Kallis's thesis - a summary

Let $\mathfrak{g}$ be a complex ~~reductive~~ semisimple Lie algebra, G its adjoint group. Let θ be an algebra automorphism of order 2 of $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding decomposition of $\mathfrak{g}$ induced by θ where $\mathfrak{k}, \mathfrak{p}$ are the +1, -1 eigenspaces of θ.

Let K_θ be the subgroup of G of the elements which commute with θ. If K is the Lie subgroup of G corresponding to the Lie algebra $\mathfrak{k}$, then K is the identity component of K_θ. In fact K_θ is a reductive algebraic group and has at most a finite number of components.

We note that there will be a real form $\mathfrak{g}_{\mathbb{R}}$ of $\mathfrak{g}$ stable under θ and hence $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}$ will be a Cartan decomposition of $\mathfrak{g}_{\mathbb{R}}$ where $\mathbb{C} \otimes \mathfrak{k}_{\mathbb{R}} = \mathfrak{k}$, $\mathbb{C} \otimes \mathfrak{p}_{\mathbb{R}} = \mathfrak{p}$. If $\mathfrak{a}_{\mathbb{R}}$ is a maximal Abelian subalgebra of $\mathfrak{p}_{\mathbb{R}}$ we let $\mathfrak{a} = \mathbb{C} \otimes \mathfrak{a}_{\mathbb{R}}$. Then $F = \{a \in \exp_G(\mathfrak{a}) \mid a^2 = 1\}$ is a finite group of order 2^r ($\dim_{\mathbb{C}} \mathfrak{a} = r$) and F normalizes K and we have in fact that $K_\theta = F \cdot K$.

There is a natural representation of K_θ on the space $\mathfrak{p}$. This extends to an action of K_θ as algebra automorphisms on $S(\mathfrak{p})$ = the affine algebra of polynomials on $\mathfrak{p}$. If J=ring of K_θ invariants in $S(\mathfrak{p})$ we know that $J = \mathbb{C}[u_1, \ldots, u_r]$ where the u_i are algebraically independent + homogeneous (indeed the K_θ invariants = K invariants in $S(\mathfrak{p})$ are determined by their restriction to $S(\mathfrak{a})^{W'}$, W'=the restricted Weyl group of $\mathfrak{a}$, $S(\mathfrak{a})^{W'}$ = W' invariants in $S(\mathfrak{a})$, i.e. $J \xrightarrow[\text{rest.}]{} S(\mathfrak{a})^{W'}$ is an algebra ~~auto~~isomorphism.)

We are going to give a detailed description of the K_θ orbits in $\mathfrak{p}$, structure of the ring J and the representation theory of K on the ring $S(\mathfrak{p})$.

I. Semisimple Elements

We recall that any $x \in \mathfrak{g}$ has a Jordan decomposition into $x = x_s + x_n$ (ad x_s is semisimple, ad x_n is nilpotent).

Then it follows that $x \in \mathfrak{p}$ implies $x_s, x_n \in \mathfrak{p}$ by uniqueness of Jordan decomposition.

Now define a <u>Cartan subspace</u> of $\mathfrak{p}$ to be a maximal Abelian subalgebra of $\mathfrak{p}$ all of whose elements are semisimple. As in the case of Cartan subalgebras in complex semisimple Lie algebras we have

<u>Proposition 1.1</u> Any two Cartan subspaces are conjugate via K_θ (in fact by K). Any semisimple element can be imbedded in a Cartan subspace. And $\mathfrak{a} = \mathbb{C} \otimes \mathfrak{a}_{\mathbb{R}}$ (defined above) is a Cartan subspace,

As a corollary we obtain the usual conjugacy of Cartan subalgebras in complex semisimple Lie algebras.

We call a K_θ orbit $\mathcal{O}$ in $\mathfrak{p}$ semisimple if it consists of semisimple elements. Let $\mathcal{O}_s$ be the collection of these orbits.

We recall the u_i defined above. We consider the algebraic morphism: (*) $u: \mathfrak{p} \longrightarrow \mathbb{C}^r$ $u(x) = (u_1(x), \ldots, u_r(x))$

Since the u_i are K_θ invariant they are constant on any K_θ orbit and hence we have

<u>Proposition 1.2</u> The map induced by (*) $u_s: \mathcal{O}_s \longrightarrow \mathbb{C}^r$ is a bijection. The (Zariski or Euclidean) closed K_θ orbits are exactly the semisimple orbits. In fact if $x \in \mathfrak{p}$ is semisimple then $K_\theta(x) = K(x)$ is a connected closed algebraic set.

II. Regular Elements of $\mathfrak{p}$

We call $X \in \mathfrak{p}$ regular if $\dim K_\theta(x) = \max_{y \in \mathfrak{p}} \dim K_\theta(y)$ (here dim is Zariski dimension). Clearly the set $\mathcal{R}$ of regular elements is an open dense subset of $\mathfrak{p}$.

For any $X \in \mathfrak{p}$ we let $\mathfrak{g}^{(x)} = \{u \in \mathfrak{g} \mid [u, x] = 0\}$, $\mathfrak{k}^{(x)} = \mathfrak{g}^{(x)} \cap \mathfrak{k}$, $\mathfrak{p}^{(x)} = \mathfrak{g}^{(x)} \cap \mathfrak{p}$. Then $\mathfrak{g}^{(x)} = \mathfrak{k}^{(x)} \oplus \mathfrak{p}^{(x)}$. We have the following criterion for regularity.

Theorem 2.1 $x \in \mathfrak{p}$ is regular iff $\dim \mathfrak{k}^{(x)} = \dim \mathfrak{m} = m$ (where $\mathfrak{m}$ = Centralizer in $\mathfrak{k}$ of $\mathfrak{a}$ of Prop. 1.1) iff $\dim \mathfrak{p}^{(x)} = r$ iff $\dim \mathfrak{g}^{(x)} = m+r$. In fact for any $x \in \mathfrak{p}$, $\dim \mathfrak{p}^{(x)} - \dim \mathfrak{k}^{(x)} = r-m$. And $x \in \mathfrak{p}$ is regular and semisimple iff $\mathfrak{p}^{(x)}$ is a Cartan subspace.

(We remark here that the fact that $\dim \mathfrak{p}^{(x)} - \dim \mathfrak{k}^{(x)} = r-m$ for any $x \in \mathfrak{p}$ follows from the fact that $\mathfrak{k}/\mathfrak{k}(x) \oplus \mathfrak{p}/\mathfrak{p}(x) = \mathfrak{g}/\mathfrak{g}(x)$ and that $\mathfrak{k}/\mathfrak{k}(x)$, $\mathfrak{p}/\mathfrak{p}(x)$ will be maximal isotropic subspaces for the nonsingular alternating form induced on $\mathfrak{g}/\mathfrak{g}(x)$ by the Killing form, i.e. $(a,b) = B([x,a],b)$, $a,b \in \mathfrak{g}$).

III. Nilpotent Elements of $\mathfrak{p}$

If η is the set of nilpotent elements in $\mathfrak{p}$ it is clear that $u^{-1}(0) = \eta$. Then η is a closed algebraic set.

If $e \neq 0$ $e \in \mathfrak{p}$ is nilpotent we recall from the Jacobson-Morosov Theorem that ther exist $x, f \in \mathfrak{g}$ so that x, e, f is the base of a three dimensional simple (TDS) Lie algebra where the bracket relations are $[x,e] = 2e$, $[x,f] = -2f$, $[e,f] = x$. We sharpen this result by choosing $x \in \mathfrak{k}$, $f \in \mathfrak{p}$. We call such a base $\{x, e, f\}$ a normal S triple.

If we let K act on the set of all normal S triples then we have a bijective correspondence between K conjugacy classes of normal S triples and K orbits of nilpotents. This implies that there will be only a finite number of K orbits in η, since for a normal S triple $\{x, e, f\}$ the eigenvalues of ad x are integral and bounded in absolute value by (dimG). Hence there are a finite number of K_θ orbits in η.

<u>Proposition 3.1</u> η is a union of a finite number of sets $\overline{O_{(e_i)}}$ i=1,...k (Zariski or Euclidean closure) where $\overline{O_{e_i}}$ is an irreducible component of η and has codimension in $\mathfrak{p}$ = r. Thus the e_i i=1,...k will be regular and nilpotent in $\mathfrak{p}$. The set $\eta_r = \eta \cap R$ of regular nilpotents will be open dense in η.

If $e \neq 0$ is regular and nilpotent we call the corresponding S triple $\{x, e, f\}$ <u>principal</u> and the TDS spanned by $\{x, e, f\}$ <u>principal</u>.

We recall from classical Lie algebra theory that if $\mathfrak{g}(\gamma) = \{u \in \mathfrak{g} \mid [x,u] = \gamma(x)u$ for all $x \in \mathfrak{a}\}$, γ a linear form on $\mathfrak{a}$, $\gamma \in \mathfrak{a}^*$=dual of then $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \sum_{\gamma \in S} \mathfrak{g}(\gamma)$, where S= $\{\gamma \in \mathfrak{a}^* \mid \mathfrak{g}(\gamma) \neq 0\}$ is a root system in $\mathfrak{a}^*$, called the "restricted root system." We let S_+ be the positive roots, Γ = simple roots (r = card Γ) ~~if ... is semisimple~~. Let $x^* \in \mathfrak{a}$ so that $\langle \gamma_i, x^* \rangle = 2$ for all $\gamma_i \in \Gamma$ (x^* is unique).

<u>Theorem 3.1</u> If $\tilde{\mathfrak{a}} = (x, e, f)$ is a principal TDS then for any $z \in \mathbb{C}^*$ $ze + \bar{z}f$ is a regular and semisimple element of $\mathfrak{p}$ and $e+f$ is conjugate (via K) to x^*. Any irreducible component with respect to the adjoint representation of $\tilde{\mathfrak{a}}$ on $\mathfrak{g}$ is odd dimensional, i.e. 0 weight appears.

This allows us to characterize the regular nilpotents in a manageable fashion. That is, if M= Centralizer in K of $A = \exp_G(\mathfrak{a})$ then $M \cdot A$ is a closed connected algebraic subgroup of G, in fact $M \cdot A$ = Centralizer in G of x^*. Then we know that $M \cdot A$ will have an open orbit Q in the space $\mathfrak{g}(\gamma_1) \oplus \cdots \oplus \mathfrak{g}(\gamma_r)$ where $\gamma_i \in \Gamma$. Then

<u>Corollary to Theorem 3.1</u> $e \in \mathfrak{p}$ is regular nilpotent iff e is G congugate to an element of Q.

Thus we can now justify the use of the group K_θ by

Theorem 3.2 K_θ acts transitively on the set $\mathfrak{n}_r$ so that $\mathfrak{n}_r$ is the unique open K_θ orbit in $\mathfrak{n}$.

IV. Structure of K_θ Orbits

We can now determine the structure of all the regular elements of $\mathfrak{p}$. For if $X \in \mathfrak{p}$ is semisimple we know that $\mathfrak{g}(x)$ is reductive and θ stable. Hence we can apply our theory to the semisimple subalgebra $\mathfrak{g}(x)' = [\mathfrak{g}(x), \mathfrak{g}(x)]$. We have that

Proposition 4.1 $X \in \mathfrak{p}$ is regular iff X_n is regular nilpotent in $\mathfrak{p}(x_s)'$ with respect to $\mathfrak{g}(x_s)'$ $\left(\mathfrak{g}(x_s)' = \mathfrak{k}(x_s)' \oplus \mathfrak{p}(x_s)'\right)$. $\mathfrak{k}(x_s)' = \mathfrak{k}(x_s) \cap \mathfrak{g}(x_s)'$

Now we call a K_θ orbit regular if it consists of regular elements. Let $\mathcal{O}_R$ be the collection of regular orbits. $\mathfrak{p}(x_s)' = \mathfrak{p}(x_s) \cap \mathfrak{g}(x_s)'$

Theorem 4.1 The map induced by (*) $u_R : \mathcal{O}_R \longrightarrow \mathbb{C}^r$ is a bijection. For any $\xi \in \mathbb{C}^r$ we have

i) $u_R^{-1}(\xi) = u^{-1}(\xi) \cap R$ is the unique open dense K_θ orbit of maximal dimension in $u^{-1}(\xi)$.

ii) each irreducible component of $u^{-1}(\xi)$ contains a regular element of $\mathfrak{p}$ and has codimension in $\mathfrak{p}$ = r.

iii) $u_{\mathcal{J}}^{-1}(\xi) = u^{-1}(\xi) \cap \mathcal{J}$ ($\mathcal{J}$ = set of semisimple elements in $\mathfrak{p}$) is the unique K_θ orbit of minimal dimension in $u^{-1}(\xi)$. It is also the unique closed orbit in $u^{-1}(\xi)$.

iv) $u^{-1}(\xi) = \{ x \in p \mid u(x_s) = \xi \}$ is a union of a finite number of K_Θ orbits.

V. Structure of the Ring J

~~We now assume $\mathfrak{g}$ is complex semisimple.~~ We will construct a crosssection for the K_Θ orbits of regular elements.

From theorem 3.1 we can assume that for the principal TDS $\{x, e, f\}$ $e+f=x^* \in \mathfrak{a}$. And if we take $e^* = \sum_{i=1}^{r} \beta_{\gamma_i}$ $f^* = \sum_{i=1}^{r} \beta_{-\gamma_i}$ so that $\{x^*, e^*, f^*\}$ is an S triple for the same TDS ($\beta_{\gamma_i} \in \mathfrak{g}^{(\gamma_i)}$ $\beta_{-\gamma_i} \in \mathfrak{g}^{(-\gamma_i)}$) it is easy to check that the $\beta_{\gamma_i}, \beta_{-\gamma_j}$ will satisfy the hypotheses of a Theorem of Serre in "Algebres de Lie Semisimple Complex (VI-19) ". This implies that

<u>Theorem 5.1</u> The subalgebra $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$ generated by $\mathfrak{a}$ and the principal TDS $\tilde{\mathfrak{a}}$ is semisimple. $\mathfrak{a}$ will be a Cartan subalgebra of $\tilde{\mathfrak{g}}$. If $S' = \{ \varphi \in S \mid \varphi/2 \notin S \}$ (see defn. of S in III) then S' will be the root system of $\tilde{\mathfrak{g}}$ with respect to $\mathfrak{a}$. $\tilde{\mathfrak{g}}$ will be stable under Θ so that there will exist a Θ stable normal real form $\mathcal{G}$ of $\tilde{\mathfrak{g}}$. And if $\tilde{G}$ is the Lie subgroup of G corresponding to $\tilde{\mathfrak{g}}$ then $\tilde{G}$ will be its own adjoint group (i.e. $\mathrm{Cent}_G \tilde{G} = \{e\}$).

Let $\tilde{\mathfrak{g}} = \tilde{\mathfrak{k}} \oplus \tilde{\mathfrak{p}}$ be the θ decomposition of $\tilde{\mathfrak{g}}$. Theorem 5.1 will now justufy the use of principal TDS for $\tilde{\mathfrak{a}} = \{x, e\ f\}$. For mow we have that $\tilde{\mathfrak{a}}$ is a principal TDS in the subalgebra (using Kostant's notation from "Lie Group Representations on Polynomial Rings."), i.e. "e" will be principal nilpotent in $\tilde{\mathfrak{g}}$. We then have that $\tilde{\mathfrak{g}}^{(e)} = \tilde{\mathfrak{p}}^{(e)} = \mathfrak{p}^{(e)}$. This means thAT by applying Kostants theory in the adjoint case we can determine the degrees of the homogeneous polynomials u_i defined above. We get the following commutative diagram:

$$\begin{tikzcd}
S(\tilde{\mathfrak{g}})^{\tilde{G}} \arrow[d] \arrow[dr] & \\
S(\tilde{\mathfrak{p}})^{\tilde{K}} \arrow[r] & S(\mathfrak{a})^{W'} \\
S(\mathfrak{p})^{K} \arrow[u] \arrow[ur] &
\end{tikzcd}$$

whrer all the maps " $\longrightarrow$ " are restriction are algebra isomorphisms.

<u>Proposition 5.1</u> There is a basis $\nu_i{}_{\,i=1,\dots r}$ of $\mathfrak{p}^{(e)}$ so that $[x, v_i] = p_i v_i$ and so that degree $(u_i) = p_i + 1$.

Now if we consider the r-plane $f + \mathfrak{p}^{(e)}$ and define the map

(1) $J \longrightarrow S(f + \mathfrak{p}^{(e)})$, $S(f+\mathfrak{p}^{(e)}) = \text{poly. rings on } f+\mathfrak{p}^{(e)}$

by restriction we have

<u>Theorem 5.2</u> (1) is an algebra isomorphism. The plane $f + \mathfrak{p}^{(e)}$ is a cross-section for the K_θ regular orbits, i.e. every regular

element of $\mathfrak{p}$ is K_θ conjugate to one and only one element of the plane $f+\mathfrak{p}^{(e)}$.

If now we consider $\mathfrak{p}$ as a linear manifold and consider the differentials $du_1, \ldots, du_r$ we get as a corollary

Corollary to Theorem 5.2 If x is regular in $\mathfrak{p}$ then the differentials $du_{1(x)}, \ldots \; du_{r(x)}$ are linearly independent.

We note that in the adjoint case the converse of this statement is true but in general is not true as an example of a rank 1 case shows.

We let $\mathfrak{J}^{\xi}$ be the ideal in $S(\mathfrak{p})$ generated by $u_1-\xi_1, \ldots, u_r-\xi_r$ for any $\xi=(\xi_1, \ldots \xi_r) \in \mathbb{C}^r$. By the fact that every irreducible component of $u^{-1}(\xi)$ has a regular element and by the above corollary we conclude that the ideal $\mathfrak{J}^{\xi}$ is radical from the following Lemma.

Lemma Let X be a n dim. vector space over $\mathbb{C}$, $S(X)$ = poly. ring over X, $f_1, \ldots f_k \in S(X)$ $k \leq n$, and $P(\xi) = \{x \in X \mid f_i(x) = \xi_i \;\; i=1, \ldots k\}$. Then if in each component of $P(\xi)$ there is a point y so that the differentials $df_{1(y)}, \ldots \; df_{k(y)}$ are linearly independent then the ideal $(f_1-\xi_1, \ldots, f_k-\xi_k)$ is radical.

Let $D(\mathfrak{p})$ be the algebra of constant coefficient differential operators on $\mathfrak{p}$. K_θ acts as algebra automorphisms in $D(\mathfrak{p})$.

We let $\mathcal{J}_*$ be the invariants, $\mathcal{J}_*^+$ =ideal generated by $\partial \in \mathcal{J}_*$ with zero as constant term. Then H= $\{f \in S(p) \mid \partial f = 0 \ \forall \partial \in \mathcal{J}_*^+\}$ is a K_θ module and by a well known result $S(p) = \mathcal{J} \cdot H = \{\sum j_i h_i \mid j_i \in \mathcal{J}, h_i \in H\}$

This means that the restriction map $H \longrightarrow S(O_x) = \{f|_{O_x} \mid f \in S(p)\}$

O_x = K_θ orbit of x

is a K_θ (hence a K) ~~mean~~ module epimorphism. But since $\mathcal{J}^0 = (u_1, \ldots u_r)$ is radical we have

<u>Theorem 5.3</u> $H \xrightarrow[\text{rest.}]{} S(O_x)$ is a bijection for x regular.

Then the map $\mathcal{J} \otimes H \to S(p)$, $f \otimes g \to f \cdot g$ is a K_θ module bijection (hence a K module bijection). Hence $S(p)$ is a free J module also. And for any $\xi \in C^r$ we have the vestor space decomp--osition $S(p) = \mathcal{J}^\xi \oplus H$.

We can determine the structuee of H as follows, We know that the Killing form B of $\mathfrak{g}$ restricted to p defines a ring isomorphism of the algebra $D(p)$ onto $S(p)$, where $\langle \partial_x, B\partial_y \rangle = B(x,y)$ with $\langle\ ,\ \rangle$ as the nonsingular pairing of $D(p) \times S(p)$ defined by $\langle \partial, f \rangle = \partial f(0)$, $x, y \in p$ and ∂_x is the vector field defined on p by $\partial_x f(z) = \frac{d}{dt} f(z+tx) \Big|_{t=0}$ for all $z \in p, f \in C^\infty(p)$.

Then if η = set of nilpotents in p let $\eta' = B(\partial_x)$ for $x \in \eta$ and H_η =linear space of all powers z^m, m=0, 1, ... for all $z \in \eta'$. Then we have

Proposition 5.2 $H = H_\eta$.

VI. Representation Theory

We now work with the grpup K and the the K orbits in $\mathfrak{p}$.

We r~~f~~ecall that $R(K/K^{(x)})$= ring of regular functions on the orbit space $K/K^{(x)}$ ($K^{(x)}$ = isotropy grp of x) = $\{f \in$ holom on $K/K^{(x)} \mid (Kf)$ = linear span of left translates of f is finite dimensional $\} = \sum_{\lambda \in D} R^\lambda(K/K^{(x)})$ where D= set of equivalence classes of finite dimensional irreducible representations of K. $R^\lambda(K/K^{(x)})$= elements of R which transform according to the irreducible representation $V_\lambda : K \longrightarrow \mathrm{Aut}\, V_\lambda$.

We have that dim $R^\lambda(K/K^{(x)}) = \ell_\lambda \cdot \dim V_\lambda$ where ℓ_λ= dim. of fixed point set of $K^{(x)}$ in V^λ =dual module to V_λ under contragredient representation.

But now if $y \in \mathfrak{p}$ is semisimple we know that the K_θ orbit of y = K orbit of y so that $R(K_\theta(y)) = R(K(y))$. But K·y is closed implies that $R(K/K^{(y)}) = S(O_{(y)})$ =restriction of polynomials to the orbit K·y. Thus now if y is both regu~~ä~~lar a~~h~~nd semisimple we have by Theorem 5.3 that H and $R(K/K^{(y)})$ are equivalent K modules. But from above we may assume $y = x^* = e + f \in \mathfrak{a}$ and hence we have $K^{(y)}$ =M. Thus we determine ℓ_λ relative to M. But since M~~$~~ is reductive we know that $\dim V_\lambda^M = \dim (V^\lambda)^M$.

Proposition 6.1 $H = \sum_{\lambda\in D} H^{(\lambda)}$ where the multiplicity of (λ) in H = dimension of the fixed point set of M in V_λ. Then if $S(p) = \sum_{\lambda\in D} S^{(\lambda)}$ is the decomposition of $S(p)$ into primary components we have $S^{(\lambda)}$ is K equivalent to $J \otimes H^{(\lambda)}$.

Now if we let $a_n = \exp_{p_G}(n\cdot x)_{n\in\mathbb{Z}_+}$ (where $\{x, e, f\}$ is the principal TDS above) we consider the isotropy group $K^{a_n(e+f)}$ and it is clear that $\dim V_\lambda^{K^{a_n(e+f)}} = \dim V_\lambda^{K^{(e+f)}} = \dim V_\lambda^{M} = \ell_\lambda$. Then we consider the limit ℓ_λ space $Z_{(\lambda)}$ of the ℓ_λ spaces $V_\lambda^{K^{a_n(e+f)}}$ in the Grassmann manifold of ℓ_λ spaces on V_λ. Then we show that $Z_{(\lambda)}$ is $v_\lambda(x)$ stable and we have that

Proposition 6.2 For fixed $\lambda \in D$ the degrees in which the $H^{(\lambda)}$ irreducible components appear in H are exactly the eigenvalues of $v_\lambda(x)$ in the space $Z_{(\lambda)}$.

Now let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, and $D(\mathfrak{g})$ the symmetric algebra of $\mathfrak{g}$ which is identified to the algebra of constant coefficient differential operators on $\mathfrak{g}$. Then we have the decomposition $D(\mathfrak{g}) = D(\mathfrak{g})\cdot k \oplus D(p)$ where $D(\mathfrak{g})k$ is the ideal in $D(\mathfrak{g})$ generated by k. If $\lambda: D(\mathfrak{g}) \to U(\mathfrak{g})$ is the well known symmetrization map then λ is a G module bijection. If $B: D(\mathfrak{g}) \to S(\mathfrak{g})$ is the linear map induced by the Killing form on $\mathfrak{g}$ ($S(\mathfrak{g}) = p\mathcal{E}_\lambda$ on $\mathfrak{g}$.) then B is also a G module bijection. Then we have that if $\sigma = \lambda \circ B^{-1}$, $U(\mathfrak{g}) = U(\mathfrak{g})k \oplus \sigma(S(\mathfrak{p}))$ where $U(\mathfrak{g})k$ is the right ideal in $U(\mathfrak{g})$ generated by k.

Thus from Proposition 6.1 and Proposition 5.2 we have

<u>Proposition 6.3</u> The linear map $\sigma(J) \otimes \sigma(H) \longrightarrow \sigma(S(\mathfrak{p}))$ defined by $s \otimes t \longrightarrow \xi$ where ξ is the unique element of $\sigma(S(\mathfrak{p}))$ determined by $s \cdot t \equiv \xi \mod U(\mathfrak{g})k$ is a K module bijection. The space $\sigma(H)$ is the linear span of the elements e^k, k=0, 1,....., where e is any nilpotent element in $\mathfrak{p}$.

Thus as in Proposition 6.1 the multiplicity of a fixed K irreducible representation (λ) in $\sigma(H)$= dimension of fixed point set of M in V_λ.

<u>Steinberg</u>: Galois coh. of alg. lin. gp.

K alg closed field
G conn. affine alg. gp $/k$.
$\exists! R$ max. solv + conn. + normal, then
G/R semi-simple
R has comp. series with quotients $\mathbb{G}_m$ and $\mathbb{G}_a$

k perfect, G/k, $K = \bar{k}$ $\Gamma = Gal(K/k)$.

<u>Defn</u>: a <u>cocycle</u> $x_\gamma \in G$ $\gamma \in \Gamma$

(1) $x_\gamma \gamma(x_\delta) = x_{\gamma\delta}$

(2) there exists a finite extension k_1 of k in K so that $x_\gamma = 1$ for $\gamma \in Gal(K/k_1)$

equivalence relation: G acts on cocycles $x_\gamma \cdot a = a^{-1} x_\gamma \gamma(a)$

$H^1(k,G)$ = equivalence classes

Classifies structures $/k$ which ~~becomes~~ isomorphic over $\bar{k}$.

<u>Examples</u>: k arb. $G = Gl_n, Sl_n, Sp_n,$ $H^1(k,G) = 0$.

<u>Thm A</u>: k perfect, G conn $/k$. Assume G contains a Borel (max. conn. solv.) subgp $/k$. Then every element of $H^1(k,B)$ can be reduced to a torus $/k$.

<u>Proof</u>: (1) k finite. Lang: q = card k, σ = Frob, then $\forall x \in G$ $\exists y \in$ $x = y \cdot \sigma(y^{-1})$. $\longrightarrow$ $H^1(k,G) = 0$.

2. k infinite. will assume G s.s., simply-conn.

(a) Rosenlicht density thm: G_k dense in G. (any conn G k inf perf)

Regular elements Any element whose centralizer is a torus.

(b) Regular elements $\supseteq$ non-empty open sets. (semi-simple)

(c) Every regular class defined $/k$ contains an element $/k$. (use B defd $/k$)

Consequences $\dim k \leq 1 \Leftrightarrow Br(k) = 0$. Examples: finite, local with alg closed residue field.

Thm B: k perfect.
If $\dim k \leq 1$ then $H^1(k, G) = 0$ for every conn. lin alg. gp$/k$ and conversely.

Thm. C: Suppose k local field residue field $\dim \leq 1$.
G semi-simple + simply-conn. Then $H^1(k, G) = 0$.
(original to Kneser, new proof to Borel-~~Tits~~).

Thm. D: Suppose k an alg. no. field. G semi-simple s.c. $/k$.

$$H^1(k, G) \longrightarrow \prod_v H^1(k_v, G) \quad \text{is bijective.}$$

only real v occur. by C.

not yet proved for E_8

J. Wolf colloquium

$\chi > 0$ or finite fund. gp.

complex flag manifold = compact homogeneous Kähler manifold

= homogeneous complex projective variety

= semi-simple gp/parabolic subgp.

Example $\quad S^2 = SL(2,\mathbb{C}) / \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

Generalizations

first $\quad G$ complex semi simple

B maximal solv. conn. subgp. $\quad$ Borel subgp

G/B

2nd $\quad$ compact hermitian symmetric space

G compact semi-simple, $\quad \sigma$ auto of order 2

K = fixed point set of σ $\quad G_u/K$

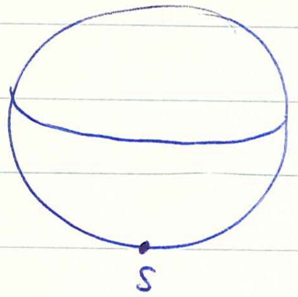
assume $\exists$ complex structure.

G complex ss $\quad P$ complex $\supseteq B$ $\quad$ "parabolic"

$G/P = X$

G_u compact real form of G

Then G_u transitive on X and $\quad X = G_u / \underbrace{G_u \cap P}_{\text{centralizer of torus.}}$



$\Big\}$ disk $= SL(2,\mathbb{R}) \Big/ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

Easy to generalize this for compact hermitian space

$$\mathfrak{g}_u = k + p$$

define $\mathfrak{g}_0 = k + \sqrt{-1}\, p$. Then G_0/K is a non-compact hermitian space. There is standard embedding of G_0/K in compact one. Always have

$$\overbrace{(\text{nonc}) \subset (\text{euclid}) \subset (\text{compact})}$$

This doesn't generalize.

<u>General Case</u>: $X = G/P$ $\quad$ G cx. semi-simple (centerless), P parab.

G_0 a real form of G. $\quad \mathfrak{g} = \mathfrak{g}_0 + \sqrt{-1}\,\mathfrak{g}_0$

$$G \supset G_0$$

(This defn of real form not same as real points.) $\quad G_0$ acts on X

Study orbits of action

$$\begin{array}{ccc} \tilde{X} & = & G/B \\ \downarrow & & \downarrow {\scriptstyle P/B} \\ X & = & G/P \end{array}$$

<u>Easy Case</u> Take $P = B$.

$B = N(B)$ hence 1-1 corres between $gB \leftrightarrow gBg^{-1}$! Hence can think of X as space of all Borel subgps with G acting by conjugation. Let τ be complex conjugation of $\mathfrak{g}$; it extends to centerless gps and ~~which~~ G_0 is connected component of fixed points of τ.

Take $x \in X$ let $B =$ stabilizer of x so $G_0(x) = G_0/G_0 \cap B$.
$G_0 \cap B = G_0 \cap B \cap \tau B = S_x$ Now $\tau\mathfrak{b}$ a Borel subgp
$\Rightarrow \mathfrak{b} \cap \tau\mathfrak{b} \supset$ Cartan subalg $\mathcal{H}$. So

$$\mathfrak{b} = \mathcal{H} + \sum_{\alpha \in \Delta^+} \mathcal{G}_\alpha \qquad \Delta = \text{roots}$$

$$\Delta^+ = \text{pos. roots}$$

$$\tau\mathfrak{b} = \mathcal{H} + \sum_{\alpha \in \tau\Delta^+} \mathcal{G}_\alpha$$

So

$$\mathfrak{b} \cap \tau\mathfrak{b} = \underbrace{\mathcal{H}}_{\text{reductive}} + \underbrace{\sum_{\alpha \in \Delta^+ \cap \tau\Delta^+} \mathcal{G}}_{\text{nilpotent}}$$

L.A of $S_x = \mathfrak{s}_x = \mathfrak{g}_0 \cap \mathfrak{b} \cap \tau\mathfrak{b}$ is a real form of $\mathfrak{b} \cap \tau\mathfrak{b}$

So

$$S_x \text{ has L.A. } \quad \mathcal{G}_0 \cap \Big(\mathcal{H} + \sum_{\Delta^+ \cap \tau\Delta^+} \mathcal{G}_\alpha\Big)$$

$\mathcal{O}$ orbit $\rightsquigarrow \mathfrak{b} \cap \tau\mathfrak{b} \rightsquigarrow$ conjugacy class $\mathcal{H}_0 = \mathcal{H} \cap \mathcal{G}_0$ of $\mathcal{G}_0$.

$|W| =$ number of ways of ordering roots
$\geq$ # orbits for a given conjugacy class of Cartan's in $\mathcal{G}_0$.

$\therefore$ Only a finite no. of orbits

Closed ~~[illegible]~~ orbit $G_0(x)$ compact

$$G_0 = KAN$$

$$G_0(x) = G_0/\underbrace{S_x}_{\text{contains } AN} \Rightarrow K \text{ transitive}$$

Let M = centralizer of A in K

$\Longrightarrow$ (Cartans $\supset A$) = (Cartans of K) $\times A$

$$\therefore\ H_K = A \cdot N \subset S_x.$$

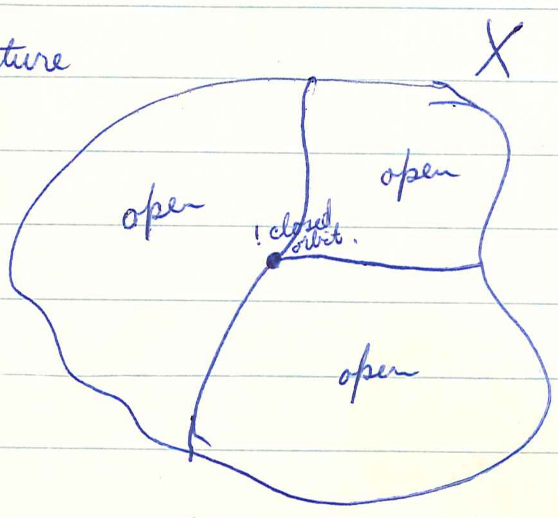
One can show that $S_x = F \cdot H_K A N$

where F elementary 2 abelian. $= M/M_0$.
$H_k A N$ minimal parabolic.

~~Thus~~ Furthermore any two such S_x are conjugate

<u>Theorem</u>: $\exists !$ closed orbit

Picture



Thus $\exists$ open orbit. $\exists$ closed orbit on topological boundary of every orbit.

Suppose rank $G = \ell$, $r = \text{card}\,\Delta^+$, $\Rightarrow \dim_{\mathbb{C}} B = \ell + r \Rightarrow \dim_{\mathbb{C}} G = \ell + 2r$
$\Rightarrow \dim_{\mathbb{C}} X = r. \Rightarrow \dim X = 2r.$ To find open orbit

$$\underbrace{\dim G_0}_{\ell + 2r} - \underbrace{\dim S_x}_{\ell + \text{card}\,\Delta^+ \cap \tau\Delta^+} = 2r$$

$$\Updownarrow$$

$$\Delta^+ \cap \tau\Delta^+ = \phi$$

and hence $\boxed{\tau\Delta^+ = \Delta^- \text{ iff } \exists \text{ open orbit}}$

$$\mathcal{H}_0 = \mathcal{H} \cap \mathcal{G} = \underbrace{\mathcal{H}_T}_{\text{toral}} + \underbrace{\mathcal{H}_V}_{\text{vector}}$$

Roots lie in $\sqrt{-1}\,\mathcal{H}_T + \mathcal{H}_V$. $\tau\Delta^+ = \Delta^- \Rightarrow \sqrt{-1}\,\mathcal{H}_T$ has a regular element $\Rightarrow \mathcal{H}_0$ is as compact as possible.

<u>Example</u>. $SL(n, \mathbb{R}) \supseteq SO(n, \mathbb{R}) \supset T^{[\frac{n}{2}]}$
$\cup$
V^{n-1}

So ~~must~~ complete $T^{[\frac{n}{2}]}$ to a Cartan subalg. ~~to~~

Thus taking a maximally compact $\mathcal{H}_0$ and dividing into chambers and ~~[illegible]~~ forming a suitable quotient due to roots coming from H_T one gets number of open orbits.

boundary components of a

bdded symmetric domain $D = G_0/K$

Γ discrete subgp. of G_0 $\ni$

$$\mu(D/\Gamma) < \infty \quad \text{but} \quad D/\Gamma \ \underline{\text{not}} \text{ compact.}$$

eg. $\odot$.

Then bdry components must be added to obtain compactification

Zelobenko Dokl 7 1403.

February 23, 1968.

Each element $\sigma \in W$ gives us a functor

$$F_\sigma V = V/\mathfrak{n}^\sigma V$$

from $(\mathfrak{g}-k)$ to $(M \times \mathfrak{a})$ and the basic conjecture is that there exist canonical ~~[illegible]~~ natural transformations

$$V/\mathfrak{n} V \otimes \mathfrak{g}^{-1} \longrightarrow V/\mathfrak{n}^\sigma V \otimes \sigma\mathfrak{g}^{-1}.$$

of $M \times \mathfrak{a}$ modules.

Approaches to problem.

(i) Russian

(ii) Bruhat - integration over a Shubert cell

(iii) Kostant

One should work out all of these approaches for $sl(2,R)$.

Method (i): ~~[illegible]~~ The problem is to ~~[illegible]~~ compute

$$\mathrm{Hom}_{\mathfrak{g}}(I\mathfrak{J}_1, I\mathfrak{J}_2) = \mathrm{Hom}_{\mathfrak{g}}(J \underset{M\mathfrak{a}}{\otimes} \mathfrak{J}_1, I(\mathfrak{J}_2))$$

$$= \mathrm{Hom}_{M\mathfrak{a}, M\mathfrak{a}}(1 \underset{\mathfrak{n}}{\otimes} J, \mathrm{Hom}(\mathfrak{J}_1, \mathfrak{J}_2)).$$

Und[illegible]tand class 1 representations.

Recall for a given value of λ we get a homomorphism $C(1,1) = S(\mathfrak{a})^W \longrightarrow \mathbb{C}$. The irreducibility of the ~~induced~~ ~~mod~~ coinduced module $(U(\mathfrak{g}) \otimes_k 1) \otimes_{\mathcal{E}_1} \lambda$ is then equivalent to the non-degeneracy of the pairing

$$\lambda \underset{\mathcal{E}_1}{\otimes} C(1,\Lambda) \otimes C(\Lambda,1) \underset{\mathcal{E}_1}{\otimes} \lambda \longrightarrow \lambda .$$

The point is that $C(\Lambda,1)$ is a free module over $\mathcal{E}_1$ with basis ~~the~~ having dim $\mathrm{Hom}_k(\Lambda, \cancel{U(\mathfrak{g})\otimes_k 1}\, H) = \mathrm{Hom}_M(\Lambda, 1) = \ell(\Lambda)$ elements. ~~Thus one gets so~~ The same is probably true for $C(1,\Lambda)$ so consequently the pairing

$$C(1,\Lambda) \otimes C(\Lambda,1) \longrightarrow C(1,1)$$

is given by an $\ell(\Lambda) \times \ell(\Lambda)$ matrix whose determinant is what we must calculate!

Check this.

$$C(\Lambda,1) = \mathrm{Hom}_k(\Lambda, U(\mathfrak{g}) \otimes_k 1)$$

Proposition:

∃ canonical isomorphism

$$\Phi : \mathcal{E}_1 \otimes \underset{\substack{\| \wr \\ \mathrm{Hom}_{\mathfrak{g}}(\Lambda, 1)}}{\mathrm{Hom}_k(\Lambda, H)} \xrightarrow{\sim} \mathcal{C}(\Lambda, 1)$$

of $\mathcal{E}_1$ modules given by sending $\varphi : \Lambda \xrightarrow[k]{} H$ into

$$\Lambda \xrightarrow{\varphi} H \hookrightarrow S(\mathfrak{p}) \xrightarrow{e} U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{k} \cong U(\mathfrak{g}) \otimes_{\mathfrak{k}} 1$$

Proof: Consider associated graded map.

$$S(\mathfrak{p})^{k} \otimes \mathrm{Hom}_k(\Lambda, H) \longrightarrow \mathrm{Hom}_k(\Lambda, S(\mathfrak{p})) .$$

Clearly an isomorphism by the formula $S(\mathfrak{p}) = S(\mathfrak{p})^{k} \otimes H$.

Why is $\mathcal{C}(1, \Lambda)$ a right free $\mathcal{E}_1$ module?

$$\mathcal{C}(1, \Lambda) = \mathrm{Hom}_k(1, U(\mathfrak{g}) \otimes_k \Lambda).$$

$S(\mathfrak{p}) \otimes \Lambda$.

Thus take $\mathrm{Hom}_k(1, H \otimes \Lambda) = \mathrm{Hom}_k(\Lambda', H)$. So if $\varphi : \Lambda' \xrightarrow[k]{} H$ then get an inv. in $H \otimes \Lambda$ hence one in $U(\mathfrak{g}) \otimes_k \Lambda$. and its OKAY

Proposition: $\exists$ canonical right $\mathcal{E}_1$ module isom.

$$\mathrm{Hom}_k(\Lambda', H) \otimes \mathcal{E}_1 \xrightarrow{\sim} \mathcal{C}(1,\Lambda)$$

where $\varphi \in \Lambda' \to H$ goes into the element

$$1 \longrightarrow \sum_i e(\varphi\hat{\lambda}_i) \otimes \lambda_i \quad \in \cancel{e}S(p)\otimes\Lambda = U(g)\otimes_k \Lambda .$$

where λ_i is a basis for Λ and $\hat{\lambda}_i$ is the dual basis.

Proof: $\varphi \otimes a \longmapsto a \cdot \sum e(\varphi\hat{\lambda}_i) \cdot \lambda_i$ $\qquad a \in U(g)^k$, $\varphi \in \mathrm{Hom}_k(\Lambda', H)$.

Passing to associated graded we have to show

$$\mathrm{Hom}_k(\Lambda', H) \otimes S(p)^k \longrightarrow \mathrm{Hom}_k(1, S(p))$$

is an isomorphism, but this is clear by $S(p) = S(p)^k \otimes H$.

So I now have to calculate comp. $\mathcal{E}(1,\Lambda)\otimes\mathcal{E}(\Lambda,1) \to \mathcal{E}(1,1)$
So take $\varphi : \Lambda \to H$ and $\psi : \Lambda' \to H$, then if $\tilde{\varphi} \in \mathcal{E}(\Lambda,1)$ and $\tilde{\psi} \in \mathcal{E}(1,\Lambda)$ are the associated elements by the above propositions, we have

$$(\tilde{\varphi} \circ \tilde{\psi})(1) = \tilde{\varphi}\left(\sum_i e(\psi\hat{\lambda}_i)\lambda_i\right) = \sum_i e(\psi\hat{\lambda}_i)\, e(\varphi\lambda_i) \otimes 1$$

which verifies Bert's formula.

Next stage is to apply the functor F.

$$\begin{tikzcd}
\mathcal{C}(\Lambda,1)\otimes\mathcal{C}(1,\Lambda) \arrow[r] \arrow[d, "F\otimes F"] & \mathcal{C}(1,1) \arrow[d, "F"] \\
{[U(\alpha)\otimes \mathrm{Hom}_M(\Lambda,1)]\otimes[U(\alpha)\otimes\mathrm{Hom}_M(1,\Lambda)]} \arrow[r] & {[U(\alpha)\otimes\mathrm{Hom}_M(1,1)]}
\end{tikzcd}$$

Therefore $F\mathcal{C}(\Lambda,1)\subset U(\alpha)\otimes\mathrm{Hom}_M(\Lambda,1)$ is free of rank $\ell(\Lambda)$ over $F\mathcal{E}_1\simeq U(\alpha)^W$

ditto for $F\mathcal{C}(1,\Lambda)$.

If you knew what the image of each generator is you would be done! Thus can you determine the image of the map

$$\mathrm{Hom}_k(\Lambda,H)\longrightarrow\mathcal{C}(\Lambda,1)\xrightarrow{F}U(\alpha)\otimes\mathrm{Hom}_M(\Lambda,1)$$

i.e.

$$\begin{tikzcd}
\mathrm{Hom}_k(\Lambda,H) \arrow[r] \arrow[d, "\text{eval at }\xi?"] & \mathcal{C}(\Lambda,1) \arrow[d, "F"] \\
\mathrm{Hom}_M(\Lambda,1) \arrow[r, "?"] & U(\alpha)\otimes\mathrm{Hom}_M(\Lambda,1)
\end{tikzcd}$$

Thus you get ~~$\ell(\Lambda)$ elements in $U(\alpha)\otimes\mathrm{Hom}_M(\Lambda,1)$~~ an $\ell(\Lambda)\times\ell(\Lambda)$ matrix in $U(\alpha)$; similarly from $\mathcal{C}(1,\Lambda)$ you get an $\ell(\Lambda)\times\ell(\Lambda)$ matrix. The product of these two matrices is what you are after.

February 24, 1968.

Two operations: $\varphi(kan) = \nu(a)$.

1. If φ is a function on G $\ni$ ~~$\varphi(gb^{-1}) = \varphi(g)$ for~~ ~~be~~

$$\hat{\varphi}(g) = \int \varphi(gk)\,dk \qquad ?$$

2. $f(kgk^{-1}) = f(g)$

$$F_f(a) = e^{\rho(\log a)} \int_N f(an)\,dn$$

$$\int_K \underbrace{e^{\nu(H(xk))}}\,dk$$

where $xk = \underline{k_x} \cdot H(x,k) \cdot N(x,k)$

$$\therefore \quad \varphi(gk) = \varphi(H(xk)).$$

~~Then~~ Better: If ν function on ~~$\mathfrak{a}$~~ $\mathfrak{a}$ set

$$\varphi_\nu(x) = \int_K \underbrace{e^{(\nu-\rho)(H(xk))}}_{\varphi(x)}\,dk$$

then
$$\varphi(kx) = \varphi(x)$$
$$\varphi(xn) = \varphi(x).$$

hoping ~~to make~~ φ to be a section of the induced repn.

Take a section S of the principal series associated to an element $\lambda \in \mathfrak{a}'$. Then S is same as a function

$$S : G \longrightarrow \mathbb{C}$$

with
$$S(g(man)^{-1}) = e^{\lambda(\log a)} S(g).$$

What are spherical functions?

Let V be a class 1 representation with k fixed vector $v_0 \neq 0$. Then by decomposing over k we may define the projection onto $\mathbb{C}v_0$ operator E. If $x \in G$ or $U(\mathfrak{g})$ then one can consider

$$tr\{E\pi(x)E\} = \varphi(x)$$

which is a scalar. Clearly

$$\varphi(k_1 x k_2) = tr\,(\pi(k_1)E\pi(x)E\pi(k_2))$$
$$= tr\, E\pi(x)E = \varphi(x)$$

so φ is ~~…~~ biinvariant for K. Also if $z \in Z$, then

If $x \in U(\mathfrak{g})$ set

$$\varphi(x) = E\pi(x)E \in \mathrm{Hom}(\mathbb{C}v_0, \mathbb{C}v_0).$$

Then $\varphi : U(\mathfrak{g}) \longrightarrow \mathbb{C}$ is linear. If ~~…~~ $y \in U(\mathfrak{g})^k$ then $E\pi(y) = \pi(y)E$ so

$$\varphi(xy) = E\,\pi(x)\pi(y)E$$
$$= E\,\pi(x)E\,\pi(y)$$
$$= \varphi(x)\cdot\varphi(y)$$

Similarly $\boxed{\varphi(yx) = \varphi(y)\varphi(x).}$

If $y \in U(k)$, then $E\pi(y) = \pi(y)E$. Moreover $\varphi(y) = \varepsilon(y)$

__Lemma:__ Let V be a $\mathfrak{g}$-k module and let Ω be a ~~class~~ set of isomorphism classes of finite simple k modules. Then $V = V^{\Omega} \oplus V^{-\Omega}$ where for all $\Lambda \in \Omega$ we have $\mathrm{Hom}_k(\Lambda, V) = \mathrm{Hom}_k(\Lambda, V^{\Omega})$ and for all $\Lambda \notin \Omega$ we have $\mathrm{Hom}_k(\Lambda, V) = \mathrm{Hom}_k(\Lambda, V^{-\Omega})$. V^{Ω} is stable under the ring $U(\mathfrak{g})^k \cdot U(k) \subset U(\mathfrak{g})$, hence the projection operator E_Ω onto V^{Ω} belongs to $\mathrm{Hom}_{U(\mathfrak{g})^k \cdot U(k)}(V, V)$.
~~Finally there is a canonical isomorphism~~

$$\cancel{U(\mathfrak{g})^k \otimes_{Z(k)} U(k) \longrightarrow U(\mathfrak{g})^k \cdot U(k).}$$

__Conjectures:__ $U(\mathfrak{g})^k \otimes_{Z(k)} U(k) \xrightarrow{\sim} U(\mathfrak{g})^k \cdot U(k)$. ?

$\sum a_i h_i = 0$ $\quad a_i \in U(\mathfrak{g})^k$ $\quad h_i$ harmonic ind in $U(k)$.

Assume now that V comes from a Banach space representation of the group G, so that now $\varphi(x)$ is defined for all distributions on G of compact support. ~~[illegible]~~
Does $\exists$ any relation between φ as a function on G and φ as a linear function on $\mathcal{U}(\mathfrak{g})$, ~~[illegible]~~ ? Yes if you are given a function f on the group ~~on interprets is~~ one has

$$\langle f, T\rangle = \int f T \, d\,\mathrm{Haar}$$

which defines $f(T)$ for $T \in C_0(G)$ and hence by limits for all $\mathcal{D}_c(G)$. ~~It is important to note that if $Z \in \mathcal{U}(\mathfrak{g})$ then~~ We have seen that

$$f(xy) = f(x) \cdot f(y) \quad \text{if one of } x \text{ or } y \text{ commutes with } E.$$

In particular if $g \in G$ and $x \in \mathcal{U}(\mathfrak{g})^k$

$$\cancel{(x * f)(g) = f(x g)}$$

$$(g * f)(x) = f(xg) = f(x) \cdot f(g)$$

Recall interpretation of $x \in \mathcal{U}(\mathfrak{g})$ as a **left** inv. DO. ie.

$$(x * f)(g) = f(gx) = f(g x g^{-1} \cdot g)$$

?

Problem: You are given the algebra $\mathcal{D}$ of distributions with compact support which you think of as the group ring of the group G. A function f on the group ~~is therefore~~ therefore gives rise to a linear function on $\mathcal{D}$. Thus we can speak of $f(x)$ where $x \in \mathcal{D}$. Now define

not the left inv. of realization

$$(g*f)(g*x) = f(x)$$

ie
$$(g*f)(x) = f(g^{-1}*x).$$
Taking linear combinations we get

$$(y*f)(x) = f(\check{y}*x)$$

$\check{y}$ = ~~transpose~~ antipode of y.

Returning to the f defined by

$$f(x) = E\,\pi(x)\,E.$$

Then
$$(y*f)(g) = f(\check{y}g) = f(\check{y})\,f(g) \qquad \text{if } y \in U(g)^k.$$

In other words

$$\boxed{yf = f(\check{y})\cdot f \qquad y \in \underline{U(g)^k U(k)}}$$

which shows among other things that the function f we have defined is an eigenfunction for the invariant D.O.'s.

Conclusion: If V is ~~irreducible~~ class 1 representation of G, then the spherical function

$$f(x) = E\,\pi(x)\,E$$

E = proj. on k inv.

$\pi(x)$ = action of $x \in G$ on V

is a K-biinvariant function on G. which is an eigenfunction for the operators of $U(\mathfrak{g})^k$.

spherical function = K inv. fn. on G/K which is an eigenfunction for the G-invariant D.O. on G/K.

Back to Harish-Chandra transforms.

~~[illegible]~~ Basic formula

$$\int_G \tilde{\varphi}(x)\, f(x)\, dx = \int_A \varphi(a)\, \hat{f}(a)\, da$$

$$\parallel$$

$$\int_G dx \int_K dk\ \varphi(xk)\, f(x)$$

$$\parallel \quad f \text{ biinv.}$$

$$\int_K dk \int_G \varphi(xk)\, f(xk)\, dx$$

$$\parallel$$

$$\int_G \varphi(x)\, f(x)\, dx = \int_{KAN} \varphi(kan)\, f(kan)\, e^{2\rho(\log a)}\, dk\, da\, dn$$

$\varphi(kg) = \varphi(g)$

$$= \int_{AN} \varphi(a) f(an)\, e^{2\rho \log}\, da\, dn$$

$$= \int_A \varphi(a)\, e^{2\rho(\log a)} \int_N f(an)\, dn \quad =$$

Thus

$$\int_G \left(\int_K \varphi(xk)\, dk\right) f(x)\, dx = \int_A \varphi(a) \left(e^{2\rho(\log a)} \cdot \int_N f(an)\, dn\right) da$$

where

$$\varphi(kxn) = \varphi(x)$$

$$f(k_1 x k_2) = f(x).$$

<u>Summary</u>:

1.) spherical function = K-invariant function on G/K which is an eigenfunction for the invariant D.O.'s on G/K. (necessarily analytic)

2.) If V ~~repn. of~~ a Banach repn. of V with V^K of dim 1, then $\varphi(x) = E\pi(x)E$ is a spherical function.

3.) The spherical functions are given by $\varphi_\lambda(x) = \int_K e^{(\lambda-\rho)(H(xk))} dk$ where $\lambda \in \mathfrak{a}'$ and $\varphi_\lambda = \varphi_\mu \Leftrightarrow \lambda = \mu^s$ for some $s \in W$.

<u>basic idea:</u> A spherical fn φ defines a hom. $\mathcal{E}_1 \to \mathbb{C}$, because it is ~~the~~ an eigenfunction. On the other hand a homomorphism $\mathcal{E}_1 \to \mathbb{C}$ is an irred $\mathcal{E}_1$ module and it determines an irred invariant Diff equation on the trivial bundle on G/K. Thus the <u>problem</u> is: Can you relate spherical functions to solutions of differential eqns on G/K?

Relation diagram

$$\begin{array}{ccc}
(\text{inv D.E. on bundle } G\times_K \Lambda' \to G/K) & & \\
\| & & (\text{solutions of D.E.}) \\
(\text{surjection } U(\mathfrak{g})\otimes_k \Lambda \longrightarrow M) & \nearrow & \\
\downarrow & & \\
(\mathcal{E}^0_\Lambda \text{ module } \mathrm{Hom}_k(\Lambda, M)) & &
\end{array}$$

Recall that ~~if~~ a section of the bundle $G\times_K \Lambda' \to G/B$ may be identified with a map $G \xrightarrow{s} \Lambda'$ which is K equivariant. ie.

$$\Gamma(\mathcal{U}, G\times_K \Lambda') = \mathrm{Hom}_k(\Lambda, C^\infty(\pi^{-1}\mathcal{U}))$$

$\pi: G \to G/K$ canon. proj.

~~i.e. if $\varphi: \Lambda \to C^\infty(\pi^{-1}\mathcal{U})$ then $s(\varphi)(g)(\lambda) = \varphi(\lambda)(g)$.~~

~~Check~~

~~[illegible]~~

~~$[s(\varphi)(kg)](\lambda) = \varphi(\lambda)(kg) = [k\cdot\varphi(\lambda)](g) = \varphi$~~

To $\varphi : \Lambda \longrightarrow C^\infty(\pi^{-1}U)$ we associate ~~$s(u)$~~ $u \mapsto s(u)$ ~~where~~ given by $s(u)(\lambda) = \varphi(\lambda)(u^{-1})$. Then

$$\cancel{[s(ku)](\lambda) = \varphi(\lambda)(u^{-1}k^{-1}) = [k^{-1}\cdot\varphi(\lambda)](u^{-1})}$$

$$\cancel{= \varphi(k^{-1}\lambda)}$$

$$s(uk^{-1}) \overset{?}{=} k\cdot s(u)$$

ie
$$[s(uk^{-1})](\lambda) \overset{?}{=} [k\cdot s(u)](\lambda)$$

$$\varphi(\lambda)(ku^{-1}) \qquad s(u)(k^{-1}\lambda) = \varphi(k^{-1}\lambda)(u^{-1}) =$$

$$\Gamma(U, G\times_K \Lambda') = \mathrm{Hom}_K(\Lambda, C^\infty(\pi^{-1}U))$$

~~where if $f \in C^\infty(\pi^{-1}U)$ we define~~

$$(k\cdot f)(u) = f(uk)$$

<u>Proof:</u> $\varphi : \Lambda \longrightarrow C^\infty(\pi^{-1}U)$ defines a function

$$s : \pi^{-1}U \longrightarrow \Lambda' \quad \text{with} \quad s(uk) = k^{-1}s(u)$$

by
$$s(u)(\lambda) = \varphi(\lambda)(u)$$

<u>Check:</u>
$$s(uk)(\lambda) \overset{?}{=} [k^{-1}s(u)](\lambda) = s(u)(k\lambda) = \varphi(k\lambda)(u)$$

$$s(uk)(\lambda) = \varphi(\lambda)(uk)$$

$$\varphi(k\lambda)(u) = [k\cdot\varphi(\lambda)](u) = \varphi(\lambda)(uk)$$

✓

$$\boxed{\begin{aligned}\Gamma(G/K, G\times_K \Lambda') &= \mathrm{Hom}_K(\Lambda, C^\infty(G)) \\ &\text{where } (kf)(g) = f(gk).\end{aligned}}$$

The Solutions of my invariant D.E. given by M are

$$\boxed{\Gamma(G/K, \mathcal{S}_M) = \mathrm{Hom}_{\mathfrak{g}}(M, C^\infty(G))}$$

Now suppose you give yourself

$$M = (\mathcal{U}(\mathfrak{g}) \otimes_k \Lambda) \otimes_{\mathcal{E}_\Lambda} \xi$$

i.e.

$$\begin{aligned}\Gamma(G/K, \mathcal{S}_M) &= \mathrm{Hom}_{\mathfrak{g}}((\mathcal{U}(\mathfrak{g}) \otimes_k \Lambda) \otimes_{\mathcal{E}_\Lambda} \xi, C^\infty(G)) \\ &= \mathrm{Hom}_{\mathcal{E}_\Lambda}(\xi, \mathrm{Hom}_{\mathfrak{g}}(\mathcal{U}(\mathfrak{g}) \otimes_k \Lambda, C^\infty(G))) \\ &= \mathrm{Hom}_{\mathcal{E}_\Lambda}(\xi, \Gamma(G/K, G\times_K \Lambda')).\end{aligned}$$

If $\Lambda = 1$ and $\xi = \chi : \mathcal{E}_1 \to \mathbb{C}$, then

$$\Gamma(G/K, \mathcal{S}_M) = \mathrm{Hom}_{\mathcal{E}_1}(\chi, \Gamma(G/K, 1)) = \{ f \text{ smooth on } G/K \ni Df = \chi(D) f \quad \forall D \in \mathcal{E}_1 \}$$

Therefore irreducibility means:

Proposition: Let $\chi : \mathcal{E}_1 \to \mathbb{C}$ be a homomorphism. Then the module $(U(\mathfrak{g}) \otimes_k \mathbb{1}) \otimes_{\mathcal{E}_1} \chi$ is simple $\iff$ given any function f on G/K with $Df = \chi(D)f$ for all $D \in \mathcal{E}_1$, there is a finite linear combination of G-translates of f whose K-average is the spherical function associated to χ.

NO

This would be true in the unitary case.

Proposition: Let $\chi : \mathcal{E}_1 \to \mathbb{C}$ be a homomorphism. Then the module $(U(\mathfrak{g}) \otimes_k 1) \otimes_{\mathcal{E}_1} \chi$ is simple $\iff$ every function

? f on G/K which is an eigenfunction for $\mathcal{E}_1$ with eigenvalue χ is the limit of linear combinations of G translates of the spherical function φ_χ.

Proof: Let $M = (U(\mathfrak{g}) \otimes_k 1) \otimes_{\mathcal{E}_1} \chi$. Then the solutions S_M of the global invariant D.E. on functions on G/K defined by M are the functions f on G/K with $Df = \chi(D) \cdot f$ for all $D \in \mathcal{E}_1$. Let V be the closed invariant subspace of the solution space S_M generated by the spherical function φ_χ. ~~Note that S_M is~~ We want to conclude that $0 < V < S_M \Rightarrow$ $\exists$ diff operator $\mu : G \times_K 1 \to G \times_K 1$ non-zero on S_M but 0 on V

$\Rightarrow$ M ~~the~~ reducible. Also we want to have that if M'' is a proper non-zero quotient module of M, then M and M'' have enough solutions so that V must consist of solutions of M'' and therefore be different from S_M.

However all this is probably OKAY because we throughout the whole discussion restrict attention to K-finite things. ~~We can produce lots of solutions by duality. Thus given $0 \to M' \to M \to M'' \to 0$ with $M''M' \not\equiv 0$ let $\varphi : U(\mathfrak{g}) \otimes_K \Lambda \to M'$ and choose θ~~

~~Then get differential operator~~

$$\cancel{\theta : G \times_K \Lambda \to G \times_K \Lambda'}$$

~~which annihilates the solutions of M' but is non-zero on M.~~

Claim that any D.E. has lots of global K finite solutions e.g.

$$\mathrm{Hom}_K(\Lambda, C^\infty(G/K)) = \text{sections of } G \times_K \Lambda'$$

Suppose you give $\Lambda_1 \longrightarrow \Gamma(G \times_K \Lambda')$ K equivariant

Idea being that $(U(\mathfrak{g}) \otimes_K 1)' =$

k finite-Hom$(U(\mathfrak{g}) \otimes_K 1, \mathbb{C})$

k-finite $-\operatorname{Hom}(\mathcal{U}(\mathfrak{g})\otimes_k 1, \mathbb{C})$
this gives the exponentials of the polynomial fns on $\mathfrak{p}$. which is not stable under G-translation.

It would seem to follow by $\int$ that the k-finite solutions are always dense.

?

Conjecture: There should be some way of decomposing the functions on G/K into pieces transforming by a K repn. Λ and a eigencharacter χ of $\mathcal{E}_\Lambda$. Thus the K invariant functions can be decomposed into spherical functions

---

To generalize the notion of spherical function.

~~1) Let ξ be a finite dimensional $\mathcal{E}_\Lambda$ module. We wish to consider~~

2) Let V be a Banach repn. of G such that $\xi = \operatorname{Hom}_K(\Lambda, V)$ is a simple right $\mathcal{E}_\Lambda$ mod. Let E be the projection of V onto V^Λ. ~~Then if $x \in G$ and if $v \in \xi$ get~~

$$\boxed{E\pi(x)E : G \longrightarrow \operatorname{Hom}(V^\Lambda, V^\Lambda).}$$

~~old idea was to~~ this makes sense for any $x \in \mathcal{D}_c$

$\operatorname{Hom}_K(\Lambda, V$

$\boxed{\operatorname{trace} E\pi(x)E}$? Godement

$$\varphi(x) = E\,\pi(x)\,E$$

$$\varphi(k_1 x k_2) = \sigma(k_1)\,\pi(x)\,\sigma(k_2)$$

First of all a spherical fn. should be assoc. to an irred $\mathcal{E}_\Lambda$ module ξ. $\mathcal{E}_\Lambda$ operates on sections of bundle $G \times_K \Lambda'$. So I can speak of eigenfunctions of type ξ ie
$$\mathrm{Hom}_{\mathcal{E}_\Lambda}(\xi, \Gamma(G \times_K \Lambda'))$$
$$\|$$
$$\mathrm{Hom}_{\mathfrak{g}}(M, C^\infty(G))$$

So this is our notion of ~~...~~ eigenfunction. Among the solutions of the equation are those transforming with certain K representations. Certainly we should ~~...~~ require it to transform under k in some way.

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Ideas for further work.

1. Zhelobenko's idea of constructing the symmetry operators for simple roots using transitivity of induction.

2. Harish-Chandra's generic irreducibility thm. - this probably amounts to a determination of the category tensored with the quotient field of $S(\mathfrak{a})^W$.

3. Relation between differential equations on G/K and the induced modules $I(\zeta)$. Here's a typical case. Suppose we have that $\nu = 0$ and that the map $(U(\mathfrak{g})\otimes_k 1)\otimes_{\mathcal{E}_1} \xi \longrightarrow I(\zeta)$ $(\xi = \mathrm{Hom}_k(1, \zeta) = \lambda.)$ is an isomorphism, in other words the cyclic submodule of $I(\zeta)$ generated by the k-invariants is of full multiplicity. Let $M = (U(\mathfrak{g})\otimes_k 1)\otimes_{\mathcal{E}_1} \xi$. We know that M defines an invariant DE on G/K, whose solutions over an open set U are

$$\mathrm{Hom}_{\mathfrak{g}}(M, C^\infty(\pi^{-1}U)) = \mathrm{Hom}_{\mathcal{E}_1}(\xi, \Gamma(U, G\times_k 1))$$

i.e. the functions on U which are eigenfunctions for $\mathcal{E}_1$ with eigencharacter ξ. But in the set of global solutions there is a distinguished one, namely the unique K invariant one - i.e. the spherical function. Now $\mathfrak{g}$ acts on the solutions so we obtain a cyclic $\mathfrak{g}$ module with a k-invariant with eigencharacter λ. Thus get map $M \longrightarrow \mathrm{Hom}_{\mathfrak{g}}(M, C^\infty(G))$ which is probably injective $\Leftrightarrow$ M irreducible. In any case we now have two maps

$$\begin{tikzcd} M \arrow[r] \arrow[d] & I(\xi) \\ \mathrm{Hom}_{\mathfrak{g}}(M, C^\infty(G)) & \end{tikzcd}$$

and we can try to construct a transform among the ends.

4. One knows that the center Z acts on $C(\Lambda_1, \Lambda_2)$ and that we can decompose the simple modules with respect to their eigenvalues over Z. Given λ, ν you can reconstruct the eigenvalues on Z

Can decompose your simple modules over W orbits of ξ's in such a way that

<u>Problem:</u> For $sl(2,\mathbb{R})$ you have

$$I(\nu) = \sum (\delta_\sigma) \qquad e^{2\pi i\sigma} = \nu$$

$$\begin{cases} X\delta_\sigma(\lambda) = \frac{1}{\sqrt{2}}(\lambda+\sigma)\,\delta_{\sigma+1}(\lambda) \\ Y\delta_\sigma(\lambda) = \frac{1}{\sqrt{2}}(\lambda-\sigma)\,\delta_{\sigma-1}(\lambda) \\ H\delta_\sigma = \sigma\,\delta_\sigma \end{cases}$$

<u>Check:</u>

$$C = \tfrac{1}{2}(H^2 + XY + YX)$$

$$= \tfrac{1}{2}(H^2 + H + 2YX)$$

$$2C\delta_\sigma = \left(\sigma^2 + \sigma + 2\tfrac{1}{\sqrt{2}}\tfrac{1}{\sqrt{2}}(\lambda+\sigma)(\lambda-\sigma-1)\right)\delta_\sigma$$

$$C\delta_\sigma = \tfrac{1}{2}\left(\cancel{\sigma^2} + \cancel{\sigma} + \lambda^2 - \cancel{\sigma^2} - \lambda - \cancel{\sigma}\right)\delta_\sigma$$

$$= \tfrac{1}{2}(\lambda^2 - \lambda)\,\delta_\sigma$$

$\therefore$ eigenvalue of $C = \frac{1}{2}\left[\left(\lambda - \frac{1}{2}\right)^2 - \frac{1}{4}\right]$

Therefore define

$$\Phi\{\delta_\sigma(\lambda)\} = \frac{a(\sigma)}{b(\sigma)}\,\delta_\sigma(1-\lambda)$$

$$\Phi\{X\delta_\sigma(\lambda)\} = X \cdot \Phi(\delta_\sigma(\lambda))$$

$$\cancel{\frac{a(\sigma)}{b(\sigma)}} \qquad \frac{a(\sigma)}{b(\sigma)} \times \delta_\sigma(1-\lambda)$$

$$\Phi\left\{\frac{1}{\sqrt{2}}(\lambda+\sigma)\,\delta_{\sigma+1}(\lambda)\right\}$$

$$\parallel$$

$$\frac{1}{\sqrt{2}}(\lambda+\sigma)\;\frac{a(\sigma+1)}{\cancel{b(\sigma+1)}}\,\delta_{\lambda+1}(1-\lambda) = \frac{a(\sigma)}{\cancel{b(\sigma)}}(1-\lambda+\sigma)\,\delta_{\sigma+1}(1-\lambda)$$

$$b(\sigma) = 1$$

$$\boxed{\frac{a(\sigma+1)}{a(\sigma)} = \frac{1-\lambda+\sigma}{\lambda+\sigma}} =$$

$$\frac{a(\sigma)}{a(\sigma+1)} = \frac{\sigma+\lambda}{\sigma}$$

<u>Problem</u>: Represent $a(\sigma)$ in a nice form.

$$\frac{a(\sigma,\lambda)}{a(\sigma+1,\lambda)} = \frac{\lambda+\sigma}{1-\lambda+\sigma}$$

$$\frac{a(\sigma,1-\lambda)}{a(\sigma+1,1-\lambda)} = \frac{1-\lambda+\sigma}{\lambda+\sigma} = \frac{a(\sigma+1,\lambda)}{a(\sigma,\lambda)}$$

$$\therefore\quad a(\sigma,1-\lambda)\,a(\sigma,\lambda) = a(\sigma+1,1-\lambda)\,a(\sigma+1,\lambda)$$

$$\therefore \quad a(\sigma,\lambda)\, a(\sigma, 1-\lambda) = f(\lambda, e^{2\pi i\sigma}) = f(\lambda, \nu)$$

$$\frac{a(\sigma,\lambda)}{a(\sigma+1,\lambda)} = \frac{\lambda+\sigma}{1-\lambda+\sigma}$$

Try quotient of two Γ function. Recall

$$\Gamma(s+1) = s\,\Gamma(s). \qquad \Gamma \text{ meromorphic.}$$

$$\therefore \quad \boxed{a(\sigma,\lambda) = \frac{\Gamma(\sigma+1-\lambda)}{\Gamma(\sigma+\lambda)}}$$

Check

$$\frac{a(\sigma,\lambda)}{a(\sigma+1,\lambda)} = \frac{\Gamma(\sigma+1-\lambda)}{\Gamma(\sigma+\lambda)} \Big/ \frac{\Gamma(\sigma+1+1-\lambda)}{\Gamma(\sigma+1+\lambda)}$$

$$= \frac{\sigma+\lambda}{\sigma+1-\lambda} \quad \checkmark$$

Also

$$a(\sigma,\lambda)\, a(\sigma, 1-\lambda) = \frac{\cancel{\Gamma(\sigma+1-\lambda)}}{\Gamma(\sigma+\lambda)} \; \frac{\Gamma(\sigma+\lambda)}{\cancel{\Gamma(\sigma+1-\lambda)}} = 1.$$

<u>Proposition</u>: Let $E \to X$ be ~~an~~ oriented vector bundle over a smooth manifold X. Then there is a class $\alpha \in H_c^*(E)$ such that

$$\text{Index } f^{-1}X = \int_M \mathcal{L}(\tau_M)\cdot f^*\alpha$$

for any map $f: M \to E$ transversal to 0 section where M is compact oriented and smooth.

<u>Proof</u>: Let $U \in H_c(E)$ be a Thom class for E so that for any $\beta \in H(X)$ we have

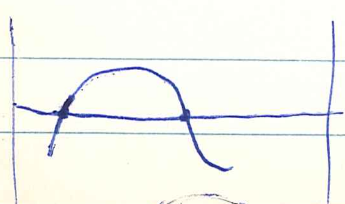
$$0_*\beta = U\cdot \pi^*\beta$$

where $0: X \to E$ is the zero-section embedding. As f is transversal to 0-section f^*U is a Thom class for $f^{-1}X$ in M. Thus if $j: f^{-1}X \to M$ is the inclusion

$$j_*(\gamma) = f^*U\cdot p^*\gamma$$

where p is a retraction of support f^*U onto $f^{-1}X$. Thus

$$\text{Index } f^{-1}\{X\} = \int_{f^{-1}X} \mathcal{L}(\tau_{f^{-1}X}) = \int_{f^{-1}X} j^*\mathcal{L}(\tau_M)\cdot \tilde{f}^*\mathcal{L}(E)^{-1}$$



$$= \int_M \mathcal{L}(\tau_M)\, f^*(U)\cdot p^*\tilde{f}^*\mathcal{L}(E)^{-1}$$

$$\begin{tikzcd} f^{-1}X \arrow[r, hook, "j"] \arrow[d, "\tilde{f}"] & M \arrow[d, "f"] \arrow[l, bend right, "p"'] \\ X \arrow[r, hook, "0"] & E \arrow[l, bend left, "\pi"] \end{tikzcd}$$

$$= \int_M \mathcal{L}(\tau_M)\, f^*\{U\cdot \pi^*\mathcal{L}(E)^{-1}\}.$$