(18-1)

MIT Topology Seminar 4/1/68

Ralph Reid (Princeton): Isotopy and embedding of polyhedra in euclidean space; the deleted join and π-manifolds as criteria.

Any n-dimensional polyhedron imbeds in $\mathbb{R}^{2n+1}$ and unknots (there is only one isotopy class of embeddings) in $\mathbb{R}^{2n+2}$ (for manifolds, these dimensions lower by 1). Examples are Q^n = n-skeleton of σ^{2n+2}, the $(2n+2)$-simplex, and an n-complex "3^n" defined as follows: let 3 be the complex of 3 points, and $3^n = 3 * \cdots * 3$ n times. These do not imbed in $\mathbb{R}^{2n}$. They have in common the property that if a top-dimensional simplex is removed, the remainder is a sphere.

Thm (Shapiro–Wu) If $H^n(K^n)=0$, K^n embeds in $\mathbb{R}^{2n}$.

Their method uses the deleted product $\tilde{K} = K \times K - d(K)$, or more precisely $K \times K - \{\sigma^p \times \sigma^q \mid \overline{\sigma^p} \cap \overline{\sigma^q} \neq \emptyset\}$, where the set removed is a neighborhood of the diagonal. It was noted in the 1930's that K^n embeds in $\mathbb{R}^q \Rightarrow \exists f : \tilde{K} \xrightarrow{eq} S^{q-1}$, there is an equivariant map $\tilde{K} \to S^{q-1}$ ($\tilde{K}$ has an obvious involution; so does S^{q-1}); given by $f(x,y) = \frac{y-x}{|y-x|} = -f(y,x)$ (note $y \neq x$). Embedding results involve trying to construct this map. Shapiro and Wu defined the first obstruction to an embedding as an obstruction to this map; and also defined a second obstruction and proved with much work that when it vanished, $K^n \subset \mathbb{R}^{2n-1}$. Haefliger and his student Claude Weber proved the

Thm If $q \geq \frac{3n+3}{2}$, then Ⓔ $\exists (\tilde{K} \xrightarrow{eq} S^{q-1}) \Rightarrow K \subset \mathbb{R}^q$

Ⓘ equivalence classes of isotopies (each level equivariant) $\longleftrightarrow$ Isotopy classes $K \hookrightarrow \mathbb{R}^q$ $(q > \frac{3n+3}{2})$.

Now Cone$(Q^{n-1}) = cQ^{n-1}$ imbeds in $\mathbb{R}^{2n}$, but although all its cohomology vanishes it does not imbed in $\mathbb{R}^{2n-2}$, since a small sphere about the cone point intersects the cone in a copy of Q^{n-1}.

The deleted product may not be the most natural tool. Another space that can be used is $J(K^n) = K^n * K^n - \{\sigma^p * \sigma^q \mid \overline{\sigma^p} \cap \overline{\sigma^q} \neq \emptyset\}$, of dimension $2n$, called the <u>deleted join</u>.

<u>Thm 1</u> (E) $K \hookrightarrow R^q \iff \exists f : J(K) \xrightarrow{eq} S^q$ $\qquad q \geq \frac{3n+3}{2}$.

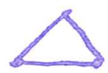
$\iff \exists g : \tilde{K} \xrightarrow{eq} S^{q-1}$

(I) isotopy result equivalent to Weber's using $J(K)$. //

<u>Cor</u> Embedding and isotopy of $Q^n \subset \mathbb{R}^{2n+1}$ (and everywhere).

<u>Lemma 1</u> $J(Q^n) \simeq S^{2n-1}$

<u>Pf</u> $J(Q^n) = J(n\text{-skeleton of } \sigma^{2n+2})$; Thus $Q^n \subset \text{bd}\,\sigma^{2n+2} \subset S^{2n+1}$ triangulated as $\text{bd}\,\sigma^{2n+2}$. There is a dual n-skeleton $Q^{n\prime}$ in S^{2n+1} also; it is homeomorphic to Q^n, and the deleted join of the two is all of S^{2n+1}.

(e.g. Q^0 = ∴ $\longrightarrow$  $= S^1$; dual 0-skel. $Q^{0\prime}$ = . Every point of S^1 is on a unique segment from a point of Q^0 to one of $Q^{0\prime}$.)

To calculate equivariant maps $K \to S^q$, use a fibration $S^q \to K \times S^q \xrightarrow{pr_1} K$. Factor by the involution; $S^q \to (K \times S^q)_{\mathbb{Z}_2} \to (K)_{\mathbb{Z}_2}$ is a fibration in which a section corresponds to an equivariant map $K \to S^q$. In the case at hand, $S^q \to S^q * S^q \to P^q$, which has obstructions to sections lying in $H^q(P^q; \mathcal{B}(\pi_q(S^q)))$. The top cohomology group is $\mathbb{Z}$; hence $\text{Iso}(Q^n \subset \mathbb{R}^{2n-1}) \longleftrightarrow$ (equivar. classes) $\longleftrightarrow H^{2n+1}(P^{2n+1}; \pi_{2n+1}(S^{2n+1}))$

$\iff \mathbb{Z}$.

Any equivariant map lies in an odd homotopy class and one can prove

$\iff \pi_{2n+1}(S^{2n+1})_{odd}$.

<u>Lemma 2</u> $J(A * B) = J(A) * J(B)$ //

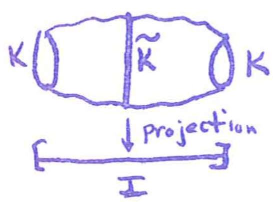
<u>Remark</u> $\tilde{K} \times \tilde{L} \neq \widetilde{K \times L}$ and similarly for the join; this is one difficulty with the reduced product that the reduced join solves.

Cor a) $J(3^n) = J(3) * \cdots * J(3) = S^1 * \cdots * S^1 \cong S^{2n+1}$

b) $3 = Q^0$; take $K^n = Q^{i_1} * \cdots * Q^{i_k}$. Then $J(K) = S^{2i_1+1} * \cdots * S^{2i_k+1} = S^{2n+1}$.

In the 30's, Kuratowski proved any graph not embeddable in $\mathbb{R}^2$ contains Q^1 or $Q^0 * Q^0$ as a subcomplex. One might hope in dimension 3 that $Q^2, Q^1 * Q^0$, and $(Q_0)^3$ would characterise non-embeddable polyhedra; we have a conjecture related to that, below.

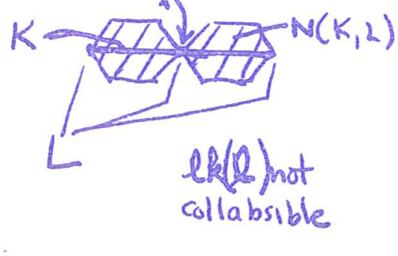
Other properties of J: to stop a complex K^n embedding in $\mathbb{R}^{2n}$ one needs top cohomology in $J(K)$; notice $J(3^n)$ and $J(Q^n)$ give exactly enough, namely have top-dimensional cohomology $\mathbb{Z}$ and all others 0 (since $\simeq S^{2n+1}$). However think of $J(K)$ as containing 2 copies (on the ends of the join) of K; then $K^n \cup K^n \longrightarrow J(K) \longrightarrow \Sigma(\tilde{K})$, the suspension of $\tilde{K}$, showing

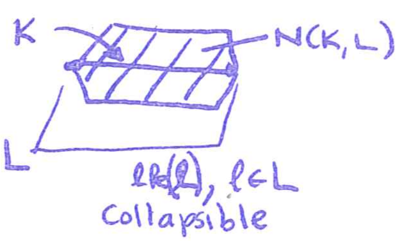


that the cohomology of $\tilde{K}$ looks like that of $J(K)$ above about the middle dimension; however it contains in the middle dimension two copies of K which have nothing to do with embeddings.

Thm 2 If $H^n(K^n) \simeq \cdots \simeq H^{n-k+1}(K^n) \simeq 0$ and $2n-k+1 \geq \frac{3n+3}{2}$, then $K^n \times D^k$ embeds in $\mathbb{R}^{2n+1}$ $\Leftrightarrow$ K^n embeds in $\mathbb{R}^{2n+1-k}$.

Pf If $(K,L) \subset A$ are subcomplexes, the relative regular neighborhood $N(K,L)$ of (K,L) in A is defined to be all simplexes touching $K-L$. We assume $K-L$ dense in K. If for all simplexes $\ell \in L$, $\mathrm{link}(\ell)$ in K is collapsible, then $N(K,L)$ is a manifold with boundary, and L is contained in the boundary.





If $(K,L) = (\text{manifold } M^n, \partial M^n)$, $lk(\ell)$ is collapsible for all $\ell \in L$.

Suppose $K^n \times D^k$ embeds in $\mathbb{R}^{2n+1}$. The pair $(K^n \times D^k, K^n \times S^{k-1})$ is link collapsible; in fact products of link collapsible spaces are link collapsible. Thus one gets a manifold, the relative regular nbd. of this pair (say, W^{2n+1}) with $\partial W^{2n+1} = M^{2n}$ and $K^n \times D^{k-1} \subset K^n \times S^{k-1} \subset M^{2n} \subset R^{2n+1}$. Continuing the process with $(K^n \times D^{k-1}, K^n \times S^{k-2})$ and so forth we finally have $K \subset M^{2n+1-k} \subset \cdots \subset M^{2n-1} \subset M^{2n} \subset R^{2n+1}$. Each manifold is embedded with codimension 1 in the next; all are π-manifolds, so one can do surgery to insure M^{2n} is $(n-1)$-connected, M^{2n-1} is $(n-2)$-connected, etc., and finally M^{2n+1-k} is $(n-k)$-connected. Once the surgery is done, we throw away the chain of manifolds and use just $K^n \subset M^{2n+1-k}_{(n-k)}$. We want to engulf K in a disk; to do so we must homotop K to a point. Obstructions to a homotopy of K to a point lie in $H^i(K^n; \pi_i(M_{(n-k)}))$. For $i \le n-k$, the coefficients vanish; for $i \ge n-k+1$, the cohomology with Z coefficients vanishes, hence the group of obstructions vanishes by the universal coefficient theorem. Reversing the homotopy one engulfs K in D^{2n+1-k}.

There is a conjecture that any π-manifold embeds in Euclidean space of 3/2 its dimension. If so, the construction above of a chain of π-manifolds gives additional information.

If there exists p with $H^n(K^n - p) \simeq \cdots \simeq H^{n-k+1}(K^n - p) \simeq 0$, the conclusion of <u>Thm 2</u> still holds. The only problem in proof is homotoping K to a point. If $p \in K \subset M$, take $\text{star}_M p$ out of M; let $K - \text{Star}_K p = K_0$, $M - \text{Star}_M p = M_0$; $K_0 \subset M_0$. But M_0 is $(n-k)$-connected, so the puncturing does not matter. This yields the homotopy $K \rightsquigarrow *$.

We can now prove the Shapiro-Wu results, and an improvement.

<u>Lemma</u> $K^n \times I$ embeds in R^{2n+1} for any n-complex K^n.

<u>Proof</u> take a general position map $K \to S^{2n}$; it has only double points, and is an immersion. Inside one top simplex σ^n take out a sphere around an image of 2 points. Cut out a sphere $S^{n-1} \times D^{n+1}$ = regular neighborhood, and replace it by $D^n \times S^n$; this surgery gets rid of the double point by running one of the preimages over the handle thus created. Then $K \subset S^n \times S^n \# S^n \times S^n \# \cdots \# S^n$.

Thus K embeds as required. //

Thus we have an embedding theorem if $H^n(K^n)=0$; or if $H^n(K^n)=\mathbb{Z}_{p^r}$. One can find an action on isotopy classes, and improve this result to the case $H^n(K^n)$ = any cyclic group. It is hard to prove for $H^n=\mathbb{Z}$; one proves first for any product of odd torsion.

<u>Thm 3</u> Define $\operatorname{Sing} K^n = \cup\{p \in K$ which are not manifold points$\}$ or

or $\cup\{p \in K$ whose link is not a homology sphere$\}$.

Then if $H^{n-1}(\operatorname{Sing} K^n)=0$, $K^n \subset \mathbb{R}^{2n}$. ($\operatorname{Sing} K^n$ is a complex of dimension $\leq n-1$). //

<u>Conjecture</u> K^n embeds in $\mathbb{R}^{2n}$ unless rank $H^n(K^n;\mathbb{Z}_2) \geq 2^{n+1}$.

Justification: Q^n is essentially the wedge of $\binom{2n+2}{n+1}$ spheres; for if you collapse the star of a vertex, all simplexes not in this star have their boundaries contained in the star. Now $\binom{2n+2}{n+1} > 2\binom{2n}{n}$ (in fact $\binom{2n+2}{n+1}$ approaches $\frac{4^{n+1}}{\sqrt{n+1}}$). A case of rank 2^{n+1} is $(Q^0)^{n+1}$ (join with itself n+1 times).

Kamber–Tondeur

Theorem: Let G be a Lie group with $\pi_0 G$ finite. Assume $H^*(\mathfrak{g}) \longrightarrow H^*(G)$ onto in dimension > 0. Then for any homomorphism $\gamma: \pi \to G$ where π is a discrete group, the map $\gamma^*: H^*(BG) \to H^*(B\pi)$ is 0.

Proof. If G^0 = connected component of G and $\Gamma = \gamma^{-1}G^0$, then have

$$\begin{array}{ccc} H(B\Gamma)^{\pi/\Gamma} & \longleftarrow & H(BG^0)^{G/G^0} \\ \uparrow\cong & & \uparrow\cong \\ H(B\pi) & \longleftarrow & H(BG) \end{array}$$

so we may assume that G is connected. Consider map of spectral sequences

$$\begin{array}{ccc} H^*(G) & = & H^*(G) \\ \uparrow i & & \uparrow \\ H^*(E\pi \times_\pi G) & \longleftarrow & H^*(pt.) \\ \uparrow & & \uparrow \\ H^*(B\pi) & \longleftarrow & H^*(BG) \end{array}$$

As the transgression takes primitive elts of $H^*(G)$ onto the generators for $H^*(BG)$ we must prove that i is onto. But the point is that we have

$$\begin{array}{ccccc} H^*(\mathfrak{g}) & \longrightarrow & H^*_\pi(G) & \longrightarrow & H^*(G) \\ & & \updownarrow pr_1^* & & \uparrow \\ & & H^*_\pi(E\pi \times G) & \xrightarrow{\cong} & H^*(E\pi \times_\pi G) \end{array}$$

so done.

Wall's formulation of Swan's theorem

(i) If π is finite of period $2q$, $q > 1$ and $g \in H^{2q}(\pi, \mathbb{Z}) \simeq \mathbb{Z}_\ell$ (ℓ = order π) is a maximal generator, then there exists an oriented $(2q-1)$-dimensional Poincare complex $Y(g)$, ~~with~~ which is dominated by a CW complex and has $\pi_1 Y(g) \simeq \pi$ and universal covering ~~[illegible]~~ of the homotopy type of S^{2q-1}. Further ~~[illegible]~~ $Y(\pi_1, g_1)$ and $Y(\pi_2, g_2)$ are homot. equiv $\iff \exists$ isom $\alpha : \pi_1 \to \pi_2 \ni \alpha^* g_2 = g_1$.

(ii) conversely $Y \ni \tilde{Y} = S^n$, Y ~~[illegible]~~ finitely dominated Poincaré cx. of dim n
n even $\Rightarrow Y \sim S^n$ or $\mathbb{R}P_n$, $n = 2q-1$, $q > 1$ then $Y \sim Y(g)$ g = k-invariant

(iii) $Y(g^r)$ a finite cx.

G' conn. simply-conn. nilp. Lie gp., center Z', dim. n
Γ' discrete uniform subgp., $A = \Gamma' \cap Z'$ rank $A = m$.

$G = G'/A$ $\Gamma = \Gamma'/A$

$\alpha : \Gamma \longrightarrow G$ induced map

Theorem (Kamber): $\alpha^* : H(BG, Q) \longrightarrow H^*(B\Gamma, Q)$ is zero $\Longleftrightarrow G \simeq R^n$

$$H(BG,Q) = H(B(Z'/A), Q) \simeq Q[x_1, \dots, x_m]$$

(Z'/A: torus) $\deg x_i = 2$.

Theorem: X compact manifold with flat G structure on its tangent bundle. If G compact or complex reductive, then $\chi(X) = 0$.

better Th

Theorem: If G either compact or complex reductive ($\pi_0 G$ finite) and $\gamma : \pi \to G$ is a homomorphism where π is discrete, then $\gamma^* : H^*(BG, Q) \longrightarrow H^*(B\pi, Q)$ is zero.

April 6, 1968:

Ultimate problem is to ~~[illegible]~~ find some analogue of the $G, \overline{W}$ situation in the differentiable category. ~~[illegible]~~ The goal is to classify in some sense smooth proper maps over a base X, in the way that the fundamental group classifies the smooth finite maps.

Ideas: A. Cobordism classification

Definition:

$$\begin{array}{ccc} Y & & Z \\ & \searrow \; \swarrow & \\ & X & \end{array}$$

are cobordant if $\exists$ smooth proper W with bdry over X etc.

homotopy classification: Embed

$$\begin{array}{ccc} Y & \xhookrightarrow{i} & X \times \mathbb{R}^N \\ & \searrow & \downarrow pr_1 \\ & & X \end{array}$$

~~Regular Immersions? over each pt. of X~~ where i is an embedding ~~of the fiber~~ over each point of X. Form fiberwise tubular nbd. and Thom space, so one gets a map

$$X \times S^{d+N} \longrightarrow MG(N).$$

I believe that $[X, \underline{\mathrm{Hom}}(S^{d+\infty}, MG(\infty))]$ gives the cobordism classification as usual.

Let F be a compact manifold fixed. For any manifold X we wish to classify somehow proper smooth maps $\pi : E \to X$ with fiber F. This is a regidification problem. The old method is to put rigidification data on such a map π so that the isomorphism classes functor becomes representable or at least representable by an ind. object. However another method might be to apply a functor and classify the result.

~~Honestly represen~~

$$C_N^q(X, A) = \mathrm{Hom}(X, N^{-1}\{A[q+1, q]\})$$

certainly a representable functor!!

$$0 \longrightarrow A \to C_N^0(\cdot, A) \longrightarrow C_N^1(\cdot, A) \longrightarrow C_N^2(\cdot, A) \longrightarrow \cdots$$

resolution of the constant sheaf A by representable sheaves.

Let X be a topological spaces. Can we find representable sheaves resolving the constant sheaf functorial in X. The answer is no since the final object is a point there are only dull sheaves over a point.

Question: Let X be a smooth manifold with a basepoint x_o. Consider the functor from the category of Lie groups to the category of sets which associates to a Lie group G the set of isomorphism classes of principal G bundles with group G over X together with a basepoint over x_o and a connection. Is this functor pro-representable?

Example: If we consider only discrete groups G, then the functor is represented by $\pi_1(X,x_o)$.

Discussion: The question is probably false, because one would expect that if α represented the functor, then $\pi_*\alpha$ would be the homotopy groups of X. But π_2 of any Lie group is 0. Nevertheless we know that the set of reduced paths in X form a group (reduced = piecewise smooth parameterized according to arc lenthg and no retracing) although not it seems in a continuous way.

The real problem is the classification of smooth proper maps over X, i.e. the Barratt-Guggenheim-Moore theory.or Stasheff theory if such exists.. The problem is to rigidify things in such a way that they become representable. Let's review Milnor's theory. Milnor studies principal bundles over a topological space together with trivializations over a numerable covering. Let the bundle be $E \xrightarrow{\pi} X$ and the covering (U_i) and the partition of unity ρ_i, and finally let $\alpha_i: \pi^{-1}U_i \to G$ be a fibre coordiante over U_i. Milnor defines the universal bundle EG to be the set of formal sums $\sum t_i g_i$ where $0 \leq t_i \leq 1$ $\sum t_i = 1$ and $g_i \in G$, and where we identify two formal sums if they are the same except for zeros. Now can define $\partial: E \xrightarrow{\partial} EG$ by $\partial x = \sum \rho_i(x)\alpha_i(x)$ observing that this sum makes perfect sense.

Thus the data in Milnor's theory are the following: A principal bundle $\pi: E \to X$, a partition of unity $\sum \rho_i = 1$, and trivializations of E over U_i where $U_i = \rho_i^{-1}(0,1]$. The data in the differential theory are a principal bundle and a connection. Question: In the differential category is there a classifying space in the strict sense, i. e. a manifold with a principal bundle over it? For example if G is discrete, then a connection is trivial and unique and so if the answer were to be affirmative, then it would be possible to define in a rather canonical way a universal bundle for a discrete group.

Let G be a compact Lie group. To find a natural candidate for BG. The case of line bundles as studied by Kostant. <u>Theorem</u>: Equivalence of categories betwenn complex line bundles over X with a connection and closed 2-forms on X given by taking a line bundle with connection into its curvature form! So the correspondecnce we are after is perfect in this case.

connection <-> lifting function

curvature <-> twisting function

Can anything be done for complex vector bundleds? Standard rigidification consists in choosing a family of generating sections. Return to line bundles. A family of generating sectionsdefines a connection namely the pullback of the connection from projective space.

Onthe category of topologiscal spa ces consider the functor which associates the double coverings, say pointed connected spaces. Why isn't this functor representable?

For a pointed connected space X, Let D(X) be the set of isomorphism classes pointed double coverings, i.e. $H^1(X,x_0,Z/2Z)$. Why isn't this functor representable? Answer: because of disconnected equivalenee relations. Thus if $Y \to X$ is a finite covering map, we don't have

$$H^1(X,Z/2Z) = H^1(Y,Z/2Z)^G$$

but instead an exact sequence:

$$0 \to H^1(G, H^0(Y,Z/2Z) \to H^1(X,Z/2Z) \to H^0(G,H^1(Y,Z/2Z)) \to H^2(G,Z/2Z)$$

Still confused. Problem Classify the proper smooth maps with target X. Example: Problem: classify the finite smooth maps with target X has been solved by Grothendieck in the case X is connected. The answer is to rigidify with a fibre functor in which case the category becomes equivalent to the category of finite G sets where G is a profinite group. The problem of clssifying the locally trivial maps of simplicial sets with a reduced base X has been solved by BGM. Rigidify with lifting functions and one gets the category of G-simplicial sets, where G = GX.

$A^1(X)$ is not representable. Therefore unlikely that connected bundles should be representable either.

In the category of simplicial sets cohomology can be calculated by representable functors. Let X be a topological space. Can we find a resolution of the constant sheaf by coh. trivial absolutely representable sheaves. Note that a sheaf determines an etale space over X and therefore a functor of a special type on the category of spaces over X. Seems to be false.

Localization revisited: Let A be a ring and let S be a multiplicative system in A. Then there is a universal property which one expresses in solving the equations $T_s s = 1$. To get the correct spirit one should think only of localizing a finite number of elements. Etale schemes if done correctly should consist of writing down equations $F_i(X_j) = 0$ which are totally etale(perhaps only syntonic) and taking this as the basis for localization. Ultimately what we are trying to do is to consider the dual of the category of rings and invert in some sense the etale surjective maps. We wish to consider contravariant functors on the dual of the category of rings which are sheaves for the etale topology and which are covered in an honest way by real schemes, i.e. there exists a disjoint union of affine achemes and an etale equivalence relation such that the quotient of sheaf is the functor. Description of the etale scheme by means of points.

Cohomology should be calculated by a sequence of representable functors. <u>Problem</u>: Is there some Lubkin-Cech prodedure for calculating the cohomolgy of a space such that before passage to the limit one has representable functors for the cochains?

<u>Idea</u>: The problem of classifying vector bundles over a compact space is solved by rigidifying by means of a family of generating sections. Maybe one can solve the manifold classification problem in a similar way. In other words if F is a compact smooth manifold of dimension d one considers a rigidification of a smooth map

$\pi: E \to X$ to be a map $i: E \to X \times \mathbb{R}^N$ such that $pr_1 \cdot i = \pi$ and such that for each x in X, i induces an embedding of $\pi^{-1}x \to \mathbb{R}^N$. If we are studyong vector bundles, then when X is a point the set of rigidification data is the ways of embedding a d-plane into N-space, i.e. the Stieffel manifold.or the total sapce of the principal bundle. Here the principal bundle is the set of embeddings of F into $\mathbb{R}^N$ and the base is the quotient by the group of automorphisms of F. It seems clear that basic results on infinite-dimensional manifolds will permit one to conclude that we have solved the classification problem.in some sense. A basic question is whether the classification methods of Chern-Weil type can be developed here. Here the Lie algebra is the algebra of vector fields on the manifold F. Any idea of what kind of invariant polynomial functions there might be?

April 7. 1968

Summary of yesterday's work.

If F is a compact smooth manifold, then in analogy to the Grassmanian we constructed a principal bundle for the group of diffeomorphisms of F by considering the set of embeddings of F into a high Euclidean space. We would now like to define characteristic classes for smooth maps with fibre F. As a first start we want characteristic classes for rational homology if such exist. Example: If F is a sphere of dimension d odd, then there is the obstruction to finding a section, namely the Euler class. Obvious generalization of this is to look at the first nonvanishing cohomology group of F and where it goes under the differential of the spectral sequence. In other words we are looking for universally transgressive elements in $H^*(F)$.

Example: Let F be complex projective space of dimension n, and consider only bundles over simply-connected manifolds X. The first cohomology obstruction is in $H^3(X)$ and is d^3c, where c is the generator of $H^2(F)$. Note that if the bundle came from a complex vector bundle, then $d^3c = 0$. Also if F is the Gaussian sphere, then d^3c is the Euler class and is of order 2. If the bundle E is a $PGl(n,\mathbb{C})$ bundle, then it is reasonable to assume that d^3c is the invariant constructed by Serre in his classification of the topological Brauer groups and hence is a torsion element. $0 \to \mu_n \to Gl(n,\mathbb{C}) \to PGl(n,\mathbb{C}) \to 0$. Conjecture: Any $\mathbb{C}P^n$ bundle over a simply-connected manifold has the property that for rational cohomology the generator of $H^2(\mathbb{C}P^n)$ is transgressive. If true, then we can define using

$$0 \to H^2(X) \to H^2(E) \to H^2(F) \to H^3(X) \to 0$$

Chern classes provided that $H^2(X) = 0$.

Conclusion: For a 3-connected space X we can define Chern classes for any $\mathbb{C}P^{n-1}$ bundle over X. If the conjecture is true, then we can already do this for 1-connected spaces in rational cohomology. (see scratchwork to show that we can always choose x so that $c_1=0$.)

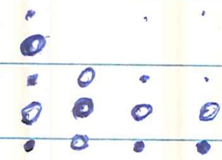
Important problem: Take F to be simply-connected. Compare the rational homotopy classification of F bundles with the rational geometric classification, where by the latter we mean the rational homotpy type of the classifying space of Aut F.

Idea: Somehow we are only understanding one side of the problem. Thus we are getting at the construction of BG but not at the construction of GX. We are thinking of a manifold in terms of its group of autos; we should be thinking of it in terms of the group of loops. It is as if we started with a minimal DG de Rham algebra and took its Lie algebra of derivations in order to classify fibrations with it as fiber. Somehow what we want to do is to get at the loop Lie algebra, which might by some chance be shifted one dimension away. It seems imperative to see if there is any relation one can understand between the automorphism group of a manifold and the Loop group. Somehow the automorphisms are the Lie algebra of vector fields. If we add in the ring of functions, then we get a mixed situation, and the loop group is somehow the ring of differential operators on the manifold.with its natural Hopf structure.

Is it possible to use Grothendieck's automorphisms of a functor idea. to construct the necessary G?

Return to your principal bundle.. Given / $\pi: E \to X$, the associated principal bundle is Isom(F,E)

Classify $\mathbb{C}P^{n-1}$ bundles over a (simply-connected) space X:



$$0 \to H^2(X) \to H^2(E) \to H^2(F) \to H^3(X) \to H^3(E) \to 0$$

$$H^2(F) = \mathbb{Q}$$

If $d_3 : H^2(F) \to H^3(X) = 0$ then choose $x \in H^2(E)$ ~~mapping onto c~~ $\ni$ $\iota^* x = c$, and we have that $1, x, \dots, x^{n-1}$ form a free basis for $H^*(E)$ over $H^*(X)$ & can define c_i

$$x^n - c_1 x^{n-1} + \cdots + (-1)^n c_n = 0.$$

Can change x by $x + \lambda$ where $\lambda \in \operatorname{Pic} X$. Then

c_1 goes into $c_1 + n\lambda$

$c_2 \qquad c_2 + c_1\lambda + \lambda^2$ $\qquad n = 2$

i.e. if $c_1 = \mu_1 + \mu_2$ $\qquad c_1' = \mu_1 + \mu_2 + 2\lambda$

$c_2 = \mu_1 \cdot \mu_2$ $\qquad c_2' = \mu_1\mu_2 + c_1\lambda + \lambda^2$

consequently c_1 is without meaning. But

$$4c_2' - c_1'^2 = 4\lambda^2 + 4c_1\lambda + 4c_2 - c_1^2 - 2\cdot 2c_1\lambda - 4\lambda^2$$

$$= 4c_2 - c_1^2 \qquad \text{makes sense.}$$

and if $i:E \to X \times R^N$ is an embedding over X, then we get a map $Isom(F,E) \to Emb(F,R^N)$ and factoring out by Aut F a map X $\to$ base of the principal bundle. Therefore the rigidification of π consists of an embedding into Euclidean space over X.

On to understand motives. Grothendieck tries to classify all coverings of X. The rigidification consisits of a fibre functor.and he considers automorphisms of the fibre fundtor. (Any chance that though the deRham cochain fuctors are not representable absolutely, there is an intermediate category with an internal Hom such that they become representable?)

Let us consider all smooth maps $/\pi:E \to X$ with base X and define a rigidification to be a map $p:E \to R^N$ which gives rise to an embedding over X. <u>Question</u>: Is the functor on the category of smooth proper manifolds over Xwhich associated to a map π the set of all embeddings into Euclidean spaces representable in any way? This means can we find a'manifold' over X and an embedding This is not a functor! Another important property of the covering maps is that the fiber functor is conservative besides being faithful.

Have discovered new phenomenon: In the category of simplicial sets where we know what happens thanks to BGM,we get a functor from the category of simplicial GX-sets to the category of simplicial sets over X which are locally trivial. However note that we do not get an equivalence of categories only a fully faithful functor such that every object in the categoryis isomorphic to one constructed.from a GX-simplicial set. Return to vector bundles

Looked at Milnor's ~~mxinx~~Universal Bundles ~~II~~ I where he constructs a GX for a simplicial complex X which is countable. The idea was to consider paths as being finite sequences each consecutive pair being ~~inxx~~ contained in a simplex and where one is allowed to delete repetitions and retracings. Milnor shows ~~hxx~~ how to topologize this so that it becomes a topological group in fact a countable CW complex. It seems clear that some variant of this should work for a smooth manifold. In other wordsLet X be a paracompact countable smooth manifold and choose a Riemannian metric for which X is complete and convex indistances less than 1. Consider paths ~~xxxhxxxixxxfxwhixh xxxxixhxxxf~~ to be finite sequences of points each consecutive pair being within distance α with the Milnor identifications so that one gets a topological group. This group doesn't seem to be a manifold in any reasonable way.because of singularities which occur when points coallesce. ~~Hxxxxx~~ However if ~~E~~ is a principal bundle over X with group G, then one may ~~byxthexdexxxixgxhxxxxkpx~~ as with Milnor construct a homomorphism GX → G inducing the bundle. It would seem reasonable that Chen's group of piecewise ~~linxxx~~ regular reduced paths forms a similar kind of topological ~~gxxxxp~~ group. Maybe this whole business can be improved on the Hopf algebra level. First observation: ~~Lxxxx~~ It is possible using the Riemannian structure to introduce L^2 paths. If one considers the monoid of paths in the sense of Moore., then one obtains a NO manifold whose tangent space at ~~thexxxxigixxixx~~ a given path is the infinitesimal ~~xxxixkixn~~ variations along the path. Thus the tangent space to the origin is?

<u>Question</u>: Consider over a space X the functor $G \mapsto H^1(X,x_0;G)$, where G runs over the category of topological groups. Is this functor representable? Probably not so instead~~consider xxx the~~

rigidify with a symmetric slicing function over a regular nbd of $\triangle$ in the sense of Milnor. Then the resulting functor should be represented by Milnor's construction.

Conclusion: In Milnor's Universal Bundles papers, he chooses two different ways of rigidifying a principal bundle. For the construction of BG he rigidifies a principal bundle using a partition of unity ρ_i together with trivializations μ_i over $Cl\ \rho_i^{-1}(0,1]$. In the construction of GX he uses a symmetric slicing function over a regular neighborhood of the diagonal. Finally in the case of vector bundles another rigidification is used namely a family of generating sections. There is a standard rigidification of a principal bundle namely a local trivialization.

We owe to Milnor the following:

$\mathrm{Hom}(X,BG)$ = Isomorphism classes of collections P,ρ_n,u_n $n\in\mathbb{N}$ where $\pi:P\to X$ is a principal bundle with group G, ρ_n is a partition of unity on X, and u_n is a trivialization of P over $CL\ \rho_n^{-1}(0,1]$.

Ideas: Any hope of getting an adjoint tothe functor $G|\to BG$? Any hope that this formula for BG will yield a good definition of higher Whitehead groups? Extremely important to notice that this formula when G is a discrete abelian group yields a representable functor for 1-cocycles.

Lemma: Topologize the infinite simplex $\triangle[\infty]$ as the inductive limit of the finite simplices. Then for any compact space X, $\mathrm{Hom}\ (X,\triangle[\infty])$ is the set of partitions of unity on X indexed by the set of natural numbers $\mathbb{N}$.

It seems to be false that the usual Cech method for calculating $H^1(X,G)$ provides a ~~xixxixxxxxxxxxx~~ representable functor. Thus ~~if~~ ~~XXXXXXXXXXXX~~ the usual refinement problem ~~xxxxx~~ doesn't permit one to take an inductive limit and define a cochain functor. Godement defines ~~xxxx~~ Cech cochains by indexing coverings by the points of X in which case ~~xxxxxxxxxxxxxxx~~ $C^1(X,G) = \underset{\longrightarrow}{\mathrm{Lim}}_U \coprod_x G(U_x)$. Unfortunately it doesn't seem to be true that the functor is represeantable or even ind-representable absolutely because of the appearance of the U_x which depend on the space X. For the same reason Lubkin's cochains must also fail to be absolutely representable functors.

Problem: If A is a discrete group, is it possible to calculate the cohomology of a space X with values in A by means of absolutely representable functors? YES take realization of exact sequence

$0 \to K(A,0) \to L(A,0) \to L(A,1) \to L(A,2) \to \cdots$

In the Milnor situation, Hom(X,BG) is endowed with an equivalence relation, namely homotopy, and the resulting homotpy calsses are H^1. Thus it would be nice if $G \to EG \to BG$ were an exact sequence of topological ~~xxxxxx~~ abelian groups for then $0 \to H^0(X,G) \to (X,EG) \to (X,BG) \to H^1(X,G) \to 0$ would be exact.

April 8,1968

Question: Is there any relation between Milnor's G construction, which is an inductive limit of n-step paths as $n \to \infty$, and Alexander -Spanier cochains?

Start as follows. Let N be a neighborhood of the diagonal in X x X and assume it to be symmetric. Then we can define Alexander q-chains to be the set of finite sequences $x_o, \dots, x_q$ where each consecutive pair is in U. Unfortunately this is not closed under faces. Thus Alexander-Spanier cohomology cannot be defined with a fixed neighborhood of the diagonal. Kan has propesed that we use the construction of a CW loop group following Toda. This probably works in the obvious way. Namely if P is a principal bundle with group G over a CW complex X, then given a cell e one trivializes the bundle over each cell e. A given cell e gives rise to a boundary map $S^m \to X$ whose buundary is map up of other cells f_i. On each f_i one obtains a different trivialization and hence a function

Question: Is there any hope of using a Morse function instead ofa CW division in the case of a manifold?

Review the reduced product space of James. Let X be a pointed space, let EX be the reduced suspension of X and Let FX be the free monoid generated by $X - x_o$. Equivalently let FX be the infinite wedge of the iterated joins No. Let W_n be the n-fold wedge of X, IβE. i.e. sequences $x_1, \dots x_n$ of points which are not the base-point together with the basepoint. Then the topology of FX is defined so that the a sequence of finite sequences the last coordinates of which converge to the basepoint converges to the sequence of lower order. In other words, we take the set of all infinite sequences mostly at the basepoint and identify them if we can delete copies of the basepoint and then we put on the quotient topology.

A CW decomposition is not functorialand neither is Milnor's regular neighborhood of the diagonal. However one may fix them and let the group vary. The problem is to find the invariant of the trivialzation. If one attaches a cell e to a space Y to form $X = Y \cup_f e^d$ and one has a quasi-trivialzation of P over Y, then $f:S^{d-1} \to Y$ gives us a bundle over S^{d-1} which is trivial. Somehow the quasi-trivialization over Y should give us another trivialization of the bundle over S^{d-1} and the difference should be the map of S^{d-1} into G that we want. This seems absolutely ridiculous.

Onto understanding Morse functions. Let E be a differentiable fibre bundle over a manifold X. Endow E with a connection and X with a Riemmanian metric (probably a connection will already do) and let f be a Morse function on X. Then the gradient lines of f would give us a really nice trivialization of the fiber bundle except for the existence of critical points. This is an important point, namely one can prove the homotppy axiom for differentiable bundles by chooding a connection and using the parallelism along lines.

See if you can make any sense out of last nights dreams.
Ideas: Ed Brown suggests that Milnor construction is to the bar construction as ? is to the cobar construction. As the suspension of the smash is the join, the dual of the join is something like the loops on the cosmash i.e. fiber of the map $\Omega(X \vee Y) \to \Omega X \vee \Omega Y$. Of course this is what is wrong with the Milnor construction that it gives the bar construction rather than the deRahm cohomology.
*It is perhaps not crucial that you use groups and perhaps you can

get away with monoids. The reason is that the relevant completion process always gives a group. Thus suppose that we start with a manifold X with a basepoint and we consider the monoid of parameterized ~~paths~~ piecewise smooth paths of variable parametrization so that we obtain a monoid. Then we wish to consider the monoid ring of chains on the loop space and complete and take the group-like elements.

completion converts a monoid into a Malcev gp.

April 9, 1968

Summary of ~~my~~ work since April 1.

The ultimate problem: See if you can relate your cochain functor to the de Rham cochain functor and thus get a hand on rational elements in the de Rham cohomology.

1. Tried to ~~study~~ understand the $G, \bar{W}$ formalism ~~[illegible]~~ in the hopes of getting a generalization to the smooth category. Rather disappointing. ~~The~~
One starts with the bifunctor $Pr(X, G)$ of isom. classes of principal bundles with base X and group G. The problem is to represent this functor

$$[GX, G]_{gr} = Pr(X, G) = [X, BG]$$

where $[\,,\,]$ and $[\,,\,]_{gr}$ are homotopy classes and group homotopy classes in a suitable sense. In good situations we may rigidify the functor Pr to a functor $\mathcal{T}$ so that

$$Hom_{gr}(GX, G) = \mathcal{T}(X, G) = Hom(X, BG)$$

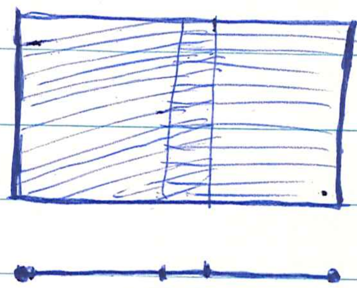
Here are some good cases:

i). simplicial sets. One restricts ~~to~~ connected X. Given a principal bundle one rigidifies it by choosing a lifting function in which case it is determined up to isomorphism by the associated twisting function.

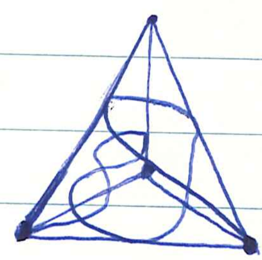
ii) DG coalgebras.

Unfortunately for topological principal bundles there doesn't seem to be a good situation. Milnor in constructing BG rigidifies using a map $X \to \Delta(\infty)$ and a trivialization over the closures of the open sets of the covering. To construct BG he uses another rigification namely a symmetric slicing function over a regular nbd. N of diagonal. The resulting GX depends on this regular neighborhood, so denote it $G_N(X)$. Then the inverse system $\{G_N(X)\}$ (if $N \subset N'$ then $G_N X \subset G_{N'} X$) is a functor on the category of pointed topological spaces. It is also possible to define a $G_u(X)$, where $u: X \to \Delta(\infty)$, using results due to some Russian on the existence of a free topological group functor. One would like to make $\{G_u(X)\}_{u: X \to \Delta(\infty)}$ into a functor of X to some kind of inverse system of topological groups, but one runs into the usual Čech refinement problem.

In the differentiable category, it seems that ~~the~~ ~~local trivialization~~ connection is an excellent replacement for local trivialization, as it enables us to lift homotopies uniquely. Observe that topologically a local trivialization does not allow us to lift homotopies uniquely

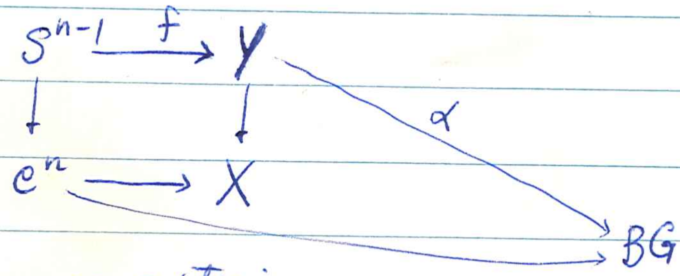


because of overlap. $G * G \longrightarrow \Delta(1)$



CW decompositions and principal bundles.

Suppose $X = Y \cup_f e^n$.



co-cartesian

Therefore if we have α we pull back to αf and then extend to e^n. In our case we have a differentiable principal bundle P over a manifold X and a proper function φ on X with only non-degenerate critical points which we shall assume to be separated. The problem we pose is to investigate what happens as we pass through a critical value

$$[GX, G]_{Hsp} = Pr(X, G) = [X, BG] \checkmark$$

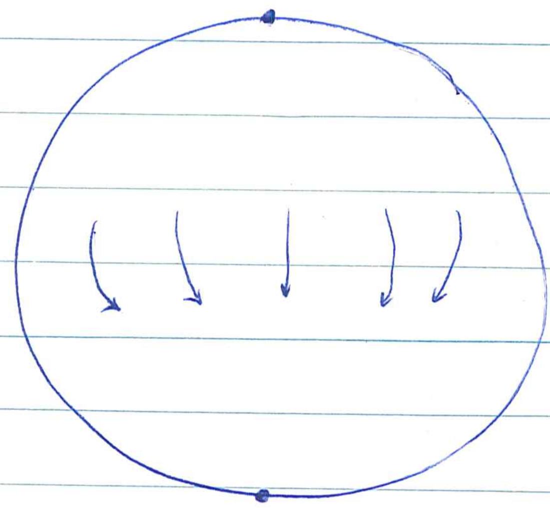
$$\underset{sgps.}{Hom}(GX, G) = \mathcal{P}(X, G) = \underset{d.sets}{Hom}(X, BG) \checkmark$$

Question: Let φ be a (proper real-valued) function on X with non-degenerate critical points. Can one obtain a de Rham cochain complex which is free as an algebra and with one generator for each critical point of φ.

Hörmander's idea: Construct a parameterix h_ε using φ such that $dh_\varepsilon + h_\varepsilon d = 1 - S_\varepsilon$ where S_ε is a smooth operator which concentrates at the critical points as $\varepsilon \longrightarrow 0$.

Can use $\nabla\varphi = \text{grad}\,\varphi$

$$i(\nabla\varphi)\,d + d\,i(\nabla\varphi) = \Theta(\nabla\varphi)$$



S^2 excellent case because my de Rham cx is

$\mathbb{Q} \quad 0 \quad \mathbb{Q}u \quad \mathbb{Q}v \quad \cdots$

$dv = u^2$.

So from my point of view I must know about the rational homotopy of my manifold!!!!! This is unreasonable so instead I should look for something group-like using the critical points. Thus as a first step, we should somehow see how to calculate the cup product structure as in the twisting function case

Suppose we have a critical point z of index m.

April 15, 1968.

Program: To see if it is possible to prove a Verdier duality theorem for simplicial sets.

Review of simplicial sets:

A. A simplicial set X has an associated ~~site~~ topos $\mathcal{S}/X$. sheaf of sets = contravariant functor on Δ/X. For any map of simplicial sets $f: Y \to X$ we have adjoint functors

$$\mathcal{S}/X \;\underset{f_*}{\overset{f_!}{\leftleftarrows}}\!\!\xrightarrow{f^*} \mathcal{S}/Y$$

where $f^*(U \to X) = U \times_X Y \longrightarrow Y$

$f_!(V \xrightarrow{u} Y) = V \xrightarrow{fu} X.$

~~The same holds for abelian sheaves.~~ B. Note that if F is ~~an ab~~ a sheaf over X, then

$$H^0(X,F) = \Gamma(X,F) = \varprojlim_{\Delta/X} F/(\Delta/X).$$

and

$$H^q(X,F) = R^q \varprojlim F/(\Delta/X).$$

In other words we are in the ~~...~~ situation mentioned by Sevans.

C. If ~~$f: Y \to X$~~ $f: X \to Y$ and F is an abelian sheaf on X, we may define $R^q f_* : Sh(X) \longrightarrow Sh(Y)$. As f^* is exact f_* preserves injectives, so we obtain the Leray spectral sequence

$$E_2^{pq} = R^p g_* \circ R^q f_* \Longrightarrow R^{p+q}(g \circ f)_* .$$

Also

$$R^q f_*(F)(\sigma) = H^q(f^{-1}\sigma, F)$$

where

$$\begin{array}{ccc} f^{-1}\sigma & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \Delta(k) & \xrightarrow{\sigma} & Y \end{array}$$

is cartesian.

D. Let $j: U \to X$ be a map of simplicial sets. Then j is injective $\Longleftrightarrow$ $j^* j_* = id \Longleftrightarrow j^* j_! \simeq id$. In this case we obtain the Artin situation

$$Sh(X,U) \underset{i^!}{\overset{i^*}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{i_*}}}}} Sh(X) \underset{j_*}{\overset{j_!}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{j^*}}}}} Sh(U).$$

$j_!$ is extension by zero on $X - U$ and hence is exact.

E. The support of a section is always the complement of a subcomplex. Thus if F is a sheaf on X and if s is a section of F we define the cosupport of s to be the $\{\sigma \in X \mid s(\sigma) = 0\}$. This is a subsimplicial set of X.

F. If j is injective, we can define $j^!: Sh(X) \to Sh(U)$ left adjoint to $j_!$ as follows. Given $F \to X$ divide out by all simplices over $X - U$. Then $j^! F$ is the quotient of $j^* F$ by simplices which are in the closed of $F/X-U$.

In general we can define $f^!$ whenever f is a finite map. Thus

$$(f_! F)(\sigma) = \bigoplus_{f\tau = \sigma} F(\tau)$$

Check your variances.

$$\begin{array}{c} F \\ X \xrightarrow{\ f\ } Y \end{array}$$

$$(f_* F)(y) = \varprojlim_{fx \to y} F(x) = \varprojlim_{f^{-1}\bar{y}} F \qquad \checkmark$$

$$(f_! F)(y) = \varinjlim_{y \to fx} F(x)$$

$$\begin{array}{ccc} \Delta(k) & \longrightarrow & \Delta(\ell) \\ \downarrow y & & \downarrow x \\ Y & \xleftarrow{\ f\ } & X \end{array}$$

$\{x \mid fx$

$$\begin{array}{ccc} \Delta(k) & \longleftarrow & \cdot \\ \downarrow & & \downarrow \\ Y & \longleftarrow & X \end{array}$$

Conclusion of board calculation is that the category of maps $y \to fx$ is disjoint union over $\alpha = \{x \mid f^{-1}x = y\}$ and that if $\eta^* \bar{x} = x$ $\quad \bar{x}$ non-deg η surj, hence that

$\eta^*: f\bar{x} \longrightarrow y$ are initial objs. in each piece. Thus

$$(f_! F)(y) = \bigoplus_{\{x \mid f^{-1}x = y\}} F(\bar{x}).$$

Hence if y non-deg

we find that

$$\boxed{(f_! F)(y) = \bigoplus_{fx=y} F(x)}.$$

It follows that $f_!$ is (always) exact and commutes with infinite products provided f is dimension-wise finite.

<u>Proposition:</u> Let $f : X \to Y$ be a map of simplicial sets. If $y \in Y$, let $I(y, f)$ be the category whose objects are maps $\varphi : y \to fx$ where $x \in X$ and where a map ψ from $\varphi : y \to fx$ to $\varphi' : y \to fx'$ is a map $\psi : x \to x'$ (i.e. a simp op. $\psi \ni \psi^* x' = x$) such that

$$\begin{array}{ccc} & \overset{\varphi}{\nearrow} & fx \\ y & & \downarrow f(\psi) \\ & \underset{\varphi'}{\searrow} & fx' \end{array}$$

~~Then I(y,f) is the direct sum of categories [illegible] I(x~~

If x is a non-degenerate simplex of X, let $I(x)$ be the category $\Delta(n) \backslash (\Delta/X)$ $n = \dim x$ where $x : \Delta(n) \to X$. Then

$$I(y, f) = \coprod_{fx=y} x \backslash (\Delta/X)$$

~~[illegible] surj, x non-degenerate. [illegible]~~

Proof: Let $\varphi : y \to fu$ be an object of $I(y,f)$. Then $f(\varphi^* u) = y$. Let $x = \varphi^* u$, and $x = \eta^* \bar{x}$. If ρ is inj $\ni$ $\eta\rho = id$, then

$$\begin{cases} \rho^*\varphi^* u = \rho^*\eta^*\bar{x} = \bar{x} \\ (\varphi\rho)\eta = \varphi . \end{cases}$$

and

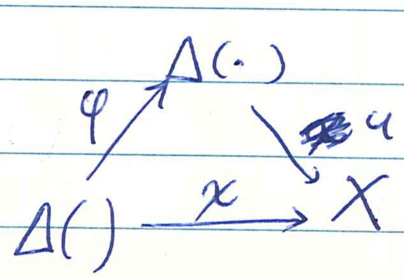
Thus $\psi = \varphi\rho : (\eta : y \longrightarrow f\bar{x}) \longrightarrow (\varphi : y \longrightarrow fu)$ and as η is surjective $\psi\eta = \varphi$, ψ is unique. We therefore see that $I(y,f)$ is the disjoint union of the categories of maps $\varphi : y \to fu$ where $\varphi^* u = x$, and x runs over $f^{-1}y$. The category of such φ, u is the same as the category

?

$$I(y,f) = \{(\varphi, u) \ni f(\varphi^* u) = y\}$$

$$= \coprod_{x \in f^{-1}y} \{ (\varphi, u) \mid \varphi^* u = x \}.$$

$\shortparallel$



$$x \backslash (\Delta / X)$$

Cor: $$(f_! F)(y) = \varinjlim_{(\varphi,u)\in I(y,f)} F(u) = \bigoplus_{fx=y} F(x)$$

Conclusion: $f_!$ always exact and it commutes with infinite products provided that f is dimension-wise finite. In latter case it has an adjoint $f^!$.

G. Cornerstone of Verdier's duality theorem is the proper supports functor $f_!$. **Properties**:

(i) f open immersion $\Rightarrow$ $f_!$ extension by zero

(ii) $f_! = \Gamma_c$ if $f: X \to pt$.

(iii) $X \xrightarrow{f} Y \xrightarrow{g} Z$ $f_!$(injective) is $g_!$ acyclic

(iv) $f_!$ commutes with filtered inductive limits

(v) $R^8 f_!$ commutes with base change ie

$$R^8 f_! (F)(y) = H^8(X_y, F/X_y).$$

He then is able to define $f^!$ by choosing a flask resolution functor C^* and setting

$$\mathrm{Hom}_Y(f_! C^\cdot(K^\cdot), L^\cdot) = \mathrm{Hom}_X(K^\cdot, f^! L^\cdot)$$

Mönday, April 15th.

Yesterday I tried to make simplicial sets behave like topological spaces. To each simplicial set X we consider the category of simplcial sets over X, S/X. With the canonical topology of S/X we obtain a topos. We obtained an analogue of Artin's theorem for an injective map j:R U→X of simplicial sets. It appears that the analogue of a closed subset of X is the complement of an injective map. This leads to a curious feature namely that the skeletal filtration produces a decreasing filtration by supports in contrast to an increasing filtration which is what one would espect on topological grounds. Very important to understand cohomology with compact supports. Thus if $j:X \hookrightarrow Y$ is an open immersion of locally compact spaces ome can define $j_!: \underline{Sh}(X) \to \underline{Sh}(Y)$ to be extension by zero. If Y is compact, then $\Gamma(Y, j_! F)$ = sections of F with compeact support. More generally if $f:X \to Y$ is a map of locally compact spaces, then we can define $f_!$ to be sections with proper supprt.

April 15
gave page 1+2 to Friedlander

$$f^*(U \to Y) = (X \times_Y U \xrightarrow{pr_1} X) \checkmark$$

and $$f_*(F)(U) = \Gamma(X \times_Y U, F)$$

ie $$\boxed{\Gamma(U, f_*(F)) = \Gamma(f^*U, F)}$$

In addition we have

$$f_!(F \to X) = (F \to X \to Y)$$

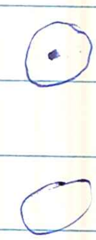
$\ni$ $$\boxed{\Gamma(f_!F, U) = \Gamma(F, f^*U).}$$

Fundamental problem — duality thms.
As Deligne has shown, the real point is the theorem of Artin.

Geometric situation:

$$Z \xrightarrow{i} X \xleftarrow{j} U$$

F

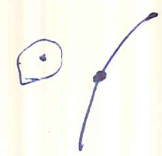


$$Sh_Z \underset{i^!}{\overset{i^*}{\underset{i_*}{\rightleftarrows}}} Sh_X \underset{j_*}{\overset{j_!}{\underset{j^*}{\rightleftarrows}}} S_U$$

$$i^* j_* : S_U \to S_Z \qquad \text{left exact}$$

$$0 \to i_* i^! F \to F \to j_* j^* F$$

$$i^! i_* = id \qquad i^* i_* = id \qquad i^! F \hookrightarrow i^* F$$



$$\mathrm{Hom}_{Sh_X}(j_! F, G) = \mathrm{Hom}_{Sh_U}(F, j^* G)$$

~~$0 \to i_* i^! F \to F \to j_! j^* F \to 0$~~

$0 \to i_* i^! F \to F \to j_* j^* F \to i_*(R^1 i^!)(F) \to 0$

$0 \to j_! j^* F \to F \to i_* i^* F \to 0$

~~$0 \to j_! j^* F \to F \to i_* i^*$~~

$$Sh_Z \;\underset{i^!}{\overset{i^*}{\leftleftarrows}}\; \xrightarrow{i_*} \; Sh_X \;\underset{j_*}{\overset{j_!}{\leftleftarrows}}\; \xrightarrow{j^*} \; Sh_U$$

horizontal compositions all 0

$$i^* i_* = i^! i_* = id$$

$$j^* j_* = id$$

$0 \to j_! G \to j_* G \to i_* i^* j_* G \to 0$

$0 \to i^! F \to i^* F \to i^* j_* j^* F \to (R^1 i_j) F \to 0$

Question: Can this situation be realized for simplicial sets? There for any map f we have $f_!$, f^*, f_* hence we want

to ~~realize~~ determine when there is an $f^!$; hopefully if f is injective.

(Topologically $f_!$ exists only for f etales and the rest of the time one tries to prove that $Rf_!$ exists.)

~~[illegible]~~

Conjecture: $f : Z \to X$ injective $\Rightarrow$ $f^!$ exists.

Proof: Define for $U \to Z$ $\quad F \to X$

$\Gamma(U, f^! F)$

$$\begin{array}{ccccc} & & & & F \\ & & & & \downarrow \\ U & \to & Z & \to & X \end{array}$$

Hom (f

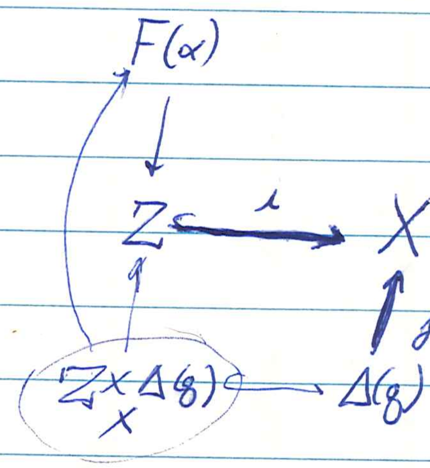
$i : Z \longrightarrow X$ injective

$$Hom_{/Z}(F, i^! G) = Hom_{/X}(i_* F, G)$$

$$(i^! G)_\sigma = Hom_{/Z}(\Delta(q) \xrightarrow{\sigma} Z, i^! G) = Hom_{/X}(i_* \sigma, G)$$

This certainly defines a ~~[illegible]~~ sheaf $i^! G$ over ~~[illegible]~~ Z so one only has to see that i_* commutes with $\varinjlim$'s.

~~[illegible]~~ $Hom\left(\underline{\Delta(q)}, i_*\{F(\alpha)\}\right) = Hom_{/Z}(\underbrace{Z \times_X \Delta(q)}_{\text{small}}, F(\alpha))$



~~[illegible]~~ Have to show i_* exact + commutes with ~~filtered [illegible] limits.~~ direct sums.

Seems always to be true that $Z \times_X \Delta(q)$ small over Z. since $\Delta(q)$ small over X.

Suppose $Z = X \times Q$.
then

$$Z \times_X \Delta(q) = (X \times Q) \times_X \Delta(q)$$

$$= \underline{Q} \times \Delta(q).$$

$$\begin{array}{c} \oplus F(\alpha) \\ \downarrow \\ \underline{Q} \times \Delta(q) \longrightarrow Z \end{array}$$

so for each element of Q   false

Lemma: If $f : Z \to X$ is finite in the sense that $(f^{-1}\sigma$ finite$)^*$ $\forall$ simplex in X, then f_* commutes with filtered inductive limits.

Proof: If F_i is a filtered inductive system of sheaves on Z, must show that for every σ in X, that

$$\varinjlim_i \Gamma(f^{-1}\sigma, F_i) \simeq \Gamma(f^{-1}\sigma, \varinjlim F_i).$$

Here $f^{-1}\sigma = \Delta(q) \times_X Z$, where $\Delta(q) \to X$ is $\tilde{\sigma}$. This is a finite simplicial set by assumption

---

* $f^{-1}\sigma = \Delta(q) \times_X Z$ finite simplicial set.
(not same as those q-simplices over σ being finite).

<u>Definition:</u> $f : X \to Y$ finite $\forall\ \alpha : F \to Y$ F finite have $X \times_Y F$ finite. (quasi-finite = dimension-wise finite).

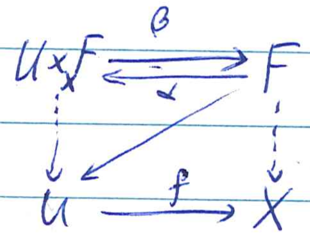
~~Proposition~~ f injective $\Rightarrow$ f_* is exact.

~~Proof~~ FALSE since Leray + $H^*(\Delta(n)) = 0$ would imply that any finite simp. cx. ~~cx has trivial~~ is cohomologically trivial

<u>Conclusion:</u> A map f like a open map, never closed.

<u>Lemma:</u> $f : U \to X$ such that $f^* f_* \xrightarrow{\sim} id$ iff f injective.

<u>Proof:</u> $Hom(Q, f^* f_* F) = Hom(f_! Q, f_* F) = Hom(f^* f_! Q, F)$, hence $f^* f_* \xrightarrow{\sim} id \iff id \xrightarrow{\sim} f^* f_!$ ie $\iff F \xrightarrow{\sim} U \times_X F$ for all $F \to U$.



Have $\beta\alpha = id_F$ so need β inj. ~~[illegible]~~ f inj $\Rightarrow \beta$ inj $\Rightarrow F \simeq f^* f_! F$.
Taking $F = U$ get $U \simeq f^* f_* U = U \times_X U \iff U \to X$ mono $\iff$ $U \to X$ injective

---

Therefore ~~[illegible]~~ if $j : U \to X$ is an injective map of simplicial sets, we may define the complement of U in X ~~to be the topos~~ which is the full subcategory of $\mathcal{S}/X$ consisting of those $F \to X$ such that $j^* F \simeq \emptyset_U$. ~~Not a topos~~

F sheaf of sets on Y
$i: Y \hookrightarrow X$ closed

$$\Gamma(U, i_* F) = \Gamma(U \cap Y, F).$$

what is the etale space of $i_* F$? ~~...~~ over U
$j^* i_* F \simeq U$.

So we start with $\mathcal{S}$ and consider the toposes one gets from pairs X, U.

What is the final object of the topos $X - U$?
it is X.

Is a pair the same as a pointed αx?

$$\begin{array}{ccc} U & \hookrightarrow & X \\ \downarrow & & \downarrow p \\ pt. & \hookrightarrow & X/U \end{array}$$

<u>Theorem:</u> $\mathcal{S}(X, U) \xleftarrow{\sim} \mathcal{S}(X/U, *) : p^*$

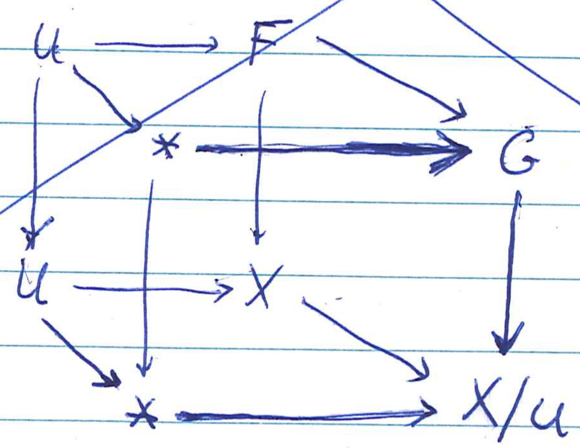
<u>Proof:</u> $\mathcal{S}(X, U)$ = full subcat of $F \to X$ such that
$\exists$ cart. square

$$\begin{array}{ccccc} U & \to & F & \to & F/U \\ \downarrow & & \downarrow & & \downarrow \\ U & \to & X & \to & X/U \end{array} \qquad \begin{array}{ccc} * & \to & G \\ \downarrow & & \downarrow \\ * & \to & X/U \end{array}$$

note that

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ * & \longrightarrow & X/U \end{array}$$

is bicartesian



Proof: $\mathcal{S}(X/U, *)$

$$\begin{array}{ccc} * & \longrightarrow & G \\ \downarrow & & \downarrow \\ * & \longrightarrow & X/U \end{array} \quad \text{cart.}$$

$$+ \quad \begin{array}{ccc} * & \longrightarrow & X/U \\ \uparrow & & \uparrow \\ U & \longrightarrow & X \end{array} \quad \text{cart}$$

$$\Longrightarrow \quad \begin{array}{ccc} U & \longrightarrow & p^*G \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array} \quad \text{cart.}$$

so p^* well-defined.

On other hand if

$$\begin{array}{ccc} U & \longrightarrow & F \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array} \quad \text{cart so is} \quad \begin{array}{ccc} * & \longrightarrow & F/U \\ \downarrow & & \downarrow \\ * & \longrightarrow & X/U \end{array}$$

so get inverse maps.

The fundamental principle: A simplicial set with basepoint is more than just a simplicial set.

Problem A: Given a pointed simplicial set $(X, *)$ find a formula for $H_*(|X| - *)$.

method: Recall that $* = U$, an open subset of X. Therefore we should calculate $i_*\mathbb{Z}$ where $i: Z \to X$ is the complement of U.

$$0 \longrightarrow j_! \mathbb{Z}_U \longrightarrow \mathbb{Z}_X \longrightarrow i_* \mathbb{Z}_Z \longrightarrow 0$$

$$Hom_X(j_!\mathbb{Z}_U, F) = Hom_U(\mathbb{Z}_U, j^*F).$$

$$j : \Delta(0) \longrightarrow X$$

$$\mathbb{Z} \times \Delta(0) \cup 0 \times X$$

$$\therefore \quad i_*\mathbb{Z} = \mathbb{Z} \wedge X = \frac{\mathbb{Z} \times X}{\mathbb{Z} \times \{*\} = 0 \times \{*\}}$$

$$\boxed{H^*(Z, \mathbb{Z}) = H^*(X, i_*\mathbb{Z}).} \qquad X$$

Question: Can you determine which pointed simplicial sets come from sphere ~~bases~~ fibrations?

Thus if $(X, *)$ is a pointed simplicial set we want to be able to determine when $\tilde{H}^*(X)$ is a free module over $H^*(X - *)$.

Caution: If $X = \Delta(q)$ we know that $H^+(X, \cdot) \equiv 0$ so if $H^*(Z, F) = H^*(Z, \iota_* F)$ must be false. Thus ι_* cannot preserve injectives and so ι^* is not exact contrary to the case of sheaves.

Artin's theorem: Let $j : U \to X$ be an injective

$S(X) = S/X \qquad Sh(X) = (S/X)_{ab}$

<u>Artin's theorem</u>: Let $j : U \to X$ be an injective map of simplicial sets and let $S(X, U)$ be the full subcategory of $S(X)$ consisting of F such that $j^*F \simeq U$. Then there are functors

$$Sh(X,U) \;\underset{i^!}{\overset{i^*}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\;i_*\;}}}}}\; Sh(X) \;\underset{j_*}{\overset{j_!}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\;j^*\;}}}}}\; Sh(U)$$

~~Where i^* is left adjoint to i_* the inclusion. Moreover these six functors~~ every functor is left adjoint to the one immediately below. ~~Each horizontal arrow is compo~~

$$i^* j_! = j^* i_* = i^! j_* = 0$$

$$i^* i_* = i^! i_* = id$$

$$j^* j_* = id = j^* j_!$$

*
$$\begin{cases} 0 \to i_* i^! F \to F \to j_* j^* F \\ j_! j^* F \to F \to i_* i^* F \to 0 \end{cases}$$

~~Remark. In the topological situation $j_!$ and i^* are exact. Here they need not be.~~

<u>Proof:</u> Keeping in mind that ι_* is the inclusion, define $\iota^!$ and ι^* by the exact sequences *

This is legitimate since $j^* j_! = j^* j_* = id$ and since j^* is exact. Using that $j^* \iota_* = 0$, one obtains that $\iota^* \iota_* = \iota^* \iota_! = id$, and $\iota^* j_! = \iota^! j_* = 0$.

$$\mathrm{Hom}_{Sh(X,U)}(\iota^* F, G) = \mathrm{Hom}_{Sh(X)}(\iota_* \iota^* F, \iota_* G) \qquad \text{(defn)}$$

$$= \mathrm{Ker}\left\{ \mathrm{Hom}_{Sh(X)}(F, \iota_* G) \longrightarrow \mathrm{Hom}_{Sh(X)}(j_! j^* F, \iota_* G) \right\}$$

$$\mathrm{Hom}_{Sh(X)}(j_! j^* F, \iota_* G) = \mathrm{Hom}_{Sh(U)}(j^* F, j^* \iota_* G), \quad j^* \iota_* G = 0$$

$$= \mathrm{Hom}_{Sh(X)}(F, \iota_* G).$$

$$\mathrm{Hom}_{Sh(X,U)}(G, \iota^! F) = \mathrm{Hom}_{Sh(X)}(\iota_* G, \iota_* \iota^! F)$$

$$= \mathrm{Ker}\left\{ \mathrm{Hom}_{Sh(X)}(i_* G, F) \to \mathrm{Hom}_{Sh(X)}(i_* G, j_* j^* F) \right\}$$

(with $\mathrm{Hom}_{Sh(X)}(i_* G, j_* j^* F) = 0$)

$$= \mathrm{Hom}_{Sh(X)}(i_* G, F) \quad \checkmark.$$

This completes proof of the theorem.

<u>Proposition:</u> The following are equivalent:

- a) i^* exact
- b) $j_! j^* F \to F$ injective

- b′) $i^!$ exact
- c′) $F \to j_* j^* F$ surjective.

<u>Proof</u>: a) $\Rightarrow$ b) Let

$$0 \to i^* Q \to i^* j_! j^* F \to i^* F, \quad i^*Q = 0,\; i^* j_! j^* F = 0$$

$$0 \to j^* Q \to j^* j_! j^* F \simeq j^* F, \quad j^*Q = 0$$

$$j_! j^* Q \to Q \to i_* i^* Q \to 0, \quad j_! j^* Q = 0,\; i_* i^* Q = 0$$

$$\therefore\ Q = 0.$$

b) $\Rightarrow$ a) Let $0 \to F' \to F \to F'' \to 0$ be exact in $Sh(X)$.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & j_! j^* F' & \longrightarrow & F' & \longrightarrow & i_* i^* F' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \gamma & & \\
0 & \longrightarrow & j_! j^* F & \longrightarrow & F & \longrightarrow & i_* i^* F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & j_! j^* F'' & \xrightarrow{\alpha} & F'' & \longrightarrow & i_* i^* F'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

$\therefore$ b) $\Rightarrow$ a) ~~injective~~

~~Remark: In contrast to topological situation i^* is not necessa~~

$$
\begin{array}{ccc}
U \times_X F & \longrightarrow & F \\
 & & \downarrow \\
U & \longrightarrow & X
\end{array}
$$

April 16, 1968

Let X be a space, k field. If X is a smooth manifold we can form the algebra of differential operators

$$D = \varinjlim_{n} \mathrm{Hom}_A (A \otimes A / J^n, A)$$

where A = smooth functions on X.

Ultimate idea: Take Amitsur complex

$$A \qquad A \hat{\otimes} A \qquad A \hat{\otimes} A \hat{\otimes} A \qquad \cdots$$

and interpret it as a kind of cobar construction.

Fredholm theory: review

Start with a finite rank endomorphism $A : V \to V$
$A \in V \otimes V^*$. ~~To find eigenfns of~~

~~$(TA)(x) = \int K(x,y) f(y)\, dy$~~

Calculate by Cramer's rule the resolvent of A.

$$(1 - zA)^{-1} = \frac{\operatorname{cof}(1 - zA)}{\det(1 - zA)}$$

$$\boxed{\det(1 - zA) = 1 - z \operatorname{tr} A + z^2 \operatorname{tr} \Lambda^2 A - \cdots}$$

~~[illegible]~~ ~~[illegible]~~

Recall Cayley-Hamilton thm.

~~$A^n - A^{n-1}\cdot \operatorname{tr} A + A^{n-2}\cdot \operatorname{tr} \Lambda^2 A - \cdots + (-1)^n \operatorname{tr} \Lambda^n A = 0$~~ ~~$\det A$~~

$$\sum_{k=0}^{\infty} \varepsilon^k \det{}^k(A; B) = \det(A + \varepsilon B)$$

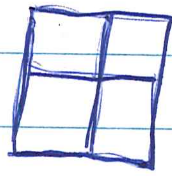
$$\det{}^k(I; B) = \operatorname{tr} \Lambda^k B$$

$$\frac{\det(A+\varepsilon B)}{A+\varepsilon B}$$

$$\det(A+\varepsilon B) = \det A + \varepsilon\, \mathrm{tr}\,(\mathrm{cof}\,A)B + O(\varepsilon^2).$$

ε · n · det.

$$\frac{d}{d\varepsilon}\det(A+\varepsilon B) = \sum(-1)^{i+j}A_{ij}B_{ij} = (\mathrm{cof}\,A)_{ji}\, b_{ji}$$



$$(\mathrm{cof}\,A)_{ij} = (-1)^{i+j}A_{ji}$$

since

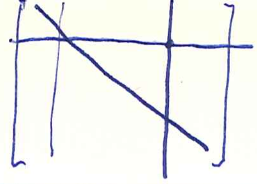
$$\sum_j a_{ij}(-1)^{i+j}A_{jk} = \delta_{ik}.$$

$$d(\det)_A\,(B) = \frac{d}{d\varepsilon}\det(A+\varepsilon B)\Big|_{\varepsilon=0} = \mathrm{tr}\,(\mathrm{cof}A)\cdot B.$$

$$A = \sum a_{ij}\, e_i \hat{e}_j$$

$$\det(1-zA) = \det\left(\sum(\delta_{ij}-z a_{ij})\, e_i\hat{e}_j\right)$$

$$= 1 - z\sum_i a_{ii} + \frac{z^2}{2!}\sum_{i,j}\begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj}\end{vmatrix} - \frac{z^3}{3!}\sum_{i,j,k}\begin{vmatrix} a_{ii} & & \\ & & \\ & & \end{vmatrix}.$$



$$\left(\operatorname{cof}(1-zA)\right)_{ij} = \delta_{ij} +$$

$$\operatorname{cof}(1-zA) = \frac{\det(1-zA)}{1-zA} = (1+zA+z^2A^2+\cdots)(1-z\sum a_{ii}+$$

$$= I + z(A - \operatorname{tr}A) + z^2(A^2 - A\cdot\operatorname{tr}A + \operatorname{tr}\Lambda^2 A) + \cdots$$

$$\delta_{ij} + z\Big(a_{ij} - \sum_k a_{kk}\Big) + z^2\Big(\sum_k a_{ik}a_{kj} - a_{ij}\, a_{kk} + \sum_{\iota,\iota'} \begin{vmatrix} a_{\iota\iota} & \\ & a_{\iota'\iota'}\end{vmatrix}\Big)$$

$$(R_z f)_x = f_x + z \sum_y \big(K(x,y) - K(x,x)\big) f_y$$

$$+ z^2 \quad K(x,z)K(z,y) - K(x,y)K(z,z) + \begin{vmatrix} K(x,x) & K(x,z) \\ K(z,x) & K(z,z)\end{vmatrix}$$

$$\operatorname{cof}(I-zA) = \frac{\det(I-zA)}{I-zA} = zA + z^2(A^2 - A\cdot\operatorname{tr}A)$$

$$+ \det(1-zA)\cdot I$$

$$\left(\mathrm{cof}\,(I-zA) - \det(I-zA)\,I\right)_{xy} = \frac{z}{0!}K(x,y) + \frac{z^2}{1!}\begin{vmatrix} K(x,1) & K(x,y) \\ K(1,1) & K(1,y)\end{vmatrix}$$

$$+ \frac{z^3}{2!}\begin{vmatrix} K(x,1) & K(x,2) & K(x,y) \\ K(1,1) & K(1,2) & K(1,y) \\ K(2,1) & K(2,2) & K(2,y)\end{vmatrix} + \cdots\cdots$$

$$\mathrm{cof}\,(I-zA) = \det(I-zA)\cdot I + R_z(A)$$

$$\boxed{C_A(z) = D_A(z)\cdot I + R_A(z)}$$

Lemma: $f(z)$ entire fn. of exp. type.

$$-\frac{1}{2\pi i}\oint_{\text{large circle}} \frac{f'}{f}\frac{dz}{z}$$

$$\frac{1}{I-zA} = I + ~~\frac{R_A(z)}{I-zA}~~ Q_A(z)$$

$$\frac{zA}{1-zA} = Q_A(z) = \frac{R_A(z)}{\det(I-zA)}$$

$$\frac{d}{dz}\det(1-zA) = -\mathrm{tr}\{\mathrm{cof}\,(1-zA)\cdot A\}$$

$$\frac{-1}{2\pi i}\int\frac{dz}{z}\,\frac{\frac{d}{dz}\det(1-zA)}{\det(1-zA)} = +\mathrm{tr}\int ~~\mathrm{cof}\,(1-zA)~~ \left\{\frac{A}{1-zA}\right\}\frac{dz}{2\pi i}$$

$$= +\mathrm{tr}\,A.$$

Fredholm theory:

$$D(z) = 1 - z\,\mathrm{tr}\,A + z^2\,\mathrm{tr}\,\Lambda^2 A - \cdots$$

$$C(z) = \frac{D(z)}{1-zA} = (1 + zA + z^2A^2 + \cdots)(1 - z\,\mathrm{tr}A + \cdots)$$

$$= 1 + z(A - \mathrm{tr}A) + z^2(A^2 - A\cdot\mathrm{tr}\,A + \mathrm{tr}\,\Lambda^2 A) + \cdots$$

$$= D(z) + E(z)$$

$$E(z) = zA + z^2(A^2 - A\,\mathrm{tr}\,A) + z^3(A^3 - A^2\mathrm{tr}A + A\,\mathrm{tr}\,\Lambda^2 A) + \cdots$$

<u>Fredholm:</u> $E(z,x,y)$ entire with values in smooth kernels. $D(z)$ entire fn. Moreover

$$\frac{1}{1-zA} = \frac{1}{D(z)}E(z)$$

gives the analytic continuation.

<u>Lemma 1:</u> Let $V_z = \{v \mid \exists n \ni (1-zA)^n v = 0\}$. Then

$\dim V_z$ = order of $D(z)$ at z.

~~[illegible] $P_z = \frac{1}{2\pi i}\oint \frac{dz}{1-zA}$~~

Lemma 2: $$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{|z|=R} \frac{D'(z)}{D(z)} \frac{dz}{z} = 0$$

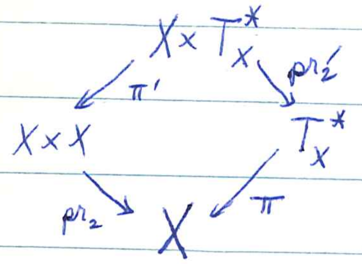
Proof:

This should follow from fact that $D(z)$ is of exponential type!

By contour integration one finds that $\sum \frac{1}{z} \dim V_z = \operatorname{tr} A$.

invariant Fourier integral.

basic diag.



$\pi' = 1 \times \pi$

We conjecture that for a parametrix P & $dP + Pd = 1 - S$ that S can be represented as a Fourier integral

$$S(x,y) = \int \sigma(y,\xi)\, \rho(x,y)\, e^{i\varphi(x,y)\xi}\, d\xi\, dy$$

This has to be interpreted as follows: Recall that connections have been chosen on the bundles $\pi^* E^\bullet$ and that φ is equivalent to an exponential map. Here

$$\sigma \in \operatorname{Hom}_{T_X^*}(\pi^* E^\bullet, \pi^* E^\bullet)$$

~~If $(x,y) \in X \times X \ni \rho(x,y) \neq 0$, then given ξ the straight line from x,~~

$$\sigma(y,\xi) \in \operatorname{Hom}(E_y^\bullet, E_y^\bullet)$$

If $(x,y) \in X \times X \ni \rho(x,y) \neq 0$ want an element

$$\alpha \in \operatorname{Hom}(E_y^\bullet, E_x^\bullet)$$

so we want an isom. $E_y \xrightarrow{\simeq} E_x$ possibly depending on ξ. But the connection gives this. In effect ~~we may interpret~~ we lift the ~~isomorphism $E_y \to E_x$ obvious on along~~ straight line **joining** (y, ξ) to (x, ξ).

I want to determine $\operatorname{tr} S/\Delta$

$$\boxed{\int \operatorname{tr}_{E_x} \sigma(x, \xi)\, d\xi\, dy}$$

for ξ very large $\operatorname{tr}_{E_x} \sigma(x, \xi) = 0$ because σ is an endo. of the complex and the cx is acyclic.

$\therefore$ Somehow $\operatorname{tr}_{E_x}\{\sigma(x, \xi)\}\, d\xi\, dy = \boxed{ch(\sigma) \cdot A.}$

$ch(\sigma) = \operatorname{tr} e^{K}$ $\qquad$ $K =$ curv. of nice con.

$$A = \det \left(\frac{K_1}{e^{K_1} - 1}\right) = \sum \operatorname{tr} \Lambda^{8} \{Z\}.$$

$\mathrm{ch}(\sigma) = \mathrm{tr}\, e^{K}$

$$S = \int \pi_*' \{ s(y,\xi)\, \rho(x,y)\, e^{i\varphi(x,y)\xi}\, dy\, d\xi \}$$

$$\in \Gamma(X \times X,\ pr_2^* E' \otimes pr_1^! E)$$

$$\mathrm{tr}(S|\Delta) = \pi_*\left(\mathrm{tr}_{E_x} s(x,\xi) \cdot dy\, d\xi\right).$$

your problem is how to relate this expression to invariants calculated by means of the curvature tensors, Todd, etc!!!!

Back to de Rham. You should be able to get the Pffaffian as an integral over ~~the~~ T^*.

Problem: Construction of the parametrix. Even Bott had to use a reduction of bundle to orth. gp! Somehow Bott constructed $U \in A_c(N) \ni pr_{1*} U = 1$. Then the parametrix was

$$P\alpha = pr_{1*} U \cdot h\, pr_2^* \alpha$$

~~the homot~~ $dh + hd = pr_1^* - pr_2^*$

Somehow I want to write

$$U = \pi'_* \; s(y,\xi)\, \rho(x,y)\, e^{i\varphi(x,y)\xi}\, dy\, d\xi$$

~~Therefore I want s(y,ξ) to be an n form~~

$$U \in \Gamma_0(X \times X, pr_2^* E^{\cdot\prime} \otimes pr_1^! E^{\cdot})$$

$$pr_2^* \Lambda^{\cdot} T^{*\prime} \otimes pr_1^* \Lambda^{\cdot} T^* \otimes pr_2^* \Lambda^n T^*$$

$$\bigoplus_{i+j=n} \Lambda^i pr_2^* T' \otimes \Lambda^j pr_1^* \Lambda^{\cdot} T'$$

$$\therefore \quad U \in A^n_{\mathbb{R}_c}(X \times X).$$

$$s(y,\xi) \in \mathrm{Hom}_{T^*_X}(\pi^* \Lambda^{\cdot} T', \pi^* \Lambda^{\cdot} T')$$

~~Using Riemannian structure~~

Bott's construction gives

$\pi^{-1}\Lambda T_X^*$

$$U = \begin{cases} ch\{p^* \Lambda \cdot 0\} & \text{in } N \\ 0 & \text{outside} \end{cases}$$

U closed $\qquad U/\Delta = c_n(T^{01})$

so integrates to χ

You are interested in showing that the last ~~term~~ coeff of a polynomial is positive. Better to use duality and show first term is positive!!!!!!

Thus given T

$$\pi^{-1}\Lambda^0 T^* \xrightarrow{e} \pi^{-1}\Lambda^1 T^* \xrightarrow{e} \pi^{-1}\Lambda^2 T^* \xrightarrow{e} \pi^{-1}\Lambda^3 T^* \qquad \text{exact off } 0$$

Have metric g on T. gives isom $T = T^*$.

~~[illegible]~~ Now I want to construct the character of this cx. Must choose a connection. Natural one comes from Riemann metric on T^*.

Over $T^* - 0$ the Riemann structure splits this canonically. In effect

$$i(\lambda)e(\lambda) + e(\lambda)i(\lambda) = |\lambda|^2 .$$

On other hand we get the standard connection pulled back from X. ∋ fibers are flat. For that conn.

$$\sum_q (-1)^q \, ch \, \Lambda^q T^* = 0 \qquad \text{for dimensional reasons.}$$

$$ch \, \lambda_{-1} E = \prod 1 - e^{x_i} = (-1)^n \, (\text{Todd} E)^{-1} \cdot c_n E .$$

any identity in the symmetric functions of a matrix will remain valid in any commutative ring.

~~Problem~~ Would like to combine this connection from the base with one from the fibers to get a connection whose curvature is 0 at ∞.

Algebra. Put trivial connection on $\pi^{-1}\Lambda^0 T^*$ so then only have to connect $\pi^{-1}\Lambda^1 T^*$.

First worry about fibers!

If ~~at~~ $\lambda \in T^*$, then

Problem: Let X be a ~~manifold~~ Riemannian manifold of dimension n. We know that T_X^* has an almost complex structure. Moreover we can form a Koszul cx. Calculate $ch(\sigma)$ as a form in $A_c^{2n}(T_X^*)$.

The complex is $\pi^{-1}(\Lambda^\cdot T_X^*)$ with d ~~dy~~ symbol.

Idea: $\pi^{-1}P$ Let P = Pfaffian on X, ~~then~~ calculated wrt some connection. Then if $\pi_0 : E \longrightarrow X$

$$\pi_0^{-1}P \sim 0$$

so can write $\pi_0^{-1}P = d\,Q$ Q explicitly ~~constructed~~ formed by differentiating P.

~~By Stokes~~ Have to calculate $\int_{S^{n-1}} Q = 2$.

E cx. dim n.

$$\begin{array}{c} E \\ \downarrow \pi \\ X \end{array}$$

~~$ch(\lambda_{-1}E) \in A_c^{2n}(E)$~~

~~$\pi^* ch(\lambda_{-1}E) = \ldots$~~

$$\pi^* ch(\lambda_{-1} E) \sim U \in A_c^{2n+\cdots}(E).$$

$s^*\pi^*$ Thom class.

$$\begin{array}{c} E \\ \downarrow \pi \\ X \end{array}$$

X smooth manifold compact

E cx vector bundle dim n endowed with connection which is then extended to $\Lambda^{\cdot}E$ in the natural way.

Can form $\sum (-1)^{q} ch\, \Lambda^{q} E$ form on X beginning in dim $2n$. By Koszul the complex $\pi^* \Lambda^{\cdot} E$ is acyclic off zero section. Hence

$$\pi^*\left(\sum (-1)^{q} ch\, \Lambda^{q} E\right) = dK$$

$K \in A^{odd}(E - 0)$. K is constructed by the Bott-Chern method. If ρ is a smooth on E $\equiv 0$ near 0 section $+ \equiv 1$ off a compact then

$$ch(\sigma) = \pi^*\left(\sum (-1)^{q} ch\, \Lambda^{q} E\right) - d\rho K$$

$$\in A_c(E) \qquad \text{begins in dim } 2n$$

Prove that

$$\pi_* \left(ch(\sigma)\, Todd\, E\right) = \pm 1 .$$

<u>Line bundle:</u> E line bundle with real curvature form ω (= locally $= \frac{i}{2\pi} d\Theta_s$ $\quad Ds = s \cdot \Theta_s$)

$$ch\, E = e^{\omega}$$

$$1 - e^{\omega} = ch(\lambda_{-1} E).$$

$\pi^* E.$ $\quad \pi^* 1 \xrightarrow{t} \pi^* E,$ $\quad t$ canonical section.

$$t(e)(1_{\pi e}) = e \in E_{\pi e}$$

Then we get two connections on $\pi^* E \,|\, E - 0$ which imply that $\pi^*\omega$ is a boundary

$$\pi^*\omega = d\lambda$$

2

$$\lambda = \frac{i}{2\pi}\pi^*\left(\Theta_s + \frac{dz_s}{z_s}\right)$$

here

$$z_s(e) \cdot s(\pi e) = e.$$

$\therefore$ given s, s' $\quad s' = fs$ $\quad f \neq 0$

$$z_{s'}(e) \cdot s'(\pi e) = e = z_s(e)\, s(\pi e)$$

$$z_{s'}(e) \cdot f(\pi e)\, s(\pi e)$$

$$z_{s'} \cdot \pi^* f = z_s$$

$$\text{Todd } x = \frac{x}{e^x - 1}$$

$$Ds' = df.\, s + f.\, s\theta$$

$$\theta' f s$$

$$\theta' - \theta = \frac{df}{f}$$

$$\pi^*\theta' - \pi^*\theta = \pi^*\frac{df}{f}$$

$$= \pi^*\left(\frac{dz}{z} - \frac{dz'}{z'}\right)$$

$$\pi^*\theta' + \frac{dz'}{z'} = \pi^*\theta + \frac{dz}{z}$$

so

$$\pi^*\{1 - e^{\omega}\} = d\left\{\frac{1-e^{\omega}}{\omega}\cdot\lambda\right\} = \sout{d(\text{Todd } E}$$

$$= -\pi^*(\text{Todd } E)^{-1}\cdot \pi^*\omega$$

$$\therefore\ \pi^*\{1 - e^{\omega}\} = -\pi^*(\text{Todd } E)^{-1}\cdot \underline{\pi^*\omega}$$

$$ch\ \sigma = -\pi^*(\text{Todd } E)^{-1}\cdot d(\rho\lambda)$$

$\rho \equiv 0$ near 0 $\quad \equiv 1$ near ∞.

$$\therefore\ \pi_*(ch\,\sigma)\cdot(\text{Todd } E) = -\pi_*(d\rho\lambda) = 1$$

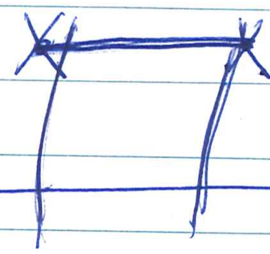
$$-\int_0 \lambda = \frac{1}{2\pi i}\int \frac{dz}{z} = 1$$

$K = \frac{i}{2\pi} \cdot$ curv. matrix on E

$$ch\, \lambda_{-1} E = \pm (\text{Todd } E)^{-1} \cdot c_n(E)$$

$$\pi^*\{ch\, \lambda_{-1} E\} = \pi^*(\text{Todd } E)^{-1} \cdot d\lambda$$

where $d\lambda = \pi^* c_n(E)$? ? ? ? ? ?



det$(K; h)$!!!!

F — complexes

$A \xrightarrow{F} B$

$1 - S$

$\text{tr}\,(1 + zS) = 1 + \text{tr}\, S + \cdots$

April 20, 1968:

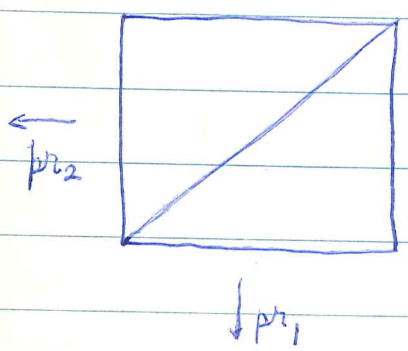
~~h(x, y)~~ ~~∫~~

$$(hf)(x) = \int p(x,y,\xi)\, e^{i\varphi(x,y,\xi)} f(y)\,dy\,d\xi.$$

Here $\varphi : N \longrightarrow T$

$\varphi(x,y,\xi) = \langle \varphi(x,y), \xi \rangle$ linear in ξ.

Express parametrix for d in this way.



Given form α have on ~~E~~ N

$$pr_1^* \alpha \sim pr_2^* \alpha$$

~~α~~

$$(pr_2)_*(U \cdot pr_2^* \alpha) = \alpha.$$

Thus if we choose h operator on ~~A E~~ $A(N)$ of degree -1 so that

$$-\beta + pr_2^* \Delta^* \beta \overset{+}{=} (dh + hd)\beta$$

we get

$$\alpha = (pr_2)_*(U \cdot pr_2^* \alpha) = (pr_2)_* \{U \cdot (pr_1^* \alpha + (dh + hd) pr_1^* \alpha)\}$$

$$= (pr_2)_* \{U \cdot pr_1^* \alpha\} + d\{(pr_2)_* U \cdot h\, pr_1^* \alpha\} + \{pr_{2*}(U \cdot h\, pr_1^*)\} d\alpha$$

Therefore $S = (pr_2)_* \{U \cdot pr_1^* \alpha\}$.

$P = (pr_2)_* \{U \cdot h \cdot pr_1^* \alpha\}$.

The key therefore is in the nature of ~~the operator~~ U.
U is a d-closed form dim n support in $N \ni pr_{2*} U = 1$

Review Botts construction of U. ~~[illegible]~~
Case 1: dim E odd. $= 2n+1$

$$\begin{array}{c} E \\ \downarrow \pi \\ X \xleftarrow{\rho} E - O \end{array} \qquad \rho^* E = 1 \oplus Q. \quad (Q: 2d)$$

He lets $\Phi \in A^{2d}(E-O)$ Pfaffian of curvature
$d\Phi = 0$ of Q rel. to some conn.

Then he chooses C^∞ fn. ρ $\int_0^\infty \rho(r)\, dr \neq 0$
ρ constant $\neq 0$ near 0 $\rho \in C_0^\infty$ and sets

$$U = d\rho \cdot \Phi \in A^{2d+1}(E). \qquad \rho = \rho(r^2).$$

The point is that U is closed and one checks that

$$\pi_* U = 1$$

by restriction to a fiber + observing correct answer here.

<u>All this is absolute magic</u>, and uses very heavily the fact that we are working with forms, etc. for which we have good control.

<u>Reformulate problem:</u> Let Y be a submanifold of X with oriented normal bundle of dim d. Construct the Thom class in $H_c^d(N)$ where N = tubular nbd. of Y in X.

<u>First case:</u> Suppose Y is of codimension 1. Then normal bundle is trivial so write $N \cong Y \times [-1,1]$ and choose ~~a function $\varphi(t) \in C_0^\infty(-1,1)$ with $\int_{-\infty}^{\infty} \varphi(t)\,dt = 1$. The Thom class is $\pi^*[\varphi(t)dt]$~~ $\omega \in A_c^1(-1,1)$

with $\int \omega = 1$ and let $U = pr_2^* \omega \in A_c^1(N)$.

Reduction to this case by blowing up!

$$\begin{array}{ccc} \mathbb{S}\nu & \xrightarrow{\ j\ } & \tilde{X} \\ \downarrow f & & \downarrow g \\ Y & \xrightarrow[\text{codim } d]{\ i\ } & X \simeq N. \end{array}$$

$g_* = 1.$ $\qquad g^* : H(X) \hookrightarrow H(\tilde{X})$

$g_* 1 = 1$

$g^* U$ d form on $\tilde{X}$	should $\exists$ a class in $\mathbb{S}\nu$ of

By Gysin ∃

$$H^*(Y) \xrightarrow{f^*} H^*(SD) \xrightarrow{f_*} H^{*-d+1}(Y) \xrightarrow{\cup e} H^{*+1}(Y)$$

$$H^{d-1}(S\nu) \longrightarrow H^0(Y) \xrightarrow{\cup e} H^d(Y)$$

If d odd so that $e = 0$ then get class $\in H^{d-1}(S\nu)$
$\chi \ni f_* \chi = 1$. Then I take

$j_* \chi$ (has) suppose off critical set, so

$$j_* \chi = g^* V$$

&

$$g_* j_* \chi = \iota_* f_* \chi = \iota_* 1 = u.$$
$$\parallel$$
$$g_* g^* V = V.$$

Let's see if we can construct u in this way.
Gysin should be a special case of Lefschetz?

Y Dν Sν

a priori clear this won't work.

Fourier transforms on manifolds.

Let $\varphi : N \longrightarrow T$ be the inverse of an exponential isom, let $\rho \in C_0^\infty(N)$ be $\equiv 1$ near Δ. Consider operator on $C^\infty(X)$

$$f \longmapsto (2\pi)^{-n} \int \rho(x,y)\, e^{i\langle\varphi(x,y),\xi\rangle} f(y)\, dy\, d\xi$$

where $dy\, d\xi$ is the canonical volume element on T^*.

Euclidean space. Take $\varphi(x,y) = (x, x-y)$

$$(Kf)(x) = (2\pi)^{-n} \int \rho(x,y)\, e^{i\langle x-y,\xi\rangle} f(y)\, dy\, d\xi$$

By Fourier

$$(2\pi)^{-n} \int e^{i\langle x-y,\xi\rangle} f(y)\, d\xi\, dy = f(x).$$

Thus

$$((Kf) - f)(x) = (2\pi)^{-n} \int (\rho(x,y) - 1)\, e^{i\langle x-y,\xi\rangle} f(y)\, dy\, d\xi$$

Want to show ~~left hand~~ right side is a smooth operator

$$= (2\pi)^{-n} \int (\rho(x, x+y) - 1)\, e^{i\langle y,\xi\rangle} f(x+y)\, dy\, d\xi$$

Put in $\sum y_i^2 / \|y\|^2$ and integrate

$$y_i e^{i\langle y, \xi\rangle} = \frac{1}{i}\frac{d}{d\xi} e^{i\langle y,\xi\rangle}$$

get 0

On Euclidean space with $\varphi(x,y) = (x - y)$)

$$(Kf)(x) = (2\pi)^{-n} \int \rho(x,y)\, e^{i\langle x-y,\xi\rangle} f(y)\, dy\, d\xi$$

$f(y)$ ↑ compact support.

integral first by y then by ξ makes sense + by Fourier get

$$\rho(x,x) f(x) = f(x).$$

On a manifold

$$(2\pi)^{-n} \int \rho(x,y)\, e^{i\langle\varphi(x,y),\xi\rangle} f(y)\, dy\, d\xi$$

(dy ↑ first, $d\xi$ ↑ 2nd.)

If you are permitted to change order of integration, then formally you get

$$(2\pi)^{-n} \int e^{i\langle \varphi(x,y),\xi\rangle} d\xi = \begin{cases} 0 & \\ \infty & x=y \end{cases}$$

$$\rho(x,x)\, f(x) = f(x).$$

Lemma:

$$\lim_{\varepsilon \to 0+} (2\pi)^{-n} \int \rho(x,y)\, e^{i\langle \varphi(x,y), \xi \rangle - \varepsilon |\xi|^2} f(y)\, dy\, d\xi = f(x).$$

$$\langle x-y, \xi \rangle$$

Better check this carefully.

~~Want~~

F = compact Lie gp.

~~Abstract~~

Lie (F) = ~~x~~ $A^0(F) \otimes_{\mathbb{R}} \mathfrak{g}$ = L

~~Ex~~ As a Lie algebra it is an infinite direct sum no a Nasty tensor product

Calculate L_{ab} in case $\mathfrak{g}$ is abelian.

$$\sum a_i X_i =$$

$a_1 X_1$ $\qquad$ aX

$[\mathfrak{g}, \mathfrak{g}]$

circle $\quad [a\frac{\partial}{\partial x}, b\frac{\partial}{\partial x}] = \left(a\frac{\partial b}{\partial x} - b\frac{\partial a}{\partial x}\right)\frac{\partial}{\partial x}$

Therefore consider $\int a \longleftarrow a\frac{\partial}{\partial x}$

$$\frac{\partial}{\partial x}$$

Conclusion: For S^1 we have found a non-trivial linear functional on $L = Lie(S^1)$ which vanishes on commutators.

$$a\frac{\partial}{\partial x} \longmapsto \int a\,dx$$

$$\left[a\frac{\partial}{\partial x}, b\frac{\partial}{\partial x}\right] = \left(a\frac{\partial b}{\partial x} - b\frac{\partial a}{\partial x}\right)\frac{\partial}{\partial x}$$

$$\int \frac{\partial c}{\partial x}\,dx = 0.$$

[illegible] $e^{2\pi i n x}\frac{d}{dx}$ $n \in \mathbb{Z}$ top. base for L.

$$\left[e^{2\pi i n x}\frac{d}{dx}, e^{2\pi i m x}\frac{d}{dx}\right] = \left[(2\pi i m)e^{2\pi i m x} - (2\pi i n)e^{2\pi i n x}\right]\frac{\partial}{\partial x}$$

Changing $\frac{d}{dx}$ to $\frac{1}{2\pi i}\frac{d}{dx}$ we find

~~[illegible]~~

$$[u_m, u_n] = nu_n - mu_m.$$

$$\left[\frac{d}{dx}, u_m\right] = mu_m \qquad \text{Cartan subalg.}$$

$u_0 = \frac{d}{dx}$ maximal abelian subalg.

$$u_n = e^{2\pi i n x} \frac{1}{2\pi i} \frac{d}{dx}$$

$$[u_n, u_m] = (m-n)\, u_{m+n}$$

$$\begin{cases} [u_1, u_{-1}] = (-1-1)\, u_0 = -2u_0 \\ [u_0, u_1] = u_1 \\ [u_0, u_{-1}] = -u_{-1} \end{cases}$$

Thus get a union

$$\left[e^x \frac{d}{dx}, e^{-x} \frac{d}{dx}\right]$$

$$= e^x\left(-e^{-x}\frac{d}{dx}\right) - e^{-x}\left(e^x \cdot \frac{d}{dx}\right)$$

$$= -2\frac{d}{dx}$$

$$\frac{a\frac{\partial b}{\partial x} - b\frac{\partial a}{\partial x}}{a^2} = \frac{\partial}{\partial x}\left(\frac{b}{a}\right)$$

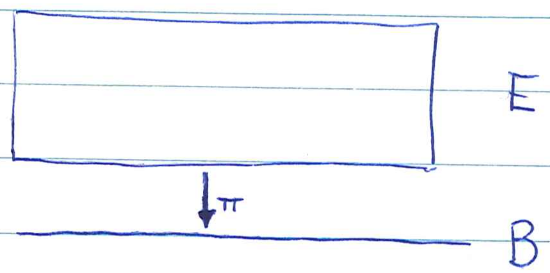
$$a\frac{\partial b}{\partial x} - b\frac{\partial a}{\partial x} = a^2 \frac{\partial}{\partial x}\left[\frac{b}{a}\right] = f$$

~~what is~~

Thus f must vanish somewhere

Thus if f has ~~an [illegible]~~ an even vanishing F. coeff
OKAY.

~~Topologically [illegible]~~ differentiably oriented sphere bundles are trivial ~~[illegible]~~ iff they have sections. Obstruction is a two-dimensional class. So there should be a 1-dimensional coh. class for the Lie algebra. But there isn't.



[E oriented smooth fiber bundle with S^1 for fiber]

I want a differential form procedure for defining the obstruction to finding a cross-section.

Chern method fails for S^1 bundles. Reason is that for the group $\mathrm{Aut}^\circ F$ we have

$$0 \longrightarrow H^1(\mathfrak{g}) \longrightarrow H^1(\mathrm{Aut}^\circ F) \longrightarrow H^1_{gp}(\mathrm{Aut}^\circ F)$$

$$H^1(\mathfrak{g}) = 0, \qquad H^1(\mathrm{Aut}^\circ F) \neq 0$$

In fact topologically we have a retraction of $\mathrm{Aut}^\circ F$ onto its maximal compact subgroups of rotations.

We understand Fredholm theory for a smooth kernel. It remains to understand ~~what~~ the relation between this and the homotopy given by a parametrix.

First note that the basic Fredholm theory formulas still hold

$$D(z) = \det(1 - zA) = 1 - z\,\mathrm{tr}\,A + z^2 \Lambda^2 A - \cdots \qquad \text{function}$$

$$C(z) = \det(1-zA)I + \underbrace{zA + z^2(A^2 - A\,\mathrm{tr}\,A) + \cdots}_{E(z)} \qquad \text{operator}$$

and
$$\frac{C(z)}{D(z)} = \frac{1}{1 - zA}$$

$$C(z) = D(z)\cdot I + E(z) \qquad \frac{E(z)}{D(z)} = \frac{zA}{I - zA}$$

Now suppose that A is an endomorphism of a complex $V^\bullet$. Then if we change A by a homotopy $D(z)$ doesn't change. This is because if $A - B = dP + Pd$, then
$$\mathrm{tr}\,A - \mathrm{tr}\,B = \mathrm{tr}\,(\text{commutator}) = 0.$$
We would like to find a formula for $\mathrm{tr}\,S$ where
$$I - S = dP + Pd.$$

<u>First question:</u> What is the significance of $\det(I - zS)$ in this case?

We know that $H_*(\det(I - zS)) = \det(I - zH_*(S)) = (1-z)^{\chi}$

where $\chi = L(1) = \operatorname{tr} S.$

Recall that in the Fredholm theory $I -$ ~~z~~S is calculated by the Neumann series.

To simplify we assume that $V^{+} \underset{d}{\overset{P}{\rightleftarrows}} V^{-}$.

$dP = 1 - S_1$

$Pd = 1 - S_0$

$$1 - S_0 = Pd$$

$$S_0 = 1 - Pd$$

$$S_0^2 = (1 - Pd)^2 = 1 - 2Pd + PdPd$$

$$(1 - zS_0)^{-1} = (1 - z(1 - Pd))^{-1}$$

$$= (1 + zPd - z)^{-1}$$

$$= [(1 - z) + zPd]^{-1}$$

$$= \frac{1}{1-z}\left(1 + \frac{z}{1-z}Pd\right)^{-1}$$

$$= \frac{1}{1-z}\left\{1 + \frac{z}{z-1}Pd + \frac{z^2}{(z-1)^2}(Pd)^2 + \cdots\right\}$$

$$(1 - zS)^{-1} = \frac{1}{1-z} + \frac{z}{(1-z)^2}(Pd + dP) + \frac{z^2}{(1-z)^3}(Pd + dP)^2 + \cdots$$

$ch\, A = tr\, e^A$ $\qquad$ $\det A = e^{tr\, A}$

$(-1)\ ch\, \Lambda^i$

The problem somehow is to take

$$D(z) = \det(1 - z\, S)$$

and express it in terms of the symbol of d or equivalently of

Tate procedure of higher order traces.

$$ch(\lambda_{-1} E) = \frac{(-1)^n c_n(E)}{Todd(E)}$$

$$Todd(E) \cdot ch(\lambda_{-1} E) = (-1)^n c_n(E)$$

K curvature of tangent bundle T

$$Todd\, T = \det \frac{K}{e^K - 1} \qquad K = \frac{i}{2\pi} K.$$

Prove complex R-R thm.

$$\int_X Todd\,(T) = \sum (-1)^n \dim H^n(X, \mathcal{O})$$

April 22, 1968.

We want to try for a parametrix proof of Atiyah-Singer. $\pi : T^*_X \longrightarrow X$. $E^\cdot, D$ elliptic complex on X. $\pi^*(E^\cdot), \sigma(D)$ complex of vector bundle on T^*_X acyclic off 0 section, defines element $\sigma \in K_c(T^*_X)$. Then

$$\mathrm{Index}\,(E^\cdot; D) = \int_X \pi_*\{ch\,\sigma\}\cdot A(X)$$

where $A(X)$ is some genus associated to the tangent bundle of X.

Technical question: Let $E^\cdot$ σ be a complex of bundles over a manifold Y each endowed with a connection. Suppose the complex is acyclic over $A \subset Y$ A closed, and let h be a homotopy operator for the complex restricted to U a nbd. of A. If ρ is a smooth fn. on Y zero outside of U $=1$ on A show that

$$\sum (-1)^q\, ch\, E^q = \omega + d\lambda$$
$$\prod \; c(E^q)^{(-1)^q}$$

where ω has support in $Y - A$. Also obtain explicit formulas for ω and λ in terms of h.

$$E^0 \xrightarrow{a} E^1 \xrightarrow{a} E^2 \longrightarrow \cdots$$

$$ha + ah = \underline{1}$$

D^0 $\quad$ D^1 $\quad$ D^2 $\quad$ conn.

K^0 $\quad$ K^1 $\quad$ K^2 $\quad$ curvatures of conn.

$$Ds = \theta \cdot s$$

(s arb. flat section relative to coordinates.

$$Ks = D^2 s = (d\theta - \theta\theta)s.$$

~~$[D,K]s = DKs - KDs$~~

~~$= (-d\theta\theta + \theta \cdot d\theta)s + (d\theta - \theta\theta)\theta s$~~

$$dK = -d\theta \cdot \theta + \theta \cdot d\theta = [\theta, d\theta]$$

$$[\theta, K] = [\theta, d\theta] - [\theta\theta, \theta]$$

$$\therefore \boxed{dK = [\theta, K]}$$

Now have a 1-parameter family of connections

$$D_t = tD_1 + (1-t)D_0$$

curvatures $K_t =$

$$\dot{D}_t \in \Gamma(E, T^* \otimes E) = \Gamma\{\mathrm{Hom}(E,E) \otimes T^*\}.$$

~~$K_t s = D_t D_t s$~~

~~$\dot{K}_t s = \dot{D}_t \dot{\theta}_t s$~~

$$K_t = d\theta_t - \theta_t \theta_t$$

$$\boxed{\begin{aligned} \dot{K}_t &= d\dot{\theta}_t - [\dot{\theta}_t, \theta_t] \\ 0 &= dK_t + [K_t, \theta_t] \end{aligned}}$$

not clear

~~D~~ a isom. $h = a^{-1}$.

$D^0 - hD^1 a$ ~~connection~~ $\in \Gamma(Hom(E^0, E^0) \otimes T^*)$

locally

$as^0 = ~~A~~ fs^1$ $s^0 = has^0 = hfs^1$

$$(hD^1 a)s^0 = hD^1(fs^1)$$
$$= h\{df \cdot s^1 + f\theta^1 s^1\}$$
$$= df \cdot f^{-1} s^0 + hf\theta^1 s^1$$

~~D(as^0)~~ $D\lambda_i s_i = \lambda_i \theta_{ji} s_j$

nothing to prevent you from doing everything for a cx.

$\therefore$ $D \in \boxed{\mathring{Hom}(\Gamma(E^\cdot), E^\cdot \otimes T^*)}$ etc.

homotopy operator

Special case: $E^0 \xrightarrow{\varphi} E^1$ φ isomorphism

$$\mathrm{tr}\, K^0 - \mathrm{tr}\, K^1 = d(?)$$

$$D_0 = D^0$$
$$D_1 = \varphi^{-1} D^1 \varphi$$

~~$D_0 - D_1 = \dot\theta$~~

$D_t s = \theta_t s$ etc.

$$\mathrm{tr}\, \dot K_t = \mathrm{tr}\, d\dot\theta_t - \mathrm{tr}\, [\dot\theta_t, \theta_t]$$
$$\overset{\parallel}{0}$$

$$\mathrm{tr}\, \dot K_t = d(\mathrm{tr}\, \dot\theta_t).$$

$$\mathrm{tr}\, K^0 - \mathrm{tr}\, K^1 = d\, \mathrm{tr}\, (D_0 - D_1).$$

check this

$$D_t s = \theta_t s = [\theta_0 + t(\overset{\eta}{\theta_1 - \theta_0})]\, s$$

$$d\theta_t - \theta_t \theta_t = d\theta_0 + t\, d\eta - \theta_0\theta_0 - t\,\theta_0\eta - t\,\eta\,\theta_0 - t^2\eta^2$$

diff $\mathrm{tr}\,\{ \text{———} \} = \mathrm{tr}\, d\eta - \mathrm{tr}\, \eta^2$ ($\mathrm{tr}\,\eta^2 = 0$)

where $\eta = D^0 - \varphi^{-1} D^1 \varphi$

~~$\varphi(K)$~~

$$\boxed{\varphi(K_t)^{\bullet} = d\ \varphi(K_t;\ \dot{\theta}_t)}$$

$\varphi(K_t)^{\bullet}$

The point is that I must determine what a complex with net curvature 0 is. Curvature tensor should be a commutator.

$E^0 \overset{\varphi}{\oplus} E^{1}$

$D^0 \qquad D^1$

$K^0 \xrightarrow{\varphi} K^1$

Keep on trying:

$\varphi : \quad \mathrm{Hom}(E,E)$

φ polynomial function on matrices

$\varphi \in S(\mathrm{Hom}(V,V)^*)$.

$$\varphi(A,B) = \frac{d}{d\varepsilon}\varphi(A+\varepsilon B)\Big|_{\varepsilon=0}$$

$$\varphi(A+\varepsilon B) = \varphi(A) + \varepsilon\varphi(B) + O(\varepsilon^2).$$

$$\varphi(K_t)^{\bullet} = \varphi(K_t, \dot K_t)$$

φ invariant $\Longleftrightarrow$ $\forall B$

$$\varphi(e^{tB}Ae^{-tB}) \equiv \varphi(A)$$

$$e^{tB}Ae^{-tB} \equiv (1+tB)A(1-tB) \qquad O(t^2)$$

$$\equiv (A+tBA)(1-tB)$$

$$\equiv A + t[B,A] \qquad O(t^2).$$

$$\varphi(A+t[B,A]) = \varphi(A) + t\,\varphi(A;[B,A])$$

Thus φ <u>invariant</u> $\Longleftrightarrow$ $\varphi(A,[B,A]) = 0$ all A, B.

Now lets take up the invariance proof.

$$\varphi(K_t)^\bullet = \varphi(K_t, \dot{K}_t)$$

$$= \varphi(K_t, d\dot{\theta}_t) - \varphi(K_t, [\dot{\theta}_t, \theta_t])$$

$$d\{\varphi(K_t, \dot{\theta}_t)\} = \varphi(K_t, dK_t, \dot{\theta}_t) + \varphi(K_t, d\dot{\theta}_t)$$

~~$\varphi([K_t, \theta_t], \dot{\theta}_t) = \varphi(K_t, [\dot{\theta}_t, \theta_t]) = 0.$~~

$$\varphi(K_t, \underbrace{[K_t, \theta_t]}_{-dK_t}, \dot{\theta}_t) - \varphi(K_t, [\dot{\theta}_t, \theta_t]) = 0.$$

$$\boxed{\varphi(K_t)^\bullet = d\{\varphi(K_t; \dot{\theta}_t)\}}$$

Back to our complex $E^\bullet$, suppose we have a homotopy operator. h ~~$dh+hd$~~ $\delta h + h\delta = 1$.

~~We get a complex~~

Given connections on $E^\bullet$

$$E^0 \underset{\delta}{\overset{h}{\rightleftarrows}} E^1 \underset{\delta}{\overset{h}{\rightleftarrows}} E^2 \qquad E^3$$

By insisting that even ~~by~~ be good we get new connection

$$\underset{\sim}{D}^a = \delta D^{a-1} h + h D^{a+1} \delta \qquad a \text{ odd}$$

$$\underset{\sim}{D}^a = D^a \qquad a \text{ even}$$

$$\eta^a = 0 \qquad a \text{ even}$$

$$\eta^a = \underset{\sim}{D}^a - D^a$$

$$E^0 \xrightarrow[\delta]{} E^1$$

$$D^0 - \delta D^1 h = \eta \in Hom(E^0, E^0) \otimes T^*$$

Lemma: net curvature is 0 if

~~(i) $D = \delta D h + h D \delta$~~

(ii) $\boxed{\delta D = D\delta \quad + \quad hD = Dh.}$

clear.

~~$1 = \delta h + h\delta$ + (i)~~

~~$\delta D = \delta D h + \delta h D \delta$ (i)~~ ~~$\delta h D \delta h \delta$~~

~~$\delta D = \delta h D \delta$~~ ~~$D\delta = \delta D h \delta$~~

$D\delta - h\delta D\delta$

Does $\exists$ a procedure for starting with δ, h + const. $\tilde{D}$.

$$\tilde{D} = \delta D h + (h\delta) D (h\delta) \quad \Rightarrow \quad \delta\tilde{D} = \tilde{D}\delta$$

$$h\tilde{D} = \tilde{D}h$$

~~$(D + \delta D h)\delta$~~

~~$(1 + \delta D h)$~~

$$\tilde{D} - D = \delta D h + h\delta D h\delta$$

$$- D(h\delta + \delta h)$$

$$= \{\delta D h\} + (-\delta h) D h\delta - D\delta h$$

Let X be a Zariski space with ~~[illegible]~~ chain condition. ~~[illegible]~~ Then if Φ^p is the family of ~~supports~~ closed sets of dim $\leq p$ have

$$0 \to \Gamma_{\Phi^{p-1}} \to \Gamma_{\Phi^p} \to \Gamma_{\Phi^p/\Phi^{p-1}} \to 0$$

for any injective sheaf and

$$\Gamma_{\Phi^p/\Phi^{p-1}} = \bigoplus_{d(x)=p} \Gamma_x$$

where $\Gamma_x = \Gamma_{\overline{\{x\}}, \overline{\{x\}}-\{x\}}$

Proof Enough to test on injective sheaves so given $s \in I$ $\exists$ ~~[illegible]~~ $\operatorname{Supp} s =$.

---

hence get spectral sequence

$$\boxed{E_2^{pq} = \bigoplus_{d(x)=q} H_x^{p+q}(F) \Longrightarrow H^{p+q}(X,F)}$$

$$H^{p+q}_{\Phi^{q-1}}(X,F) \longrightarrow H^{p+q}_{\Phi^q}(X,F)$$

$$\nwarrow \qquad \swarrow$$

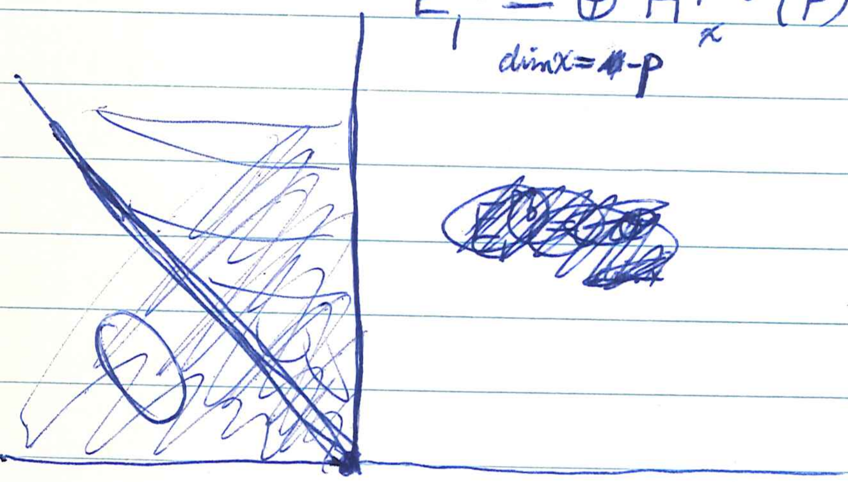
$$E_2^{pq} = \bigoplus_{d(x)=q} H_x^{p+q}(F)$$

not a good rep.

$$E^{pq} \qquad H^{p+q+1}_{\Phi^{q-1}} \longrightarrow H^{p+q+1}_{d(x)=q-1} = E^{p+2\ \ q-1}$$

$$\bigoplus_{\dim x = r} H^{n}_{x}(F) \longrightarrow \bigoplus_{\dim y = r-1} H^{n+1}_{y}(F)$$

$$E_1^{pq} = \bigoplus_{\dim x = -p} H^{p+q}_{x}(F) \longrightarrow H^{p+q}(X,F)$$



April 26, 1968 Summary.

Problems worked on since April 1:

A. Connect up your rational homotopy functor with de Rham cohomology.

B. Rational Pontryagin classes for topological manifolds

C. Proof of index theorem using Fourier representation of the parametrix on T^*.

D. Characteristic classes for smooth bundles.

E. A rational view of manifolds and surgery.

Progress:

Summary April 28, 1968

Ideas:

A. On char classes for smooth manifold bundles: (explicit for BAut F using embeddings) ① Chern doesn't work for S^1 bundle. The problem again is the maximal compact subgp. one. Raises general question of relating for a top. gp. G the cohomology

$$H^*_{top}(G),\ H^*_{G\text{-equiv.}}(G),\ H^*(BG),\ H^*_{gp.}(G),\ H^*_{G\text{-equiv.}}(EG).$$

should be spectral sequences

$$E_2^{pq} = H^p_{gp} \otimes H^q_{top} \Longrightarrow H^{p+q}_{G\text{-equiv}}$$

$$E_2^{pq} = H^p_{gp.}(G, H^q(X)) \Longrightarrow H^{p+q}(X/G) \quad \text{if } G \text{ acts freely.}$$

② Integration of ~~[illegible]~~ of a char. class of tangent bundles to fibers gives some char. classes. These generalize ~~[illegible]~~ Pontryagin numbers and are fiber cobordism invariants. Not all = e.g. Atiyah's ~~[illegible]~~ mistake + CP^n bundle over 3 conn. spaces. Similar classes constructed from associated manifold bundles to the tangent bundle do not seem to yield anything new. (however maybe get something by acting on connections?)

③ Algebraic model for surgery using rational de Rham algebras. Can homology classes be represented by a torus. False if X not simply-connected, e.g. X = nilmanifold-

I want a preliminary model for rational manifold theory. Ingrediants

(i) definite dimension n

(ii) surgery instead of attaching cells.

(iii) L classes.

① Naive approach will be to see what cell attaching does.

② Lefschetz duality — locally compact manifolds, point at ∞

etc.

Let A be a DG Poincare alg. connected for simplicity! ~~Take a cycle + attach~~

Suppose A zero, Then choose we add x of degree k such that $dx = 0$ we must also put in y of degree $n-k$ with $dy = 0$ and $\int x \cdot y = 1$

model of $S^k \times S^{n-k}$. How to think of

$$\partial\{D^p \times D^q\} = S^p \times D^q \cup D^p \times S^q$$

$D^p \times D^q$

boundary is $S^{p+q-1} = S^{p-1} * S^{q-1}$

Q. D^p: $0 \quad x^{p-1} \quad dx^{p-1} \quad 0 \quad 0 \quad 0$

D^q $\quad y^{q-1} \quad dy^{q-1}$

$D^p \times D^q$

Lefschetz duality: $H^*_c(X)$ $\quad X = \partial Y$

$H^*_c(X) = H^*(X, \partial X)$

a) $H^*(X, \partial X) \xrightarrow{j} H^*(X) \xrightarrow{i} H^*(\partial X) \xrightarrow{\Delta\varepsilon} H^*(X, \partial X)$

b) $H^*(X, \partial X)$ module over $H^*(X)$

i ring homomorphism, j module homs. $/ H^*(X)$.

c) $\int : H^{n+1}(X, \partial X) \to \mathbb{Q}$

$(\alpha, \beta) = \int \alpha\beta$ non-deg. pairing: $H^*_c(X) \times H^*(X) \to \mathbb{Q}$

$0 \to A(X, \partial X) \to A(X) \to A(\partial X) \to 0$

Conjectures: and questions:

a) If X is a space, then any n-dimensional homology class of X may be realized by a map of $T^n \to X$, where T^n is the n-dimensional torus, at least rationally.

b) Let M be a smooth manifold and let $\pi: P \to M$ be a manifold bundle associated to the tangent bundle of M, e.g. the a Grassman bundle. Assuming that P is oriented over M we can take characteristic classes of it push them down to M via π_*. Can the resulting cohomology classes always be expressed in terms of the characteristic classes of M?

Answer to b) is yes. Thus if $P \to X$ is the principal bundle associated to the tangent bundle of X, with structural group G, and if F is a smooth compact manifold on which G acts, then there is a commutative diagram

$$\begin{array}{ccc} P \times_G F & \xrightarrow{f'} & E_G \times_G F \\ \downarrow p & & \downarrow q \\ M & \xrightarrow{f} & B_G \end{array}$$

where f is the classifying map for P. Taking a characteristic class of $E_G \times_G F$ we can write it as the sum of products; of characteristic classes of the base and of the tangent bundle along the fibers. As integration along the fibers commutes with base change one sees that all characteristic classes obtained in the form p_* (char. cl.) are in the image of f^*.

Characteristic classes of representations of discrete groups. Let V be a complex vector space of dimension n. Let G be a group acting on V. Them we know how to define Chern classes of the representaion ~~ixix~~ $c_q(V)$ in $H^{2q}(G,\mathbb{Z})$ by use of equivariant cohomology. Clearly one should take ~~xllxofxthisxgroup~~ G = Gl(V) with the discrete topology in order to get universal chern classes. for representations of disctete groups.

Review the proof of Novikov's theorem: If you have a vector bundle E over a sphere S which is topologicall trivial, then E homoemorphic to SxR^q so by first theorem E diffeomorphic to TxR^n, where ~~X~~ T is a homotopy sphere. We then have maps $E \xrightarrow{i} E \xrightarrow{p} S$ which are smooth. ~~Hence~~ ~~thextangentxbundlexto T x~~. The normal bundle of the embedding i is trivial, the tangent budle of T is stably trivial since T is a homotopy sphere, hence ~~IX~~ i^* tangent bundle to E is ~~stabl~~ trivial. But the tangent bundle to E is the sum of p^*E and p^* tangent bundle to S which is stably-trivial. Thus ~~R~~ $(pi)^*E = 0$ and so E=0. Now the first ~~kxk~~/step in Novikov's theorem is to note that if the dimension of E is 1 and if S is simply-connected ~~simplyxconnectedxand~~ of dimension ≥ 5 with vanishing K ~~Wh~~π_1, then the result follows from the h-cobordism theorem. ~~ofxSmale~~. combined with Siebenmann's result on adding boundaries to E. Still do not understand the inductive step of Novikov's proof.

Surgery for a DG algebra: Let $A = \oplus A^q$ $0 \leq q \leq n$ be a DG anti-commutative over Q graded finite dimensional algebra/endowed with a linear functional $\int: H^nA \to Q$ which gives Poincare duality, i.e. $(\alpha,\beta) = \int\alpha\beta$ is a non-degenerate bilinear form on H^*A. Actually we shall assume already that this bilinear form is non-degenerate on A. I now wish to define what is meant by surgery of A. So I start with a map of a sphere $S^p \to A$.

Problem: Define rational Pontryagin classes for rational homology n-manifolds generalizing Milnor's definition in the case that the space is a simplicial complex. Use Atiyah's approach. Suppose that a discrete group π acts on X^n. Then we can form the bundle $Px_G X$ where P is a princpal bundle on which G acts whose base B is a manifold. Then by the index theorem for this map $p: Px_G X \to B$ we know $p_*(L(T_p)) = \text{ch Sign } p$, where Sign p is of course calculated from the π action on the middle dimension of X. Unfortunately this appraach doesn't yield anything unless X has ~~dimension~~ even dimension. It does however seem to give information about the components of $L(T_p)$, other than those at the top. Check this Thus we are given a map $B\pi \to BSO(p,q)$enabling us to define ch(sign) which is a characteristic class . If dim X = 4i, ~~then we get the~~ and B is of dimension q, then we get characteristic classes of dim q-4i. Atiyah has shown that these other classes are non-zero.in general. The problem therefore is to determine just how non-trivial they can be, and whether ~~in any~~ there is any chance that they determine the Pontryagin classes of X. Char sign might be very non-trivial. Start with a discrete group G acting on a manifold X. Then it must preserve the Pontryagin classes, however it needn't preserve them as cocycles, ie it should not be possible to construct a G-invariant connection. It seems clear that you must find a method which works even when the dimension of X is not dividible by 2. Recall that Haefliger showed at Cornell how to define the normal spherical fibre space of an embedding by imitating the procedure of Nash.- all paths issuing form the submanifold whichdo not return.

~~Sxx~~ Suppose we identify a ~~submanifold~~ submanifold of X with the sheaf it generates. How does one go about constructing the duall cohomology class to a submanifold? Recall that a cohomology class of dimension r is a map $Z[r] \to Z$ in to derived category. Method. Consider a fixed resolution $K^\bullet$ of Z apply $\Gamma_Y(K^\bullet)$. Local calculations show that this has a canonical class of dimension equal to the co-dimension of the submanifold Y. One then considers the image of this in the cohomolggy of X. Next question is how to construct Pontryagin classes, and more generally how to cope with the normal bundle. The point is that the same prodecure should yield a K theoretic orientation class.

What Sullivan claims is that if X is a Pl manifold of dimsnsion n, then he can define ~~a~~ in a nice way a ~~xxxx~~ stable vector bundle V over the Thom complex associated to the stable normal bundle of X. In other wrods if X is embedded in a high Euclidean space wihh normal microbundle λ and if T is the thom spa ce of λ, then there is a complex of vector bundles on T which is acycli off X. Incase that X is ~~mmm~~smooth this complex is not associated to the normal bundle but instead to some L functor of the normal bundle ~~which~~ whose denominators allow one to ~~impossible~~ extend the definition to microbundles.

Question: ~~Can you define the index of a submanifold~~

The Pontyagin classes which after all are equivalent to the L-genus are a collection of cohomology classes whose knowledge is equivalent not only to the Index of the manifold X, but also to the indices of various submanifolds of X. More precisely if I know the L-genus of X, then I cancalculate the index of a submifold Y in terms of its dual cohomolgoy class. More presisely if Y is of dimension $4i$, then some multiple of Y is cobordant to the inverse image of a regular point of a smooth map f of X to S^{n-4i}, hence the index of Y is $\int_X L_i f^*\alpha$ dived by that multiple. Of course $f^*\alpha$ is just the dual cohomology class of Y. This is nonsense as the normal bundle of Y in X nedd not be trivial and therefore will contribute to the index. In fact the index of Y is L_i tangent bundle to Y which is L_i of the tangent bundle to X when the normal bundle is trivial. The L genus of X can be claculated from the indices of the submanifolds of X which have trivial normal bundles.

To what extent can we mimic DG constructions using the cohomology ring instead of the ring of functions. Thus instead of differential operators we should consider $H^*_\Delta(X\times X)$ acting on $H^*(X)$ and take the associated de Rham cohomology. I wonder what the heck this is? $\mathrm{Diff}\,(A) = \mathrm{limind}\ \mathrm{Hom}_A(A\otimes A/J^k, A)$

$$\mathrm{limind}\ \mathrm{Ext}^q_{A\otimes A}(A\otimes A/J^n, A) = 0 \quad q>0.$$

Thus if we take just $H(X\times X) = HX\otimes HX$ acting on HX we don't get anything interesting. In the topological situation $H_\Delta(X\times X)$ is the cohomology of the Thom space of the tangent bundle, hence has a canonical class U of dimension n which acts as the identity, $U^2 = \chi\, U$.

Milnor's version of the combinatorial invariance of the rational Pontryagin classes: He defines $\mathcal{L}_i = L_i(p_1,..,p_i)$ in X^n for $n>8i+2$ by the formula

$$\int_X \mathcal{L}_i f^*\alpha = I(f) \qquad \alpha = \text{generator of } H^{n-4i}(S^{n-4i})$$

for any map $f: X^n \to S^{n-4i}$, where $I(f)$ is the index of f and is defined by taking a simplicial approximation of f and taking the inverse image of a generic point and then the index of the inverse image. In order to know that this determines $\mathcal{L}_i$ it is of course necessary to show that every element of $H^{n-4i}(X,\mathbb{Q})$ can be represented in the form $f^*\alpha$ for some $f: X^n \to S^{n-4i}$. This will certainly be the case if n is sufficiently large, In fact if $n<2(n-4i)$ or $n>8i$. In fact for n odd it is always true, and for n even, the problem is that we get a map $X^n \to K(n-4i)$ and trouble arises in dimension $2(n-4i)$.

This is Sullivan's basic idea: One should be able to generalize the above rational calculations of Milnor using homology to K calculations off "2" provided one is willing to use K theory and not cohomology.

Potential method: Try to give a direct definition of the index of a map of oriented manifolds. Thus the present prodedure is to take an arbitrary continuous map $f:X \to Y$ from one oriented smooth manifold to another then approximate it by a smooth map and then take the index of the inverse image of a regular value. So what we want is someway of doing this which depends only one the map f. Now Verdier's idea is to replace a triangulation by an isomorphism in the derived category of the chains on the space with a finite complex. Another possibility might be to consider the critical values of the application in the sense of Fary, and to see if there is a generic one.