Tate: On $K_2(F)$ where F is a field   April 22, 1969

p prime

Theorem: $\left.\begin{array}{ll} p \neq \operatorname{char} F & cd_p F \leq 1 \\ \text{or } p = \operatorname{char} F, & [F:F^p] \leq p \end{array}\right\} \Longrightarrow K_2(F) = pK_2(F)$.

Key lemma: Suppose E is a finite F-algebra $a = x^m$ $b = N_{E/F}\, y$ $x, y \in E$, then $\{a, b\} \in mK_2(F)$. ($\{\,,\}: F^* \times F^* \longrightarrow K_2(F)$ universal symbol)

Proof: $\{a, b\} = \{a, Ny\} = N\{a, y\}$ proj. formula of Bass

$$= N\{x^m, y\} = (N\{x, y\})^m .$$

Proof of thm. $p \neq \operatorname{char} F$, ~~more generally suppose m an integer > 1 prime to char F. Then Claim we may assume that $\mu_m \subset F$. In effect~~ Claim we may assume $\mu_p \subset F$. In effect if $F' = F[\mu_p]$ then $[F', F] = m$ is prime to p, F' has $cd_p \leq 1$ and by proj fmla

$$K_2(F)/pK_2(F) \xrightarrow{\sim} K_2(F')/pK_2(F').$$

If $\mu_p \subset F$, then $E = F(a^{1/p})$ is cyclic (of order p (if $a^{1/p} \in F$ trivial)) over F so by periodicity

$$F^*/N_{E/F}(E^*) \simeq H^2(E/F, E^*) = \operatorname{Ker} Br(F) \longrightarrow Br(E)$$

Thus $F^*/N_{E/F}\, E^* \longhookrightarrow {}_pBr(F)$. But by Hilbert 90

$$H^2(F, \mu_p) = {}_pBr(F)$$

is zero as $cd_p F \leq 1$. Thus b is a norm from $F(a^{1/p})$ and so can use key lemma.

If $p = \operatorname{char} F$ + $[F:F^p] \leq p$, then either $a \in F^p$ (trivial) or $F^{1/p} = F[a^{1/p}]$ so that $b = y^p = Ny$ so done again.

The theorem admits an almost converse.

Examples of symbols:

① $K_2(F) \longrightarrow \Omega^2_{F/F_0}$ $\quad$ F_0 prime field

$$\{a,b\} \longmapsto \frac{da}{a} \wedge \frac{db}{b} = (a,b)_{\text{diff}}$$

In characteristic p, $(a,b)_{\text{diff}} = 0 \implies da, db$ dependent $\implies b^{1/p} \in F(a^{1/p}) \implies \{a,b\} \in pK_2(F)$. This shows that $[F:F^p] \le 1$ necessary for $K_2(F) = pK_2(F)$.

② Suppose m prime to char F. Then

$$F^*/(F^*)^m \xrightarrow[\cong]{\delta} H^1(F, \mu_m)$$

so get $(a,b)_m = \delta b \cup \delta a \in H^2(F, \mu_m^{\otimes 2})$. Tate shows this is a symbol as follows. First he shows that if E is a finite (separable?) F-algebra, then

$$\begin{tikzcd} K_2(E) \arrow[r, "\lambda_m^E"] \arrow[d, "N"] & H^2(E, \mu_m^2) \arrow[d] \\ K_2(F) \arrow[r, "\lambda_m^F"] & H^2(F, \mu_m^2) \end{tikzcd}$$

commutes. Then one takes $E = F[x]/(x^m - a)$ and notes that

$$1 - a = N_{E/F}(1-x)$$

Thus $(a, 1-a)_m$ is divisible by m so is zero.

The basic conjecture

$$\lambda_m : K_2(F)/mK_2(F) \overset{?}{\hookrightarrow} H^2(F, \mu_m^2) \quad \text{is injective}$$

If $\mu_m \subset F$, then

$$H^2(F, \mu_m^2) = H^2(F, \mu_m) \otimes \mu_m = {}_m Br(F) \otimes \mu_m$$

and

(Corps Locaux) $(a,b)_m = (a,b)_z \otimes z$ $\qquad z$ gen of μ_m

where $(a,b)_z$ is the "cyclic" algebra $\quad x^m = a,\ y^m = b,\quad yxy^{-1} = zx.$

One knows well that

$$(a,b)_m \neq 0 \Longrightarrow b \in N\, F(a^{1/m})^*$$
$$\xRightarrow[\text{lemma}]{\text{key}} \{a,b\} \in m K_2(F).$$

Thus

$$\lambda_m : K_2(F)/mK_2(F) \longrightarrow H^2(F, \mu_m^2)$$

is injective on decomposable elements, hence injective if Ker λ_m generated by decomposable

<u>Lemma (linear alg):</u> X, Y v.s. over a field k, then any bilinear map $X \otimes Y \longrightarrow k$ has this property.

<u>Cor:</u> λ_m injective if m is a prime $\neq$ char F and if $\dim_{\mathbb{F}_p} H^2(F, \mu_p^2) \leq 1$, more generally if $\dim_p Br(F[\zeta_p]) \leq 1$

Using this Tate proves

<u>Thm:</u> F local field ($\mathbb{R}, \mathbb{C}$, $p$adic) then for all m

$$K_2(F)/mK_2(F) \xrightarrow{\cong} \mu_F/m\mu_F$$

~~[illegible]~~ +

<u>Thm</u>: $\lambda : K_2(F) \longrightarrow \hat{\mu}_F$ Hilbert symbol (surjective)

a) $\mathrm{Ker}\,\lambda$ divisible

b) $\mathrm{Ker}\,\lambda$ generated by $\{U, U\}$ for arb. small open U in F

c) Any continuous symbol $F^* \times F^* \longrightarrow A$, A loc. comp factors through μ_m (C. Moore).

Later remarks:

$$K_2(F) = F^* \otimes F^* / \text{subgp gen. by } [a] \otimes [1-a] \qquad \begin{array}{ll} [a] = a & \text{if } a \in F^* \\ [a] = 0 & \text{if } a \in F - F^* \end{array}$$

In effect the antisymmetry follows from

$$f \neq 1 \Rightarrow 1 = \{f^{-1}, 1-f^{-1}\} = \{f^{-1}, (f-1)f^{-1}\} = \{f, 1-f\}^{-1}\{f^{-1}, -f^{-1}\} = \{f, -f\}$$

(true also for $f = 1$ by bilinearity).

whence

$$1 = \{fg, -fg\} =$$

$$= \{f, -fg\}\{g, -fg\} = \{f, -f\}\{f, g\}\{g, f\}\{g, -g\}$$

$$= \{f, g\}\{g, f\}.$$

<u>Lemma</u>: If $T^m = f$ splits completely in $F[T]$, all f then $K_2(F)$ is uniquely divisible by m.

<u>Proof</u>: F^* divisible by $m \overset{!}{\Longrightarrow} F^* \otimes F^*$ uniquely divisible by m (seems to use structure theory of f.g. abelian groups; Rests on showing that

$${}_m(A \otimes B) = \mathrm{Im}\ {}_mA \otimes B + A \otimes {}_mB \quad \text{true by passage to limits)}$$

Thus enough to show that R gen. by $[a] \otimes [1-a]$ is divisible by m

But if $T^m - a = \prod (T - x_i)$ then

$$[a] \otimes [1-a] = \sum{}' [a] \otimes [1-x_i] = \sum m[x_i] \otimes [1-x_i] \in mR.$$

<u>Corollary</u>: F algebraically closed $\Longrightarrow$ $K_2(F)$ is a $\mathbb{Q}$ vector space.

April 27^{+28}, 1969. On category schemes.

§1. A,A cogebras and bigebras.

1.1 Let A be a ring not necessarily commutative. By an A,A cogebra we mean an A-bimodule P endowed with maps of A-bimodules

$$P \xrightarrow{\varepsilon} A$$

$$P \xrightarrow{\Delta} P \otimes_A P$$

satisfying the counit and coassociativity identities. Here the tensor product is taken with respect to the right (resp. left) A-module structure on the first (resp. second) factor; it is endowed with the left (resp. right) A-module structure coming from the left (resp. right) A-module structure of its first (resp. second) factor.

1.2. Example. Let T: Mod $A \longrightarrow$ Mod A be a (co?) triple endowed with structural maps

$$\begin{cases} T \longrightarrow \mathrm{id} \\ T \longrightarrow TT \end{cases}$$

If T is compatible with inductive limits, then

$$T(M) = P \otimes_A M$$

where $P = T(A)$ is an A-bimodule. Then P is an A,A cogebra.

1.3. By a comodule for an A,A cogebra P we mean an A-module M endowed with a map of A-modules

$$M \xrightarrow{\Delta} P \otimes M$$

such that the diagrams

$$\begin{tikzcd} M \arrow[r, "\Delta"] \arrow[dr, "1\otimes\cdot"', "\cong"] & P\otimes M \arrow[d, "\varepsilon\otimes id"] \\ & A\otimes M \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\Delta"] \arrow[d, "\Delta"] & P\otimes M \arrow[d, "\Delta\otimes id"] \\ P\otimes M \arrow[r, "id\otimes\Delta"] & P\otimes P\otimes M \end{tikzcd}$$

commute. The category $Com(P)$ of P-comodules forms an additive category which is abelian if P is right A-flat.

<u>Proposition 1.3.1:</u> The forgetful functor

$$h: Com(P) \longrightarrow Mod\,A$$

has as right adjoint $g(M) = P\otimes M$.

<u>Proof:</u> Given a P-comodule X and a map $X \xrightarrow{u} M$ of A-modules, the corresponding map of P-comodules is

$$X \xrightarrow{\Delta} P\otimes X \xrightarrow{id\otimes u} P\otimes M.$$

The triple associated to the pair

$$Com(P) \underset{g}{\overset{h}{\rightleftarrows}} Mod\,A$$

is $$TM = hgM = P\otimes M.$$

~~Thus given P we have a pair of adjoint~~

1.4. Suppose that

$$\mathcal{A} \underset{g}{\overset{h}{\rightleftarrows}} Mod\,A$$

are additive adjoint functors, $\mathcal{A}$ being an additive category. ~~The~~ Suppose that hg commutes with inductive limits so that

$$hg(M) = P \otimes M$$

for an A,A cogebra P. Then ~~there~~ for any $X \in \operatorname{Ob} \mathcal{A}$ there ~~[illegible]~~ is an adjunction map

$$hX \longrightarrow hghX = P \otimes hX$$

and hence ~~a~~ h induces a functor

$$\mathcal{A} \xrightarrow{\tilde{h}} \operatorname{Com}(P).$$

~~[illegible]~~ Using the ~~[illegible]~~ faithfully flat descent argument one obtains the following conditions for $\tilde{h}$ to be an equivalence.

<u>Theorem 1.4.1: Let</u> ~~[illegible]~~ $\mathcal{A}$ be an abelian category and let

$$\mathcal{A} \underset{g}{\overset{h}{\rightleftarrows}} \operatorname{Mod} A$$

be adjoint functors. Assume that h is exact and faithful and that ~~hg commutes with~~ g commutes with inductive limits. Then the pair $(\mathcal{A}, h)$ is equivalent to the category of P-comodules and the forgetful functor for some A,A cogebra P which is right A-flat.

<u>Proof sketch:</u> hg commutes with $\varinjlim$'s so

$$hg\,M = P \otimes M$$

where P is an A,A cogebra, and we have

$$\tilde{h} : \mathcal{A} \longrightarrow \operatorname{Com}(P).$$

Note that P is right flat since h, g are exact. $\tilde{h}$ is fully faithful; this follows from the fact that for $X \in \mathrm{Ob}\, \mathcal{A}$

$$X \longrightarrow ghX \rightrightarrows ghghX$$

is exact (apply h, it becomes contractible). $\tilde{h}$ is essentially surjective, since if $X = \mathrm{Ker}\ gM \rightrightarrows ghgM$, then $\tilde{h}X \cong M$.

1.5. Here's another way of recovering P from

$$h : \mathrm{Com}(P) \longrightarrow \mathrm{Mod}\, A$$

not using g. Let F be an A,A-module and let

$$h_F : \mathrm{Com}(P) \longrightarrow \mathrm{Mod}\, A$$

$$h_F(M) = F \otimes M$$

Then

Prop. 1.5.1: $\boxed{\mathrm{Hom}(h, h_F) \cong \mathrm{Hom}_{A,A\,\mathrm{mod}}(P, F)}$

Proof: Given $\theta : h \longrightarrow h_F$ define $\theta^{\#}$ to be the composition

$$P \xrightarrow{\theta(P)} F \otimes P \xrightarrow{id \otimes \varepsilon} F \otimes A \cong F$$

~~Mixes~~ This is obviously a left A-module homomorphism. It is also a right one since right multiplication by a is an endo. of P in $\mathrm{Com}(P)$. Conversely given $\varphi : P \longrightarrow F$ an A,A-hom. define $\varphi^b(M)$ to be the composition

$$M \xrightarrow{\Delta} P\otimes M \xrightarrow{\varphi\otimes id} F\otimes M$$

Obviously $(\varphi^b)^\# = \varphi$. To show $(\theta^\#)^b = \theta$ we ~~[illegible]~~ use the diagram

$$\begin{tikzcd}
M \arrow[r, "\Delta"] \arrow[d, "\theta(M)"] \arrow[rr, bend left, "(\theta^\#)^b"] & P\otimes M \arrow[r, "\theta^\#\otimes id"] \arrow[d, "\theta(P)\otimes id"] & F\otimes M \arrow[dd, "\cong"', "id\otimes 1\otimes id"] \\
F\otimes M \arrow[r, "id\otimes\Delta"] \arrow[drr, "\cong", "id\otimes 1\otimes id"'] & F\otimes P\otimes M \arrow[dr, "id\otimes\varepsilon\otimes id"] & \\
 & & F\otimes A\otimes M
\end{tikzcd}$$

Everything is clearly commutative except the upper left square. To see this one commutes note that $\Delta: M \longrightarrow P\otimes M$ is a morphism in $Com(P)$ (the map $M \longrightarrow gh\, M$) so the first square of

$$\begin{tikzcd}
M \arrow[r, "\Delta"] \arrow[d, "\theta(M)"] & P\otimes M \arrow[d, "\theta(P\otimes M)"] & P \arrow[l, "id\otimes m"'] \arrow[d, "\theta(P)"] \\
F\otimes M \arrow[r, "id\otimes\Delta"] & F\otimes P\otimes M & F\otimes P \arrow[l, "id\otimes id\otimes m"']
\end{tikzcd}$$

is commutative. For any element m of M the map $id\otimes m: P \longrightarrow P\otimes M$ is a map in $Com(P)$, hence the second square commutes for all $m\in M$. Thus $\theta(P\otimes M) = \theta(P)\otimes id_M$ and so we are done.

Corresponding to the identity $h \to h = h_A$ we get from 1.5.1 the map

$$\varepsilon : P \longrightarrow A$$

and $\underbrace{\Delta \text{ corresponds}}$ to the composition

$$h \xrightarrow{c} \underset{\substack{\| \\ h_P}}{P \otimes h} \xrightarrow{id \otimes c} P \otimes P \otimes h$$

~~is~~ where $c : h \longrightarrow h_P$ is the canonical map ~~giving~~ corresponding to $id_P : P \to P$.

Remark 1.5.2: So given a category $\mathcal{A}$ (not nec. additive) and a functor

$$h : \mathcal{A} \longrightarrow \text{Mod}\, A$$

such that

$$F \longrightarrow \text{Hom}(h, h_F)$$

is representable, we obtain a A, A cogebra. Conversely any P is obtained in this way with $\mathcal{A} = \text{Com}\, P$ and $h =$ forgetful functor.

One can next ask when

$$\tilde{h} : \mathcal{A} \longrightarrow \text{Com}(P)$$

is an equivalence. Necessary and sufficient conditions are due to Beck for an arbitrary triple. Sufficient practical conditions are given by theorem 1.4.1.

1.6. Let $P(A)$ be the category of projective A-modules of finite type, let $\mathcal{A}$ be an additive category and let

$$h : \mathcal{A} \longrightarrow P(A)$$

be an additive functor. We suppose that the functor $X \longmapsto \mathrm{Hom}(hX, A)$ is ind-representable. Then in fact there are adjoint functors

$$\mathrm{Ind}\,\mathcal{A} \underset{g}{\overset{h}{\rightleftarrows}} \mathrm{Ind}\, P(A) \;\overset{\text{by Lazard}}{=}\; \mathrm{Flat}(A)$$

In effect the category $\mathrm{Ind}\,\mathcal{A}$ is closed under direct sums, retracts, and filtered inductive limits, hence the category of M in $\mathrm{Flat}(A)$ for which $X \longmapsto \mathrm{Hom}(hX, M)$ is ind-representable is closed under direct sums, retracts, and filtered inductive limits, so is all of $\mathrm{Flat}(A)$. Moreover

(1.6.1) $$hgM \xleftarrow{\sim} hgA \otimes M$$

for the same reason. (One sees that g is compatible with filtered inductive limits since for all $X \in \mathrm{Ob}\,\mathcal{A}$

$$\left(\varinjlim_i g M_i\right)(X) = \varinjlim \mathrm{Hom}(hX, M_i) = \mathrm{Hom}(hX, \varinjlim M_i)$$

$$= g(\varinjlim M_i)\,(X) \,.)$$

~~Proposition 1.6.1: Let $h : \mathcal{A} \longrightarrow P(A)$ be an additive functor such that $X \longmapsto \mathrm{Hom}(hX, A)$ is ind-representable,~~

Let $P = hg(A)$. Then P is an A,A comodule which is flat as a left A-modules. Moreover by 1.6.1 P is an A,A coalgebra in a natural way. Finally I claim that the formula 1.5.1 holds. In effect given $\varphi: P \to F$ one defines $\varphi^b : h \to h_F$ ~~as follows. First we note that it suffices to define~~ ~~$\varphi^b: hX \to F\otimes hX$ for X of the form~~ [illegible] ~~flat A-module. In effect there is a natural~~

to be the composition

$$hX \longrightarrow hghX \simeq P\otimes hX \xrightarrow{\varphi\otimes id} F\otimes hX.$$

Given $\theta: h \to h_F$ ~~and define~~ it extends to the ind categories so we can define $\theta^{\#}$ to be the composite

$$P = hg(A) \xrightarrow{\theta(gA)} F\otimes hg(A) \xrightarrow{id\otimes\varepsilon} F\otimes A \simeq F.$$

Again $(\varphi^b)^{\#} = \varphi$ is obvious. To show $(\theta^{\#})^b = \theta$ we follow the proof of 1.5.1, the key point being to show that

$\theta(\cancel{h}ghX)$ and $\theta(gA)\otimes id \; : \; P\otimes hX \longrightarrow F\otimes P\otimes hX$ coincide, or more precisely that

$$\begin{tikzcd}
hgA \otimes hX \arrow[r, "\simeq"] \arrow[d, "\Theta(gA)\otimes id"'] & hghX \arrow[d, "\Theta(ghX)"] \\
F\otimes hgA \otimes hX \arrow[r, "\simeq"] & F\otimes hghX
\end{tikzcd}$$

commutes. We may replace hX by any M in $\mathrm{Flat}(A)$ and again fix an element $m \in M$ and use the same argument as before. Thus we have proved

<u>Proposition 1.6.2:</u> Let $h : \mathcal{A} \longrightarrow \mathcal{P}(A)$ be an additive functor such that $X \longmapsto \mathrm{Hom}(hX, A)$ is ind-representable, say

$$\mathrm{Hom}(hX, A) = \varinjlim_i \mathrm{Hom}(X, E_i)$$

Then $\underbrace{P = \varinjlim h(E_i) \text{ is an } A,A \text{ cogebra and}}$ for all A,A modules F we have a canonical isomorphism

$$\mathrm{Hom}(h, h_F) \cong \mathrm{Hom}_{A,A\ \mathrm{mod}}(P, F).$$

In particular P is an A,A-cogebra in a natural way.

Of course h induces a functor $\bar{h} : \mathcal{A} \longrightarrow \mathrm{Com}(P) \cap \mathcal{P}(A)$. We would like to know when $\bar{h}$ is an equivalence of categories, but to carry out the descent argument of 1.4 I need the hypotheses of 1.4.1. This forces me to assume A is a sfield.

<u>Theorem:</u>

1.7. <u>Let</u> A <u>be a sfield</u>, <u>let</u> $\mathcal{A}$ <u>be an abelian category, and let</u>

$$h : \mathcal{A} \longrightarrow \mathrm{Modf}(A)$$

be a faithful exact functor. Then the pair $(\mathcal{A}, h)$ is equivalent to the pair $(\mathrm{Com}(P) \cap \mathrm{Modf}(A), \text{forget})$ where P is the A, A coalgebra given by

$$(1.7.1) \qquad \mathrm{Hom}_{A,A\,\mathrm{mod}}(P, F) = \mathrm{Hom}(h, h_F).$$

Moreover the category of such functors h is equivalent to the category of A, A cogebras.

Proof: ~~[illegible]~~ Every object of $\mathcal{A}$ is of finite length, hence every object of $\mathrm{Ind}\,\mathcal{A}$ is strictly ind-representable. ~~[illegible] that~~ In fact $\mathrm{Ind}\,\mathcal{A}$ is the locally noetherian category ~~[illegible]~~ associated to $\mathcal{A}$ by Gabriel. As h is exact and $\mathcal{A}$ is abelian $X \rightsquigarrow \mathrm{Hom}(hX, V)$ is left exact, hence ind-representable. Thus we are in the situation of 1.6 and we have adjoint functors

$$\mathrm{Ind}\,\mathcal{A} \underset{g}{\overset{h}{\rightleftarrows}} \mathrm{Mod}(A)$$

$$hgV \xleftarrow{\sim} hgA \otimes V$$

where $hgA = P$ is an A, A cogebra. ~~[illegible]~~ Claim that (the extended) h is exact and faithful; as h is a left adjoint it suffices to show h transforms a non-zero injection into a non-zero injection. This follows from the fact that every ind-representable functor is strictly ind-representable.

By 1.4.1

$$\tilde{h} : \operatorname{Ind} \mathcal{A} \longrightarrow \operatorname{Com}(P)$$

is an equivalence of categories. Thus

$$\tilde{h} : \mathcal{A} \longrightarrow \operatorname{Com}(P) \cap \operatorname{Modf}(A)$$

is fully faithful. The small $\tilde{h}$ is essentially surjective since a finite dimensional P-comodule is ~~an Artinian~~ a noetherian object of Com(P) hence ~~[illegible]~~ corresponds to a noetherian object of Ind $\mathcal{A}$, which by Gabriel is isomorphic to an object of $\mathcal{A}$. Thus the small $\tilde{h}$ is an equivalence of categories. By 1.6.2 P is the A,A-cogebra satisfying 1.7.1, so the first assertion of the theorem is proved.

For the second assertion we must make precise the category $\mathcal{Q}$ whose objects are the faithful exact h with target Modf(A). Thus we define a morphism from $h : \mathcal{A} \to \operatorname{Modf}(A)$ to $h' : \mathcal{A}' \to \operatorname{Modf}(A')$ to be a pair consisting of an additive functor $F : \mathcal{A} \to \mathcal{A}'$ modulo the isomorphisms of F with other functors, together with an isomorphism $h' \circ F \simeq h$. Then we have a functor

$$(A,A \text{ cogebras}) \longrightarrow \mathcal{Q}$$

$$P \longmapsto (\operatorname{Com} P) \cap \operatorname{Modf}(A) \overset{\text{defn}}{=} \operatorname{Comf}(P)$$

Suppose given

$$\begin{tikzcd} \operatorname{Comf} P \arrow[rr, "F"] \arrow[dr, "h"'] & \arrow[d, phantom, "h \xrightarrow{\gamma} h'"] & \operatorname{Comf} P' \arrow[dl, "h'"] \\ & \operatorname{Modf}(A) & \end{tikzcd}$$

Modulo ~~[illegible]~~ an isomorphism of

For the second assertion we must know that for any A,A cogebra P there are enough objects in $\mathrm{Com}(P) \cap \mathrm{Modf}(A) \overset{\mathrm{defn}}{=} \mathrm{Comf}(P)$. Now in fact $\mathrm{Comf}(P)$ is the (full subcategory of) noetherian objects of $\mathrm{Com}(P)$ and $\mathrm{Com}(P) \cong$ Ind $\mathrm{Comf}(P)$ in virtue of the following

__Lemma__ (1.7.2): Any P-comodule M is the union of its subcomodules which are of finite type as A-modules.

__Proof__: Suppose that e_i is a basis for P as a right A-module and let ~~...~~ $\varphi_i : M \longrightarrow M$ be defined by

$$\Delta m = \sum e_i \otimes \varphi_i(m) \quad \in P \otimes M$$

~~...~~ If $a \in A$ and $s(a)\, e_i = \sum_j e_j\, t(a_{ij})$ where s (resp. t) refers to the left (resp. right) A-module structure of P, then

$$\Delta(am) = \sum_{i,j} e_j \otimes a_{ij}\, \varphi_i(m).$$

This shows that $\varphi_i(am)$ is an A-linear combination of the $\varphi_i(m)$. Thus if V is a finite dimensional subspace of M and $V = \sum A v_\ell \quad 1 \le \ell \le m$, then

$$\bar{V} = \sum_i A \cdot \varphi_i(V) = \sum_{\ell, i} A\, \varphi_i(v_\ell)$$

is also finite dimensional. But $\bar{V}$ is a subcomodule of M, indeed writing

$$\Delta e_i = \sum e_j \otimes e_k\, t(a^i_{jk})$$

we have that

$$\sum_i e_i \otimes \Delta \varphi_i(v_\ell) = (id \otimes \Delta)\Delta v_\ell = (\Delta \otimes id)\Delta v_k = \sum e_j \otimes e_k \otimes a^i_{jk} \varphi_i(v_\ell)$$

showing that $\Delta \varphi_i(v_\ell) \in P \otimes \bar{V}$. QED for the lemma.

We now make precise the category of faithful exact functors $h : \mathcal{A} \longrightarrow \underbrace{\mathrm{Modf}(A)}_{\text{used in thm 1.7}}$. ~~[illegible]~~ Start with the 2-category whose objects are such pairs $(\mathcal{A}, h)$, whose 1-morphisms ~~[illegible]~~ from $(\mathcal{A}, h)$ to $(\mathcal{A}', h')$ are pairs (F, u) where $F : \mathcal{A} \longrightarrow \mathcal{A}'$ is an additive functor and u is an isomorphism $h \longrightarrow h' \circ F$, and whose 2-morphisms from (F, u) to (F_1, u_1) are isomorphisms of functors $\theta : F \longrightarrow F_1$ such that

$$\begin{tikzcd}
h'F(X) \arrow[r, "h(\theta(X))"] & h'F_1(X) \\
h(X) \arrow[u, "u", "\wr"'] \arrow[ur, "u_1"', "\wr"] &
\end{tikzcd}$$

commutes for all $X \in \mathrm{Ob}\, \mathcal{A}$. The category with objects $(\mathcal{A}, h)$ and isomorphism classes of 1-maps (F, u) is the category $\mathcal{Q}$ we have in mind. Now we have a functor

$$(A, A \text{ cogebras}) \longrightarrow \mathcal{Q}$$

$$P \longmapsto \mathrm{Comf}(P)$$

It's pretty clear that what we have done above shows this functor is an equivalence of categories. QED theorem.

1.8. Suppose (from now on) that A is a commutative ring. If P and Q are A,A cogebras, we let $P * Q$ be the A,A-cogebra

$$P * Q = P \underset{A\otimes A}{\otimes} Q \qquad\qquad P \underset{A\otimes A}{\otimes} Q \xrightarrow{\varepsilon * \varepsilon} A \underset{A\otimes A}{\otimes} A = A$$

$$\begin{array}{c} P \underset{A\otimes A}{\otimes} Q \xrightarrow{\Delta\otimes\Delta} (P \otimes_A P) \underset{A\otimes A}{\otimes} (Q \otimes_A Q) \\ \Big\downarrow \\ (P \underset{A\otimes A}{\otimes} Q) \otimes_A (P \underset{A\otimes A}{\otimes} Q) \end{array}$$

(The upper is universal for multilinear maps $\Phi : P \times P \times Q \times Q \to$ such that

$$\Phi(xa, y, z, w) = \Phi(x, ay, z, w)$$
$$\Phi(x, y, za, w) = \Phi(x, y, z, aw)$$
$$\Phi(ax, y, z, w) = \Phi(x, y, az, w)$$
$$\Phi(x, ya, z, w) = \Phi(x, y, z, wa)$$

The lower satisfies the additional conditions

$$\Phi(xa, y, z, w) = \Phi(x, y, za, w)$$
$$\Phi(x, ay, z, w) = \Phi(x, y, z, aw)$$

which are equivalent granted the others. This accounts for the fact that there are 4 tensors over A in the upper and 5 in the lower)

Suppose that

$$h : \mathcal{a} \longrightarrow \text{Mod } A$$

$$h_1 : \mathcal{a}_1 \longrightarrow \text{Mod } A$$

are (not nec. additive) functors whose endos. are represented by P and Q.

Prop. ~~1.8.1~~ : Let $h \otimes h_1 : \mathcal{a} \times \mathcal{a}_1 \longrightarrow \text{Mod } A$ be the functor

$$X, Y \longmapsto hX \otimes hY_1$$

Then

$$\text{Hom}\left(h\otimes h_1, (h\otimes h_1)_F\right) \cong \text{Hom}_{A,A \text{ mod}}\left(P * Q, F\right)$$

Proof (sketch): Suppose given $\Theta : h\otimes h_1 \longrightarrow F \otimes (h \otimes h_1)$. Fixing $x \in hX$ we get a transformation Θ_x

$$h_1 Y \xrightarrow{x \otimes ?} hX \otimes h_1 Y \xrightarrow{\Theta} F \otimes hX \otimes h_1 Y$$

hence a map

$$\Theta_x^{\#} : P_1 \longrightarrow F \otimes hX$$

~~[illegible]~~ Fixing an element $z \in P_1$ we obtain $x \longmapsto \Theta_x^{\#}(z)$, which is a natural transf.

$$hX \longrightarrow F \otimes hX$$

which is A-linear since $\Theta_{ax} = a\Theta_x$. Hence we get a map

$$\Phi : P \times P_1 \longrightarrow F.$$

One checks that $\Phi(az, w) = a\Phi(z,w) = \Phi(z, aw)$ and similarly on the right, hence we get a map

$$\Phi : P * P_1 \longrightarrow F$$

of A,A-modules. It remains to check that the composition of functors induces the cogebra ~~structure~~ on $P * P$, describe in 1.8, but this (sauf erreur) offers no difficulties.

1.8.2. Suppose now $h : \mathcal{A} \to \mathrm{Mod}\, A$ is a functor and that $\mathcal{A}$ is endowed with an operation $\otimes$ and a compatibility with h. Thus we have

$$\begin{tikzcd}
\mathcal{A} \times \mathcal{A} \arrow[rr, "\otimes"] \arrow[dr, "h \otimes h"'] & \arrow[d, phantom, "\overset{\simeq\, u}{\Leftarrow}"] & \mathcal{A} \arrow[dl, "h"] \\
& \mathrm{Mod}\, A &
\end{tikzcd}$$

$$u : h(X \otimes Y) \simeq hX \otimes hY$$

If $\mathrm{End}\, h$ is represented by a cogebra P, then by 1.8.1 $\mathrm{End}\, h \otimes h$ is represented by $P * P$, so the pair $(\otimes, u)$ gives rise to a cogebra map

$$\mu : P * P \longrightarrow P.$$

If $\mathcal{A}$ is provided with an object 1 and ~~isomorphism~~ a unit isom. $v : 1 \otimes X \simeq X$ ~~...~~ and if h is provided with a compatibility of this ~~...~~ structure, i.e.

$$h1 \simeq A$$

$$\begin{array}{ccc}
h(1 \otimes X) & \simeq & h(X) \\
\wr\| & & \wr\downarrow \\
h1 \otimes hX & \simeq & A \otimes h(X)
\end{array}$$

commutes, then we have ~~morphism~~ a composition morphism

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "X \mapsto (1,X)"] \arrow[dr, "h"'] & \mathcal{A}\times\mathcal{A} \arrow[r, "\otimes"] \arrow[d, "h\otimes h"] & \mathcal{A} \arrow[dl, "h"] \\
 & \mathrm{Mod}\,\mathcal{A} &
\end{tikzcd}$$

(this is $X \longmapsto 1\otimes X$ + isomorphism

$h(1\otimes X) \simeq h1 \otimes hX \simeq A\otimes hX \simeq hX$)

which is isomorphic via $\nu : 1\otimes X \simeq X$ to the identity of $(\mathcal{A}, h)$. The pair consisting of 1 and $h1 \simeq A$ can be viewed as a map

$$\begin{tikzcd}
e \arrow[r, "\cdot \mapsto 1"] \arrow[dr, "A"'] & \mathcal{A} \arrow[d, "h"] \\
 & \mathrm{Mod}\,\mathcal{A}
\end{tikzcd}$$

hence gives a cogebra map $A \xrightarrow{\eta} P$. The isomorphism of functors (just described) tells us that

$$P \xrightarrow{\simeq} A\otimes P \xrightarrow{\eta\otimes id} P\otimes P \xrightarrow{\mu} P$$

is the identity of P, hence η is a left unit for P.

Therefore we see that if $\mathcal{A}$ is provided with a tensor operation + unit and if h is provided with a compatibility with this structure, then P has a unit. (In a similar manner the existence of (resp. commutativity) an associativity isomorphism for $\otimes$ which is compatible with h will imply that P is associative and commutative as a ring.

<u>Proposition 1.8.3:</u> Let $h: \mathcal{a} \longrightarrow \text{Mod } A$ be a functor such that End h is representable by an A,A-cogebra P. Suppose $\mathcal{a}$ endowed with a (unitary associative and commutative) tensor product operation and that h is endowed with a compatibility with (respect to) this tensor product; hence P is an A,A-bigebra (commutative). Then for any A,A algebra R

$$\text{Hom}^{\otimes}(h, h_R) \simeq \text{Hom}_{A,A\text{-alg}}(P, R).$$

<u>Proof:</u> Given $\theta: h \longrightarrow h_R$, θ is a $\otimes$ functor iff $\forall X, Y$ in $\mathcal{a}$ the diagram

$$\begin{tikzcd}
hX \otimes hY \arrow[r, "\theta(X)\otimes\theta(Y)"] \arrow[d, "\wr"'] & (R\otimes hX) \otimes_R (R \otimes hX) \arrow[d, "\wr"] \\
h(X\otimes Y) \arrow[dr, "\theta(X\otimes Y)"] & R \otimes hX \otimes hY \arrow[d, "\wr"] \\
& R \otimes h(X \otimes Y)
\end{tikzcd}$$

commutes. The upper path from $h\otimes h$ to $R\otimes h\otimes h$ is represented by the composition

$$P * P \xrightarrow{\theta^{\#} * \theta^{\#}} R \underset{A\otimes A}{\otimes} R \xrightarrow{\mu_R} R.$$

The lower path is represented by

$$P * P \xrightarrow{\mu_P} P \xrightarrow{\theta^{\#}} R.$$

Thus θ is a tensor functor iff $\theta^{\#}$ is a ring homomorphism.

<u>Corollary 1.8.4</u>: Let $\mathcal{a}, h$ be as in 1.8.3 and let $\underline{\mathrm{End}}^{\otimes} h$ be the covariant functor from (rings) to $\mathcal{C}at$ given by

$$\mathrm{Ob}\,(\underline{\mathrm{End}}^{\otimes} h)(R) = \mathrm{Hom}_{(\text{rings})}(A, R)$$

$$\mathrm{Hom}_{(\underline{\mathrm{End}}^{\otimes} h)(R)}(u, v) = \mathrm{Hom}^{\otimes}(h_u, h_v)$$

where if $u : A \to R$ is a ring homomorphism, then

$$h_u : \mathcal{a} \longrightarrow \mathrm{Mod}\, R$$

$$X \longmapsto R \otimes_u hX$$

Then $\underline{\mathrm{End}}^{\otimes} h$ is represented by the co-category objects in rings

$$\begin{tikzcd} A \arrow[r, shift left, "\text{left}"] \arrow[r, shift right, "\text{right}"'] & P \arrow[l, bend right, "\varepsilon"'] \end{tikzcd}, \quad \Delta : P \longrightarrow P \otimes_A P$$

<u>Proof</u>: It suffices to note that a natural transformation $h_u \longrightarrow h_v$ is the same as one $h \longrightarrow R \otimes h$ where R is an A,A-algebra via u, v. Hence by 1.8.3 $\mathrm{Hom}^{\otimes}(h_u, h_v) \cong \mathrm{Hom}_{A,A\text{ alg}}(P, {}_uR_v)$. The rest is checking.

Combining 1.8.4 and 1.4.1 we have

<u>Corollary 1.8.5</u>: Let $\mathcal{a}$ be an abelian category with tensor product (unitary, associative and comm.) and let $h : \mathcal{a} \longrightarrow \mathrm{Mod}\, A$ be an exact faithful functor compatible with the tensor product. Assume that h has a right adjoint g which commutes with inductive limits.

Then $(\mathcal{A},h)$ is equivalent to the $\otimes$category of P-comodules and the forgetful functor, where P is the A-A bigebra representing $\underline{\mathrm{End}}^{\otimes}h$. Every P which is flat as a right A-module occurs in this way.

Defn: $\otimes$-category = an additive category endowed with a unitary, assoc., comm. tensor product.

Combining 1.8.4 and 1.6.2 we have

Corollary 1.8.6: If $h: \mathcal{A} \to P(A)$ is a $\otimes$-functor where $\mathcal{A}$ is a $\otimes$-category and $X \mapsto \mathrm{Hom}_A(hX,A)$ is ind-representable, say by $\{E_i\}$, then $\underline{\mathrm{End}}^{\otimes}h$ is represented by an A,A-bigebra P, given as a left A-module by

$$P = \varinjlim_i h(E_i).$$

Finally combining 1.8.4 and 1.7 we have

Corollary 1.8.7: Let A be a field, let $\mathcal{A}$ be an abelian $\otimes$-category and let $h: \mathcal{A} \to \mathrm{Modf}(A)$ be a faithful exact tensor functor. Then $\mathcal{A}$ with h is equivalent to the $\otimes$category $\mathrm{Com}(P) \cap \mathrm{Modf}\,A$, with the forgetful functor, where P is the A,A bigebra representing $\underline{\mathrm{End}}^{\otimes}h$. Conversely if P is an A,A-bigebra, then P represents $\mathrm{End}^{\otimes}h$ where h is the forgetful functor on $\mathrm{Com}(P) \cap \mathrm{Modf}(A)$.

1.9. In the situation of 1.8.6 we have the following criterion for $\underline{\mathrm{End}}^{\otimes} h$ to be a groupoid scheme.

<u>Proposition 1.9.1</u>: Let $h : \mathcal{A} \longrightarrow P(A)$ be as in 1.8.6. Suppose that every object X of $\mathcal{A}$ has a dual DX provided with maps

$$1 \xrightarrow{\Phi} DX \otimes X \xrightarrow{\Phi} 1 \qquad (1.9.2)$$

inducing isomorphisms

$$\mathrm{Hom}_{\mathcal{A}}(Z, DX \otimes Y) \cong \mathrm{Hom}_{\mathcal{A}}(X \otimes Z, Y)$$

for all Y, Z in $\mathcal{A}$. Then $\underline{\mathrm{End}}^{\otimes} h$ is an affine groupoid scheme.

<u>Proof</u>: Given $u, v : A \rightrightarrows R$ and $\theta : h_u \to h_v$, we must show that θ is an isomorphism. By assumption

$$X \cong X \otimes 1 \xrightarrow{id \otimes \Phi} X \otimes DX \otimes X \xrightarrow{\Phi \otimes id} 1 \otimes X \cong X$$

is the identity. Applying h we have that

$$hX \xrightarrow{id \otimes h(\Phi)(1)} hX \otimes hDX \otimes hX \xrightarrow{h\Phi \otimes id} hX$$

is the identity. Thus if

$$h(\Phi)(1) = \sum_{i=1}^{n} \lambda_i \otimes v_i \qquad \begin{array}{l} v_i \in hX \\ \lambda_i \in hDX \end{array}$$

and if

$$\langle v, \lambda \rangle = (h\Phi)(v \otimes \lambda) \qquad v \in hX,\ \lambda \in hDX$$

then

$$v = \sum_{i=1}^{n} \langle v, \lambda_i \rangle v_i \qquad \text{for all } v \in hX.$$

Similarly

$$DX \cong 1 \otimes DX \xrightarrow{\Psi \otimes id} DX \otimes X \otimes DX \xrightarrow{id \otimes \Phi} DX \otimes 1 \cong DX$$

is the identity and we find that

$$\lambda = \sum_i \lambda_i \langle v_i, \lambda \rangle \qquad \text{for all } \lambda \in hDX.$$

Thus we see that

$$\langle\, , \rangle : hX \otimes hDX \longrightarrow A$$

is a perfect duality in $P(A)$ and that $h(\Psi)1$ ~~is the~~ corresponds to "identity ~~[illegible]~~ transformation under the isomorphism $hX \otimes (hX^\vee) \simeq \mathrm{Hom}(hX, hX)$.

Now apply Θ to the maps Ψ, Φ

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & h_u X \otimes h_u DX \arrow[r] \arrow[d, "\Theta(X) \otimes \Theta(DX)"] & R \\ & h_v X \otimes h_v(DX) \arrow[ur] & \end{tikzcd}$$

observing that the ~~[illegible]~~ above paragraph holds with h replaced by h_u and h_v. ~~Thus we have the following situation~~ Thus denoting $h_u X$ by V and $h_v X$ by W, we have a map $\varphi \overset{\Theta(X)}{=} : V \longrightarrow W$ in $P(R)$ and a map $\psi : V^\vee \longrightarrow W^\vee$ such that

$$\begin{tikzcd} & V \otimes V^\vee \arrow[dr, "ev"] \arrow[dd, "\varphi \otimes \psi"] & \\ R \arrow[ur, "id"] \arrow[dr, "id"'] & & R \\ & W \otimes W^\vee \arrow[ur, "ev"'] & \end{tikzcd}$$

commutes. The second triangle shows that

$$\langle \varphi v, \psi\lambda \rangle = \langle v, \lambda \rangle$$

for all $v \in V$, $\lambda \in V^\vee$ hence that $\psi^t \varphi = id_V$. The first triangle shows that

$$\sum_i \varphi e_i \otimes \psi \lambda_i \qquad \text{where } id_V = \sum e_i \otimes \lambda_i$$

is the identity transformation of W, i.e. for all $w \in W$, $\mu \in W^\vee$

$$\begin{aligned} \langle w, \mu \rangle &= \sum \langle \varphi e_i, \mu \rangle \langle w, \psi \lambda_i \rangle \\ &= \sum \langle e_i, \varphi^t \mu \rangle \langle \psi^t w, \lambda_i \rangle \\ &= \langle \psi^t w, \varphi^t \mu \rangle, \end{aligned}$$

that is, that $\varphi \psi^t = id_W$. Thus $\varphi = \Theta(X)$ is an isomorphism. QED.

<u>Remarks</u>. 1. The first ~~part~~ of the proof shows that when ~~[illegible]~~ every object of $\mathcal{A}$ has a dual (even though $\mathcal{A}$ isn't additive), then $hX \in P(A)$ and $h(DX) = (hX)^\vee$ for ~~[illegible]~~ any functor $h : \mathcal{A} \to \text{Mod}\, A$ compatible with $\otimes$.

2. If P is ~~[illegible]~~ an A,A bigebra with antipode, that is, P is an affine groupoid scheme ~~[illegible]~~ with object ring A, then define

$$\text{Hom}_A(M,N) \longrightarrow \text{Hom}_A(M, P \otimes N)$$

for two P-comodules M, N ~~[illegible]~~ by sending f into the composition

$$M \xrightarrow{\Delta} P \otimes M \xrightarrow{id \otimes f} P \otimes N \xrightarrow{\tau \otimes \Delta} ({}_sP) \otimes P \otimes N \xrightarrow{\mu_P \otimes id} P \otimes N.$$

If $M \in \text{Ob}\, P(A)$, then this makes $\text{Hom}(M,N)$ into a P-comodule undoubtedly enjoying the properties of an **internal** <u>**Hom**</u>. It is clear (sauf erreur) that ~~[illegible]~~ for $M \in P(A) \cap \mathrm{Com}(P)$ its dual is $M^\vee = \mathrm{Hom}_A(M, A)$ with the comodule structure just described.

Comments on rewriting

Let $\mathcal{Q}$ be the category with objects functors $h: \mathcal{A} \to$ Mod A and where a morphism $(\mathcal{A}, h) \longrightarrow (\mathcal{A}', h')$ is a pair $F: \mathcal{A} \to \mathcal{A}'$ $u: h \simeq h'F$ modulo isomorphisms of F with an F'. Then we have

$$(A,A\,\mathrm{Cog}) \longrightarrow \mathcal{Q}$$
$$P \longmapsto \mathrm{Com}\,P$$

Conversely given h suppose $\exists$ A,A module $P \ni$

$$\mathrm{Hom}_{A,A\,\mathrm{mod}}(P, F) = \mathrm{Hom}(h, h_F)$$

for tensor products it seems desirable to prove the stronger statement that $\mathrm{Hom}_{A,A\ \mathrm{mods.}}(G \otimes P, F) = \mathrm{Hom}(h_G, h_F)$

Then show P is an A,A cog and

$$\mathrm{Hom}_{\mathcal{Q}}(\mathcal{A}, \mathrm{Com}(Q)) = \mathrm{Hom}_{\mathrm{Cog}}(P, Q)$$
$$\Big\|\ \text{always}$$
$$\{\eta \in \mathrm{Hom}(h, h_Q) \mid (id \otimes \eta)\eta = (\Delta \otimes id)\eta\}.$$

Now if A commutative and if End h on $\mathcal{A} = P$
End h' on $\mathcal{A}' = P'$, then

$$\text{End } h \otimes h' \text{ on } \mathcal{A} \times \mathcal{A}' \text{ is } P * P_1,$$

the canonical map $h \otimes h' \longrightarrow (h \otimes h')_{P * P'}$ being

$$hX \otimes h'Y \longrightarrow {}_{\cdot}(P \otimes hX) \otimes {}_{\cdot}(P' \otimes h'Y) \cong ({}_{\cdot}P \otimes {}_{\cdot}P') \otimes_{A \otimes_{\mathbb{Z}} A} (hX \otimes_{\mathbb{Z}} h'Y)$$
$$\downarrow$$
$$(P * P') \otimes_A (hX \otimes_A h'Y)$$

Another method of proving 1.9.1 is as follows:
Given $h: \mathcal{A} \longrightarrow P(A)$ we can also consider $h^\circ: \mathcal{A}^\circ \longrightarrow P(A)$, $h^\circ(X) = h(X)'$. Then if P represents End h, one shows that P but with left and right A-module structures reversed represents End h°. Hence if one has a duality functor

$$D: \mathcal{A}^\circ \longrightarrow \mathcal{A}^*$$

with $h(DX) = h(X)'$, then one gets a map $\tau: P \longrightarrow P$ reversing left and right A-modules structures.

Concerning 1.7.2 one can prove that P is the union of its lattice of left finite dimensional sub A,A-cogebras. Indeed we know already that P is the union of its finite dimensional subcomodules. Pick one V. Then V' is a right module ~~of~~ over the ring $Q = \mathrm{Hom}_{A^\circ}(P, A)$ with product as in 2.2.6. The annihilator $\mathfrak{a}$ of V' is an ideal in Q of finite ~~left~~ right codimension. ~~Then whose annihilator in P is a subcogebra of finite right dimension~~ Since V is a subcomodule of P, V' is a quotient Q° module of Q, hence V' is a quotient of $Q/\mathfrak{a}$. So V is a subcomodule of $\mathfrak{a}^\perp$ which is a subcogebra of P of finite left dimension.