Conjecture: $U(\mathfrak{g})\otimes_{\mathfrak{b}}\lambda$ irreducible $\Longleftrightarrow$ For no $\alpha\in\Sigma$ is $2\frac{\lambda(H_\alpha)}{\alpha(H_\alpha)}$ an integer ≥ 0.

($\Longrightarrow$) Suppose $2\frac{\lambda(H_\alpha)}{\alpha(H_\alpha)} = \ell$ integer ≥ 0. ~~and let~~

~~$\tilde{\mathfrak{b}} = \mathfrak{b} + (Y_\alpha)$. Then $\tilde{\mathfrak{b}}$ is a subalgebra of $\mathfrak{g}$~~ ? ~~and we are going to define a finite dimensional representation of $\tilde{\mathfrak{b}}$. Note that $\tilde{\mathfrak{b}}$ = semi-direct product of $(Y_\alpha, H_\alpha, X_\alpha)$ and the~~

first suppose $\ell = 0$, and let $\mathfrak{g}_1$ be the centralizer of λ as an element of $\mathfrak{h}'$ ie

$$\mathfrak{g}_1 = \{x\in\mathfrak{g} \mid \lambda([x,y]) = 0 \text{ all } y\in\mathfrak{g}\}.$$

then
$$\mathfrak{g}_1 \supset \mathfrak{h} + (X_\alpha) + (Y_\alpha)$$

In fact the centralizer of λ is ~~the set of all~~ spanned by those root vectors $\alpha \ni \lambda(H_\alpha) = 0$.

Be careful – this is the first non-abelian ve situation
Suppose we study $sl(3)$. rank 2.

$$\begin{bmatrix} d_1 & X_1 & X_\alpha \\ Y_1 & d_2 & X_2 \\ Y_\alpha & Y_2 & d_3 \end{bmatrix}$$

$$\left[\begin{bmatrix} d_1 & \\ & d_2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right] = (d_1 - d_2)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

roots

$$\alpha_1(d) = d_1 - d_2$$

$$\alpha_2(d) = d_2 - d_3$$

$$\alpha(d) = d_1 - d_3 \qquad \alpha = \alpha_1 + \alpha_2$$

~~assume~~

Killing form ~~$\mathrm{tr}(X, Y)$~~

$$\mathrm{tr}(\mathrm{ad}\, d)^2 = \sum_{i,j} (d_i - d_j)^2 = \sum_{\underline{i,j}} (d_i^2 + d_j^2 - 2 d_i d_j)$$

$$\langle X, Y \rangle = 2n\, \mathrm{tr}(XY)$$

$$\underline{X_\alpha, Y} \qquad \boxed{\begin{array}{c} X_1, X_2 \\ X_1 + X_2 \end{array}}$$

~~$[X_i, X_j]$~~ $\langle X_i, X_j \rangle$

Feb 13.

Checks last nite's calculations. Use standard base $X_\alpha, Y_\alpha, H_\alpha$
$\alpha(H_\alpha)=2$

Theorem: Suppose that $\lambda(H_\alpha)$ is a integer ≥ 0 where α is a simple positive root. Then $U(\mathfrak{g})\otimes_{\mathfrak{b}}\lambda$ is reducible.

Proof. (i) If $\lambda(H_\alpha)=\ell$ then the element $(Y_\alpha)^{\ell+1}\otimes 1$ in $U(\mathfrak{g})\otimes_{\mathfrak{b}}\lambda$ is a dominant weight vector. e.g.

$$[X_i, Y_\alpha^{\ell+1}]=0 \qquad \text{if} \quad i\neq\alpha \qquad \text{because } \alpha \text{ simple}$$

$$[X_\alpha, Y_\alpha^{\ell+1}] = H_\alpha Y_\alpha^{\ell} + Y H_\alpha Y_\alpha^{\ell-1} + \cdots \qquad Y_\alpha^{\ell} H_\alpha$$

$H_\alpha Y_\alpha = Y_\alpha(H-2)$

$$= Y_\alpha^{\ell}\left(H_\alpha + (H_\alpha-2) + \cdots + (H_\alpha-2\ell)\right)$$

$$= Y_\alpha^{\ell}\left((\ell+1)H_\alpha - \ell(\ell+1)\right)$$

$$\therefore\ X_\alpha(Y_\alpha^{\ell+1}\otimes 1) = Y_\alpha^{\ell}(\ell+1)\otimes[\lambda(H_\alpha)-\ell]1 = 0.$$

(ii) Let $\tilde{\mathfrak{b}}=\mathfrak{b}+(Y_\alpha)$. As α is simple $\tilde{\mathfrak{b}}$ is a subalgebra. ~~with~~ The radical of $\tilde{\mathfrak{b}} = (X_\beta,\ \beta\in\Delta-\{\alpha\})$.
As $\lambda(H_\alpha)=\ell$ integer ≥ 0, there is a finite dimensional rep. V of $\tilde{\mathfrak{g}} = (H_\alpha, X_\alpha, Y_\alpha)$ with dominant weight λ. Consider V as a $\tilde{\mathfrak{b}}$ module with $\mathfrak{n}$ acting trivially. Then get

$$U(\tilde{\mathfrak{g}})\otimes_{\mathfrak{b}}\lambda \xrightarrow{\text{onto}} V$$

so

$$\begin{tikzcd} U(\mathfrak{g})\otimes_{\tilde{\mathfrak{b}}}(U(\tilde{\mathfrak{b}})\otimes_{\mathfrak{b}}\lambda) \arrow[r, "\text{onto}"] \arrow[d, phantom, "\shortparallel"] & U(\mathfrak{g})\otimes_{\tilde{\mathfrak{b}}} V \\ U(\mathfrak{g})\otimes_{\mathfrak{b}}\lambda \arrow[r, two heads] & U(\mathfrak{g})\otimes_{\tilde{\mathfrak{b}}} V \end{tikzcd}$$

examination of [illegible] Hilbert poly shows not an iso.

Example to show that α simple is necessary.

sl_3

$$\boxed{\begin{array}{ccc} & X_1 & X_\alpha \\ Y_1 & & X_2 \\ Y_\alpha & Y_2 & \end{array}}$$

$$H_i = [X_i, Y_i] \qquad i = 1, 2, \alpha$$

$$H_1 = \begin{bmatrix} 1 & & \\ & -1 & \\ & & \end{bmatrix} \qquad H_2 = \begin{bmatrix} & & \\ & 1 & \\ & & -1 \end{bmatrix}$$

$$H_\alpha = \begin{bmatrix} 1 & & \\ & & \\ & & -1 \end{bmatrix}$$

$\alpha_1(d) = d_1 - d_2$

$\alpha_2(d) = d_2 - d_3$

$\alpha_\alpha(d) = d_1 - d_3$

note that $\alpha_i(H_{\alpha_i}) = 2$.

Any linear fn on h can be rep in the form

$$\lambda(d) = \lambda_1 d_1 + \lambda_2 d_2 + \lambda_3 d_3 \qquad \text{where} \qquad \lambda_1 + \lambda_2 + \lambda_3 = 0$$

because then $= \lambda_1(d_1 - d_2) + \lambda_3(d_3 - d_2) = \lambda_1\alpha_1 - \lambda_3\alpha_2$.

We assume that $\lambda(H_\alpha) = \lambda_1 - \lambda_3 = 0$

ie $\lambda_1 = \lambda_3 = \mu$ &

$$\lambda(d) = \mu\alpha_1 - \mu\alpha_2.$$

A necessary condition that $U(g) \otimes_b \lambda$ be reducible is that for some $\sigma \in W$ we have

$$(\lambda + g) - \sigma(\lambda + g) = n_1\alpha_1 + n_2\alpha_2$$

where n_1, n_2 are int ≥ 0 not both 0.

$$g = \tfrac{1}{2}(\alpha_1 + \alpha_2 + \alpha) = \alpha.$$

$(\lambda+\alpha)(d) =$ ~~$\mu(d_1) - 2\mu($~~

$$\mu(d_1 - 2d_2 + d_3) + (d_1 - d_3)$$

$$(\lambda+g)(d) = (\mu+1)d_1 + (-2\mu)d_2 + (\mu-1)d_3$$

Weyl group acts as permutations.

123 $\underline{(\mu-1)d_1 + (\mu+1)d_2 + (-2\mu)d_3}$

$(2)d_1 + (-3\mu-1)d_2 + (3\mu-1)d_3$ ✗

132 $(-2\mu)d_1 + (\mu-1)d_2 + (\mu+1)d_3$

$(3\mu+1)d_1 + (-3\mu+1)d_3 + (-2)d_3$ ✗

12 $(-2\mu)d_1 + (\mu+1)d_2 + (\mu-1)d_3$

$(3\mu+1)d_1 + (-3\mu-1)d_3 + (0)d_3$ ✗

23 $(\mu+1)d_1 + (\mu-1)d_2 + (-2\mu)d_3$

$(0)d_1 + (-3\mu+1)d_2 + (3\mu-1)d_3$ ✗

13 $(\mu-1)d_1 + (-2\mu)d_2 + (\mu+1)d_3$

$(2)d_1 + (0)d_2 + (-2)d_3$ ✓

Thus only element of Weyl with necessary requirements for generic μ is s_α

$$(\lambda+g) - s_\alpha(\lambda+g) = g - s_\alpha g = 2\alpha$$

Important correction:

$s_\alpha g = g - \alpha$ holds for α simple.

e.g. $sl(3)$ here $s_\alpha(\alpha_1) = -\alpha_2$ not a pos. root.

$s_\alpha(\alpha_2) = -\alpha_1$

$\therefore$

$s_\alpha g = -g$ here.

~~So the old argument that $s_\alpha(\lambda+g) - (\lambda+g)$ is pos sum is incorrect.~~

So the bad weight if it $\exists$ is of the for

$$\tilde{\lambda} + g = s_\alpha(\lambda + g)$$

$$\tilde{\lambda} = s_\alpha(\lambda+g) - g$$

$$= \lambda + s_\alpha g - g = \underline{\underline{\lambda - 2\alpha}}$$

not $\lambda - \alpha$ as you thought.

The weight space in $U(n^-) \otimes \lambda$ of weight $\lambda - \underline{2\alpha}$ has basis

$$\boxed{Y_\alpha^2 \otimes 1,\ Y_1 Y_2 Y_\alpha \otimes 1,\ Y_1^2 Y_2^2 \otimes 1}$$

$2\alpha = \alpha + \alpha$

$= \alpha + \alpha_1 + \alpha_2$

$= 2\alpha_1 + 2\alpha_2$

Question: can we find

$$\rho Y_\alpha^2 + \sigma Y_1 Y_2 Y_\alpha + \tau Y_1^2 Y_2^2$$

killed by X_1 and X_2.

note $[Y_1, Y_2] = -Y_\alpha$

$$[X_1, Y_\alpha] = -Y_2 \qquad = -[X_1,[Y_1,Y_2]] = -[H_1, Y_2] = -Y_2$$

$$[X_1, Y_1] = H_1 \qquad \underset{-(d_3-d_2)(H_1)\cdot Y_2}{\overset{\|}{\quad}}$$

$$[X_1, Y_2] = 0$$

$$[H_1, Y_2] = -Y_\alpha$$

$$\boxed{[X_1, Y_\alpha^2] = -Y_2Y_\alpha - Y_\alpha Y_2 = -2Y_2Y_\alpha}$$

$$[X_1, Y_1Y_2Y_\alpha] = \underline{H_1Y_2}Y_\alpha + Y_1Y_2(-Y_2)$$

$$H_1Y_2 = Y_2H_1 + [H_1, Y_2] = Y_2H_1 + Y_2$$

$$= (Y_2H_1 + Y_2)Y_\alpha - Y_1Y_2^2 = Y_2H_1Y_\alpha + Y_2Y_\alpha - Y_1Y_2^2$$

$$H_1Y_\alpha = Y_\alpha H_1 + [H_1, Y_\alpha]$$

$$= Y_\alpha H_1 + -(d_1-d_3)(H_1)Y_\alpha$$

$$= Y_\alpha H_1 - Y_\alpha$$

$$= Y_2(Y_\alpha H_1 - \cancel{Y_\alpha}) + \cancel{Y_2Y_\alpha} - Y_1Y_2^2$$

$$= Y_2Y_\alpha H_1 - Y_1Y_2^2$$

X_1, Y_1

~~$[X_1, Y_\alpha^2] = Y_2Y_\alpha H_1 - Y_1Y_2^2$~~

$$\boxed{[X_1, Y_1Y_2Y_\alpha] = Y_2Y_\alpha H_1 - Y_1Y_2^2}$$

$$[X_2, Y_\alpha] = +[X_2, [Y_2, Y_1]] = [H_2, Y_1] = -(d_1 - d_2)(H_2)Y_1$$
$$= Y_1$$

$$[X_2, Y_\alpha] = Y_1$$

$$H_1 Y_2 = Y_2(H_1 + 1)$$
$$[H_1, Y_2] = Y_2$$

$$[X_1, Y_1^2 Y_2^2] = (H_1 Y_1 + Y_1 H_1) Y_2^2$$
$$= (Y_1(H_1 - 2) + Y_1 H_1) Y_2^2$$
$$= Y_1 (2H_1 - 2) Y_2^2$$
$$= Y_1 Y_2 (2H_1) Y_2$$
$$= Y_1 Y_2^2 (2H_1 + 2)$$

$$\boxed{[X_1, Y_1^2 Y_2^2] = Y_1 Y_2^2 (2H_1 + 2)}$$

$$\boxed{\begin{aligned} [H_2, Y_1] &= Y_1 \\ [H_2, Y_2] &= -2Y_2 \\ [H_2, Y_\alpha] &= -Y_\alpha \end{aligned}}$$

$$[H_2, Y_\alpha] = -(d_1 - d_3)(H_2) Y_\alpha$$
$$= -Y_\alpha$$

$$[H_2, Y_1] = -(d_1 - d_2)(H_2) Y_1$$
$$+ Y_1$$

check $[H_2, [Y_2, Y_1]] = -2Y_\alpha + [Y_2, +Y_1] = -$ [illegible] $- Y_\alpha$

$H_{2,}$

$$\boxed{[X_2, Y_\alpha^2] = Y_1 Y_\alpha + Y_\alpha Y_1 = 2 Y_1 Y_\alpha}$$

$$[X_2, Y_1 Y_2 Y_\alpha] = Y_1 (H_2 Y_\alpha + Y_2 Y_1)$$

$$= Y_1 (Y_\alpha H_2 + [H_2, Y_\alpha] + Y_1 Y_2 - [Y_1, Y_2])$$

$$= Y_1 (Y_\alpha H_2 - \cancel{Y_\alpha} + Y_1 Y_2 + \cancel{Y_\alpha})$$

$$\boxed{[X_2, Y_1 Y_2 Y_\alpha] = Y_1 Y_\alpha H_2 + Y_1^2 Y_2}$$

$$[X_2, Y_1^2 Y_2^2] = Y_1^2 (H_2 Y_2 + Y_2 H_2)$$

$$= Y_1^2 (Y_2 (H_2 - 2) + Y_2 H_2)$$

$$\boxed{[X_2, Y_1^2 Y_2^2] = Y_1^2 Y_2 (2H_2 - 2)}$$

$$\boxed{\begin{aligned} 2\rho + \sigma \lambda(H_2) &= 0 \\ \sigma + \tau(2\lambda(H_2) - 2) &= 0 \end{aligned}}$$

make $\tau = 1$

dimension of solution space is 1.

$$-2\rho + \sigma \lambda(H_1) = 0$$

$$-\sigma + \tau(2\lambda(H_1) + 2) = 0$$

$\sigma \neq 0 \implies \lambda(H_1) + \lambda(H_2) = 0$

~~$\tau \neq 0 \implies \lambda(H_1) + \lambda(H_2) = 0$~~

ie $\boxed{\lambda(H_\alpha) = 0}$

<u>Theorem</u>: There is a canonical isom. $\gamma: U(\mathfrak{g})^{\mathfrak{g}} \longrightarrow U(\mathfrak{h})^{W}$

<u>Proof</u>: ~~[illegible]~~ I will define a map of spec's. Let $\lambda \in \mathfrak{h}'$ ~~Choose a set of positive~~ be such that $\lambda(H_\alpha) \in \mathbb{Z} - 0$ for every ~~positive~~ root α and let $\Sigma' = \{\alpha / \lambda(H_\alpha) > 0\}$. so that Σ' is a system of positive roots. Let $\alpha_1, \ldots, \alpha_\ell$ be the simple roots in Σ'. Then $\lambda(H_i)$ is an integer > 0, so $(\lambda - g)(H_i)$ is an integer ≥ 0.

$$g - g(H_i)\alpha_i = S_{\alpha_i}(g)$$

$$\shortparallel$$

$$g - \alpha_i \qquad \therefore \quad g(H_i) = 1.$$

Hence there is a finite dimensional irred. repn. V of $\mathfrak{g}$ with dominant wgt. $\lambda - g$ and whose character χ_V is given on $U(\mathfrak{h})$ by

$$\chi_V(h) = \left\langle h, \ \frac{\det e^{\lambda}}{\det e^{g}} \cdot \frac{\prod_{\alpha \in \Sigma'} \langle g, \alpha \rangle}{\prod_{\alpha \in \Sigma} \langle \lambda, \alpha \rangle} \right\rangle$$

In other words

$$\chi_V = \chi_{\lambda - g}$$

so we get a maximal ideal in $U(\mathfrak{g})^{\mathfrak{g}}$.

<u>Nice proof</u>: Given max ideal in $U(\mathfrak{h})^{W}$ coming from $\lambda \in \mathfrak{h}'$ choose Σ' + take induced rep with weight $\lambda - g$ get a character on Z.

<u>Theorem (Harish-Chandra)</u>. There is a canonical isomorphism

$$\gamma : U(\mathfrak{g})^{\mathfrak{g}} \longrightarrow U(\mathfrak{h})^{W}$$

<u>Proof</u>: Given a maximal ideal $\mathfrak{m}$ in $U(\mathfrak{h})^W$ choose $\lambda \in \mathfrak{h}'$ ~~arising~~ giving rise to it, choose a $\mathfrak{b}$ and consider the induced rep with dominant weight $\lambda - g_{\mathfrak{b}}$ and let $\chi_{\lambda,\mathfrak{b}}$ be the resulting character. Claim independent of the choices of $\lambda + \mathfrak{b}$.

a) Independence of λ. If λ' is another $\exists \sigma \in W$ $\ni$ $\sigma\lambda' = \lambda \implies \chi_{\lambda,\mathfrak{b}} = \chi_{\lambda',\mathfrak{b}}$.

b) Independence of $\mathfrak{b}$. If $\mathfrak{b}'$ is another $\exists\ \sigma \in W$ with $\mathfrak{b}' = \sigma\mathfrak{b}$. Then σ is an autom. of $U(\mathfrak{g})$ which comes from an inner auto of $\mathfrak{g}$ so σ is trivial on Z. Thus

$$U(\mathfrak{g}) \otimes_{\mathfrak{b}} (\lambda - g_{\mathfrak{b}}) \text{ and } U(\mathfrak{g}) \otimes_{\sigma\mathfrak{b}} (\sigma\lambda - \underset{\overset{\shortparallel}{g_{\sigma\mathfrak{b}}}}{\sigma g_{\mathfrak{b}}})$$

have the same character, i.e.

$$\chi_{\lambda,\mathfrak{b}} = \chi_{\sigma\lambda,\sigma\mathfrak{b}} = \chi_{\lambda,\sigma\mathfrak{b}} \quad \text{by a).}$$

But H-C has **defined** an isom with above properties! so γ canonical.

Theorem (~~PRV~~): Let $\mathfrak{m}_\lambda$ be the maximal ideal of Z corresponding to $\lambda \in \mathfrak{h}'$ under the isomorphism of H-C. Then $\mathfrak{m}$ generates a maximal ideal in $U(\mathfrak{g})$ iff

$$\lambda(H_\alpha) \notin \mathbb{Z} - 0$$

for every root α.

$\Longleftarrow$ (due to PRV). ~~In other words, they~~

According to PRV the principal series representation is irreducible hence the natural map

$$U(\mathfrak{g})/\mathfrak{m} \longrightarrow \pi_{\lambda,0}$$

must be onto. But now count multiplicities!

$\Longrightarrow$. Suppose that $\lambda(H_\alpha) \in \mathbb{Z} - 0$ for some root α. ~~Consider~~ Let $\Sigma_1 = \{\alpha \mid \lambda(H_\alpha) = \ell,\ \ell \in \mathbb{Z}, \ell > 0\}$ Σ_1 is a set of roots closed under addition and not meeting $-\Sigma_1$, hence may assume ~~it consists of the roots of some reductive subal~~ it is generated by $\Sigma_1 \cap \Pi = \Pi_1$ in which case Σ_1 is generated by Π_1. Let $\mathfrak{g}_1$ be generated by $\mathfrak{h}, X_\alpha, X_{-\alpha}, \alpha \in \Sigma_1$ so that $\mathfrak{g}_1$ is reductive and $\mathfrak{b}_1 = \mathfrak{g}_1 + \mathfrak{b}$ is a parabolic group. Then we know that $(\lambda - g)(\alpha_i)$ is an integer ≥ 0 for any $\alpha_i \in \Pi_1$ so there is a finite dimensional irreducible representation V of $\mathfrak{b}_1$ with dominant weight $\lambda - g$ when restricted to $\mathfrak{b}$. It follows that there is a surjection

$$\mathcal{U}(\mathfrak{g})\otimes_{\mathfrak{b}}(\lambda_{\mathfrak{g}}) \longrightarrow \mathcal{U}(\mathfrak{g})\otimes_{\mathfrak{b}_1} V$$

Now let I be the annihilator in $\mathcal{U}(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})\otimes_{\mathfrak{b}_1} V$. You want to show

(i) $I > \mathcal{U}(\mathfrak{g})\mathfrak{m}_\lambda$

(ii) I is ~~the unique~~ a maximal ideal.

(iii) ~~For any subset of Π_1 there is a unique~~

There is a 1-1 correspondence between subsets of Π_1 and prime ideals containing $\mathcal{U}(\mathfrak{g})\mathfrak{m}_\lambda$ in $\mathcal{U}(\mathfrak{g})$. (not quite correct)

Corollary: Only a finite number of prime ideals containing a given $\mathfrak{m}_\lambda$.

Be more careful: If $\Sigma_1 = \{\alpha / \lambda(H_\alpha) = \ell \quad \ell \in \mathbb{Z} \ \ell > 0\}$ then $\mathfrak{g}_1 = \mathfrak{h} + \sum_{\alpha\in\Sigma_1}' (X_\alpha) + (Y_\alpha)$ is a $\overbrace{\text{subalgebra}}^{\text{reductive}}$ of $\mathfrak{g}$. Note that Π_1 can then be formed in terms of this data.

What about polynomial rings!

Again take $\lambda \in \mathfrak{h}'$ and consider $\{\alpha / \lambda(H_\alpha) = 0\}$

better to think of $\lambda = H \in \mathfrak{h}$ +

$\{\alpha / \alpha(H) = 0\}$.

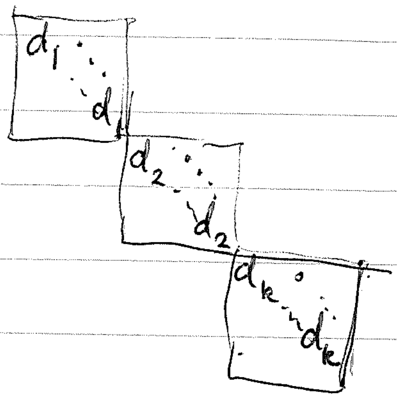
Gives rise to a group $\mathfrak{g}_1$ = centralizer of H. Choose 1 positive root system.

H $\quad d_1, \dots, d_n$

$\{\alpha\} = \quad (i,j) \quad d_i = d_j.$

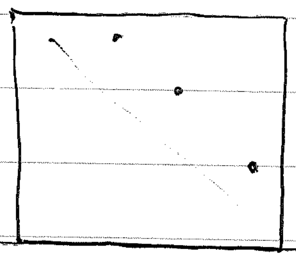
So by suitable permutation we arrange that

$H =$



$\mathfrak{g}_1$ as indicated $\quad \Pi_1$ Jordan pieces

So the various orbits are in 1-1 corresp with subsets of Π_1 X modulo W_1.



partitions.

Returning to the proof:

In order to prove (ii) we must improve PRV. to case of a parabolic gp.

(i) At the moment we are only interested in the prime ideal I being $> U(\mathfrak{g}) m_\lambda$. Hence we must produce an element of $U(\mathfrak{g})$ which is 0 on $U(\mathfrak{g}) \otimes_{\mathfrak{b}_1} V$ and non-zero on $U(\mathfrak{g}) \otimes_{\mathfrak{b}} (\lambda - g_0)$.

Go back to the case of a simple root α_i + suppose

$$\lambda(H_{\alpha_i}) = 0.$$

Then we saw that $\mathcal{U}(\mathfrak{g})\otimes_{\mathfrak{b}}\lambda$ had the weight vector $Y_i \otimes \lambda$ since

$$X_i(Y_i\otimes\lambda) = H_i\otimes\lambda = 1\otimes\lambda(H_i) = 0.$$

In the case of $sl(2)$ the means that the operator X_i is identically 0 on the irreducible representation.

So we consider $\mathcal{U}(\mathfrak{g})\otimes_{\tilde{\mathfrak{b}}}\tilde{\lambda}$

where $\tilde{\lambda}/\mathfrak{b} = \lambda$

$\tilde{\lambda}(Y_i) = 0$ OKAY since $\tilde{\lambda}[X_i, Y_i] = \lambda(H_i) = 0$

Then ~~$\mathcal{U}(\mathfrak{g})\otimes_{\mathfrak{b}}$~~ have $\mathfrak{g} = \underbrace{\sum_{\alpha\neq\alpha_i}(Y_\alpha)}_{\mathfrak{n}^-} + \tilde{\mathfrak{b}}.$

So

$$\mathcal{U}(\mathfrak{g})\otimes_{\tilde{\mathfrak{b}}}\tilde{\lambda} = \underline{\mathfrak{n}^{-}\otimes\tilde{\lambda}},$$

and

let Δ_i be the Casimir of $\mathfrak{g}_i$.

$$\Delta_i = X_iY_i + Y_iX_i \qquad ?$$

~~[illegible]~~

I need an operator $Q \in U(\mathfrak{g})$ such that

$$Q\left(\underline{U(\mathfrak{n}^-)\otimes\lambda}\right) = 0.$$

~~Suppose I take~~

$$[X_i, Y_{\alpha_0} \cdots Y_{\alpha_0}] \qquad \alpha_0 \neq \alpha_i .$$

sl(3)

$$\underline{[X_1, Y_\alpha] = -Y_2}$$

~~If~~ Is there a Q which commutes with all the Y_α.

$\boxed{sl(3)!!!}$

Casimir

~~$[X_1, H_2 + H_1, Y_1 X_1]$~~

~~Y_α~~ ~~$[H, X]$~~

$[H, X] = 2X$

$[H, Y] = 2Y$

$[X, Y] = H$

$\langle H, H\rangle = 8$

$\langle X, Y\rangle = 4$

$\mathrm{ad}X\,\mathrm{ad}Y\;H = 2H$

$\mathrm{ad}X\,\mathrm{ad}Y\;X = 2X$

$$Cas = \frac{1}{8}H^2 + \frac{1}{4}(XY + YX)$$

$$8\cdot Cas = H^2 + 4(YX) + 2H = \boxed{H^2 + 2H + 4YX}$$

Check $\quad -\underbrace{2XH}_{(HX-2X)} - 2HX \cancel{+} - \cancel{2}4X + 4HX$

$$= 0.$$

$$H_1^2 + 2H_1 + 4Y_1X_1 \qquad Y_{\cancel{1}2}$$

idea is that $\quad U(\mathfrak{g})/U(\mathfrak{g})m_\lambda \quad$ is of dimension $2r$

and $\quad U(\mathfrak{g}) \otimes_{\mathfrak{b}} \lambda \quad$ is of dimension $r-1$

hence $\quad U(\mathfrak{g})/I \quad$ ~~can be~~ is of dimension $\leq 2r-2$.

$$(H_1^2 + 2H_1 + 4Y_1X_1)(Y_2 \otimes 1)$$

$$\begin{cases} H_1^2\,(Y_2 \otimes 1) = (\lambda^{(H_1)} - \alpha_2^{(H_1)})^2 \; Y_2 \otimes 1 \\ 2H_1\,(Y_2 \otimes 1) = 2(\lambda^{(H_1)} - \alpha_2^{(H_1)}) \; Y_2 \otimes 1 \\ Y_1X_1 = 0. \quad \text{since } X_1, Y_2 = 0 \end{cases}$$

$\lambda(H_1) = 0$ recall

is of weight 0, hence will act on the $\mathfrak{h}$ weight spaces

---

$$[X_1, Y_\alpha] = \underset{\sim}{-Y_2} \qquad \underline{\underline{Y_2X_1}}$$

$$[X_1, Y_2] = 0$$

$$[X_1, \cancel{H}Y_1] = \lambda(H_1) = 0.$$

$$[X_1, Y_\alpha^k Y_2^j \overset{\downarrow}{Y_1^i}] =$$

Problem: Find an element ~~[illegible]~~ in the annihilator of $U(\mathfrak{g})\otimes_{\tilde{\mathfrak{b}}}\lambda$ which is not in the annihilator of $U(\mathfrak{g})\otimes_{\mathfrak{b}}\lambda$.

Question: Is it possible to calculate $gr(U/\mathrm{ann}\, M)$ from $gr\, M$? Thus for example can we conclude that since

$$gr\, M = S(\mathfrak{m}^-)\otimes\lambda = S(\mathfrak{g}/\tilde{\mathfrak{b}})\otimes\lambda$$

Annihilator $gr\, M$ is therefore the largest invariant ideal ~~generated by functions vanishing on~~ consisting of functions which vanish on ~~the orthogonal of~~ $\tilde{\mathfrak{b}}$ ~~[illegible]~~ should somehow be the orbit containing $\mathfrak{m}^-$ which is a species of nilpotent elements. $\mathfrak{m}^-$ generated by

Thus let $f \in U$. In order that $f = 0$ on M

February 14, 1968

Talk: Irreducible modules over enveloping algs.

Outline:

1. gen. Schur's lemma.

2. nilpotent Lie algebras.

Dixmier I. If $\mathfrak{g}$ nilpotent, I ideal in $U(\mathfrak{g})$, TFAE

$$\begin{cases} I \text{ maximal} \\ Z(U(\mathfrak{g})/I) = \mathbb{C} \\ U(\mathfrak{g})/I \simeq \mathbb{C}[p_1,\dots,p_r,q_1,\dots,q_r] \end{cases}$$

$[p_i, p_j] = [q_i, q_j] = 0$

$[p_i, q_j] = \delta_{ij}$

Corollary. If M irreducible over $U(\mathfrak{g})$, $\mathfrak{g}$ nilpotent, then $\operatorname{Ann} M$ is a maximal ideal of $U(\mathfrak{g})$.

Thus the problem of classifying irreducible modules over a nilpotent Lie algebra reduces to classifying the max ideals + then classifying irreducible modules over the Heisenberg algebra $A_r = \mathbb{C}[p_1,\dots,p_r,q_1,\dots,q_r] = \mathbb{C}[p_1,q_1] \otimes \cdots$

~~But such a module must be the tensor product of irreducible modules over [illegible] each factor by the Schur's lemma i.e. if given M consider it as an A_1 module and let~~

$\cong$ polynomial diff ops. ~~$a(x)$~~ $p(x,D)$.

Latter problem is hard e.g. such a module will be of form ~~left ideal eg~~ $A_r/(P_1,\dots,P_\ell)$. ~~where P_i's are arbitrary coeffs.~~ i.e the same as the overdetermined system

$$P_i(x,D)\,u = 0$$

but where $P_i(x,D) = \sum a^i_\alpha(x) D^\alpha$ can be highly irregular. so that little of the Cartan-Kahler-Kuranishi theory can be applied.

Thus it is known ~~Th~~

Rinehart: $r \leq hd\, A_r \leq 2r-1$ $\quad r \geq 1$

First problem has been solved.

Dixmier II: $\exists$ natural 1-1 correspondence between maximal ideals in $U(\mathfrak{g})$ and orbits of Ad $\mathfrak{g}$ in $\mathfrak{g}'$ (~~[illegible]~~ = maximally invariant ideals in $S(\mathfrak{g})$)
ad $\mathfrak{g}$ stable

Recently generalized to prime ideals by ~~[illegible]~~ Nouazé-Gabriel.

3. semi-simple Lie algebras

(By the Schur's lemma we know that irreducible modules are distinguished by the ~~[illegible]~~ homomorphisms $\chi : Z \to \mathbb{C}$ ~~$\mathfrak{Z}$~~
The corollary is false – Ann M is only a prime ideal containing ~~[illegible]~~ Ker χ ~~$\mathfrak{Z}$~~ and the structure of $U/\text{Ann}\, M$ is not yet known but where it is known the classification of all irreducible modules seems hopeless.)

There is a class of irred modules of which occur in practice ~~and for~~ which ~~there is some~~ will eventually be classified. Suppose G semi-simple Lie gp with L.A. $\mathfrak{g}_{\mathbb{R}} \ni$ $\mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}$ that $k_{\mathbb{R}}$ is a max. ~~compact~~ subalg of $\mathfrak{g}_{\mathbb{R}}$ on which the Killing form is neg. def., so that if K is the corresponding subgp of G, then ~~[illegible]~~ ~~compact~~ K/center of G is compact. ~~[illegible]~~
Let $k = k_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, $p =$ orthogonal of k so $\mathfrak{g} = k \oplus p$ and let θ be the inv. of $\mathfrak{g}$ ~~+~~1 on k -1 on p.

If V is a continuous repn of G on a TVS, then ~~[illegible]~~ set $M =$ space of K-finite vectors. Pose

Problem: Given ~~[illegible]~~ $\mathfrak{g}, \theta$, classify irred $\mathfrak{g}$ modules which as $\mathfrak{k}$ modules are sums of fin. ~~[illegible]~~ dim'l irreducibles.

Thm of H-C: Let Λ be an irreducible f.d. $\mathfrak{k}$-module and let $\mathcal{E}_\Lambda = \mathrm{Hom}_{\mathfrak{g}}(U(\mathfrak{g})\otimes_{\mathfrak{k}}\Lambda, U(\mathfrak{g})\otimes_{\mathfrak{k}}\Lambda)$. If V is an irreducible $\mathfrak{g}$ module $\ni$ $\mathrm{Hom}_{\mathfrak{g}}(U(\mathfrak{g})\otimes_{\mathfrak{k}}\Lambda, V) = \mathrm{Hom}_{\mathfrak{k}}(\Lambda, V) \neq 0$, then $\mathrm{Hom}_{\mathfrak{k}}(\Lambda, V)$ is an irreducible f.d. $\mathcal{E}_\Lambda$ module. ~~and there~~ Furthermore there is a 1-1 corresp between such irred $\mathfrak{g}$ modules V and irreducible $\mathcal{E}_\Lambda$ modules.

Suppose $\Lambda = 1$ in which case $\mathcal{E}_\Lambda$ is a polynomial ring (ring of inv. D.O. on G/K) so $\mathrm{Hom}_{\mathfrak{k}}(1, V)$ 1-dim'l. The modules of class 1. Structure then known, in particular when the module

$$\xi \underset{\mathcal{E}_\Lambda}{\otimes} (U(\mathfrak{g})\otimes_{\mathfrak{k}}\Lambda) \qquad \text{is irreducible}$$

Special case: ~~[illegible]~~ Take $\mathfrak{g}, \theta, \mathfrak{k}$ as above to be $\mathfrak{g}\times\mathfrak{g}$, $\theta(x,y)=(y,x)$, $\mathfrak{k} = \Delta\mathfrak{g}$. Then an irred. module of class 1 is a left and right $U(\mathfrak{g})$ module with an element v such that $Xv = vX$, ie of the form

$U(\mathfrak{g})/I$, where I is a maximal ~~an~~ ideal, with the obvious left + right action. It can be shown that E_Λ acts as the center of $U(\mathfrak{g})$. so

Cor 1: There is a unique maximal ideal of $U(\mathfrak{g})$ containing a maximal ideal of Z, thus 1-1 corresp. between maximal ideals of $U(\mathfrak{g})$ and max ideals of Z.

~~PRV have determined when the module $U(\mathfrak{g})$~~

Thm H-C. ~~Then~~ $\mathfrak{g} = \mathfrak{h} + \sum'_{\alpha\in\Sigma'} \mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha}$, $\mathfrak{b} = \mathfrak{h} + \sum_{\alpha\in\Sigma} \mathfrak{g}^\alpha$

$\rho = \frac{1}{2}\sum'_{\alpha\in\Sigma'} \alpha \in \mathfrak{h}'$. There is a canonical isomorphism (ind of choice of Σ)

$$\gamma : Z \xrightarrow{\sim} S(\mathfrak{h})^W$$

~~defined by~~ such that if for $\lambda \in \mathfrak{h}'$ we denote $\chi_\lambda : Z \xrightarrow{\gamma} S(\mathfrak{h})^W \xrightarrow{ev_\lambda} \mathbb{C}$, then χ_λ is the inf character of the module

$$U(\mathfrak{g}) \otimes_{\mathfrak{b}} (\lambda - \rho)$$

~~the~~ a representation with dominant weight $\lambda - \rho$.

Theorem (PRV): (Recall can choose $X_\alpha \in \mathfrak{g}^\alpha$ $\alpha \in \Delta$ such that if $H_\alpha = [X_\alpha, X_{-\alpha}]$ ~~we have~~ then $\alpha(H_\alpha) = 2$) Let $m_\lambda = \operatorname{Ker} \chi_\lambda$. Then $U(\mathfrak{g})m_\lambda$ is a maximal ideal of of $U(\mathfrak{g}) \iff$ for every root α $\lambda(H_\alpha) \notin \mathbb{Z}-\{0\}$.

Relations with $\overset{\mathrm{Ad}\,\mathfrak{g}}{\text{orbits}}$ in $\mathfrak{g}' \simeq \mathfrak{g}$ by Killing. Kostant's paper. ~~Again ∃ canonical isom.~~ Again one has an isom

~~$S(\mathfrak{g})^{\mathfrak{g}} \to S(\mathfrak{h})^W$~~

$$\tilde{\gamma} : S(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} S(\mathfrak{h})^W \quad \text{(Chevalley)}$$

~~which permits one to~~ [illegible]

Kostant theory of G orbit st in $\mathfrak{g}' \simeq \mathfrak{g}$	primes in $U(\mathfrak{g})$ which rest. to maximals of Z.
closed orbits (= orbits of s.s elts) $\leftrightarrow$ max ideal m_λ in $S(\mathfrak{h})^W$	max. ideals $\leftrightarrow$ max ideals of Z
good orbits (orbits of reg. s.s. elts) $\leftrightarrow$ (m_λ, $\lambda(H_\alpha) \neq 0$ all α) (those max ideals correspond to [illegible] in $S(\mathfrak{g})$ generated by $m \cap S(\mathfrak{g})^{\mathfrak{g}}$)	good max ideals = max. ideal m ∋ $m = U(\mathfrak{g})\, m_\lambda$. $\iff \{\lambda \mid \lambda(H_\alpha) \notin \mathbb{Z}-\{0\}$ all $\alpha\}$.
for each λ there is a ~~unique~~ dense open orbit, which is closed $\iff \lambda$ good.	∃ ! prime namely $U(\mathfrak{g})\, m_\lambda$ containing m_λ
Even if λ bad ∃ only finitely many orbits with given $\lambda \leftrightarrow$ subsets of a set of simple roots	Conjecture: Only finitely many primes containing m_λ and these are in 1-1 corres with subsets of a Π_1.

Generalized Schur's Lemma: $\mathbb{C}$ alg. closed, field

U algebra over $\mathbb{C}$ with a filtration $F_0U \subset F_1U \subset \cdots \subset$ subspaces

$\ni$ (i) $F_pU \cdot F_qU \subset F_{p+q}U$, $\mathbb{C} \subseteq F_0U$, $U = \bigcup F_pU$

(ii) $\operatorname{gr} U$ is a finitely gen. comm. alg $/ \mathbb{C}$.

If M is any irred. U module, then $\mathbb{C} \xrightarrow{\sim} \operatorname{Hom}_U(M, M)$.

Cor 1.

$\therefore$ ~~$X^m Y^m$~~

$$X^2 \cdot Y^m \cdot 1 = Y^{m-2}\left((m-1)H-(m-1)(m-2)\right)$$

Finally we get

$$\boxed{X^m \cdot Y^m = m!\,(H)(H-1)\cdots(H-m+1)}$$

$$\boxed{X^i Y^j = \frac{j!}{(j-i)!}\,(H-j+1)\cdots(H-j+i)}$$

perhaps we can write this better in $\overset{a}{\text{the}}$ form using binomial coefficients, thus

~~$(H-j+1)$~~ set $k+i=j$

~~$X^i Y^{i+k}$~~ $\cancel{i}\,(H-k)\cdots(H-j+1) \overset{?}{=} \binom{H-k}{i}$

$$\binom{H-k}{i} = \frac{(H-k)\cdots(H-k-i+1)}{1\cdots i}$$

$$\therefore \boxed{X^i Y^{i+k} = (i!\cancel{k})^2\binom{i+k}{i}\cdot\binom{H-k}{i}}$$

$$X^i Y^j = \frac{j!\,i!}{(j-i)!}\binom{H-j+i}{i}Y^{j-i}$$

Irreducibility of dominant weight reps.

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathfrak{b}} \lambda = \mathcal{U}(\mathfrak{n}^-) \otimes \lambda$$

Want to determine when reducible ie when there is a vector $v \in \mathcal{U}(\mathfrak{n}^-) \otimes \lambda$ with the property that $X_i v = 0$ for all i.

The kernel must be a certain weight space under $\mathfrak{h}$ i.e. of the form $\sum\limits_{|\gamma| = \lambda - \sigma}' Y^{\gamma}$

The problem is whether one can determine the irred. without calculating the obvious determinant!

The point is somehow that the different X_i should go to different places? Thus X_i goes from the weight μ to the weight $\mu - \alpha_i$

Problem: Can one determine irreducibility without calculating the determinant – or can one calculate this determinant easily.

For a simple root we have

$$[X_i, Y_i^m] = H_i Y_i^{m-1} + YHY^{m-2} + \cdots + Y^{m-1}H$$

$$= Y^{m-1}(H - 2(m-1) + \cdots + H - 2 + H)$$

$$= Y^{m-1}(mH - m(m-1)).$$

$$X \cdot Y^m \cdot 1 = Y^{m-1}(mH - m(m-1))$$

Proposition:
$$\boxed{\frac{X^i}{i!}\,\frac{Y^j}{j!} \equiv \frac{Y^{j-i}}{(j-i)!}\cdot\binom{H-j+i}{i}}$$
mod $U(\mathfrak{g})X$.
$i \leq j$

Proof: ~~But~~ By induction on ~~j~~ i, the case $i=0$ being clear.

$$\binom{H}{i} = \frac{H(H-1)\cdots\cdots(H-i+1)}{i!}$$

Suppose true for $i-1$, then

$$\frac{X^i}{i!}\,\frac{Y^j}{j!} = \frac{X}{i}\,\frac{Y^{j-i+1}}{(j-i+1)!}\binom{H-j+i-1}{i-1}$$

$$= \frac{1}{i}\,\frac{Y^{j-i}}{(j-i+1)!}(j-i+1)(H-j+i)\binom{H-j+i-1}{i-1}$$

$$= \frac{Y^{j-i}}{(j-i)!}\binom{H-j+i}{i}$$

QED.

Inductive step is that

$$[X, Y^m] = m.Y^{m-1}(H-(m-1)).$$

Now the point is to proceed to obtain similar formulas but when there are more roots around. Thus for example we worked modulo $U(\mathfrak{g})X$. Perhaps we can also work modulo preceding X's and Y's.

We are given an ordering of the roots. Filter $U(\mathfrak{g})$ accordingly; namely the problem is to arrange $X^\gamma Y^\delta$ in a convenient order for the purposes of calculation. So set

$$X^\gamma Y^\delta = \sum_\varepsilon Y^\varepsilon P_\varepsilon(H). \qquad \text{sum taken over } |\varepsilon| = |\delta| - |\gamma|.$$

Thus

$$\boxed{X \cdot X^\gamma Y^\delta = \sum_\varepsilon \underline{XY^\varepsilon} P_\varepsilon(H)}$$

$$XY^\varepsilon = \underline{Y^{\varepsilon-1}X}\,\cancel{X}\,\underline{Q(H)}\,P_\varepsilon(H)$$

$$+ \; Y^\varepsilon P_\varepsilon(H + \cdot)$$

But even though this way you might be able to calculate $X^\gamma Y^\delta$ it has the disadvantage that you don't get the determinant! So one still must arrange the results in a nice order

The thing to keep in mind is that we have to ~~al~~ also be able to prove the irreducibility of reps of the form $U(\mathfrak{g}) \otimes_{\tilde{\mathfrak{b}}} V$, Therefore if we ~~somehow can filter we~~ are going to have to calculate determinants with fewer Y's. How is this going to run?

So we are given $\Pi_1 \subset \Pi$ and $\mathfrak{g}_1$, $\Sigma_1 \subset \Sigma$ and an irred. rep V of $\tilde{\mathfrak{b}}_1 = \mathfrak{g}_1 \oplus \mathfrak{n}_1$. Of course we have a dominant weight vector 1 for V. I will assume that V is 1-dimensional for simplicity i.e. that

$$Y_i \cdot 1 = 0 \quad \text{for} \quad \alpha_i \in \Pi_1.$$

so that $\lambda(H_i) = 0$.

Now I want to prove irreducibility. I am given ~~on~~ the basis

$$\underline{Y^{\delta} \otimes 1} \quad \text{where } \delta \text{ contains no roots in } \Pi_1$$

As before irreducibility means no dominant wgt. i.e. ~~no~~ nothing killed by X_i. The natural thing seems to be to ~~[illegible]~~ think of δ as being part of an irred rep of $\mathfrak{g}_1$, take the contragredient rep. X. use weights. Now I can ~~[illegible]~~ probably concentrate on $Y^{\delta} \otimes 1$ that I know are killed by X_i $\alpha_i \in \Pi_1$. These ~~[illegible]~~ probably take a simple form.

Feb 15.

Program: Irred of dominant wgt rep
maximal ideal of $U(\mathfrak{g})$.

Determinant:

Situation: Given $\Pi_1 \subset \Pi$ and $\lambda \in \mathfrak{h}' \ni \lambda(H_i) = 0$ for $\alpha_i \in \Pi_1$, I want to show that the repn. $U(\mathfrak{g}) \otimes_{\mathfrak{b}_1} \lambda$ is irred where $\mathfrak{b}_1 = \sum_{\alpha \in \Sigma_1} (Y_\alpha) + h + \sum_{\alpha \in \Sigma} (X_\alpha)$.

$U(\mathfrak{g}) \otimes_{\mathfrak{b}_1} \lambda$ has basis $Y^\xi \otimes 1$ where ξ is a fn. which assigns to each $\alpha \in \Sigma - \Sigma_1$ a non-negative integer and $Y^\xi = \prod_{\alpha \in \Sigma - \Sigma_1} Y_\alpha^{\xi_\alpha}$, the product being taken in order.

First case: Show that $U(\mathfrak{g}) \otimes_{\mathfrak{b}_1} \lambda$ is reducible if $\lambda(H_j) = \ell$ int. ≥ 0 for some $\alpha_j \in \Pi - \Pi_1$. This is easy! Because then can define a $\mathfrak{g}_2 > \mathfrak{g}_1$ through which repn. factors. In fact if $\ell = 0$, then we get the element $Y_j \otimes 1$ and

$$X_j \cdot Y_j \otimes 1 = \lambda(H_j) = 0$$

$$X_i \cdot Y_j \otimes 1 = 0.$$

But now suppose that we have a root of the form $\alpha + k\alpha_j$ where $\alpha \in \Sigma_1$ and k int > 0, and that

$$\lambda(H_{\alpha + k\alpha_j}) = \ell \quad \text{int} \geq 0.$$

But note that $H_{\alpha+k\alpha_j} \neq H_\alpha + k H_{\alpha_j}$. We must study α series.

Recall that if $\alpha, \beta \in \Delta$, then $\{k \mid \beta + k\alpha \in \Delta\}$ is α-series cont. β and that if ~~the~~ ~~interval $p \le k \le q$ is contained and maximal in the α series, we get a representation of $\mathfrak{g}^\alpha + \mathfrak{h} + \mathfrak{g}^{-\alpha}$~~ $\sum\limits_{p \le k \le q} \mathfrak{g}^{\beta + k\alpha}$ is irreducible, then

assume X_α, Y_α, chosen so that $[X_\alpha, Y_\alpha] = H'_\alpha$ + $\langle X_\alpha, Y_\alpha \rangle = 1$ we get

$$\sum_{p \le k \le q} \langle \beta + k\alpha, \alpha \rangle = 0 \qquad \text{trace commutator} = 0$$

$$\therefore \quad (q - \overset{+1}{p}) \langle \beta, \alpha \rangle + (q - \overset{+1}{p}) \frac{p+q}{2} \langle \alpha, \alpha \rangle = 0$$

$$\therefore 2 \frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} = -(p+q)$$

which shows there is a single string.

Now

$$H_\alpha = \frac{2}{\langle \alpha, \alpha \rangle} H'_\alpha ,$$

thus

$$\langle \alpha + k\alpha_j, \alpha + k\alpha_j \rangle = \langle \alpha, \alpha \rangle + k^2 \langle \alpha_j, \alpha_j \rangle + 2k \langle \alpha_j, \alpha \rangle$$

$$\langle \alpha + k\alpha_j, \alpha + k\alpha_j \rangle H_{\alpha + k\alpha_j} = \langle \alpha, \alpha \rangle H_\alpha + k^2 \langle \alpha_j, \alpha_j \rangle H_{\alpha_j}.$$

$$\langle \qquad \rangle \ell = k^2 \langle \alpha_j, \alpha_j \rangle \lambda(H_{\alpha_j})$$

$-2\frac{\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle} = p+q$ where $\beta+k\alpha$, $q\le k\le p$ is the α series cont β.

Go back

We ~~suppose that~~ are trying to calculate the irreducibility of the dominant weight rep. $U(g)\otimes_b\lambda$. and we have the conjecture that if $\lambda(H_\alpha)=\ell$ no $\alpha\in\Sigma'$ its irreducible.

The problem is to determine

$$\det(X^\gamma Y^\delta)=\det(P_{\gamma,\delta}(H)) \qquad \text{where } |\gamma|=|\delta|\in h'.$$

and the method is somehow to filter the monomials X^γ, Y^δ so that this can be determined.

Kostant's formula for the multiplicity of a weight is clear **confirmation of your program**, the point being to calculate in the representation ring of h the multiplicities which should occur.

Let λ be integral + dominant ie $\lambda(H_{\alpha_i})=\ell_i\ge 0$. Then I can make the following analysis of the dominant wgt representation! **Look at**

$$\begin{tikzcd} & U(g)\otimes_b\lambda \\ U(g)\otimes(s(\lambda+g)-g) \arrow[ur] & \end{tikzcd}$$

canonical map.

A fundamental formula is that the Jordan-Hölder components of the induced representations mesh together quite nicely according to the

His formula is that

$$\cancel{\sum_{\sigma\neq 1}' \operatorname{sg}\sigma\, \{P(s(\lambda+g)}$$

$$\chi_V = \cancel{\chi(U(g)\otimes_b \lambda)}$$

$$\boxed{\chi_V = \sum_{\sigma}' \operatorname{sg}\sigma\ \chi\left(U(g)\otimes_b \{\sigma(\lambda+g)-g\}\right)}$$

$\cancel{\chi_V}$ But

$$\chi\left(U(g)\otimes_b \lambda\right) = \chi\left(U(n^-)\otimes\lambda\right)$$

$$=$$

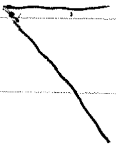
<u>Problem:</u> Find irred. repns

$$\det(X^\gamma Y^\delta)$$

We need some way of arranging things so that we can tell what will happen. It seems that we must determine when things are reducible

To order things in a natural way! we have to worry about ~~Roots~~

What is the relation of ordering + simpleness



The roots $\alpha_{ij}(d) = d_i - d_j$ is $>0 \iff i<j$.

Thus $\alpha_{ij} < \alpha_{kl}$

Take lexico ordering on simple roots. i.e.

$$\boxed{\sum \lambda_i d_i > 0}$$

~~iff~~ if first non-zero λ_i is >0.

Thus ~~α_{ij}~~

note that if we write

$$\lambda(d) = \sum_{i=1}^{n-1} g_i \alpha_{i,i+1}$$

then $\lambda > 0 \iff$ first non-zero $g_i > 0$.

Therefore perhaps it's wise to order h' lexicographically using the simple roots in order. Next ~~its wise to use~~ we want to <u>introduce an ordering on</u> monomials ξ

$$\xi = \sum \underline{n_\alpha} \alpha \quad \text{formal sum.}$$

~~Thus the root α should be arranged by size~~

A root α should be measured by how many neg. roots there are in $\boxed{s_\alpha g}$

$s_\alpha g$

Therefore to determine $$s_\alpha(\lambda+g) - g = \underset{\uparrow}{\lambda} - \underline{\underline{\mu}}$$

~~But claims that~~ the elements $$\mu = l\alpha + (g - s_\alpha g)$$

$$\{\sigma g - g \quad , \quad \sigma \in W\}$$

are all distinct. Any possibility that

$$\underline{\sigma(\lambda+g) - \cancel{g} = \tau(\lambda+g) - \cancel{g}}$$

$\Rightarrow \lambda+g$ lies on a ~~chamber~~. wall

Calculate for sl(3

Calculations for $sl(3)$

$$\boxed{\begin{array}{ccc} & X_1 & X_\alpha \\ Y_1 & \cdot & X_2 \\ Y_\alpha & Y_2 & \end{array}}$$

$$H_1 = [X_1, Y_1] = \begin{bmatrix} 1 & & \\ & -1 & \\ & & 0 \end{bmatrix}$$

$$H_2 = [X_2, Y_2] = \begin{bmatrix} 0 & & \\ & 1 & \\ & & -1 \end{bmatrix}$$

$$H_\alpha = [X_\alpha, Y_\alpha] = \begin{bmatrix} 1 & & \\ & & \\ & & -1 \end{bmatrix}$$

$$[X_1, X_2] = X_\alpha$$
$$[Y_1, Y_2] = -Y_\alpha$$

$$\begin{array}{lll} [H_1, Y_1] = -2Y_1 & [H_2, Y_1] = +Y_1 & [H_\alpha, Y_1] = -Y_1 \\ [H_1, Y_2] = +Y_2 & [H_2, Y_2] = -2Y_2 & [H_\alpha, Y_2] = -Y_2 \\ [H_1, Y_\alpha] = -Y_\alpha & [H_2, Y_\alpha] = -Y_\alpha & [H_\alpha, Y_\alpha] = -2Y_\alpha \end{array}$$

$$\begin{array}{lll} [X_1, Y_1] = H_1 & [X_2, Y_1] = 0 & [X_\alpha, Y_1] = -X_2 \\ [X_1, Y_2] = 0 & [X_2, Y_2] = H_2 & [X_\alpha, Y_2] = X_1 \\ [X_1, Y_\alpha] = -Y_2 & [X_2, Y_\alpha] = Y_1 & [X_\alpha, Y_\alpha] = H_\alpha \end{array}$$

Check $[X_2, [-Y_1, Y_2]] = -[Y_1, H_2] = [H_2, Y_1] = Y_1$ ✓

$$[[X_1, X_2], Y_1] = [H_1, X_2] = -X_2.$$
$$Y_2 \qquad [X_1, H_2] = + - [H_2, X_1] = +X_1$$

$$H_\alpha = [X_\alpha, Y_\alpha] = [[X_1, X_2], Y_\alpha] = [-Y_2, X_2] + [X_1, Y_1]$$
$$= H_2 + H_1 \quad ✓$$

The idea somehow is to determine when

$$\mathcal{U}(n^-) = \sum (Y^{\xi})$$

$$[X_i, Y^{\xi}] =$$

$$[X_1, \prod Y_\alpha^{\xi_\alpha}] = \sum_\alpha \prod [X_1, Y_\alpha^{\xi_\alpha}] \cdots$$

α_1: $[X_1, Y_1] = H_1$, $[X_2, Y_1] = 0$

α_2: $[X_2, Y_1] = 0$, $[X_2, Y_2] = H_2$

$\alpha_1+\alpha_2$: $[X_1, Y_1 Y_2] = Y_2(H_2+1)$, $[X_1, Y_\alpha] = -Y_2$; $[X_2, Y_1 Y_2] = Y_1 H_2$, $[X_2, Y_\alpha] = Y_1$

$2\alpha_1$: $[X_1, Y_1^2] = Y_1 \cdot 2(H-1)$, $[X_2, Y_1^2] = 0$

$2\alpha_2$: $[X_1, Y_2^2] = 0$, $[X_2, Y_2^2] = Y_2 \cdot 2 \cdot (H-1)$.

$2\alpha_1+2\alpha_2$:

$$[X_1, Y_\alpha^2] = -2Y_2Y_\alpha \qquad [X_2, Y_\alpha^2] = 2Y_1Y_\alpha$$

$$[X_1, Y_1Y_2Y_\alpha] = Y_2Y_\alpha H_1 - Y_1Y_2^2 \qquad [X_2, Y_1Y_2Y_\alpha] = Y_1Y_\alpha H_2 + Y_1^2Y_2$$

$$[X_1, Y_1^2Y_2^2] = Y_1Y_2^2(2H_1+2) \qquad [X_2, Y_1^2Y_2^2] = Y_1^2Y_2(2H_2-2)$$

Try induction – want to build up to g through ~~the weight spaces of~~ smaller groups. Thus one adds a simple root at a time each time getting

So the situation to examine is

$$h = g_0 \subset g_1 \subset g_2 \subset \cdots \subset g_\ell = g$$

$$[X_1, Y^\xi] = \qquad \begin{array}{cc} X_1 & Y^\xi \\ \vdots & \vdots \\ X_\ell & Y^n \end{array}$$

X_i, Y^j

How to prove irred.

$\sigma(\lambda + g) - g.$

$$\begin{tikzcd} 0 \arrow[r] & W \arrow[r] & U(g)\otimes_b \lambda \arrow[r] & W'' \arrow[r] & 0 \\ & U(g)\otimes_b \mu \arrow[u] \arrow[ur] & & & \end{tikzcd}$$

have to determine these only finitely many possibilities.

Any hope that we might be able to show that $\lambda(H_\alpha) = \ell \geq 0$ for some $\alpha \in \Sigma \implies U(\mathfrak{g}) \otimes_{\mathfrak{b}} \lambda$ by using a different Borel. I don't see any.

I don't see any results here!

Take $\alpha \in \Delta$ let $\mathfrak{g}_1 = \mathfrak{g}^{-\alpha} + \mathfrak{h} + \mathfrak{g}^{\alpha}$ and write

$$\mathfrak{g} = \mathfrak{g}_1 \oplus W_1 \oplus \cdots \oplus W_k$$

where $W_1, \ldots, W_k$ are the various α series.

Example In $sl(n)$: If α is the largest root, then there is an α-series for each positive root β starting with Y_β and the α series consists of $\mathfrak{g}^{-\beta}, \mathfrak{g}^{-\beta+\alpha}$, $\underset{\substack{\uparrow \\ \text{not a root} \\ \text{because too big.}}}{\mathfrak{g}^{-\beta+2\alpha}}$

For $sl(n)$ what do we mean by largest root? It's clearly $\lambda_1 - \lambda_n$ since for

$$(\lambda_1-\lambda_n)-(\lambda_i-\lambda_j) = \begin{cases}(\lambda_1-\lambda_i)+(\lambda_j-\lambda_n) \geq 0 \text{ with} \\ = \iff i=1,\ j=n.\end{cases}$$

Therefore for any root $\beta \in \Sigma$. we have that

$$2\frac{\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle} = \begin{cases} 2 & \beta=\alpha \\ 1 & \text{if } [X_{-\beta}, X_\alpha] \neq 0 \quad \beta<\alpha \\ 0 & \text{otherwise.}\end{cases} \qquad p+q$$

We can check this for $\mathfrak{sl}(n,\mathbb{R})$ because recall that up to a scalar $H_{(i,j)} = \delta_{ii} - \delta_{jj}$ so

$$\langle \beta_{ij};\ 1,n \rangle \sim \begin{cases} 0 & \text{if } 1<i<j<n \\ 1 & 1=i \quad j<n \\ 2 & 1=i \quad j=n \end{cases}$$

$$\langle \alpha, \alpha \rangle \sim 2.$$

Thus

$$\mathfrak{g} = \mathfrak{g}_1 + \sum_{\substack{\beta\in\Delta \\ [X_\beta, X_\alpha]=0}} \mathfrak{g}^{-\beta} + \sum_{\substack{\beta\in\Delta \\ [X_\beta X_\alpha]\neq 0 \\ \beta\neq\alpha}} \left(\mathfrak{g}^{-\beta} + \mathfrak{g}^{-\beta+\alpha}\right)$$

$$+ \sum_{\substack{\beta\in\Delta \\ [X_{-\alpha}, X_\beta] = 0}} \mathfrak{g}^{\beta}$$

centralizer of $\mathfrak{g}_1$

$-\beta+2\alpha = \alpha + (\alpha-\beta) > \alpha.$

$\therefore$ not a root

this holds in general when α is largest root

2	1	$^2X_\alpha$
1	0 centralizer of $\mathfrak{g}_1$ which is on outside in corners	1
$^2X_{-\alpha}$	1	2

Now see if this is any good for dominant wgt. reps. Suppose we ~~take~~ the corresponding decomposition of $U(\mathfrak{n}^-)\otimes\lambda$. Thus there are parts

---

Have basis for $\mathfrak{n}^-$ consisting of Y_β of three types: $\beta=\alpha$, β cent. α, β not. cent. α. ~~We can probably ignore the latter~~ $\neq\alpha$

Put $\mathfrak{g}_2 =$ cent of $\mathfrak{g}_1$.

$$\mathfrak{g} = \underbrace{\mathfrak{g}_1}_{\Delta_1} + \underbrace{\mathfrak{g}_2}_{\Delta_2} + \underbrace{\sum_{\substack{\beta\in\Delta\\ [X_\beta, X_\alpha]\neq 0\\ \beta\neq\alpha}}}_{\Delta_3} \left(\mathfrak{g}^{-\beta} + \mathfrak{g}^{\beta+\alpha}\right)$$

$\Delta_1 = \{\alpha\} \subset \Delta$

$\Delta_2 = \{\beta \mid 2\frac{\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle} = 0\}$

Now we have to look at

$\Delta_3 = \{\beta \mid 2\frac{\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle} = \pm 1\}$.

$$\mathfrak{n}^- = \mathfrak{n}_1^- + \mathfrak{n}_2^- + \sum_{\beta\in\Delta_3} \mathfrak{g}^{-\beta}$$

$$2\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} = \ell$$

Now we are given that ~~$\langle\lambda,\alpha\rangle = 0$~~ and we are trying to look for extreme vectors of weight

$$s_\alpha(\lambda+g) - g = \lambda - \ell\alpha + (s_\alpha g - g)$$

Now $g = \frac{1}{2}\sum_{\beta\in\Delta}\beta = \frac{1}{2}\left\{\alpha + \sum_{\beta\in\Delta_2}\beta + \sum_{\beta\in\Delta_3}\beta\right\}$

Case $\beta \in \Delta_2$ i.e. $\langle \beta, \alpha \rangle = 0$

$$\Longrightarrow \quad s_\alpha \beta = \beta.$$

Case $\beta \in \Delta_3$ ie $2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} = +1$ then

$$s_\alpha \beta = \beta - 2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha = \beta - \alpha$$

thus

$$g - s_\alpha g = \frac{1}{2}\Big\{\alpha - (-\alpha) + \sum_{\beta \in \Delta_2} \beta - \beta + \sum_{\beta \in \Delta_3} \beta - (\beta - \alpha)\Big\}.$$

$$= \alpha + \frac{1}{2}\sum_{\alpha \in \Delta_3} \alpha$$

and therefore

~~Note that Δ_3 is the roots of occurring in the parabolic grp~~

$$\boxed{\cancel{S_\alpha(\lambda + g) - g = \lambda - (l+1)\alpha - \frac{1}{2}\,\mathrm{card}\,\Delta_3\,\alpha}}$$

Now can you produce an interesting element of this weight?
(check with $sl(3)$ ~~$g - S_\alpha g = 2\alpha$ (I know)~~ $\overset{?}{=}$ ~~$1\alpha + \frac{1}{2}(\alpha_1 + \alpha_2)$~~.

$g = \alpha$ so $S_\alpha g = -\alpha$ $\Rightarrow$ $g - S_\alpha g = 2\alpha$.

But if $\lambda = 0$ $l = 0$ in fmla. $g - S_\alpha g = \alpha + $ ~~$\frac{1}{2}\sum_{\beta \in \Delta_3} \beta$~~

$$\boxed{S_\alpha(\lambda + g) - g = \lambda - \left(l + 1 + \frac{1}{2}\,\mathrm{card}\,\Delta_3\right)\alpha} \qquad l = 2\frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}$$

~~Can you produce an element~~ Suppose you know an element Φ of $U(\mathfrak{n}^-)$ of the weight with the property that

~~$[X_i, \Phi]$~~

~~$X_i \Phi \equiv 0 \mod U(\mathfrak{g})\mathfrak{n}^+$~~

$$X_j \Phi \in U(\mathfrak{g})(H_i - \lambda(H_i), \mathfrak{n}^+). \quad \text{for all } j.$$

February 19, 1968

Summary of preceding 2 months work on Lie algebras

The problem: Let $\mathfrak{g}$ be a semi-simple Lie alg. $/\mathbb{C}$, let θ be an involution of $\mathfrak{g}$, and let $\mathfrak{g} = k + p$ be the decomposition associated to θ. Classify simple $U(\mathfrak{g})$ modules which are inductive limits of finite dimensional k modules.

Elementary facts.

1) These are the irreducible objects in the abelian + locally noetherian category C of $\mathfrak{g}$ modules which ~~are~~ as k modules are sums of finite simple k modules. This category has a set of small projective generators $U(\mathfrak{g}) \otimes_k \Lambda$ where Λ runs over the finite simple k modules.

Thus a knowledge of $\mathrm{Hom}_{\mathfrak{g}}(U(\mathfrak{g}) \otimes_k \Lambda_1, U(\mathfrak{g}) \otimes_k \Lambda_2)$ for every Λ_1, Λ_2 together with composition should lead to solution of above problem.

~~2) **Proposition:** Let P be a projective object in an abelian category. There is a 1-1 correspondence between isom classes of simple objects V modules of P and simple $\mathrm{End}(P)^\circ$ modules given by $V \mapsto \mathrm{Hom}(P, V)$.~~

~~**Proof:** Let N be an irreducible $(\mathrm{End}\, P)^\circ$ modules, choose~~

2

with injective limits 2) <u>Proposition</u>: Let P be a projective object in ~~an~~ ~~(Grothendieck)~~ abelian category $\mathcal{A}$. Then there is a 1-1 correspondence between ~~the~~ isomorphism classes of simple objects M of $\mathcal{A}$ with $\text{Hom}(P,M) \neq 0$ and isomorphism classes of simple $(\text{End } P)^\circ$ modules given by $V \mapsto \text{Hom}(P,V)$.

<u>Proof</u>: Let N be a ~~simple~~ $(\text{End } P)^\circ$ module, let ~~choose a surjection~~ $\text{End } P \xrightarrow{\alpha} N$ be an $(\text{End } P)^\circ$ module map and let $W = \sum \text{Im}\beta \subset P$ where the sum is taken over all maps β with target P such that $\alpha \circ \text{Hom}(P,\beta) = 0$. Let $V = P/W$. Then there is a commutative diagram

$$\begin{tikzcd}
 & \text{Hom}(P,P) \arrow[dl, "\alpha"'] \arrow[dr, "\text{given by } \overbrace{\text{proj}}^{\pi : P \to V}\text{, onto as } P \text{ projective}"] & \\
N & & \text{Hom}(P,V) \arrow[ll, hook', "\text{by definition of } V"']
\end{tikzcd}$$

Suppose now that M is a simple object of $\mathcal{A}$ and that $N = \text{Hom}(P,M)$. If n is a non-zero element of N, let $\alpha : \text{Hom}(P,P) \to N$ send 1 to N. There is a commutative diagram

$$\begin{tikzcd}
 & P \arrow[dl, "n"'] \arrow[dr, "\pi"] & \\
M & & V \arrow[ll, dashed, "\theta"']
\end{tikzcd}$$

where the dotted arrow θ exists by construction of π. ~~Applying~~ ~~H~~ As M is simple θ is onto so as P is projective, $\text{Hom}(P,V) \to \text{Hom}(P,M) = N$ is onto. Thus α is onto ~~since N~~ so N is simple.

Conversely given a simple $\mathrm{Hom}(P,P)^\circ$ module N we show that. ~~[illegible] let [illegible] be surjective and~~ ~~[illegible]~~ if $\alpha : \mathrm{Hom}(P,P) \longrightarrow N$ is non-zero then the corresponding V is simple. Suppose $V \xrightarrow{\varphi} V'$ is surjective; then $N \underset{(\alpha \text{ surj})}{=} \mathrm{Hom}(P,V) \xrightarrow{\varphi_*} \mathrm{Hom}(P,V')$ is surjective. If $V' \neq 0$, then $\mathrm{Hom}(P,V') \neq 0$ (contains $\varphi\pi$, ~~surj~~ which is surj.), so N simple $\Rightarrow$ φ_* is an isom. Let $Q = \mathrm{Ker}\ \varphi\pi$ and let $\beta : Q \to P$ be the inclusion map. Then $\varphi_*(\pi\beta) = 0 \Rightarrow \pi\beta = 0$. ~~[illegible]~~ But $\pi\beta$ maps Q onto $\mathrm{Ker}\ \varphi$ so φ is an isomorphism + V is irreducible.

~~[illegible] It is clear that the isomorphism class of V is independent of the choice of α by irreducibility of N~~

Looking carefully at above two paragraphs one sees that $M \simeq V$ if $V = \mathrm{Hom}(P,M)$ and that $\mathrm{Hom}(P,V) = N$ which shows the correspondence is 1-1. QED.

3) Let $\mathcal{C}(\Lambda_1, \Lambda_2) = \mathrm{Hom}_{\mathfrak{g}}(U(\mathfrak{g}) \otimes_k \Lambda_1, U(\mathfrak{g}) \otimes_k \Lambda_2)$ if Λ_1, Λ_2 are finite semi-simple over k. Note that if $\Lambda = \Lambda_1 \oplus \Lambda_2$, then

$$\mathcal{C}(\Lambda, \Lambda) = \mathcal{C}(\Lambda_1, \Lambda_1) \oplus \mathcal{C}(\Lambda_1, \Lambda_2) \oplus \mathcal{C}(\Lambda_2, \Lambda_1) \oplus \mathcal{C}(\Lambda_2, \Lambda_2)$$

and hence a knowledge of $\mathcal{C}(\Lambda, \Lambda)$ for each Λ determines that of the whole category.

<u>Proposition:</u> $U(\mathfrak{k})\,U(\mathfrak{a})\,U(\mathfrak{k}) = U(\mathfrak{g})$.

<u>Proof:</u> Let $e : S(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$ be the canonical map. We know that $U(\mathfrak{k})\cdot e(S(\mathfrak{p})) = U(\mathfrak{g})$, ~~and~~ that ~~$e(S(\mathfrak{p}))$ is~~ $e(X^n) = X^n$, and that $S_n(\mathfrak{p})$ is spanned by elements of the form X^n where $X \in \mathfrak{p}$. Note that the ~~regular~~ span of the X^n with X regular semi-simple in $\mathfrak{p}$ ~~[illegible]~~ is a finite dimensional subspace of $S_n(\mathfrak{p})$ hence closed; ~~we~~ thus can assume X reg. semi-simple in which case X is K conjugate to an element of $\mathfrak{a}$. But $U(\mathfrak{k})U(\mathfrak{a})U(\mathfrak{k})$ is K stable so $U(\mathfrak{k})\,U(\mathfrak{a})\,U(\mathfrak{k})$ contains X^n for all n, $X \in \mathfrak{p}$ and so $U(\mathfrak{k})U(\mathfrak{a})U(\mathfrak{k}) = U(\mathfrak{g})$.

The proof also shows that $\sum\limits_{n,\,x\in\mathfrak{k}} (\operatorname{ad} x)^n \cdot U(\mathfrak{a}) = e(S(\mathfrak{p}))$.

<u>Consequently:</u> Suppose there is a map $U(\mathfrak{g}) \otimes_{\mathfrak{k}} \Lambda \hookrightarrow \mathrm{Hom}_{\mathfrak{k}}(U(\mathfrak{g}), \Lambda)$. Then

$$\mathrm{Hom}_{\mathfrak{g}}(U(\mathfrak{g})\otimes_{\mathfrak{k}}\Lambda_1, U(\mathfrak{g})\otimes_{\mathfrak{k}}\Lambda) \hookrightarrow \mathrm{Hom}_{\mathfrak{k}}(\Lambda_1, \mathrm{Hom}_{\mathfrak{k}}(U(\mathfrak{g}), \Lambda))$$

$$= \mathrm{Hom}_{\mathfrak{k}\times\mathfrak{k}}(U(\mathfrak{g}), \mathrm{Hom}(\Lambda_1, \Lambda))$$

$$\hookrightarrow \mathrm{Hom}_{\hat{M}}(U(\mathfrak{a}), \mathrm{Hom}(\Lambda_1, \Lambda)) \qquad \text{from proposition}$$

$$= \mathrm{Hom}_{W}(U(\mathfrak{a}), \mathrm{Hom}_{M}(\Lambda_1, \Lambda)).$$

February 20, 1968

4). We try to determine what we can about $C(\Lambda_1, \Lambda_2)$ by using the natural filtration on differential operators. Thus define

$$\theta \in F_n \operatorname{Hom}_{\mathfrak{g}}(U(\mathfrak{g})\otimes_k \Lambda_1, U(\mathfrak{g})\otimes_k \Lambda_2) \Leftrightarrow \theta(1\otimes\Lambda_1) \subset F_n U(\mathfrak{g})\otimes_k \Lambda_2.$$

~~Then~~

Let ~~$gr\, C(\Lambda_1,\Lambda_2) \simeq \operatorname{Hom}$~~ $\tilde{\mathfrak{g}}$ = the semi-direct product of k ~~$\mathfrak{g}$~~ and $\mathfrak{p}$ where $\mathfrak{p}$ is considered to be abelian.

Proposition: ~~$gr\, C(\Lambda_1,\Lambda_2)$~~ There is a canonical isomorphism $gr\, C(\Lambda_1, \Lambda_2) \simeq \operatorname{Hom}_{\tilde{\mathfrak{g}}}(U(\tilde{\mathfrak{g}})\otimes_k \Lambda_1, U(\tilde{\mathfrak{g}})\otimes \Lambda_2)$ which is compatible with composition.

Proof: Definition of the map. ~~If $\theta \in F_n C(\Lambda_1, \Lambda_2)$ then $\theta(1\otimes\Lambda_1) \subset F_n U(\mathfrak{g})\Lambda_2$ so we get a map [illegible] $\otimes\Lambda_1 \to$~~

$$\begin{tikzcd}
gr \operatorname{Hom}_{\mathfrak{g}}(U(\mathfrak{g})\otimes_k \Lambda_1, U(\mathfrak{g})\otimes_k \Lambda_2) \arrow[d, equal] & \\
gr \operatorname{Hom}_k(\Lambda_1, U(\mathfrak{g})\otimes_k \Lambda_2) \arrow[d, equal, "\text{complete red.}"] & \\
\operatorname{Hom}_k(\Lambda_1, gr\, U(\mathfrak{g})\otimes_k \Lambda_2) \arrow[d, equal] & \operatorname{Hom}_k(U(\tilde{\mathfrak{g}})\otimes_k \Lambda_1, U(\mathfrak{g})\otimes_k \Lambda_2) \arrow[d, equal] \\
\operatorname{Hom}_k(\Lambda_1, S(\mathfrak{p})\otimes \Lambda_2) \arrow[r, equal] & \operatorname{Hom}_k(\Lambda_1, U(\tilde{\mathfrak{g}})\otimes_k \Lambda_2)
\end{tikzcd}$$

In other words if $\theta \in C(\Lambda_1, \Lambda_2)$ carries Λ_1 into $e(F_n S(\mathfrak{p}))\cdot\Lambda_2$ then modulo $F_{n-1} U(\mathfrak{g})\Lambda_2$ we get $\bar{\theta} : \Lambda_1 \longrightarrow S_n(\mathfrak{p}) \otimes \Lambda_2$. Now for composition. Suppose that we are given

$$\theta_1 : \Lambda_1 \longrightarrow U(\mathfrak{g}) \otimes_k \Lambda_2 \qquad \text{of degree } m$$

$$\theta_2 : \Lambda_2 \longrightarrow U(\mathfrak{g}) \otimes_k \Lambda_3 \qquad \text{of degree } n$$

then we write $\theta_1(\lambda) = \sum_i a_i \varphi_i(\lambda)$ where $a_i \in e[S(\mathfrak{p})]$ + $\varphi_i \in \mathrm{Hom}(\Lambda_1, \Lambda_2)$ similarly $\theta_2(\lambda) = \sum b_i \psi_i(\lambda)$. Then by definition. $\theta_1(u \cdot \lambda) = u \cdot \theta_1(\lambda)$ if $u \in U(\mathfrak{g})$ so

$$\theta_2(\theta_1(\lambda)) = \sum' a_i\, \theta_2(\varphi_i(\lambda))$$
$$= \sum' a_i b_j \psi_j \varphi_i(\lambda)$$

Looking at terms of highest degree one sees that $\overline{\theta_2 \theta_1} = \bar{\theta}_2 \bar{\theta}_1$ QED.

<u>Remarks:</u> Thus $\mathrm{gr}\, C(\Lambda_1, \Lambda_2) = \mathrm{Hom}_{\tilde{\mathfrak{g}}}(S(\mathfrak{p}) \otimes \Lambda_1, S(\mathfrak{p}) \otimes \Lambda_2)$. Now use Rallis

$$\mathrm{Hom}_{\tilde{\mathfrak{g}}}(S(\mathfrak{p}) \otimes \Lambda_1, S(\mathfrak{p}) \otimes \Lambda_2) \simeq \mathrm{Hom}_k(\Lambda_1, S(\mathfrak{p}) \otimes_k \Lambda_2)$$

$$\cong S(\mathfrak{p})^k \otimes \mathrm{Hom}_k(\Lambda_1, H \otimes \Lambda_2) \qquad \text{as graded modules}$$

$$\cong S(\mathfrak{p})^k \otimes \mathrm{Hom}_M(\Lambda_1, \Lambda_2) \qquad \text{but the grading is incorrect.}$$

To get grading on $\mathrm{Hom}_M(\Lambda_1, \Lambda_2)$ one must take a limit over K conjugates of M.

5) <u>The map E.</u> Let $\mathfrak{g} = \mathfrak{k}+\mathfrak{a}+\mathfrak{n}$ be an Iwasawa decomposition of $\mathfrak{g}$ and let $M \subset K$ be the centralizer of $\mathfrak{a}$. Here K is the simply-connected complex group with Lie algebra $\mathfrak{k}$ and K acts on the $\mathfrak{g}$ modules under consideration. If $V \in C$ then $V/\mathfrak{n}V$ is an $\mathfrak{a}$ module and an M module and these actions commute. This is because:

<u>Proposition:</u> M normalizes $\mathfrak{n}$.

<u>Proof:</u> Let $\mathfrak{m}$ be the Lie algebra of M. Then by Iwasawa $\mathfrak{m} = \mathfrak{h}_k + \sum_{\alpha\in\Sigma''} (e_\alpha)+(e_{-\alpha})$, $\mathfrak{n} = \sum_{\alpha\in\Sigma'} (e_\alpha)$. As $[\mathfrak{h},\mathfrak{n}] \subseteq \mathfrak{n}$ and

$\Sigma' = \{\alpha\in\Sigma \mid \theta\alpha \text{ change sign}\}$ $\Sigma'' = \{\alpha \mid \theta\alpha=\alpha\}$.
$\Rightarrow$ if $\gamma = \alpha+\beta \in \Sigma$ & $\alpha \in \Sigma'$ & $\beta\in\Sigma''$, then $\gamma \in \Sigma''$. Thus $[\mathfrak{m}, \mathfrak{n}] \subset \mathfrak{n}$. Finally we note that

? | Under the hom. $M \to \operatorname{Ad}\mathfrak{g}$, the components are represented by elements in $\bar{K}\cap A$ which will normalize $\mathfrak{n}$.

So I need

<u>Lemma:</u> Let $F = \{x\in K \mid \rho x \in A\}$ where $\rho : K \to \operatorname{End}\mathfrak{g}$ is the adjoint representation and $A = \exp\mathfrak{a}$.) Then $M = F\cdot M^0$.

for adjoint group.

Assume lemma holds. Why does M action on V commute with $\mathfrak{a}$ action? The point is that if $m \in M$ and $X \in \mathfrak{a}$, then we write $m = \exp Q$ where $Q \in \mathfrak{k}$. Then

$$\exp tQ\,(X \cdot v) = [\mathrm{Ad}(\exp tQ)\, X] \cdot (\exp tQ\, v).$$

as is clear from power series which converges by $\mathfrak{k}$-finiteness. So if t such that things commute we have that $m(X v) = [(\mathrm{Ad}\, m) X] \cdot mv = X \cdot mv.$

Proof of lemma: It is clear that F contains the ~~center~~ kernel of ρ so we need only show that $\bar{M} = \bar{F} \cdot \bar{M}_0$ where $-$ denotes image under ρ. Note that $\bar{F} = \bar{K} \cap A$. But let θ be the involution of the adjoint group $\bar{G}$. Then θ preserves the centralizer C of $\mathfrak{a}$ which is connected (general fact that the centralizer of a semi-simple element x in a reductive gp is connected — in effect write such an ~~elt.~~ elt b of cent. in the form $b = a e^{\mathrm{ad}\, y}$ where a semi-simple y nilpotent + $a * y = y$. $b = e^{t\,\mathrm{ad} x}\, b\, e^{-t\,\mathrm{ad} x} = e^{t\,\mathrm{ad}x} a e^{-t\,\mathrm{ad}x} e^{\mathrm{ad}(e^{t\,\mathrm{ad}x} * y)}$. By uniqueness $a \in G^x$ $y \in \mathfrak{g}^x$. Thus b connectable to a. So may assume b semi-simple whence may embed b in a Cartan ~~subalgebra of $\mathfrak{g}$~~ subgroup of G, whose subalgebra contains x since x semi-simple. Then as Cartan subgroups are conn. done).

Now $\bar{M} = C \cap K$, and $C^\theta = \tilde{F} M_0$ where $\tilde{F} = C^\theta \cap A$. Thus $\bar{M}/M_0$ is an element 2 abelian gp.

Basic Question: Even when Λ is not simple is $U(\mathfrak{g})^k \longrightarrow C(\Lambda, \Lambda)$ onto?

NO Recall proof. Have

$$U(\mathfrak{g}) \longrightarrow \mathrm{Hom}(\Lambda, U(\mathfrak{g}) \otimes_k \Lambda)$$
$$x \longmapsto (\lambda \mapsto x \otimes \lambda)$$

This map is onto. In effect an elt on the right is of the form $\sum_i e(a_i)\varphi_i$ where a_i basis for $S(\mathfrak{p})$ and $\varphi_i \in \mathrm{Hom}(\Lambda, \Lambda)$. But as Λ simple $\varphi_i(\lambda) = b_i\lambda$ where $b_i \in U(k)$ so the elt on the right comes from $\sum e(a_i) b_i$ on left. This shows

$$U(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{a} = eS(\mathfrak{p}) \otimes U(k)/\mathfrak{a} = \mathrm{Hom}(\Lambda, U(\mathfrak{g}) \otimes_k \Lambda)$$

where $\mathfrak{a}$ = annihilator ideal of Λ. Now let k act on LHS by adjoint action + right HS same + get

$$U(\mathfrak{g})^k \twoheadrightarrow \left(U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{a}\right)^{\mathrm{Ad}\,k} \overset{\sim}{\longrightarrow} \mathrm{Hom}_k(\Lambda, U(\mathfrak{g}) \otimes_k \Lambda)$$

Remark: This shows there is something special about simple Λ. In particular maybe for simple Λ the restriction map $grE_\Lambda \to [S(\mathfrak{a}) \otimes \mathrm{Hom}_M(\Lambda, \Lambda)]^W$ might be an isomorphism. My old idea was that this was impossible because could always embed any irred. into $\mathrm{Hom}(\Lambda, \Lambda)$ for suitable Λ, however this is wrong for $\mathfrak{sl}(2)$ since only ~~odd~~ even $S_k V$ can be embedded in $\mathrm{Hom}(S_k V, S_k V)$

<u>Hope:</u> $\left[S(\mathfrak{p}) \otimes \mathrm{Hom}(\Lambda,\Lambda)\right]^{k} \xrightarrow{\sim} \left[S(\mathfrak{a}) \otimes \mathrm{Hom}_{M}(\Lambda,\Lambda)\right]^{W}$ if Λ irred.

It seems that H-C must have proved this in order to assert his formula for spherical functions.

Start with a function $f : \mathfrak{p} \to \mathrm{Hom}(\Lambda,\Lambda)$ k-equivariant where k acts on a hom. φ by $X * \varphi = X\varphi - \varphi X$. We have to show that if f vanishes on the singular semi-simple elements then f vanishes on all singular elements. ~~We can construct a counterexample by concentrating on [illegible] an orbit closure~~ We ~~[illegible]~~ note that the equivariant functions ~~we are looking at form an ideal~~ f which vanish on the regular semi-simple elements form ~~[illegible]~~ an ideal in the ring of all equivariant functions. Observe that if $f : \mathfrak{p} \to \mathrm{Hom}(\Lambda,\Lambda)$ has image generating a k rep ν then $f^{t} : \mathfrak{p} \to \mathrm{Hom}(\Lambda,\Lambda)$ has image generating a k rep ν^{*} and consequently we can look at $\mathrm{tr}\, f^{t} f$. Can also consider $\det f$

February 21, 1968

6) Theorem: (Harish-Chandra). Every simple $\mathfrak{g}k$ module appears as a composition quotient of an induced module $I(\xi)$ for some simple $M\times\mathfrak{a}$ module ξ.

Proof: Recall the functor $F: V \mapsto V/\mathfrak{n}V$ gives rise to a homomorphism

$$(\ ) \qquad F: \mathcal{E}_\Lambda \longrightarrow U(\mathfrak{a}) \otimes \operatorname{Hom}_M(\Lambda, \Lambda).$$

Also there is a can. isom

$$\operatorname{Hom}_{\mathfrak{g}}(U(\mathfrak{g}) \otimes_k \Lambda, I\xi) = \operatorname{Hom}_{M\times\mathfrak{a}}(1 \otimes_{\mathfrak{n}} U(\mathfrak{g}) \otimes_k \Lambda, \xi)$$

$$= \operatorname{Hom}_{M\times\mathfrak{a}}(U(\mathfrak{a}) \otimes \Lambda, \xi)$$

$$= \operatorname{Hom}_M(\Lambda, \xi).$$

for any finite semi-simple k module Λ and finite $M\times\mathfrak{a}$ module ξ. First claim is that this isomorphism is a dihomomorphism of right modules for the map F. (clear ~~[illegible]~~ since $H_0(\mathfrak{n}, \cdot)$ and I ~~being~~ are adjoint functors.) Need the following

Lemma 1: F is injective and both rings are finite modules over Z.

Lemma 2: If $R \hookrightarrow S$ is an injection of finite algebras over a commutative noetherian ring, then every simple R modules appears as a composition quotient of some simple S module.

Now given a simple $\mathfrak{g}$-k module V we know that $\mathrm{Hom}_k(\Lambda, V)$ is a simple right $\mathcal{E}_\Lambda$ modules. Let $\eta \in \mathrm{Hom}_k(\Lambda, V)$ be $\neq 0$ and let $J \subset \mathcal{E}^0_\Lambda$ be the ~~right~~ left ideal annihating η. Any right simple $U(\mathfrak{a}) \otimes \mathrm{Hom}_M(\Lambda, \Lambda)$ module is of the form $\mathrm{Hom}_M(\Lambda, \zeta)$ for a unique simple $M \times \mathfrak{a}$ module $\zeta = \nu \otimes \lambda$. ($M$ is reductive in K hence Λ is ampletely reducible over M, which means that $\mathrm{Hom}_M(\Lambda, \Lambda)$ is a product of matrix rings corresp. to the simple M modules ν contained in Λ). By lemma's 1 & 2 $\mathrm{Hom}_k(\Lambda, V)$ appears as a comp. quotient in $\mathrm{Hom}_M(\Lambda, \zeta)$ for some ζ. This means that there is an element $\theta \in \mathrm{Hom}_M(\Lambda, \zeta)$ whose annihilator $J' \subset \mathcal{E}^0_\Lambda$ is contained in J.

θ defines a map $\theta^\#: U(\mathfrak{g}) \otimes_k \Lambda \longrightarrow I(\zeta)$ whose image we will denote by W. Thus we have commutative triangles

$$\begin{tikzcd}
& \mathcal{E}_\Lambda \arrow[dl] \arrow[dr] & \\
\eta \cdot \mathcal{E}_\Lambda \arrow[d, phantom, "\parallel"] & & \theta^\# \cdot \mathcal{E}_\Lambda \arrow[ll, dashed] \arrow[d, phantom, "\parallel"] \\
\mathcal{E}_\Lambda / J & & \mathcal{E}_\Lambda / J'
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathrm{End}_{\mathfrak{g}}(U(\mathfrak{g}) \otimes_k \Lambda) \arrow[dl, "\eta \circ"'] \arrow[dr, "\theta^\# \circ"] & \\
\mathrm{Hom}_k(\Lambda, V) & & \mathrm{Hom}_k(\Lambda, W) \arrow[ll, dashed]
\end{tikzcd}$$

i.e. we have a commutative triangle

$$\begin{tikzcd}
& U(\mathfrak{g}) \otimes_k \Lambda \arrow[dl] \arrow[dr, "\theta^\#"] & \\
V & & W \subset I(\zeta) \arrow[ll, dashed, "\alpha"']
\end{tikzcd}$$

The reason α exists is because if one goes back to the proof of one knows that the kernel of $\Theta^{\#}$ gives a map $\beta : \operatorname{Ker} \Theta^{\#} \longrightarrow U(\mathfrak{g}) \otimes_k \Lambda$ which is such that

$$\operatorname{Hom}(P, \operatorname{Ker} \Theta^{\#}) \longrightarrow \operatorname{Hom}(P, P) \longrightarrow \operatorname{Hom}(P, V)$$

is zero and therefore by the formula for V, β goes to 0 in V.

~~i.e. $U(\mathfrak{g})\otimes_k \Lambda \twoheadrightarrow W \twoheadrightarrow V$ and so V will appear as a composition quotient in $I(\mathcal{J})$. Note that we need only assume that $\mathrm{Hom}_k(\Lambda, V)$ appears as a composition quotient of $\mathrm{Hom}_M(\Lambda, \mathcal{J})$.~~

Proof of

Lemma 2: Let $R \hookrightarrow S$ be an injective map of finite algebras over a commutative (noetherian) ring $\mathbf{Z}$. Then every simple R module appears as a Jordan-Hölder component of a simple S module.

Proof: Let Λ_1 be a simple R module; we know that it is finite over $Z/\mathfrak{m}$ for some max. ideal in $\mathbf{Z}$. By Krull $\exists\, n$ with $\mathfrak{m}R \supset R \cap \mathfrak{m}^n S$. Then $R/\mathfrak{a} \hookrightarrow S/\mathfrak{m}^n S$ where $\mathfrak{a} = R \cap \mathfrak{m}^n S$ and Λ_1 is a simple $R/\mathfrak{a}$ module. In this case we have reduced to the case where R and S are finite over a field. Note S is a faithful R module. The associated semi-simple module is faithful for $R/\mathrm{rad}\, R$ (if N is an ideal killing all comp. quotients of a faithful R module that $N^k = 0$ so $N \subset \mathrm{rad}\, R$) so every simple R module appears as a composition quotient of S. But R comp. quotients are the same as the R comp quotients of the S comp. quotients of S. QED a Miscellaneous result

~~$Z \longrightarrow \mathcal{E}_\Lambda \xhookrightarrow{F} \mathcal{U}(\mathfrak{a}) \otimes \mathrm{Hom}_M(\Lambda, \Lambda).$~~

~~both rings modules are finite over the center!~~

~~Now let Supp suppose Λ_1 irreducible over $\mathcal{E}_\Lambda$. and let $\mathfrak{a}$ be the annihilator ideal $\mathfrak{a} \subset \mathcal{E}_\Lambda$.~~

Lemma: Let $R \hookrightarrow S$ be finite algebras over $\mathbb{C}$. Then every irreducible rep. of R occurs in ~~some~~ irred. repn. of S

~~$\Longleftarrow$~~ the radical of $R = $ (the radical of $S) \cap R$

Proof: $(\Longleftarrow)$ ~~Let Λ_1 be a simple R module occurring within. Replace R, S by their~~ Then $R/N_R \hookrightarrow S/N_S$ so may assume that R and S are semi-simple. In this case S is a ~~left~~ right module hence projective and obviously faithfully flat. Thus if Λ_1 is a simple R module $S \otimes_R \Lambda_1 \neq 0$ so there will be a non-zero map $S \otimes_R \Lambda_1 \to \Lambda_2$ for some simple S module Λ_2, whence Λ_1 occurs in Λ_2.

~~$(\Longrightarrow)$ Thus is The semi-simple algebra R/N_R has a faithful set semi-simple representation V which by assumption will occur inside of some semisimple representation of S. Thus $N_R \subset N_S$~~

$\Longrightarrow$ false not even true that rad $R \subset$ rad S. e.g. take R alg. of polys in a nilp. matrix in $S = \mathrm{Hom}(V, V)$. Then every irred rep of R occurs in the irred rep. of S.

Proof of lemma 1: Recall

$$\begin{tikzcd}
\mathrm{Hom}_{\mathfrak{g}}(U(\mathfrak{g})\otimes_k\Lambda_1, U(\mathfrak{g})\otimes_k\Lambda_2) \arrow[r, "\text{induced by functor } V\mapsto V/\mathfrak{n}V."] \arrow[dr, "F"'] & \mathrm{Hom}_{M\times\mathfrak{a}}(U(\mathfrak{a})\otimes\Lambda_1, U(\mathfrak{a})\otimes\Lambda_2) \arrow[d, equal] \\
 & U(\mathfrak{a})\otimes\mathrm{Hom}_M(\Lambda_1,\Lambda_2)
\end{tikzcd}$$

Thus have that if $\theta = \sum a_i\varphi_i$ $a_i\in U(\mathfrak{n}+\mathfrak{a})$; $\varphi_i\in\mathrm{Hom}(\Lambda_1,\Lambda_2)$ then ~~$\mathrm{Hom}_k(\Lambda_1, U(\mathfrak{g})\otimes_k\Lambda_2)$~~ $F(\theta) = \sum\varepsilon(a_i)\varphi_i$ where $\varepsilon: U(\mathfrak{n}+\mathfrak{a})\to U(\mathfrak{a})$ sends $\mathfrak{n}$ to zero. We wish to determine

$$\mathrm{gr}\,F : \mathrm{Hom}_k(\Lambda_1, S(\mathfrak{p})\otimes\Lambda_2)\longrightarrow S(\mathfrak{a})\otimes\mathrm{Hom}_M(\Lambda_1,\Lambda_2).$$

So assume θ of degree n i.e. that the $a_i\in F_nU(\mathfrak{n}+\mathfrak{a})$. Note that $U(\mathfrak{n}+\mathfrak{a})$ and $e(S(\mathfrak{p}))$ are both coset representatives of $U(\mathfrak{g})$ modulo $U(\mathfrak{g})k$, hence if $a_i\in F_nU(\mathfrak{n}+\mathfrak{a})$ and $b_i\in F_n e(S(\mathfrak{p}))$ and $a_i-b_i\in U(\mathfrak{g})k$, then $a_i - b_i = \sum_j c_{ij}X_j$ where $c_{ij}\in F_{n-1}U(\mathfrak{g})$ and X_{ij} is a basis for k. Thus if $-$ denotes leading coefficient we have that

$$\sum\bar{a}_i\varphi_i(\lambda) - \sum\bar{b}_i\varphi_i(\lambda) = \sum\overline{c_{ij}}(X_j\cdot\varphi_i(\lambda)) = 0$$

Therefore $\mathrm{gr}\,F$ is the map induced by sending $\mathfrak{p} = \mathfrak{g}/k \cong \mathfrak{a}+\mathfrak{n}\to\mathfrak{a}$ ie by sending $\mathfrak{g}\to\mathfrak{g}/k+\mathfrak{n}\simeq\mathfrak{a}$. But k and $\mathfrak{n}$ are both orthogonal to $\mathfrak{a}$ by Iwasawa decomp. <u>Conclusion:</u> $\mathrm{gr}\,F$ is induced by orthogonal projection map $\mathfrak{p}\to\mathfrak{a}$.

Thus we are reduced to showing that the map $\mathrm{gr}\,F$ is injective and that both sides are finite modules over $S(\mathfrak{g})^{\mathfrak{g}}$.

The injectivity is easy; we have to show that

$$\left[S(\mathfrak{p}) \otimes \mathrm{Hom}(\Lambda_1, \Lambda_2)\right]^{k} \longrightarrow S(\mathfrak{a}) \otimes \mathrm{Hom}_{M}(\Lambda_1, \Lambda_2)$$

is injective, i.e. that a k-invariant polynomial function on $\mathfrak{p}'$ is determined by its restriction to $\mathfrak{a}'$. But that's clear because every semi-simple element is conjugate to an element of $\mathfrak{a}$, so a k equivariant function which is 0 on $\mathfrak{a}$ would be zero on $\mathfrak{p}$ regular and hence identically zero.

To show finiteness it suffices to show $S(\mathfrak{a})^{W}$ finite over $S(\mathfrak{g})^{\mathfrak{g}}$ $\simeq S(h)^{\tilde{W}}$, $\tilde{W}$ = full Weyl group. But $S(\mathfrak{a})$ finite over $S(h)$ which is finite over $S(h)^{\tilde{W}}$.

<u>Remarks</u>: Note that $S(h)^{\tilde{W}} \longrightarrow S(\mathfrak{a})^{W}$ is surjective off the singular locus of $\mathfrak{a}'$. In effect if two regular elements of $\mathfrak{a}$ are $\tilde{W}$ conjugate then as the element of $\tilde{W}$ conjugating them lies in the adjoint group of $\mathfrak{g}$ this element will normalize $\mathfrak{a}$ and so be in the baby Weyl group W.

Remarks: We conjectured at one time that the map

$$-:\left(S(\mathfrak{p})\otimes\Lambda\right)^{k}\hookrightarrow\left(S(\mathfrak{a})\otimes\Lambda^{M}\right)^{W}$$

given by orth. projection $\mathfrak{p}\to\mathfrak{a}$ would be an isom. It is always injective. ~~By but by suitable~~ Claim its onto if tensored with $\mathbb{C}[1/D]$ where D is the invariant polynomial describing the singular locus. By suitable choice of Λ we were able to produce examples where it was not onto, e.g. by finding functions in $S(\mathfrak{p})$ which vanish on the semi-simple singular elements but not all singular elements. There then arose the problem of whether these bad Λ's occur in things of the form $\mathrm{Hom}(\Lambda_1,\Lambda_1)$ where Λ_1 is irreducible. But notice if Λ is bad, then $\mathrm{Hom}(\Lambda', H)\neq 0$ so the zero weight occurs in Λ' hence also in Λ. But by PRV if the dominant weight of Λ_1 is suff. high, then Λ occurs in $\mathrm{Hom}(\Lambda_1,\Lambda_1)$ exactly $\dim \mathrm{Hom}_{\mathfrak{h}}(1,\Lambda)$ times.

Conclusion: It's false in general that

$$\left[S(\mathfrak{p})\otimes\mathrm{Hom}(\Lambda,\Lambda)\right]^{k}\longrightarrow\left[S(\mathfrak{a})\otimes\mathrm{Hom}_{M}(\Lambda,\Lambda)\right]^{W}$$

be an isomorphism for Λ irreducible.

February 22, 1968

Some conjectures:

a) $I(\wp)$ finite length.

~~b) $I(\wp)$ and $I(\sigma * \wp)$~~

b) $I(\wp_1)$ and $I(\wp_2)$ have same composition quotients if $\wp_1$ and $\wp_2$ are "related" by the Weyl group W, otherwise they are disjoint.

If the disjointness assertion is correct, then one sees that the PRV list $\hat{\pi}_\wp$ is incomplete. In effect ~~a composition~~ if $\wp$ is totally integral, then $\hat{\pi}_\wp$ is a finite dimensional repn. and the other composition quotients of $I(\wp)$ can appear nowhere else.

c) A natural map $I(\wp) \longrightarrow I(\sigma * \wp)$ should be the same as a morphism of functors $V/\mathfrak{n}V \longrightarrow V/\mathfrak{n}^\sigma V$.

7) <u>Proposition</u>: $I(\xi)$ is simple $\Leftrightarrow$ for every finite semi-simple k module Λ, $\mathrm{Hom}_M(\Lambda, \xi)$ is a simple $\mathcal{E}_\Lambda^\circ$ module.

($\Leftarrow$)
<u>Proof</u>: It **suffices** to show **that** every non-zero map $\alpha : U(\mathfrak{g}) \otimes_k \Lambda \longrightarrow I(\xi)$ with Λ simple is surjective. i.e. given $\alpha \in \mathrm{Hom}_M(\Lambda, \xi)$ $\alpha \neq 0$ and $\beta \in \mathrm{Hom}_M(\Lambda_1, \xi)$ there is a map $\theta \in \mathcal{C}(\Lambda_1, \Lambda)$ such that $\alpha \circ \theta = \beta$ ie.

$$\begin{tikzcd}
& U(\mathfrak{g}) \otimes_k \Lambda \arrow[dr, "\alpha"] & \\
U(\mathfrak{g}) \otimes_k \Lambda_1 \arrow[ur, dotted, "\theta"] \arrow[rr, "\beta"] & & I(\xi)
\end{tikzcd}$$

But consider $\mathcal{E}(\Lambda_1 + \Lambda) = \mathcal{E}_{\Lambda_1} \oplus \mathcal{C}(\Lambda_1, \Lambda) \oplus \mathcal{C}(\Lambda, \Lambda_1) \oplus \mathcal{E}_{\Lambda_2}$ acting on $\mathrm{Hom}_M(\Lambda_1 \oplus \Lambda, \xi)$. This is simple hence given $\alpha \neq 0$ and β can find φ with $\alpha\varphi = \beta$. Taking into account the grading we see that we may take $\varphi \in \mathcal{C}(\Lambda_1, \Lambda)$ so we get the desired θ.

($\Rightarrow$) We know that V simple $\Rightarrow$ $\mathrm{Hom}_k(\Lambda, V)$ simple over $\mathcal{E}_\Lambda^\circ$. But $\mathrm{Hom}_k(\Lambda, I(\xi)) = \mathrm{Hom}_M(\Lambda, \xi)$.

QED.

Berts method for constructing a map $I(\gamma) \longrightarrow I(\sigma * \gamma)$:
Think of elements of $I(\gamma)$ as sections of a ~~the~~ bundle ~~which are~~ over G/MA which are flat over the polarization given by N. Then given another polarization N' integrate with respect to the natural volume element to get ~~an element~~ [illegible] N' flat elements.

Example: Suppose the symplectic manifold is $\mathbb{R}^2$ $\Omega = dp\,dq$.
Then we should consider functions on $\mathbb{R}^2$ with

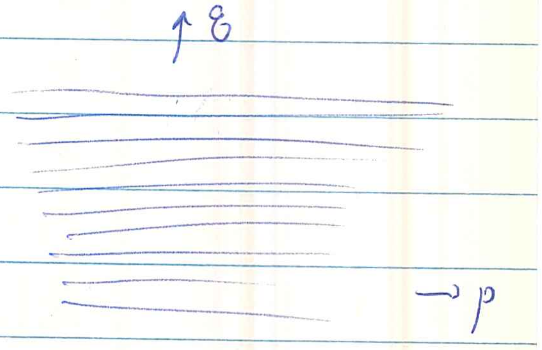
$$\nabla_X f = (X + c\eta(X))f. \qquad \eta = p\,dq$$

so that $D1 = c\eta$ and $K = D^2 1 = c\,d\eta = \Omega$.
∴ want $c = 1$.

Restrict attention to $\frac{\partial}{\partial q}$ flat functions, i.e.

$$\frac{\partial f}{\partial q} + pf = 0$$

$$f = e^{-pq} g(p).$$



Now integrate along the N' $\frac{\partial}{\partial p}$ directions.

$$f(q) = \int e^{-pq} g(p)\, dp$$

So I start with $f \in I(\xi)$ ie

$$f : G \longrightarrow \xi$$

$$f(bg) = \xi(b) f(g).$$

Thus f NUTS.

By my old work on Bruhats thesis I was led to feel that there should exist ~~a natural to~~ an automorphism $\xi \longmapsto \alpha * \xi$ of the category of $M \times \mathfrak{a}$ modules associated to each element α of K normalizing $\mathfrak{a}$, together with ~~the~~ a natural transformation

$$\Gamma_\alpha : I \longrightarrow I\alpha.$$

Thus ~~for each $\alpha \in N_K(\mathfrak{a})$~~ have a map homomorphism

$$N_K(\mathfrak{a}) \longrightarrow \text{Aut } (M \times \mathfrak{a} \text{ modules})$$

$$\alpha \qquad\qquad (\xi \longmapsto \alpha\xi).$$

$$(\alpha\beta)(\xi) = \alpha(\beta(\xi)).$$

Also need a natural transformation

$$\Gamma_\alpha : I\xi \longrightarrow I\alpha\xi\ .$$

such that

$$\begin{tikzcd}
I\xi \arrow[rr, "\Gamma_\beta"] \arrow[dr, swap, "\Gamma_{\alpha\beta}"] & & I\beta\xi \arrow[dl, "\Gamma_\alpha * \beta"] \\
& I\alpha\beta\xi &
\end{tikzcd}$$

$$\boxed{\Gamma_{\alpha\beta} = (\Gamma_\alpha * \beta) \circ \Gamma_\beta.}$$

Proposition: Γ_α equivalent to a natural transf $\theta_\alpha : F \longrightarrow \alpha^{-1} \circ F$.

Proof:

$$\begin{tikzcd}
\mathrm{Hom}_{g\text{-}k}(V, I(\xi)) \arrow[r, "\Gamma_\alpha \circ"] \arrow[d, equal] & \mathrm{Hom}_{g\text{-}k}(V, I\alpha\xi) \arrow[d, equal] \\
\mathrm{Hom}_{M\times\sigma}(FV, \xi) \arrow[dr] & \mathrm{Hom}_{M\times\sigma}(FV, \alpha\xi) \arrow[d, equal] \\
 & \mathrm{Hom}_{M\times\sigma}(\alpha^{-1}FV, \xi)
\end{tikzcd}$$

so by universal property we get $\theta_\alpha : FV \longrightarrow \alpha^{-1}FV$ natural in V. <u>QED</u>.

The transitivity property means that

$$\begin{tikzcd}
F \arrow[r, "\theta_\beta"] \arrow[d, "\theta_{\alpha\beta}"'] & \beta^{-1}F \arrow[dl, "\beta^{-1} * \theta_\alpha"] \\
\beta^{-1}\alpha^{-1}F &
\end{tikzcd}$$

$$\boxed{\theta_{\alpha\beta} = (\beta^{-1} * \theta_\alpha) \circ \theta_\beta}$$

Old Conjecture: There is some natural way of defining $\Gamma_\alpha : I\mathfrak{f} \longrightarrow I(\alpha\,\mathfrak{f})$, where

$$\alpha\cdot\mathfrak{f} = (\mathfrak{f}\otimes g^{-1})^{\alpha}\otimes g$$

where $$\mathfrak{f}^{\alpha}(x) = \mathfrak{f}(\alpha^{-1}x\alpha).$$

Check variance:

~~$(\alpha\cdot\beta\cdot\mathfrak{f})\otimes g^{-1} = (\beta\,\mathfrak{f}\otimes g^{-1})^{\alpha}$~~

~~$= ((\mathfrak{f}\otimes g^{-1})^{\beta})^{\alpha}$~~

~~$= (\mathfrak{f}\otimes g^{-1})^{\beta}(\alpha^{-1}\quad\alpha)$~~

~~$= (\mathfrak{f}\otimes g^{-1})($~~

~~$\alpha(\beta\mathfrak{f})\otimes g^{-1} = [(\beta\mathfrak{f})\otimes g^{-1}]^{\alpha} = ((\beta\mathfrak{f})\otimes g^{-1})(\alpha^{-1}\cdot\alpha)$~~

~~$= (\mathfrak{f}\otimes g^{-1})^{\beta}(\beta^{-1}\alpha^{-1}\quad\alpha\beta)$~~

$$\begin{aligned}[\alpha(\beta\mathfrak{f})\otimes g^{-1}](x) &= [\beta\,\mathfrak{f}\otimes g^{-1}]^{\alpha}(x) = (\beta\,\mathfrak{f}\otimes g^{-1})(\alpha^{-1}x\alpha)\\ &= (\mathfrak{f}\otimes g^{-1})^{\beta}(\alpha^{-1}x\alpha) = (\mathfrak{f}\otimes g^{-1})(\beta^{-1}\alpha^{-1}x\alpha\beta)\\ &= (\mathfrak{f}\otimes g^{-1})^{(\alpha\beta)}(x)\\ &= [((\alpha\beta)\mathfrak{f})\otimes g^{-1}](x).\end{aligned}$$

$$\therefore\ \alpha(\beta\,\mathfrak{f}) = (\alpha\beta)\,\mathfrak{f}.$$

OKAY

<u>**Therefore:**</u> There ~~is~~ should be a ~~[illegible]~~ natural transformation

$$\Theta_{\alpha} : F \longrightarrow \alpha^{-1} \circ F \qquad \text{ie.}$$

$$V/rV \longrightarrow \alpha^{-1}(V/rV)$$

$$V/rV \longrightarrow \cancel{(V/rV)^{\alpha}}\ (V/rV \otimes g^{-1})^{\alpha^{-1}} \otimes g$$

ie $$\boxed{V/rV \otimes g^{-1} \longrightarrow (V/rV \otimes g^{-1})^{\alpha^{-1}}}$$

~~note that~~ What is.

$$(V/rV)^{\alpha^{-1}} \quad ?$$

Let $x \in M^{\times}_{\mathfrak{A}}$ and let $\rho(x)$ be endo of V/rV. Then $(V/rV)^{\alpha^{-1}}$ is the same vector space but with

$$\tilde{\rho}(x) = \rho(\alpha x \alpha^{-1}).$$

Note that $V \xrightarrow{\alpha} V$ carries rV into $\alpha \cdot rV = \alpha \cdot r \cdot \alpha^{-1} V = r^{\alpha} V$. Thus

$$\Theta \quad V/rV \xrightarrow{\sim} \underline{V/r^{\alpha}V}$$

$$v + rV \longmapsto \alpha v + r^{\alpha}V.$$

and $$\Theta(\rho(x)(v + rV)) = \alpha(xv + rV) = \alpha x \alpha^{-1}(\alpha v + r^{\alpha}V)$$

$$= \tilde{\rho}(x)(\alpha v + r^{\alpha}V). = \tilde{\rho}(x)\, \Theta(v + rV).$$

Now let's change our definitions.

<u>Proposition:</u> Let $\alpha \in N_K(\mathfrak{a})$ and let V be a $\mathfrak{g}$-k module. Let $\rho(x)$ be multiplication by x on $V/\mathfrak{r}V$ where $x \in M \times \mathfrak{a}$, so that $\rho(x)(v + \mathfrak{r}V) = xv + \mathfrak{r}V$. Then $(V/\mathfrak{r}V)^{\alpha^{-1}}$ is the $M \times \mathfrak{a}$ module with multiplication $\tilde{\rho}(x)$ given by

$$\begin{aligned} \tilde{\rho}(x)(v+\mathfrak{r}V) &= \rho(\alpha x \alpha^{-1})(v+\mathfrak{r}V) \\ &= \alpha x \alpha^{-1} v + \mathfrak{r}V \end{aligned}$$

Let $\mathfrak{r}^{\alpha^{-1}} = \alpha^{-1}\mathfrak{r}\alpha$. Then there is an isomorphism of $M \times \mathfrak{a}$ modules

$$\theta : V/\mathfrak{r}^{\alpha^{-1}}V \xrightarrow{\sim} (V/\mathfrak{r}V)^{\alpha^{-1}}.$$

given by $$\theta(v + \alpha^{-1}\mathfrak{r}\alpha V) = \alpha v + \mathfrak{r}V.$$

<u>Proof:</u> We have only to check that

$$\begin{array}{ccc} \theta\,\rho(x)(v+\alpha^{-1}\mathfrak{r}\alpha V) & \overset{?}{=} & \tilde{\rho}(x)\,\theta(v+\alpha^{-1}\mathfrak{r}\alpha V) \\ \| & & \| \\ \theta(xv+\alpha^{-1}\mathfrak{r}\alpha V) & & \alpha x \alpha^{-1}(\alpha v + \mathfrak{r}V) \\ \| & & \| \\ \alpha x v + \mathfrak{r}V & & \alpha x v + \alpha x \alpha^{-1}\mathfrak{r}V \\ & & \| \\ & & \alpha x v + \mathfrak{r}V \end{array}$$

<u>QED.</u>

<u>Therefore</u> There should be a natural transformation

$$V/\pi V \longrightarrow V/\pi^{\alpha} V \otimes \underline{(g \otimes g^{-\alpha^{-1}})}.$$

Got any ideas?

<u>Lemma</u>: Let $C^{(+a)}$ be ~~an~~ abelian $\overbrace{\text{categories}}^{\text{Grothendieck}}$ with a ~~[illegible]~~ set $\mathcal{P}$ of $\overbrace{\text{projective}}^{\text{small}}$ generators and let F, G ~~[illegible]~~ $: C \to \mathcal{C}$ be ~~[illegible]~~ right exact additive functors which commute with inductive limits. ~~Then given maps $\theta_P: F(P) \to G(P)$ for all $P \in \mathcal{P}$~~ Then there is a bijection ~~from the [illegible]~~ between natural transf. $\theta: F \to G$ and natural transf. $\varphi: F|\mathcal{P} \to G|\mathcal{P}$, given by $\theta \mapsto \theta|\mathcal{P} = \varphi$.

<u>Proof:</u> ~~[illegible]~~ Given φ and an object X of C choose a $\mathcal{P}$-resolution of X, that is, a resolution

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0.$$

where each P_i is a sum of members of $\mathcal{P}$. Then define $\theta: FX \longrightarrow GX$ to be the 0th homology of $\varphi: FP_0 \to GP_0$ where we define φ on ~~[illegible]~~ each P_0 by direct sums. This is accomplished by smallness

$$\begin{tikzcd}
F(\oplus P_i) \arrow[d, dashed, "\text{Define } \theta"'] & \oplus F(P_i) \arrow[l, "\sim"'] \arrow[d, "\text{given}"] \\
G(\oplus P_i) & \oplus G(P_i) \arrow[l, "\sim"']
\end{tikzcd}$$

Now this is OKAY because a given map $P_i \longrightarrow \oplus P_j'$ will factor thru finitely many terms of the sum and so we can check naturality. Thus can extend φ from P to ΣP and then by right exactness to all of $\mathcal{A}$. <u>QED</u>

<u>Revision:</u> ~~Replace the functor~~ Want a natural transf

$$\begin{tikzcd} V/\mathfrak{n}V \otimes g^{-1} \arrow[r] \arrow[d, equal] & V/\alpha(\mathfrak{n})V \otimes \alpha(g^{-1}) \\ \mathbb{F}(V) & \mathbb{F}_\alpha(V). \end{tikzcd}$$

$$\begin{tikzcd} (\mathfrak{g}\text{-}k) \arrow[r, bend left, "\mathbb{F}"] \arrow[r, bend right, "\mathbb{F}_\alpha"'] & (M, \alpha) \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{C}(\Lambda_1, \Lambda_2) \arrow[r, bend left, "\mathbb{F}"] \arrow[r, bend right, "\mathbb{F}_\alpha"'] & U(\alpha) \otimes \mathrm{Hom}_M(\Lambda_1, \Lambda_2) \end{tikzcd}$$

Bruhat should therefore yield a map

$$V/\mathfrak{n}V \otimes g^{-1} \longrightarrow V/\sigma(\mathfrak{n})V \otimes \sigma g^{-1}$$

for $\sigma \in N_K(\mathfrak{a})$. The problem is how to construct it.

---

we calculated for the induced module $I(\xi) = \bigoplus_{e^{2\pi i\sigma} = \nu} \delta_\sigma$

$$\bar{X}\delta_\sigma = (\lambda+\sigma)\delta_{\sigma+1}$$
$$\bar{Y}\delta_\sigma = (\lambda-\sigma)\delta_{\sigma-1} \qquad \bar{X}, \bar{Y}, \bar{H} \text{ Chevalley basis}$$
$$\bar{H}\delta_\sigma = 2\sigma\delta_\sigma$$

$$\boxed{\begin{aligned} X\delta_\sigma &= \tfrac{1}{\sqrt{2}}(\lambda+\sigma)\delta_{\sigma+1} \\ Y\delta_\sigma &= \tfrac{1}{\sqrt{2}}(\lambda-\sigma)\delta_{\sigma-1} \\ H\delta_\sigma &= \sigma\delta_\sigma \end{aligned}}$$

Assume λ, ν bad i.e.

say $\nu = 1$ and $\lambda = l+1$ ~~an integer~~ l integer ≥ 0

Then

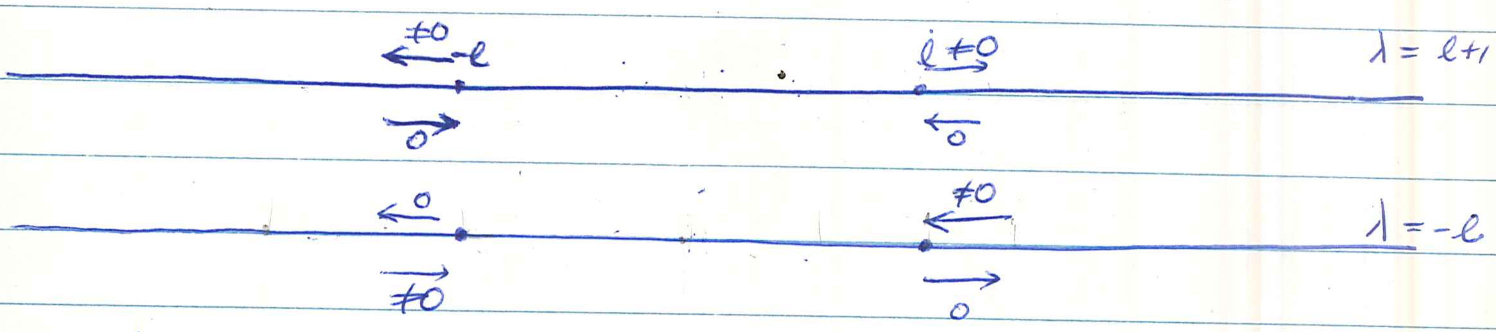
$$\begin{tikzcd} \delta_{-l-1} \arrow[r, shift left, "X=0"] & \delta_{-l} \arrow[l, shift left, "Y=(2l+1)\neq 0"] & \cdots\cdots & \delta_{l} \arrow[r, shift left, "X=(2l+1)\neq 0"] & \delta_{l+1} \arrow[l, shift left, "Y=0"] \end{tikzcd}$$

The two wings are submodules and the fin. diml. repn. is the quotient rep. This is the finite dimensional rep of dominant wgt l.

Say $\nu = 1$ and $\lambda = -\ell$ $\quad$ ℓ integer ≥ 0

$$\begin{tikzcd}
\delta_{-\ell-1} \arrow[r, shift left, "X=-2\ell-1 \neq 0"] & \delta_{-\ell} \arrow[l, shift left, "Y=0"] & \cdots & \delta_0 & \cdots & \delta_\ell \arrow[r, shift left, "X=0"] & \delta_{\ell+1} \arrow[l, shift left, "Y=-2\ell-1 \neq 0"]
\end{tikzcd}$$

__Conclusion:__ For $\overset{\nu=1}{}$, $\lambda = \ell+1$ ℓ integer ≥ 0 wings are subrepresentations and for $\lambda = -\ell$, the finite-dimensional repn. is the subrepresentation.



Thus

$$\dim \operatorname{Hom}(I(\ell+1,1), I(-\ell,1)) = 1.$$

$$\dim \operatorname{Hom}(I(-\ell,1), I(\ell+1,1)) = 2.$$

__Question:__ Are $I(\rho_1)$ and $I(\rho_2)$ disjoint if ρ_1 and ρ_2 are not Weyl conjugate? Yes because of the character which is $(\lambda - \frac{1}{2})^2$ and because ν's must be same.

$$\dim \operatorname{Hom}(I(\ell+1,1), I(\ell+1,1)) = 1$$

$$\dim \operatorname{Hom}(I(-\ell,1), I(-\ell,1)) = 1$$

Conclusion $\dim H_0(\mathfrak{n}, I(\ell+1,1)) = 2$ $\quad$ $\dim H_0(\mathfrak{n}, I(-\ell,1)) = 3$

$H_1(\mathfrak{n}, I(\ell+1,1)) = 0$

$H_1(\mathfrak{n}, I(\ell,1)) = 1$