes \left({}_2hX \otimes {}_2h'X'\right)$$

In other words $\mathrm{End}\,(h\otimes h')$ is represented by ${}_1P_2 \otimes {}_1P'_2$ as claimed.

According to prop. ( ) ${}_1P_2 \otimes {}_1P'_2$ has a dicogebra structure. The reader may check it is given by the maps

$$\varepsilon: \; {}_1P_2 \otimes {}_1P'_2 \xrightarrow{(\varepsilon\otimes\varepsilon')} A \otimes A \simeq A$$

$$\Delta: \; {}_1P_2 \otimes {}_1P'_2 \xrightarrow{\Delta\otimes\Delta'} \left({}_1P_3 \otimes {}_3P_2\right) \otimes \left({}_1P'_4 \otimes {}_4P'_2\right) \xrightarrow{id\otimes T\otimes id} \left({}_1P_3 \otimes {}_1P'_3\right) \otimes \left({}_3P_2 \otimes {}_3P'_2\right)$$

<u>Definition:</u> $P * P'$ will denote the dicogebra which as an A, A module is ${}_1P_2 \otimes {}_1P'_2$ and with ε, Δ defined as above. It will be called the tensor product of P and P'

May 25, 1969 + June 11, 1969

Unoriented cobordism and formal group laws.

§1. Formal group laws of height ∞ in characteristic p

Let p be a prime number. A formal group law F over a $\mathbb{Z}_{(p)}$-algebra R is said to be of height ∞ if

$$p_F(X) = \sum_{1\le i\le p}^{F} X$$

has all its coefficients in pR. According to a theorem of Lazard (Fröhlich, p67) F is of height ∞ if and only if it is isomorphic to the additive law $X+Y$, or equivalently if it has a logarithm series $\ell(X) = X + \cdots$ with coefficients in R.

Suppose now that R is of characteristic p.

Lemma 1: Any typical group law F over R of height ∞ is equal (not just isomorphic to) $X+Y$.

Proof: Let $\ell(X)$ be a logarithm for F, i.e

$$(\ ) \qquad \ell(F(X,Y)) = \ell(X) + \ell(Y).$$

Then $f \mapsto \ell\circ f$ is an isomorphism of the groups of curves (resp. typical curves) of F with that of G_a. As $\gamma_0(X) = X$ is typical for F, $\ell(X)$ is typical for G_a. This implies that

$$\ell(X) = \sum_{\nu\ge 0} a_\nu X^{p^\nu}$$

hence since R is of characteristic p that

$$\ell(X+Y) = \ell(X) + \ell(Y),$$

whence applying ℓ^{-1} to ( ) that $F(X,Y) = X+Y$. q.e.d.

Consider now the following functors and natural transformations from $\mathbb{F}_p$-algebras to sets.

$$(\ )\qquad \mathcal{H} \overset{i}{\hookrightarrow} \mathcal{G} \xrightarrow{\ \pi\ } \mathcal{L}_\infty$$

$$\mathcal{L}_\infty(R) = \{\text{group laws of height } \infty \text{ over } R\}$$

$$\mathcal{G}(R) = \{\text{power series } \sum_{n\geq 1} a_n X^n,\ a_n \in R,\ a_1 = 1\}$$

$$\mathcal{H}(R) = \{\text{power series } \sum_{\nu \geq 0} b_\nu X^{p^\nu},\ b_\nu \in R,\ b_0 = 1\}$$

$$\pi(f)(X,Y) = f(f^{-1}X + f^{-1}Y)$$

i = obvious inclusion

Note that $\mathcal{G}$ (is a group-valued functor under composition which) acts on $\mathcal{L}_\infty$ by $f, F \longmapsto f * F$, and $\mathcal{H}$ is the subgroup stabilizing the additive law. Moreover $\mathcal{G}/\mathcal{H} \xrightarrow{\sim} \mathcal{L}_\infty$ since every law of height ∞ is equivalent to the additive law.

We define a map

$$s : \mathcal{L}_\infty \longrightarrow \mathcal{G}$$

by sending a law F into $s(F) = c(F)^{-1}$, where $c(F)$ is the Cartier typical coordinate change.

Define $\mathcal{G} \to \mathcal{H}$ by sending $f = \sum_{n \geq 1} a_n X^n$ into $\bar{f} = \sum_{\nu \geq 0} a_{p^\nu} X^{p^\nu}$.

Lemma 2. (i) $\pi s = id$

(ii) $s\pi f = f \circ \bar{f}^{-1}$.

<u>Proof</u>: (i) The basic property of c is that $c(F) * F$ is typical; hence the additive law by lemma 1. Thus

$$c(F) * F = F_a \quad \text{so}$$

$$\pi s(F) = c(F)^{-1} * F_a = F.$$

(ii) Recall that

$$c(F)^{-1} = \underbrace{\left\{ \prod_{q \neq p} \left(1 - \frac{V_q F_q}{q} \right) \right\}}_{\text{taken in the group } Cur(F)} \gamma_0 \quad \text{where } \gamma_0(X) = X.$$

If ℓ is a logarithm for F, then $f \longmapsto \ell \circ f$ is an isomorphism of $Cur(F)$ with $Cur(F_a)$. Thus

$$\ell \circ c(F)^{-1} = \underbrace{\left\{ \prod_{q \neq p} \left(1 - \frac{V_q F_q}{q} \right) \right\}}_{\text{taken in } Cur(F_a)} (\ell \gamma_0),$$

where $\ell \gamma_0$ is the power series $\ell(X)$. Thus

$$\ell \circ c(F)^{-1} = \bar{\ell}$$

so

$$s(F) = \ell^{-1} \circ \bar{\ell}$$

if ℓ is a logarithm for F.

Now given f in $\mathcal{G}(R)$; $(f * F_a)(X, Y) = f(f^{-1}X + f^{-1}Y)$ has $f^{-1}(X)$ for a logarithm so

$$s(\pi f) = f \circ \overline{f^{-1}}$$

q.e.d.

<u>Proposition 3</u>: Every $F \in \mathcal{L}_\infty(R)$ has a unique logarithm l_F of the form

$$l_F(X) = \sum_{n\geq 0} a_n X^{n+1} \qquad \text{with} \quad \begin{array}{l} a_0 = 1 \\ a_{p^\nu - 1} = 0 \qquad \nu > 0. \end{array}$$

Moreover

$$\begin{array}{ccc} \mathcal{L}_\infty(R) & \longrightarrow & \{ f \in R[[X]]^+ \mid \bar{f} = id \} \\ F & \longmapsto & l_F \end{array}$$

is an isomorphism of functors.

<u>Corollary 4</u>: Let F_∞ over L_∞ be a universal law of height ∞ over an $\mathbb{F}_p$-algebra. Let

$$l_{F_\infty}(X) = X + \sum_{n \neq p^\nu - 1} \lambda_n X^{n+1}$$

with $\lambda_n \in L_\infty$. Then L_∞ is a polynomial ring over $\mathbb{F}_p$ with generators λ_n, $n \neq p^\nu - 1$.

<u>Proof of prop. 3</u>: By lemma 2 (i) $\mathcal{L}_\infty(R) \xrightarrow{\sim} s\mathcal{L}_\infty(R)$, which by (ii) is isomorphic to the set of $f \in R[[X]]^+$ with $\bar{f} = id$. More precisely given a law F, then $s(F)^{-1} = c_F$ satisfies by (i)

$$c_F^{-1} * F_a = F$$

hence c_F is a logarithm for F. Moreover $c_F^{-1} = s\pi c_F^{-1} = c_F^{-1} \circ \overline{c_F}$ so that c_F is the desired logarithm l_F. qed.

Complements on grading: Suppose now that R is a graded ring $R = \bigoplus_{n\geq 0} R_n$ and that we consider ~~group~~ laws $F(X,Y) = \sum a_{k\ell} X^k Y^\ell$ with $a_{k\ell} \in R_{k+\ell-1}$ and power series $f(X) = \sum a_n X^{n+1}$ with $a_n \in R_n$. In other words we have a $\mathbb{G}_m$ action on R ~~and on power series~~ $(\lambda \cdot a_n = \lambda^n a_n$ if $a_n \in R_n)$ and on power series $(\lambda \cdot f(X) = \lambda f(\lambda^{-1} X))$ and we are considering the invariants for the ~~group~~ combined action. Observe that now the appropriate family of curves is such power series invariant under $\mathbb{G}_m$ and that if F is an invariant law so is c_F. Hence the above proof shows that

$$\mathcal{L}^0_\infty(R) \xrightarrow{\sim} \left\{ f(X) = \sum_{n\geq 0} a_n X^{n+1} \;\middle|\; \begin{array}{l} a_0 = 1,\ a_{p^\nu - 1} = 0 \quad \nu > 0 \\ a_n \in R_n \end{array} \right\}$$

$$F \longmapsto c_F$$

and hence that L_∞ is a polynomial ring over $\mathbb{F}_p$ with generators λ_n, $n \neq p^\nu - 1 \quad \nu \geq 0$ of degree n.

Remark: F a group law of height ∞ over an $\mathbb{F}_p$-algebra R, then its logarithm ℓ without ~~coefficients~~ terms of degree a power of p is given by

$$\ell_F^{-1}(X) = \underbrace{\left[\prod_{\substack{q \neq p \\ q \text{ prime}}} \left(1 - \frac{V_q F_q}{q}\right) \gamma_0 \right]}_{\text{taken in group of curves of } F}(X)$$

where γ_0 is the coordinate curve.

In effect $\ell_F^{-1} : D \to G$ is a curve killed by all F_n hence is a 1-parameter subgroup!

June 11, 1969

§ 2. Unoriented cobordism theory

Let $\mathcal{N}^*$ be unoriented cobordism theory. Then there is a theory of Chern classes for real vector bundles with $c_i(E) \in \mathcal{N}^i(X)$ where if $\dim E = n$

$$c_n(E) = \iota^* \iota_* 1 \qquad \iota : X \to E \text{ zero section.}$$

Again there is a group law $F^{\mathcal{N}}(X,Y) \in \mathcal{N}^*(pt)[[X,Y]]$ ∋

$$c_1(L \otimes L') = F^{\mathcal{N}}(c_1 L, c_1 L').$$

Since $L \otimes L = 0$ for a real line bundles we have

$$F^{\mathcal{N}}(X,X) = 0.$$

Thus $F^{\mathcal{N}}$ is a law of height infinity over the $\mathbb{F}_2$-algebra $\mathcal{N}(pt)$, and

~~[illegible]~~

hence by the preceding section it has a unique logarithm of the form

$$l(X) = X + \sum_{n \neq 2^i - 1} a_n X^{n+1} \qquad a_n \in \mathcal{N}_n(pt)$$

We are going to show that $\mathcal{N}(pt)$ is a polynomial ring over $\mathbb{F}_2$ with generators a_n, $n \neq 2^i - 1$, or equivalently that $F^{\mathcal{N}}$ is a universal law of height ∞ over an $\mathbb{F}_2$-algebra. To this end let (A, P) be the affine category associating to each R over $\mathbb{F}_2$ its category of laws of ∞ height, i.e.

$$A(R) = \left\{ \text{laws } \sum a_{ke} X^k Y^e, \ a_{ke} \in R_{k+e-1} \right\}$$

$$\mathrm{Hom}(F, F') = \left\{ \text{series } \varphi(x) = \sum a_n X^{n+1}, \ a_n \in R_n, \ a_0 \in R^*, \ \varphi_* F = F' \right\}$$

[Let $G(R) = \{\varphi(X) = \sum_{n\geq 0} a_n X^{n+1} \mid a_n \in R_n,\ a_0 \in R^*\}$ under composition]

We wish to make (A,P) act on $\mathcal{N}$. First ~~suppose~~ ~~give~~ we have a morphism

$$c : A \longrightarrow \mathcal{N}(pt)$$

given by $c(F_{univ}) = F^{\mathcal{N}}$. Suppose given $v: A \longrightarrow R$ with associated group law F_v and also given a power series $\varphi(X) = \sum r_n X^{n+1}$ with $r_n \in R_n$, $r_0 \in R^*$. Then there is a unique ~~multiplicative transformation~~ natural transf. (comp. with f^*)

$$\hat{\varphi}: \mathcal{N} \longrightarrow R \underset{v\,A}{\otimes} \mathcal{N} = \mathcal{N}_v$$

such that for any proper $f: X \longrightarrow Y$ and $x \in \mathcal{N}(X)$ we have

$$\hat{\varphi}(f_*^{\mathcal{N}} x) = f_*^{\mathcal{N}_v}(\hat{\varphi}x \cdot \tilde{\varphi}(\nu_f)),$$

where $\tilde{\varphi}$ is the unique multiplicative char. class

$$\tilde{\varphi}: K \longrightarrow R \otimes_A \mathcal{N}$$

given on line bundles L by

$$\tilde{\varphi}(L) = \sum_{n\geq 0} r_n \otimes (c_1 \xi L)^n.$$

Note that

$$\hat{\varphi}(c_1 L) = \sum_{n\geq 0} r_n \otimes (c_1 \xi L)^{n+1}$$

$$= \varphi'(c_1'' \xi L \xi)$$

where φ' (resp $c_1'' L$) denotes the image of φ (resp $c_1 L$) under the maps

$$R \xrightarrow{\otimes 1} \mathcal{N}_v \xleftarrow{1 \otimes} \mathcal{N}$$

Hence

$$\hat{\varphi} F^{n} = \varphi' * F^{n''}$$
$$= \varphi' * F_v'$$

where the last step results from commutativity of

$$\begin{tikzcd} A^{F_{univ}} \arrow[r, "c"] \arrow[d, "v"] & \mathcal{N}^{F^n} \arrow[d, "''"] \\ R^{F_v} \arrow[r, "'"] & \mathcal{N}_v \end{tikzcd}.$$

Thus

$$A \longrightarrow \mathcal{N}(pt) \xrightarrow{\hat{\varphi}} R_v \otimes_A \mathcal{N}(pt) \xleftarrow{\cong} R$$

sends F_{univ} into $\varphi * F_v$ and so if u is this map, $\hat{\varphi}$ induces a multiplicative transformations

$$\hat{\hat{\varphi}} : R_u \otimes_A \mathcal{N} \longrightarrow R_v \otimes_A \mathcal{N}$$

In this way to a pair of laws F_u, F_v over R and a φ such that $\varphi * F_v = F_u$ we associate a multiplicative transformation $\hat{\hat{\varphi}}$.

<u>Lemma</u>: (i) If $u, v, w : A \longrightarrow R$ and $\varphi, \psi \in G(R)$ are such that $\varphi * F_v = F_u$, $\psi * F_w = F_v$, then the triangle

$$\begin{tikzcd} R_u \otimes_A \mathcal{N} \arrow[r, "\hat{\hat{\varphi}}"] \arrow[dr, "\widehat{\widehat{\varphi\psi}}"'] & R_v \otimes_A \mathcal{N} \arrow[d, "\hat{\hat{\psi}}"] \\ & R_w \otimes_A \mathcal{N} \end{tikzcd}$$

is commutative.

(ii) If $u = v$ and $\varphi = 1$, i.e. the series X, then $\hat{\varphi} = \mathrm{id}$ of $R_u \otimes_A \Omega$.

Proof: (ii) is clear as $\tilde{\varphi}(x) = 1$

(i) ~~To check identities we calculate effect on Chern classes.~~

$$\hat{\psi}(\hat{\varphi}\; c_1''L) = \hat{\psi}(\varphi'(c_1''L))$$

$$= \varphi'(\hat{\psi}\, c_1''L) \qquad \text{since } \hat{\psi} \text{ is a } R\text{-linear ring homomorphism}$$

$$= (\varphi'\psi')(c_1''L)$$

$$= \widehat{\varphi\psi}\,(c_1''L)$$

By Riemann-Roch an operation is determined by its effect on Chern classes so $\hat{\psi}\cdot\hat{\varphi} = \widehat{\varphi\psi}$.

I want to give a proof without using Riemann-Roch:
We know that

$$\eta \xrightarrow{\hat{\varphi}} R_v \otimes_A \eta$$

is the unique natural transformation with

$$\hat{\varphi}(f_*^{\eta} x) = f_*^{\eta_v}(\hat{\varphi} x \cdot \tilde{\varphi}(\nu_f)).$$

~~[illegible]~~

It follows that $\hat{\varphi}$ is the unique R-linear nat. transf. with

$$\hat{\varphi}(f_*^{\eta_u} x) = f_*^{\eta_v}(\hat{\varphi} x \cdot \tilde{\varphi}(\nu_f))$$

Similarly for $\hat{\psi}$ and $\widehat{\varphi\psi}$. But

$$\hat{\hat{\psi}}(\hat{\hat{\varphi}}(f_*^{\mathcal{N}_u}x)) = \hat{\hat{\psi}}\left(f_*^{\mathcal{N}_v}(\hat{\hat{\varphi}}x \cdot \tilde{\varphi}(\nu_f))\right)$$

$$= f_*^{\mathcal{N}_w}\left(\hat{\hat{\psi}}\hat{\hat{\varphi}}x \cdot \hat{\hat{\psi}}(\tilde{\varphi}(\nu_f)) \cdot \tilde{\psi}(\nu_f)\right)$$

Therefore we must show that the char classes are equal:

$$\widetilde{\varphi\psi}(x) \overset{?}{=} \hat{\hat{\psi}}(\tilde{\varphi}(x)) \cdot \tilde{\psi}(x)$$

As they are both multiplicative ~~must~~ need only check on line bundles

$$\hat{\hat{\psi}}\,\tilde{\varphi}(L) \cdot \tilde{\psi}(L) = \dot{\varphi}'(\hat{\hat{\psi}}\,c_1'L)\,\dot{\psi}'(c_1''L)$$

$$= \dot{\varphi}'(\psi'(c_1''L)) \cdot \dot{\psi}'(c_1''L)$$

where $'$ denotes $(?)\otimes 1$ and $''$ denotes $1\otimes(?)$ and where the dot means $\varphi(X) = X\dot{\varphi}(X)$. Thus we need to know that

$$\dot{\varphi\psi}(X) = \dot{\varphi}(\psi(X)) \cdot \dot{\psi}(X)$$

which is clear.

<u>Corollary:</u> $\hat{\hat{\varphi}} : \mathcal{N}_u \longrightarrow \mathcal{N}_v$ is an isomorphism.

Because the inverse is $\widehat{\varphi^{-1}}$.

Your notation is abominable. Try the following improvement - Observe that Ω_u^n has the **universal** property ~~among those with~~ for Chern theories with values in R-algebras with the group law F_u [illegible]. What this means is that

$$F_u\left(c_1^{\mathcal{N}_u}(L), c_1^{\mathcal{N}_u}(L')\right) = c_1^{\mathcal{N}_u}(L\otimes L')$$

Then whenever $\varphi * F_v = F_u$ with $\varphi \in G(R)$ one obtains

$$\hat{\varphi} : \mathcal{N}_u \longrightarrow \mathcal{N}_v$$

unique natural transformation $\ni$

$$\hat{\varphi}\left(f_*^{\mathcal{N}_u} x\right) = f_*^{\mathcal{N}_v}\left(\hat{\varphi} x \cdot \tilde{\varphi}(\nu_f)\right)$$

where $\tilde{\varphi} : K \longrightarrow \mathcal{N}_v$ is the additive char class given by

$$\tilde{\varphi}(L) = \frac{\varphi(c_1^{\Omega_v} L)}{c_1^{\Omega_v} L}$$

The rest is as before but easier.

<u>Summary:</u> If $\varphi * F_v = F_u$, then have

$$\hat{\varphi} : \mathcal{N}_u \longrightarrow \mathcal{N}_v$$

and $$\hat{\psi} \circ \hat{\varphi} = \widehat{\varphi\psi} \qquad < \text{slightly unpleasant twist.}$$

Therefore ~~we will change notation and denote by~~

we define an action of (A,P) on Ω by associating ~~to~~ $\varphi \ni \varphi * F_u = F_v$ the map

$$\widehat{\varphi^{-1}} : \mathcal{N}_u \longrightarrow \mathcal{N}_v$$

i.e.

$$\operatorname{Hom}_{\mathcal{L}(R)}(F_u, F_v) \longrightarrow \operatorname{Hom}^{\otimes}_R(\mathcal{N}_u, \mathcal{N}_v)$$

$$\varphi \longmapsto \widehat{\varphi^{-1}}$$

$\mathcal{L}$ = laws of height ∞

This works!

Suppose now we use the fact that any law of height ∞ is ~~equivalent~~ isomorphic to the additive law. Thus we have ~~an equivalence of affine categories~~ that $\mathcal{L}_\infty$ is equivalent to the full subcategory consisting of the law $X+Y$ and the series $\varphi \ni \varphi * (X+Y) = X+Y$, i.e. $\ni$

$$\varphi(X) = {\sum_{\nu \geq 0}}' a_\nu X^{2^\nu} \qquad a_\nu \in R_{2^\nu}, \quad a_0 \in R^*.$$

Thus we have the functor of inclusion

$$f : (A, P) \longrightarrow (\mathbb{F}_2, Q)$$

and the quasi-inverse functor which associates to any law F the law $X+Y$ and an isomorphism of these laws, that is, a logarithm l for F. We know that l is uniquely determined by the condition that it have no terms of degree a power of 2. So there are multiplicative isomorphisms

$$\mathcal{N} = A_{id} \otimes_A \mathcal{N} \xrightarrow[\hat{l}^{-1}_{univ}]{\sim} A_{gf} \otimes_A \mathcal{N} \cong A_g \otimes_{\mathbb{F}_2} (\mathbb{F}_{2\,f} \otimes_A \mathcal{N})$$

Here $l_{univ} * F_{univ} = (X+Y)$, and $g : \mathbb{F}_2 \longrightarrow A$ is the unique map possible, so that $gf : A \longrightarrow A$ gives the law $X+Y$ over A. We state our conclusions as follows.

Theorem: Let F_{univ} over A be a universal law of height ∞ over a ring A of char 2 and let $c : A \longrightarrow \mathcal{N}$ be the homomorphism carrying F_{univ} to $F^{\mathcal{N}}$. Let $f : A \longrightarrow \mathbb{F}_2$ give the law $X+Y$ and set

$$\overline{\mathfrak{N}}(X) = (\mathbb{F}_2)_f \otimes_A \mathfrak{N}(X).$$

Then $\overline{\mathfrak{N}}$ is the universal Chern theory with group law $X+Y$. Let l be the unique logarithm of F_{univ} without term of degree ~~a~~ power of 2, and let

$$\gamma : \overline{\mathfrak{N}}(X) \longrightarrow \mathfrak{N}(X)$$

be the unique map with

$$\gamma(f_*^{\overline{\mathfrak{N}}} x) = f_*(\cancel{\gamma}x \cdot \tilde{\gamma}(\nu_f))$$

where $\tilde{\gamma} : K \longrightarrow \mathfrak{N}$ is the char. class with

$$\gamma(L) = \frac{l(c_1^{\mathfrak{N}} L)}{c_1^{\mathfrak{N}} L}$$

Then the A-linear extension of γ is an (natural for f^*) isomorphism.

$$\boxed{A \otimes_{\mathbb{F}_2} \overline{\mathfrak{N}}(X) \xrightarrow{\sim} \mathfrak{N}(X)}$$

Now we apply results of Thom which up to now we have not used. ~~Then We know in fact that~~ From the above we have that

$$\mathfrak{N}(pt) \cong A \otimes_{\mathbb{F}_2} \overline{\mathfrak{N}}(pt)$$

By Thom $\mathfrak{N}(pt)$ is a poly ring with one generator of each degree $\neq 2^i - 1$; but the same is true for A, so $\overline{\mathfrak{N}}(pt) = \mathbb{F}_2$. However ~~thus~~

the decomposition (A) shows that $\bar{\mathcal{N}}$ is a generalized coh. theory, hence by uniqueness $\bar{\mathcal{N}}(X) \xrightarrow{\sim} H^*(X)$, homology mod 2. Conclude.

<u>Theorem:</u> (i) $F^{\mathcal{N}}$ over $\mathcal{N}(pt)$ is a **universal** group law of height ∞ over a ring of char 2.

(ii) $H^*(X)$ is universal Chern theory on manifolds with law $X+Y$.

(iii) There is a canonical isomorphism

$$\mathcal{N}(X) \simeq \mathcal{N}(pt) \otimes H^*(X).$$

(iv) The ~~affine category~~ affine groupoid of operations on $H^*(X)$ ~~[illegible]~~ associates to each $\mathbb{F}_2$ algebra R the group of autos of $\hat{\mathbb{G}}_a$ over R (i.e. power series $\varphi(X) = \sum_{\nu \geq 0} r_\nu X^{2^\nu}$ $r_\nu \in R_{2^\nu}$, $r_0 \in R^*$). (In other words Steenrod operations give all stable cohomology operations)

<u>Remarks:</u> To prove Thom's theorem on the structure of $\mathcal{N}(pt)$ one, of course, must use (iv).

Relation to Dold generators

$$P(m,n) = CP_n \times S^m / \mathbb{Z}/2$$

fiber bundle over $\mathbb{RP}_m$
fiber $\mathbb{CP}_n$

$$H^*(P(m,n)) = \mathbb{F}_2[\underset{1}{c}, \underset{2}{d}] / c^{m+1}=0, d^{n+1}=0$$

$$w(P(m,n)) = (1+c)^m (1+c+d)^{n+1}$$

If $(i+1) = 2^r(2s+1)$ then we take

$$P(i) = \begin{cases} P(i,0) = \mathbb{RP}_i & i \text{ even} \\ P(2^r-1, s2^r) & i \text{ odd} \end{cases}$$

Dold claims it is enough to prove that the char. number ~~associated~~ to $\sum_{\nu=1}^{i} t_\nu^i$ is non-zero for $P(i)$ whence it follows that the element $P(i)$ are generators

Dold defines $s^h(\underline{t}) = \sum_{\nu=1}^{t} t_\nu^h$ Newton

and its image under the map $H^*(BO) \longrightarrow H(P(i))$ to be $\Phi_{i,t}(s^h(\underline{t}))$. Then he shows that

$$\Phi_{ii}(s^i(i)) = \Phi_{i,i+2}(s^i(i+2))$$

$$= (n+1) \sum_{p+2q=i}' \binom{p+2q-1}{q} c^p d^q + m c^i$$

and shows $\neq 0$.