Feb 8

Theorem: (Harish-Chandra). If V is an irreducible $\mathfrak{g}, k$ module then for some irreducible $M, \mathfrak{a}$ module ζ, V is a composition quotient of $I(\zeta)$.

Proof: Let Λ be an irreducible finite k module such that $\xi = \mathrm{Hom}_k(\Lambda, V) \neq 0$. Then ξ is an irreducible Ω_Λ° module. We shall make the following assumption which will be verified later:

Change

Hypothesis: There is an irreducible $M, \mathfrak{a}$ module ζ such that ξ is the restriction of $\mathrm{Hom}_M(\Lambda, \zeta)$ under the natural homomorphism $\Box : \Omega_\Lambda^\circ \longrightarrow U(\mathfrak{a}) \otimes \mathrm{Hom}_M(\Lambda, \Lambda)$.

Consider diagram

$$\begin{array}{ccc} (U(\mathfrak{g}) \otimes_k \Lambda) \otimes_{\Omega_\Lambda^\circ} \xi & \xrightarrow{\ \alpha\ } & I(\zeta) \\ \downarrow \beta & & \\ V & & \end{array}$$

Definition of β: Recall $\xi = \mathrm{Hom}_k(\Lambda, V)$ so

$$\beta(u \otimes \lambda \otimes \omega) = u \cdot \omega(\lambda).$$

Definition of α: By universal property of LHS α given by a map

$$\xi \longrightarrow \mathrm{Hom}_k(\Lambda, I(\zeta))$$

of Ω_Λ° modules.

But $\quad \mathrm{Hom}_k(\Lambda, I(\xi)) \rightrightarrows \mathrm{Hom}_M(\Lambda, \xi)$ is a dihomomorphism wrt the map

$$\Omega_\Lambda^\circ \longrightarrow U(\mathfrak{a}) \otimes \mathrm{Hom}_M(\Lambda,\Lambda),$$

and by hypothesis we can find such a ~~dihom~~ non-zero dihom. $\xi \longrightarrow \mathrm{Hom}_M(\Lambda, \xi)$. So we get α which is non-zero.

Problem is to construct a non-zero map

$$\mathrm{Hom}_k(\Lambda, V) \longrightarrow \mathrm{Hom}_M(\Lambda, \xi)$$

which is a right module dihomomorphism for the map

$$\Omega_\Lambda \longrightarrow U(\mathfrak{a}) \otimes \mathrm{Hom}_M(\Lambda,\Lambda).$$

But you are now reduced to the following lemma:

Lemma: Suppose $R \xrightarrow{f} S$ map of rings, ~~module~~, Λ_1 irred R module. Then there is a non-zero f dihom $\Lambda_1 \longrightarrow \Lambda_2$ where Λ_2 is an irreducible S module.

Proof: Choose an irreducible quotient of $S \underset{R}{\otimes} \Lambda_1$ which is possible since $\Lambda_1 \simeq R/L \Rightarrow S \otimes_R \Lambda_1 \simeq S/SL$ finitely type.

NO $\quad S \otimes_R \Lambda_1$ might be 0

$$F : \Omega_\Lambda \longrightarrow U(\mathfrak{a}) \otimes \mathrm{Hom}_M(\Lambda, \Lambda). \qquad \text{ring homom.}$$

$$\begin{array}{ccc} \Omega_\Lambda & & U(\mathfrak{a}) \otimes \mathrm{Hom}_M(\Lambda,\Lambda) \\ \| & & \| \\ \mathrm{End}_{\mathfrak{g}}(U(\mathfrak{g}) \otimes_k \Lambda) & \longrightarrow & \mathrm{End}_{M,\mathfrak{a}}(1 \otimes_{\mathfrak{n}} U(\mathfrak{g}) \otimes_k \Lambda) \end{array}$$

now an irreducible right module over $U(\mathfrak{a}) \otimes \mathrm{Hom}_M(\Lambda,\Lambda)$ is of the form $\lambda \otimes \mathrm{Hom}_M(\Lambda, \nu) = \mathrm{Hom}_M(\Lambda, \xi)$. In the case where $I(\xi)$ is irreducible, then I know in principle how to define an isom

$$I(\xi_1 \otimes g) \longrightarrow I(\xi_1^{\alpha_s} \otimes g).$$

$\xi = \xi_1 \otimes g.$

$\xi = \lambda \otimes \nu$

$\xi_1 = (\lambda - g) \otimes \nu$

$\xi_1^{\alpha_s} = (\lambda' - g) \otimes \nu$

$-(\lambda - \tfrac{1}{2}) = \lambda' - \tfrac{1}{2}$

$\lambda - 1 = \lambda'$. ✓

~~and of course the solar diagram~~

$$\cancel{I(\xi_1 \otimes g) \longrightarrow I(\xi_1^{\alpha_s} \otimes g)}$$

which of course means that

$$\mathrm{Hom}_M(\Lambda, \xi_1 \otimes g) \xrightarrow{\sim} \mathrm{Hom}_M(\Lambda, \xi_1^{\alpha_s} \otimes g)$$

as Ω_Λ modules.

In other words

$$\cancel{\mathrm{Hom}_M(\Lambda, \xi_1 \otimes g) \simeq \mathrm{Hom}_M(\Lambda}$$

recall $\xi^{\alpha_s}(t) = \xi(\alpha_s^{-1} t\, \alpha_s)$

$$\cancel{\lambda^{\alpha_s}(H) = \lambda(\mathrm{Ad}\,\alpha_s^{-1} \cdot H)}$$

$$F: \Omega_\Lambda \longrightarrow U(\mathfrak{a}) \otimes \mathrm{Hom}_M(\Lambda,\Lambda) \qquad \text{ring hom.}$$

Claim $\exists$ an Ω_Λ isom.

$$\mathrm{Hom}_M(\Lambda, V) \otimes \lambda_1 \otimes g \longrightarrow \mathrm{Hom}_M(\Lambda, V^s) \otimes \lambda_1^{\alpha_s} \otimes g$$

Let a group W act on a ring R ~~and R module M~~.
Let V be an R module

R ~ M $\qquad R =$

Let $s \in W$ then s defines

$$U(\mathfrak{a}) \otimes \mathrm{Hom}_M(\Lambda,\Lambda) \longrightarrow U(\mathfrak{a}) \otimes \mathrm{Hom}_M(\Lambda,\Lambda)$$

$$p \otimes \varphi \qquad\qquad p^s \otimes \varphi^s$$

---

where $\varphi^s(\lambda) = \cancel{\varphi(\alpha_s^{-1}\lambda\alpha_s)} \quad \alpha_s \varphi(\alpha_s^{-1}\lambda)$

all $m\varphi m^{-1} = \varphi$ this is independent of the choice of α_s

and

$p^s(\lambda) = ?$

Conjecture: $F: \Omega_\Lambda \longrightarrow \left[U(\mathfrak{a}) \otimes \mathrm{Hom}_M(\Lambda,\Lambda)\right]^W$ is an isomorphism.

Definition of F:

$$
\begin{tikzcd}
\Omega_\Lambda = \mathrm{End}_{\mathfrak{g}}(U(\mathfrak{g})\otimes_k \Lambda) \arrow[r, "1\otimes_{\mathfrak{a}}"] \arrow[dd, dashed, "F"] & \mathrm{End}_{M,\mathfrak{a}}(U(\mathfrak{a})\otimes\Lambda) \arrow[d, "\wr"] \\
 & U(\mathfrak{a}) \otimes \mathrm{Hom}_M(\Lambda,\Lambda) \arrow[d, "\beta\otimes 1"] \\
\left[U(\mathfrak{a})\otimes \mathrm{Hom}_M(\Lambda,\Lambda)\right]^W \arrow[r, hook] & U(\mathfrak{a})\otimes \mathrm{Hom}_M(\Lambda,\Lambda)
\end{tikzcd}
$$

$A^2 - A$

$\beta(A) = A + g(A)$

Evidence: We know that generically. $I(\lambda\otimes g) \simeq I(\lambda^{\alpha_s}\otimes g)$ hence

$$\lambda\otimes g\otimes \mathrm{Hom}_M(\Lambda,\nu) \simeq \lambda^{\alpha_s}\otimes g\otimes \mathrm{Hom}_M(\Lambda,\nu^{\alpha_s}).$$

~~$F(\Omega_\Lambda) \subseteq U(\mathfrak{a})\otimes$~~ $G(\Omega_\Lambda) \subset U(\mathfrak{a})\otimes \mathrm{Hom}_M(\Lambda,\Lambda)$ right modules.

$$\Downarrow$$

$$\lambda\otimes \mathrm{Hom}_M(\Lambda,\nu) \simeq \lambda^{\alpha_s}\otimes \mathrm{Hom}_M(\Lambda,\nu^{\alpha_s})$$

as $F(\Omega_\Lambda)(\subset U(\mathfrak{a})\otimes \mathrm{Hom}_M(\Lambda,\Lambda))$ right modules.

$$P(\lambda+g) = (\beta P)(\lambda) \qquad \underline{\beta A = A + g(A).}$$

$$\lambda(A) + g(A) = (\beta A)(\lambda) \qquad A^2 - A \mapsto (A+\tfrac{1}{2})^2 - (A+\tfrac{1}{2}) = A^2 - \tfrac{1}{4}$$

Be Cautious: We know that $F(\Omega_\Lambda) \subset$ ~~Hom~~ $U(\mathfrak{a}) \otimes \mathrm{Hom}_M(\Lambda, \Lambda)$ is such that there is ~~[illegible]~~ an isom.

$$\lambda \otimes \mathrm{Hom}_M(\Lambda, \nu) \simeq \underline{\lambda^{\alpha_s} \otimes \mathrm{Hom}_M(\Lambda, \nu^{\alpha_s})}$$

of right $F(\Omega_\Lambda)$ modules.

We know that W acts on $U(\mathfrak{a}) \otimes \mathrm{Hom}_M(\Lambda, \Lambda)$ by

$$s(P \otimes \varphi) = (sP) \otimes s\varphi$$

$$\text{where} \quad (sP)(\lambda) = P(s^{-1}\lambda)$$

$$(s\varphi) = \alpha_s \varphi \alpha_s^{-1}.$$

$$(\lambda \otimes \nu)^{\alpha_s}(t) = (\lambda \otimes \nu)(\alpha_s^{-1} t \alpha_s) \qquad t \in MA.$$

$$t = me^a$$

$$= (\lambda \otimes \nu)(\alpha_s^{-1} m \alpha_s \cdot e^{\alpha_s^{-1} a \alpha_s})$$

$$= \nu(\alpha_s^{-1} m \alpha_s) \cdot e^{\lambda(s^{-1} a)}$$

Problem: ~~Calculate the module structure of~~ Show that ~~if we mod the module~~ if $V = \lambda \otimes \mathrm{Hom}_M(\Lambda, \nu)$ ~~then~~ as a right $U(\mathfrak{a}) \otimes \mathrm{Hom}_M(\Lambda, \Lambda)$ module and we define V^s by

$$(P \otimes \varphi)(v)^s = (P \otimes \varphi)^{s^{-1}}(v)$$

$$\begin{tikzcd} V \arrow[d, "s"] \arrow[dr, "\bar{\varphi}"] & \\ V^s \arrow[r, "\varphi"] & {} \end{tikzcd}$$

Then $V^s \simeq \lambda^{s} \otimes \mathrm{Hom}_M(\Lambda, \nu^{\alpha_s})$

A map $\varphi : V^s \longrightarrow W$ same as a map $\bar{\varphi} : V \to W$ $\ni$

$$\bar{\varphi}(zv) = \varphi(z^s v^s) = z^s \bar{\varphi}(v).$$

so I want a map

$$\chi : \lambda \otimes \mathrm{Hom}_M(\Lambda, \nu) \longrightarrow \lambda^s \otimes \mathrm{Hom}_M(\Lambda, \nu^{\alpha_s})$$

Such that $\quad \chi(vz) = \chi(v) z^s.$

~~[illegible]~~

Definition: of $F : \Omega_\Lambda \longrightarrow U(\mathfrak{a}) \otimes \mathrm{Hom}_M(\Lambda,\Lambda)$

$$\begin{tikzcd}
\Omega_\Lambda = \mathrm{End}_{\mathfrak{g}}(U(\mathfrak{g})\otimes_{\mathfrak{p}}\Lambda) \arrow[r, "\otimes"] & \mathrm{End}_{M,\mathfrak{a}}(U(\mathfrak{a})\Lambda) := U(\mathfrak{a})\otimes \mathrm{Hom}_M(\Lambda,\Lambda) \arrow[d, "\beta\otimes 1"] \\
 & U(\mathfrak{a})\otimes \mathrm{Hom}_M(\Lambda,\Lambda)
\end{tikzcd}$$

$$\beta(A) = A + g(A).$$

$$g = \frac{1}{2}\sum_{\alpha\in\Sigma'} \alpha$$

Conjecture: F induces an isomorphism

$$\Omega_\Lambda \longrightarrow \left[U(\mathfrak{a})\otimes \mathrm{Hom}_M(\Lambda,\Lambda)\right]^W$$

This should not be that difficult.
Alternative definition of F. Know that

$$(U(\mathfrak{g})^k)^o \longrightarrow \Omega_\Lambda \quad \text{onto}$$

$$a \longmapsto (u\otimes\lambda \mapsto ua\otimes\lambda)$$

$$\psi_a(u\otimes\lambda) = ua\otimes\lambda$$

$$\psi_{ab}(u\otimes\lambda) = (uab\otimes\lambda) = \psi_b(\psi_a(u\otimes\lambda)).$$

take

$$\begin{tikzcd}
U(\mathfrak{g})^k \arrow[d, hook] & & \\
U(\mathfrak{g}) \arrow[r] & U(\mathfrak{g})/\mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})\,\partial(\Lambda) \simeq U(\mathfrak{a})\otimes\mathrm{Hom}(\Lambda,\Lambda). & \arrow[from=2-2, phantom, "\uparrow \beta\otimes 1"]
\end{tikzcd}$$

$u \in U(\mathfrak{g})^k$ write

$$u = u_+ + u_0 + u_-$$

where $u_+ \in \mathfrak{n} U(\mathfrak{g})$

$u_0 \in U(\mathfrak{a}) \otimes \mathrm{Hom}(\Lambda, \Lambda)$

$u_- \in U(\mathfrak{g}) \mathcal{I}(\Lambda)$

$\cancel{\mathfrak{n} U(\mathfrak{g})}$ $\qquad U(\mathfrak{g}) \otimes_k \mathrm{Hom}(\Lambda, \Lambda)$

$$0 \longrightarrow \mathcal{I}(\Lambda) \longrightarrow U(k) \longrightarrow \mathrm{Hom}(\Lambda, \Lambda) \longrightarrow 0$$

$$v \qquad \lambda \mapsto v\lambda$$

$$1 \underset{\mathfrak{n}}{\otimes} U(\mathfrak{g}) \underset{k}{\otimes} \mathrm{Hom}(\Lambda, \Lambda)$$

$$\parallel$$

$$U(\mathfrak{n}) U(\mathfrak{a}) U(k) \otimes$$

$$\parallel$$

$$U(\mathfrak{a}) \otimes \mathrm{Hom}(\Lambda, \Lambda).$$

Given ~~u let~~ ~~the~~

~~let~~ $\pi : U(\mathfrak{g}) \longrightarrow U(\mathfrak{a}) \otimes \mathrm{Hom}(\Lambda, \Lambda)$

$$U(\mathfrak{g}) = U(\mathfrak{n}) \otimes U(\mathfrak{a}) \otimes U(k) \xrightarrow{\varepsilon \otimes id \otimes \pi_\Lambda} U(\mathfrak{a}) \otimes \mathrm{Hom}(\Lambda, \Lambda)$$

Then claim that if $u \in U(\mathfrak{g})^k$ π is a homom.

$$\pi(u \cdot v)$$

$$U(\mathfrak{g}) \xrightarrow{\pi_1} U(\mathfrak{a})\,U(\mathfrak{k}).$$

$$u \equiv u_0 \quad \text{mod.} \quad \mathfrak{r}U(\mathfrak{g}) + U(\mathfrak{g})\partial(\Lambda)$$

where $u_0 \in U(\mathfrak{a})\,T$ $\qquad U(\mathfrak{k}) = T + \partial(\Lambda)$

$$u - u_0 \in \mathfrak{r}U(\mathfrak{g}) + U(\mathfrak{g})\partial(\Lambda).$$

$$v - v_0 \in$$

$$uv - u_0v_0 = \underline{(u-u_0)v} + \underline{u_0(v-v_0)}$$

$$[\mathfrak{r}U(\mathfrak{g}) + U(\mathfrak{g})\partial(\Lambda)]v + \underline{U(\mathfrak{a})T}[\mathfrak{r}U(\mathfrak{g}) + U(\mathfrak{g})\partial(\Lambda)]$$

won't work.

$$U(\mathfrak{g})^k \longrightarrow U(\mathfrak{g})^k/(U(\mathfrak{g})\mathfrak{l})^k$$

$$\downarrow$$

$$(U(\mathfrak{g})^{k})^{\circ} \longrightarrow \Omega_\Lambda$$

Let $v \in U(\mathfrak{g})^{k}$

$$\psi_v : \Lambda \longrightarrow U(\mathfrak{g}) \otimes_k \Lambda \xrightarrow{\varepsilon \otimes 1 \otimes \mu} U(\mathfrak{a}) \otimes \Lambda$$

$$\underline{\psi_v(\lambda) = v \otimes \lambda.}$$

$$v \qquad \underline{U(\mathfrak{n}) \otimes U(\mathfrak{a}) \otimes U(k)} \qquad U(\mathfrak{a}) \otimes_{U(\mathfrak{a}+\mathfrak{n})} U(\mathfrak{a}+\mathfrak{n}) U(k) \otimes_k \Lambda = U(\mathfrak{a}) \otimes \Lambda.$$

$$\lambda \longmapsto (\varepsilon \otimes id \otimes id) v \cdot \lambda$$

take v

So take v apply $\varepsilon_{\mathfrak{n}} \otimes 1$ get in $U(\mathfrak{a})U(k)$ then apply map $U(k) \to Hom(\Lambda, \Lambda)$.

this seems to be ~~quite~~ correct.

$$\underline{uv} \qquad u \longmapsto \quad \lambda \mapsto w_1 w_2 \lambda$$

$$U(\mathfrak{g})^{k} \longrightarrow U$$
$$uv \longmapsto \pi v \cdot \pi u.$$

$$U(k) \longrightarrow Hom(\Lambda, \Lambda)$$
$$x \longmapsto (\lambda \mapsto x\lambda)$$

$$\pi v \cdot \pi u = \pi(uv).$$

$$\pi u = (\varepsilon_{\mathfrak{n}} \otimes 1) u.$$

problem is to ~~figure~~ make ~~[illegible]~~ W act.

Take an element $\alpha_s \in W$. Then how to make α_s act ~~on W~~

First problem — how to show image of F lies where it should. H-C's method is to define a fn. φ_λ on G $\ni$

$$D\varphi_\lambda = \langle \gamma(D), e^\lambda \rangle \varphi_\lambda$$

and to show that $\varphi_{s\lambda} = \varphi_\lambda$. Then

$$\langle \gamma(D), e^{s\lambda} \rangle = \langle \gamma(D), e^\lambda \rangle$$
$$\parallel$$
$$\langle \gamma(D)^{s^{-1}}, e^{\lambda} \rangle$$

$\therefore$ $\gamma(D)^s$ $\gamma(D)$ have same values and so are equal.

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next he ~~defines~~ considers the filtered map

$$\Omega_\Lambda \xrightarrow{F} \underline{[U(\mathfrak{a}) \otimes \mathrm{Hom}_M(\Lambda, \Lambda)]^W}$$

$$gr\,\Omega_\Lambda = \mathrm{Hom}_{\mathfrak{p},k}(S(\mathfrak{p}) \otimes \Lambda, S(\mathfrak{p}) \otimes \Lambda) = \mathrm{Hom}_k(\Lambda, S(\mathfrak{p}) \otimes \Lambda)$$
$$= J \otimes \mathrm{Hom}_k(\Lambda, H \otimes \Lambda)$$

$$\begin{array}{ccc} gr\,\Omega_\Lambda & \longrightarrow & [S(\mathfrak{a})\otimes \mathrm{Hom}_M(\Lambda,\Lambda)]^W \\ \wr\| & & \\ [S(\mathfrak{p})\otimes \mathrm{Hom}(\Lambda,\Lambda)]^k & \xrightarrow[\approx]{} & [S(\mathfrak{a})\otimes \mathrm{Hom}_M(\Lambda,\Lambda)]^N \end{array}$$

Suppose $\mathrm{Hom}_{\mathfrak{g}}(U(\mathfrak{g})\otimes_k \Lambda_1, U(\mathfrak{g})\otimes_k \Lambda_2)$

$$\wr\|$$

$$[U(\mathfrak{a})\otimes \mathrm{Hom}_M(\Lambda_1,\Lambda_2)]^W$$

then tensoring with $U(\mathfrak{a})^W \longrightarrow \mathbb{C}$ get.

$$\left[U(\mathfrak{a})/\mathfrak{a}_\lambda \otimes \mathrm{Hom}_M(\Lambda_1,\Lambda_2)\right)$$

---

try!! Hom

---

for $sl(2,\mathbb{R})$ how does W act on

$$\boxed{\mathrm{Hom}_M(\Lambda_1,\Lambda_2)\ ?}$$

ie given α_s $\varphi\in \mathrm{Hom}_{\underline{M}}(\Lambda_1,\Lambda_2)$

ie. $\alpha_s \in K$ so

$$\alpha_s\varphi\alpha_s^{-1} = \pm\varphi.$$

acts trivially if $\sigma_1\equiv\sigma_2\ (2)$

by -1 if $\sigma_1\equiv\sigma_2+1\ (2)$

Therefore in even ~~dimensions~~ differences we would get

$$U(\mathfrak{a})^W \otimes \mathrm{Hom}(\Lambda_1, \Lambda_2)$$

so if compatible with ~~dimension~~ composition

$$[H \otimes \mathrm{Hom}(\Lambda,\Lambda)]^k \longrightarrow [H \otimes \mathrm{Hom}_M(\Lambda,\Lambda)]^W$$

In Helgason is given a proof that

$$S(\mathfrak{p})^k \xrightarrow{\sim} S(\mathfrak{a})^W$$

using a mixture of techniques. ~~Techniques~~

injectivity s-s elts. all conjugate to things in $\mathfrak{a}$

$S(\mathfrak{a})$ integral over J direct calc.

$\therefore$ $S(\mathfrak{a})^W$ ——— J

But have same q.f. by $\mathcal{G}$ theory.

$$[S(\mathfrak{p}) \otimes \mathrm{Hom}(\Lambda,\Lambda)]^k \longrightarrow [S(\mathfrak{a}) \otimes \mathrm{Hom}(\Lambda,\Lambda)]^N.$$

injectivity. Given $\sum_i P_i \otimes \varphi_i$ $\quad$ φ_i basis for $\mathrm{Hom}(\Lambda,\Lambda)$

assume that

$$\sum_i P_i(a) \cdot \varphi_i(\lambda) = 0 \quad \text{all } \lambda \in \Lambda, \ a \in \mathfrak{a}'$$

$$\Longrightarrow \quad P_i(a) = 0 \quad \text{all } a \in \mathfrak{a}.$$

But $\sum_i{}' P_i(p^*) \otimes \varphi_i(\lambda) \neq 0$

Then move by K ✓.

maybe this proves injectivity in general!!! ie still have a map ~~[illegible]~~ Ought to work.

write this up.

why onto? The method of ~~Harish-Chandra first~~ Chevalley is to show that the map is ~~onto~~ integral + birational and hence an isomorphism as $S(\mathfrak{p})^k$ is normal.

Proof. That $\boxed{S(\mathfrak{p})^k \text{ is normal}}$ Let $z \in$ q.f. of $S(\mathfrak{p})^k$. ~~so~~ be integral over $S(\mathfrak{p})^k \Rightarrow z \in S(\mathfrak{p})$

~~so $pz = q$ ... $p, q \in S(\mathfrak{p})^k$~~

~~$\Rightarrow (pz)$~~ But $S(\mathfrak{p})$ int domain so

$$pz^k = p^k z^k = q^k = q \Rightarrow z^k = z. \Rightarrow z \in S(\mathfrak{p})^k.$$

Therefore consider the map

$$\cancel{S(\mathfrak{p})^k}$$

$$[S(\mathfrak{p}) \otimes \mathrm{Hom}(\Lambda, \Lambda)]^k \longrightarrow [S(\mathfrak{a}) \otimes \mathrm{Hom}(\Lambda, \Lambda)]^{\underline{N}}$$

which is gotten by restriction.

$$S(p) \otimes \mathrm{Hom}(\Lambda, \Lambda) \simeq \Gamma(p', \mathrm{Hom}(\mathcal{O}$$

<u>General fact:</u> for any finite k module Λ

$$[S(p) \otimes \Lambda]^k \xrightarrow{\sim} [S(\alpha) \otimes \Lambda]^N$$

~~Proof:~~

$$\Gamma(p', \mathcal{O} \otimes \Lambda)^k \xrightarrow{\sim} \Gamma(\alpha', \mathcal{O} \otimes \Lambda)^N \quad \text{why ?}$$

$\boxed{\text{The general problem:}}$

$$\Gamma(g')^g \longrightarrow \Gamma(p')^k \qquad \text{not an isom.}$$

but this is Rallis maybe:

$$[S(p) \otimes \Lambda]^k \longrightarrow [S(\alpha) \otimes \Lambda]^N \qquad ?$$

$$\|$$

$$J \otimes [H \otimes \Lambda]^k$$

Have to prove that

$$\underline{\mathrm{Res}: \left[S(\mathfrak{p})\otimes\Lambda\right]^{k} \xrightarrow{\cong} \left[S(\mathfrak{a})\otimes\Lambda\right]^{N}}$$

$$\mathrm{Hom}_k(\Lambda, S(\mathfrak{p})) \xrightarrow{\cong} \mathrm{Hom}_N(\Lambda, S(\mathfrak{a})). \quad ?$$

<u>method:</u> $\mathbf{1}\otimes_J \; S(\mathfrak{p}$

$$\Gamma(\mathfrak{p}', \mathcal{O}_{\mathfrak{p}'}\otimes\Lambda)$$

over the ~~[illegible]~~ good set the fibers

Let $\mathcal{U}$ be the set of regular elts of $\mathfrak{p}'$.

~~[illegible]~~

$$\Gamma(\mathcal{U}, \mathcal{O}_{\mathfrak{p}'}\otimes\Lambda)^{K}$$

idea is that

$$\begin{array}{ccc} \mathfrak{p}'_{reg} & \longrightarrow & \mathfrak{a}'_{reg}. \\ \downarrow & & \downarrow \\ \mathfrak{p}'_{reg}/K & \cong & \mathfrak{a}'_{reg}/W \end{array}$$

geometric quotient.

so that clearly

$$\Gamma(\mathfrak{p}'_{reg}, \mathcal{O}_{\mathfrak{p}'}\otimes\Lambda)^{K} \xrightarrow{\sim} \Gamma(\mathfrak{a}'_{reg}, \mathcal{O}_{\mathfrak{a}'}\otimes\Lambda)^{N}$$

Thus get an isomorphism at generic point.

We have to redo certain ideas of H-C & Kostant

The adjoint representation: Want to show that

$$\boxed{\left(S(\mathfrak{g})\otimes\Lambda\right)^{\mathfrak{g}} \xrightarrow{\sim} \left(S(\mathfrak{h})\otimes\Lambda\right)^{N}} \qquad N$$

$$\left(S(\mathfrak{h})\otimes\Lambda^{\mathfrak{h}}\right)^{N}$$

thus the 0 weight must occur in Λ.

note that Seminar Lie ~~[illegible]~~ proves that

$$S(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} S(\mathfrak{h})^{W}$$

by means of the H-C method.

The Chevalley method: injectivity ✓

$S(\mathfrak{g})^{\mathfrak{g}}$ integrally closed ✓

$S(\mathfrak{h})$ integral over $S(\mathfrak{g})^{\mathfrak{g}}$.

quotient fields are the same.

extend Galois to modules. Thus I have a finite group W acting on X and two ~~[illegible]~~

i.e. there is a module M over A^W such that

$$A\otimes_{A^W} M \to M$$

$$\begin{array}{c} X^{\mathcal{F}} \\ f\big\downarrow \\ X/W^{\mathcal{G}} \end{array}$$

descends since flat. f.

$$\begin{tikzcd} S(h) & S(h)\otimes\Lambda^h \\ S(h)^W \arrow[u] & (S(h)\otimes\Lambda^h)^W \arrow[u, dotted] \end{tikzcd}$$ is a faithfully flat descent

however ~~look~~ at the image of ~~$S(h)\otimes$~~ $(S(g)\otimes\Lambda)^g$
+ can you show that

$$(S(g)\otimes\Lambda)^g \underset{S(h)^W}{\otimes} S(h)\cancel{\otimes\Lambda^h} \xrightarrow[\text{auto.?}]{\sim} \cancel{S(g)\otimes\Lambda}\ S(h)\otimes\Lambda^h.$$

$$(S(g)\otimes\Lambda)^g \longrightarrow S(h)\otimes\Lambda^h$$

Show that image contains Λ^h.
i.e. that ~~#~~ 0 weight space

~~This gives simple proof that $S(g)^g \xrightarrow{\sim} S(h)^W$ as follows~~

~~We know~~

~~$$S(g)^g \hookrightarrow S(h)^W$$~~

~~can show~~

~~But~~

~~$$S(g)^g \underset{S(h)^W}{\otimes}$$~~

$\boxed{\text{Why is } (S(\mathfrak{g})\otimes\Lambda)^{\mathfrak{g}} \otimes S(\mathfrak{h}) \longrightarrow S(\mathfrak{h})\otimes\Lambda^{\mathfrak{h}} \text{ onto?}}$

$(S(\mathfrak{p})\otimes\Lambda)^{K} \otimes S(\mathfrak{a}) \longrightarrow S(\mathfrak{a})\otimes\Lambda^{M}$ onto?

Take an M invariant Λ. Somehow want to induces ?

Look at 0 weight space!!! of Λ + construct

enough to ~~show~~ show

$\cancel{S(\mathfrak{g})}$ $\Lambda^{\mathfrak{h}} \subset (S(\mathfrak{g})\otimes\Lambda)^{\mathfrak{g}} + \mathfrak{h}S(\mathfrak{h})\Lambda^{\mathfrak{h}}$

Take a 0 weight element + use Naky

Λ

$$\begin{tikzcd} S(\mathfrak{g}) \\ S(\mathfrak{g})^{\mathfrak{g}} \arrow[u, "\text{f.f.}"'] \end{tikzcd}$$

<u>Conjecture:</u> $(S(\mathfrak{p})\otimes\Lambda)^k \xrightarrow{\sim} (S(\mathfrak{a})\otimes\Lambda^M)^W$

evidence for conjecture — true if $\Lambda = 1$

In fact both sides are free $S(\mathfrak{p})^k \simeq S(\mathfrak{a})^W$ modules of ~~rank~~ the same rank: left $(H\otimes\Lambda)^k = \Lambda^M$ (Rallis)

right: by Chevalley $S(\mathfrak{a})$ free of rang $|W|$ over $S(\mathfrak{a})^W$ and $S(\mathfrak{a})/F \simeq$ group ring of W.

So $(S(\mathfrak{a})/F\otimes\Lambda^M)^W \simeq \Lambda^M$.

How to prove the conjecture

given that $\mathfrak{g}$ is a semi-simple Lie algebra with inv. θ $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$. Let $\mathfrak{a}$ be a maximal abelian semi-simple subspace Choose $z\in\mathfrak{a}$ so that $T_z = (\operatorname{ad} z)^2$ is as regular as possible

$S = J\otimes H$ follows from Chevalley thm.

$S(\mathfrak{g}) \cong J$

To show that S free over J.

To show that

$S(\mathfrak{g})$ free over $S(\mathfrak{g})^{\mathcal{G}}$

using that $\quad\quad\quad\quad \Vert$

$S(\mathfrak{h})$ free over $S(\mathfrak{h})^W$

$$\begin{tikzcd} S(\mathfrak{g}) \arrow[r, phantom, "\text{free over}"] \arrow[d] & S(\mathfrak{g})^{\mathfrak{g}} \arrow[d, "\wr"] \\ S(\mathfrak{h}) \arrow[r, dash] & S(\mathfrak{h})^W \end{tikzcd}$$

$$\boxed{\begin{tikzcd} & R \arrow[dr, "\text{c.i.}"] & \\ k \arrow[ur] \arrow[rr, "\text{free}"'] & & S \end{tikzcd}}$$

$\cancel{\mathbb{L}D_{R/k} \otimes_R S \to}$

~~Let [illegible] be a basis for.~~

Thus if $\tilde{\mathfrak{n}}$ is the orthogonal of $\mathfrak{a}$ in $\mathfrak{p}$ we can ~~conclude that S is free [illegible]~~ filter R by powers of $\tilde{\mathfrak{n}}$ and because things are graded ~~we find that~~ the filtration is finite in each degree, ~~and~~ hence R is free over k.

Why is $S(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} S(\mathfrak{h})^W$?

Injectivity by conjugacy thm which one ~~proves by using the map [illegible]~~ may prove using differentials

Then must know that <u>$S(\mathfrak{h})$ integral</u> over $S(\mathfrak{g})$

Prove that $\boxed{S(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} S(\mathfrak{h})^{W}}$

(a) <u>injectivity:</u> ~~[illegible]~~

(b) $S(\mathfrak{h})$ integral over $S(\mathfrak{g})^{\mathfrak{g}}$. The point is ~~that~~ to examine the char. poly of ad x

Set
$$P(x,T) = \det(T - \operatorname{ad} x)$$
$$= T^n + p_1(x)T^{n-1} + \cdots + p_n(x)$$

where $p_i(x) \in S(\mathfrak{g})^{\mathfrak{g}}$. ~~One then sees that the eigenvalues of~~ One then sees that the eigenvalues of ad h are $\pm\alpha(h)$. Thus

$$\alpha(h)^n + p_1(h)\,\alpha(h)^{n-1} + \cdots + p_n(h) = 0$$

all x and α.

~~By Cayley Hamilton,~~ $\boxed{\cancel{P(h, \operatorname{ad} h) = 0}}$

$$P(H, \alpha(H)) = 0 \quad \text{for all } H$$

$\Rightarrow$ α integral over $\underline{S(\mathfrak{g}')^{\mathfrak{g}}}$

$\Rightarrow$ $S(\mathfrak{g}')^{\mathfrak{g}} \longrightarrow S(\mathfrak{h}')$ integral OKAY.

$\Rightarrow$ $S(\mathfrak{g})^{\mathfrak{g}} \longrightarrow S(\mathfrak{h})$ integral.

Next have to know that

© $S(\mathfrak{g}')^{\mathfrak{g}} \hookrightarrow S(\mathfrak{h}')^W$ is an iso for quotient fields.

~~Def[illegible]~~ This is easy because ~~[illegible]~~ ~~element of h~~ by the conjugacy thm every regular ~~element~~ orbit intersects $\mathfrak{h}$ transversally at ~~W points~~ as W orbit. Thus a W-function on $\mathfrak{h}$ defines a $\mathfrak{g}$-function on $\mathfrak{g}_{reg}$

~~Lemma: Let D be the regulator fn. the reg g~~

Lemma: Let D be the regulator fn. on $\mathfrak{g}$ and let $f \in S(\mathfrak{g}')[\frac{1}{D}]$ be $\mathfrak{g}$ invariant. Then if $f/\mathfrak{h}$ is regular, f is regular.

Proof: $S(\mathfrak{g}')^{\mathfrak{g}}$ is a unique fact. domain, so write $f = \frac{u}{D^j}$ in lowest terms, then if $j \neq 0$ f not regular on $\mathfrak{h}$ unless u vanishes to order j on $\mathfrak{h}$. i.e.

$$\boxed{u = \cancel{g(\ldots)}\; g_i n_i + D^j z}$$

Next I have to try to prove

$$(S(\mathfrak{g}) \otimes \Lambda)^{\mathfrak{g}} \longrightarrow (S(\mathfrak{h}) \otimes \Lambda)^N$$

again one can see that this should be an isomorphism when tensored with $\mathbb{C}\left[\frac{1}{D}\right]$, ~~so one writes~~ so take $\alpha \in (S(\mathfrak{h}) \otimes \Lambda)^N$ and write $D^j\alpha = \beta$ where $\beta \in S(\mathfrak{g}) \otimes \Lambda$ with j least. Thus

$$\alpha = \frac{\beta}{D^j} \Big| \mathfrak{h}.$$

now $\frac{\beta}{D^j}$ is regular on h ~~clearly~~ by assumption

and thus $\beta \equiv \gamma D$ mod $\underline{n}$
fns. vanishing

Go back to functions.

Given f W-invariant on h and we can put

$$D^j f = \text{res}\, u \quad \text{where } u \in S(\mathfrak{g})^{\mathfrak{I}}$$

assume j least. Then

$$\frac{u}{D} \text{ restrict to a reg. fn. on } h$$

ie $\underline{u = DV + g}$ $\quad g \in I(h)$

first note that if V N-invariant mod n
then can modify to be N invariant.

So can also assume $\underline{g \in N \text{ invariants in } I(h)}$

By $D + I(h)$ transversal, so

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$$\left(S(\mathfrak{p})\otimes\Lambda\right)^{k} \longrightarrow \left(S(\mathfrak{a})\otimes\Lambda\right)^{N}$$

<u>This is onto:</u>

a) show $\otimes\, \mathbb{C}[\frac{1}{D}]$ it is onto; here $D = \prod_{\alpha\in\Delta} \alpha$ on h

b) Suppose given $s \in (S(\mathfrak{a})\otimes\Lambda)^N$. Think of s as an N-invariant polynomial function on $\mathfrak{a}$ with values ~~on~~ in Λ. By a) ~~then~~

$$t = D^{q} s$$

where t is a k inv. function on $\mathfrak{p}$ with values in Λ.

as-injective We may assume that q is least, i.e. that $\frac{t}{D}$ is not regular on $\mathfrak{p}$. This means that at some and hence most points of $\mathfrak{p}$ where $D=0$ we have $t \neq 0$. I want to conclude that there is a point $x \in \mathfrak{a}$ with $D(x)=0$ and $t(x) \neq 0$. However the ~~set of~~ generic points of $\{x\in\mathfrak{p} \mid D(x)=0\}$ is not K conjugate to an element of $\mathfrak{a}$.

~~The hypersurface $D(x)=0$~~

Why true for functions? t is a k-invariant fn. on $\mathfrak{p}$; t restricted to $\mathfrak{p}_{\text{Sing}}$ is non-zero. Why is it true that t rest. to $\mathfrak{a}_s$ is non-zero

~~[illegible]~~ <u>Idea:</u> Choose an ~~closed~~ closure orbit $\bar{\mathcal{O}}$ such that t restricted to this orbit is $\neq 0$, $\mathcal{O} \subset \mathfrak{p}_s$. ~~Consider the function~~ ~~Idea is that $\bar{\mathcal{O}}$~~ Then $\bar{\mathcal{O}}$ is a complete intersection and its singular locus is codim 2 so $1/t$ which is regular on generic orbit must extend to a reg. fn. on $\bar{O}$; then t never 0 on $\bar{O}$.

Be more precise and work in H-C proof:

$$\underline{S(\mathfrak{p}')^k} \subset \boxed{S(\mathfrak{a}')^N}$$

int closed — integral — birational

$$\boxed{\frac{t}{D}} \text{ regular on } \mathfrak{a}$$

sheaf of functions on $\mathfrak{p}$ values in Λ ~~which are invariant~~ + have polar singularities along D

$$t = D\cdot s + g \qquad g \in I(\mathfrak{a})$$

Given $t : \bar{O} \longrightarrow \Lambda$ invariant.

Question can t be $\neq 0$ generically

0 on sing. set.

This won't work: Take a function f on $\bar{O}$ vanishing on sing set let V be its space of translates; then get

$$t : \bar{O} \longrightarrow V'$$
$$x \qquad (g \mapsto g(x))$$

equivariant + $t = 0$ on sing. set, but is not generally 0.

Can you make same technique work globally? Take the affine variety $\mathfrak{p}_{sing}$ and a function f on $\mathfrak{p}_{sing}$ which is non-zero yet vanishes on the semi-simple elements of $\mathfrak{p}_{sing}$ ($\exists$ a non-semi-simple ~~element~~ quasi-regular elt. of $\mathfrak{p}_{sing}$, call it z; then nearby any element is quasi-reg. + non-semi-simple; so can choose $f \ni f(z) \neq 0$ yet $f(s.s.) = 0$.). The space of K translates of f is finite dimensional – call it V and consider the map

$$t : \mathfrak{p}_{sing} \longrightarrow V'$$

$$x \longmapsto (f \mapsto f(x)) \qquad f \in V$$

Then t is a reg. ~~function~~ function on $\mathfrak{p}$ sing. with values in V' which is non-zero yet whose restriction to $\mathfrak{p}_{s.s.} \cap \mathfrak{p}_{sing}$ (in particular $\mathfrak{a}_{sing}$) is 0.

Can t be lifted to a fn. on $\mathfrak{p}$.

$$\mathrm{Hom}(V, S(\mathfrak{p}))^k \longrightarrow \mathrm{Hom}(V, S(\mathfrak{p})/(D))^k \longrightarrow 0$$

χ

Yes!!! Thus there exists a polynomial function

$$t : \mathfrak{p} \longrightarrow V'$$

k invariant, such that t vanishes on the irregular semi-simple elements and yet is non zero. on ~~the~~ regular non semi-simple elts.

~~regular~~ $\therefore \frac{t}{D}$ regular when rest to $\mathfrak{a}$.

Summary: The conjecture that

$$- : (S(\mathfrak{p}')\otimes\Lambda)^{k} \longrightarrow (S(\mathfrak{a}')\otimes\Lambda)^{\underline{N}}$$

is an isomorphism is false in general.

Proof: ~~Consider in $\mathfrak{p}'$ the set of elements which are~~ Let $\mathfrak{p}_{sing}\subset\mathfrak{p}$ be the subvariety of singular elements ~~and let $X \in \mathfrak{p}_{sing}$ be the subvariety of semi-simple~~ and let $Z\in\mathfrak{p}_{sing}$ be a quasi-regular element. The set of such Z is open in $\mathfrak{p}_{sing}$ so there is a non-zero function f on $\mathfrak{p}_{sing}$ which vanishes on all non quasi-regular elements and in particular the semi-simple elts of $\mathfrak{p}_{sing}$. Lift f to an element f of $S(\mathfrak{p}')$ and let V be the K subspace of $S(\mathfrak{p}')$ generated by f, and let $\Lambda = V'$ so that we get ~~then as a k invariant~~ an element $\chi\in(S(\mathfrak{p}')\otimes\Lambda)^{k}$ corresponding to the inclusion $V\to S(\mathfrak{p}')$.
Let D be the "discriminant function" on $\mathfrak{p}$ so that D is the irreducible element of $S(\mathfrak{p}')$ with $\mathfrak{p}_{sing}$ for zeroes; and D is invariant. Then by construction χ is a polynomial function on $\mathfrak{p}$ which is k invariant, which vanishes for sing semi-simple elements, and which is non-zero on some singular elements. Hence $\bar{\chi} = \chi|\mathfrak{a}$ vanishes on $\mathfrak{a}_{sing}$ so $\bar{\chi}/\bar{D}$ is a regular fn. on $\mathfrak{a}$ with values in Λ, necessarily N invariant. But as $-$ is injective if $\bar{\chi}/\bar{D}\in \mathrm{Im}\,- \Rightarrow \chi/D\in(S(\mathfrak{p}')\otimes\Lambda)^{k}$ which is false since $\chi\neq 0$ where $D=0$.

More explicitly for $sl(2,\mathbb{R})$. Here $p = (X, Y)$

$k = (H)$ where $H \cdot X = X$, $H \cdot Y = -Y$. so $e^{tH} \cdot X = e^{t}X$, $e^{tH} \cdot Y = e^{-t}Y$

and $\alpha = (X+Y) = \sqrt{2}A$ Consider

$$N = e^{\pi i n H}$$

$$\left(S(p) \otimes (X)\right)^{k} \qquad \left(S(\alpha) \otimes (X)\right)^{N}$$

$$\| $$

$$\mathbb{C}[XY]\; Y \otimes X \qquad \mathbb{C}[A^2] \cdot A \otimes X$$

no.

$$\underline{\left(S(p) \otimes (X^2)\right)^{k}} \longrightarrow \underline{\left(S(\alpha) \otimes (X^2)\right)^{N}}$$

$$\|$$

$$\mathbb{C}[XY] \cdot Y^2 \otimes X^2 \qquad \underline{\mathbb{C}[A^2] \cdot X^2}$$

map takes $\begin{cases} X \longrightarrow A \\ Y \longrightarrow A \end{cases}$

so image is $\underline{\mathbb{C}[A^2] A^2 \otimes X^2}$

not equal !!!

<u>Conjecture:</u> There is an $\overbrace{\text{function}}^{\text{analytic}}$ D such that if we stay away from the zeroes of D we get an isomorphism of categories.

We know that

$$\mathrm{Hom}_{\mathfrak{g}}\left(\mathcal{U}(\mathfrak{g})\otimes_k \Lambda_1, \mathcal{U}(\mathfrak{g})\otimes_k \Lambda_2\right) \longrightarrow \left[\mathcal{U}(\mathfrak{a})\otimes \mathrm{Hom}_M(\Lambda_1,\Lambda_2)\right]$$

is injective and hopefully we can get it to be Weyl invariant?

---

try $sl(2,\mathbb{R})$ for Weyl invariance!

Image of F in $\mathcal{U}(\mathfrak{a})\otimes \mathrm{Hom}(\Lambda_\sigma,\Lambda_{\sigma+1})$ is

$$\begin{cases} A-\sigma-1 \otimes \varphi^{\sigma}_{\sigma+1} \longmapsto \left(A-\sigma-\frac{1}{2}\right)\otimes \varphi^{\sigma}_{\sigma+1} \\ (A+\sigma)\otimes \varphi^{\sigma+1}_{\sigma} \longmapsto \left(A+\sigma+\frac{1}{2}\right)\otimes \varphi^{\sigma+1}_{\sigma} \end{cases}$$

action of W?

$\left(A+\sigma+\frac{1}{2}\right)\left(A-\sigma-\frac{1}{2}\right)$

W acts by -1 on $\mathrm{Hom}(\Lambda_\sigma,\Lambda_{\sigma+1})$

-1 on A.

$A^2-\left(\sigma+\frac{1}{2}\right)^2$

$$\left(A-\sigma-\frac{1}{2}\right)\otimes\varphi^{\sigma}_{\sigma+1} \longmapsto \left(\mp A+\sigma+\frac{1}{2}\right)\otimes\varphi^{\sigma}_{\sigma+1}$$

$$\left(A+\sigma+\frac{1}{2}\right)\otimes\varphi^{\sigma+1}_{\sigma} \longmapsto \left(+A\mp\sigma\mp\frac{1}{2}\right)\otimes\varphi^{\sigma+1}_{\sigma}$$

$\cancel{\left(A-\sigma-\frac{1}{2}\right)\left(A-\sigma-\frac{3}{2}\right) = A^2 - 2A\sigma + \sigma^2 - \frac{3}{2}A \ldots}$

$\boxed{\text{hence this will not be easy!!!!!}}$

no good at all!

Problems: A. Determine how to make the Weyl group act the image of F.

$$\left(A+\sigma+\tfrac{1}{2}\right)\varphi_{\sigma}^{\sigma+1}\left(A-\sigma-\tfrac{1}{2}\right)\varphi_{\sigma+1}^{\sigma} = A^2-\left(\sigma+\tfrac{1}{2}\right)^2 \quad \varphi_\sigma^\sigma$$

$$\underset{\text{old}}{=} \underline{\left(A \pm \tfrac{1}{2}\right)^2} - \left(\sigma+\tfrac{1}{2}\right)^2.$$

Thus $\varphi_{\sigma+1}^{\sigma}$ should go into $-\varphi_\sigma^{\sigma+1}$

$\cancel{A^2 \; A \; \sigma^2 + \sigma}$

$$\left(A-\sigma+\tfrac{1}{2}\right)\varphi_\sigma^{\sigma-1}\left(A+\sigma-\tfrac{1}{2}\right)\varphi_{\sigma-1}^{\sigma} = A^2-\left(\sigma-\tfrac{1}{2}\right)^2$$

$$\Delta = H^2 + \cancel{H} + \underline{2YX}$$

$$A^2 + A = \sigma^2 + \sigma + A^2 - A - \sigma^2 - \sigma$$

$\boxed{\text{does not seem to be true that lands in } \left[U(\mathfrak{a}) \otimes \mathrm{Hom}_M(\Lambda_1, \Lambda_2)\right]^W}$

$\underline{\Lambda_1 \oplus \Lambda_2}$

$$\mathrm{End}_{\mathfrak{g}}\left(U(\mathfrak{g}) \otimes_k (\Lambda_1 \oplus \Lambda_2)\right) \longrightarrow U(\mathfrak{a}) \otimes \mathrm{Hom}_M(\Lambda_1\oplus\Lambda_2, \Lambda_1\oplus\Lambda_2)$$

inv. under W?

$$P(A^2)\left(A-\sigma-\tfrac{1}{2}\right) = P(A^2)\left(A+\sigma+\tfrac{1}{2}\right)$$

$$P(A^2)\left(-\sigma-\tfrac{1}{2}\right) = P(A)^2\left(\sigma+\tfrac{1}{2}\right)$$

$$P(A^2)(2\sigma+1) = 0 \implies P(A^2) = 0$$

$U(\mathfrak{a}) \otimes (\Lambda_1, \Lambda_1)_M$

$U(\mathfrak{a}) \otimes (\Lambda_1, \Lambda_2)_M$

$U(\mathfrak{a}) \otimes (\Lambda_2, \Lambda_1)_M$

$U(\mathfrak{a}) \otimes (\Lambda_2, \Lambda_2)_M.$

Weyl acts on each piece

Euclidean case: $\tilde{\mathfrak{g}} = \mathfrak{k} \times_s \mathfrak{p}$. , $\mathfrak{a}$, $\tilde{\mathfrak{r}}$ = perp of $\mathfrak{a}$ in $\mathfrak{p}$

$$
\begin{tikzcd}
\mathrm{End}_{\mathfrak{g}}\{\mathcal{U}(\tilde{\mathfrak{g}})\otimes_{\mathfrak{k}}\Lambda\} \arrow[r] \arrow[d, equal] & \left(S(\mathfrak{a})\otimes \mathrm{Hom}_M(\Lambda,\Lambda)\right)^W \arrow[dd, "\cong"] \\
\mathrm{Hom}_{\mathfrak{k}}(\Lambda, S(\mathfrak{p})\otimes\Lambda) \arrow[d, equal] & \\
\left[S(\mathfrak{p})\otimes \mathrm{Hom}(\Lambda,\Lambda)\right]^{\mathfrak{k}} \arrow[r] & \left[S(\mathfrak{a})\otimes \mathrm{Hom}(\Lambda,\Lambda)\right]^{\tilde{N}}
\end{tikzcd}
$$

an isomorphism off D. $\otimes\, \mathbb{C}[D^{-1}]$. clear

Corollary: Nice description of irreducible $\tilde{\mathfrak{g}}$ modules ~~where~~ whose support doesn't intersect $D=0$.

Return to $sl(2,\mathbb{R})$

Category consists of objects σ $\qquad \sigma\in\mathbb{C}$

~~and~~ $\mathrm{Hom}(\sigma,\tau)=0$ unless $\sigma-\tau\in\mathbb{Z}$

$\mathrm{Hom}(\sigma,\sigma+n)$ free module over $\mathbb{C}[A^2]$

with a generator $(X_+)^n \qquad n\geq 0$

~~and~~ $(X_-)^{-n} \qquad n\leq 0$

such that

$$X_+ X_- = \left[A^2 - (\sigma+\tfrac{1}{2})^2\right] id_\sigma \qquad \text{in } Hom(\sigma,\sigma).$$

$$X_- X_+ = \underbrace{\left[A^2 - (\sigma-\tfrac{1}{2})^2\right]} id_\sigma$$

check: $X_+X_- - X_-X_+ = -2\sigma \, id_\sigma$ OKAY. because

$$X_+ = \sqrt{2}\, Y$$
$$X_- = \sqrt{2}\, X.$$

The fundamental idea: Choose a morphism

$$\mathbb{C}[A^2] \longrightarrow R$$

extend the base and calculate the resulting ~~[illegible]~~ category

---

It's clear that we never want $z^2 \neq (\sigma \pm \tfrac{1}{2})^2$.

$$z^2 \neq \left(\sigma_0 + n \pm \tfrac{1}{2}\right)^2$$

or $\quad z^2 \neq \left(\sigma_0 + \tfrac{1}{2} + n\right)^2.$

Take a function such as

z

R is an algebra of ~~analytic~~ entire fns. flat over $\mathbb{C}[A^2]$.

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$$

~~Proof: Take logarithmic derivatives~~

$$\cancel{\frac{\pi \cos \pi z}{\sin \pi z} - \frac{1}{z}}$$

$$\cancel{\pi \cot \pi z}$$

we want $\quad \underline{A^2 \neq (\sigma_0 + \tfrac{1}{2} + n)^2}$

what is the analytic fn with zeroes $\qquad \sigma_1 = \sigma_0 + \frac{1}{2}$

$$(\pm\, \sigma_1) + n \qquad n \in \mathbb{Z}.$$

Fix some point as the origin O.

Define a new operator Y_+ $\quad Y_-$
such that $\quad Y_+ Y_- = Y_- Y_+ = id$

$\boxed{\text{set } \sigma = 0}$ $\quad$ Change X_+ as an operator
X_- as an operator.

~~Grand Conjecture: Fix χ and ν and consider category of $\mathfrak{g}, k$ modules associated to χ and ν. Then there is a function of χ and ν~~

~~Fix χ and ν~~

You must define a map

$$\begin{array}{c} \mathrm{Hom}_{\mathfrak{g}}\left(U(\mathfrak{g})\otimes_k \Lambda_1,\ U(\mathfrak{g})\otimes_k \Lambda_2\right) \\ \big\downarrow \\ \left(R \otimes \mathrm{Hom}_M(\Lambda_1, \Lambda_2)\right)^W \end{array}$$

where R is a suitable constructed quotient ring of $U(\alpha)$.

Conjecture: Let Q be the quotient field of Z, and let $\tilde{Q}$ be the quotient field of $U(\alpha)^W$. Then the ~~different abstract category~~ following categories are equivalent:

(i) objects ~~~~: semi-simple finite k reps Λ.
morphisms $Q \underset{Z}{\otimes} \mathrm{Hom}_{\mathfrak{g}}(U(\mathfrak{g})\otimes_k \Lambda_1, U(\mathfrak{g})\otimes_k \Lambda_2)$

(ii) objects: semi-simple finite k reps. Λ
morphisms $\tilde{Q} \underset{U(\alpha)^W}{\otimes} \left[U(\alpha) \otimes \mathrm{Hom}_M(\Lambda_1, \Lambda_2) \right]^W$.

---

either true or false: Try to define a functor F.

$$F(\sigma) = \sigma \quad \text{clear.}$$

In ~~~~ (i) have X_+, X_- with relations

$$\left.\begin{aligned} X_+ X_- &= \left[A^2 - (\sigma + \tfrac{1}{2})^2\right] id_\sigma \\ X_- X_+ &= \left[A^2 - (\sigma - \tfrac{1}{2})^2\right] id_\sigma \end{aligned}\right\} \text{in } \mathrm{Hom}(\sigma,\sigma)$$

In (ii) we have Y_+, Y_- with relations

$$\left.\begin{aligned} Y_+ Y_- &= A^2\, id_\sigma \\ Y_- Y_+ &= A^2\, id_\sigma \end{aligned}\right\} \text{in } \mathrm{Hom}(\sigma,\sigma) \qquad \begin{aligned} &Y_+ = A \cdot \varphi^{\sigma}_{\sigma+1} \\ &\text{in } \mathrm{Hom}(\sigma, \sigma+1). \end{aligned}$$

Also have to define

$$F(X_+(\sigma)) = f(A,\sigma)\, Y_+(\sigma)$$

$$F(X_-(\sigma)) = g(A,\sigma)\, Y_-(\sigma)$$

$$F(P(A^2)\varphi) = P(A^2)F(\varphi)$$

so that

$$\cancel{F(X_-(\sigma+1)X_+}$$

$$F(X_-(\sigma+1)X_+(\sigma)) = F(X_-(\sigma+1))F(X_+(\sigma))$$

$$\| \qquad\qquad \|$$

$$F((A^2-(\sigma\pm\tfrac{1}{2})^2)\, id_\sigma) \qquad g(A,\sigma+1)\, Y_-(\sigma+1)\circ f(A,\sigma)\, Y_+(\sigma)$$

$$\|$$

$$[A^2-(\sigma\pm\tfrac{1}{2})^2]\, id_\sigma \overset{?}{=} g(A,\sigma+1) f(A,\sigma)\, A^2\, id_\sigma .$$

first conditions $\boxed{g(A,\sigma+1)\, f(A,\sigma) = \dfrac{A^2-(\sigma\pm\frac{1}{2})^2}{A^2}}$

$$F(X_+(\sigma-1)\, X_-(\sigma)) = F(X_+(\sigma-1))\cdot F(X_-(\sigma))$$

$$\| \qquad\qquad \|$$

$$F([A^2-(\sigma\mp\tfrac{1}{2})^2]\, id_\sigma) \qquad f(A,\sigma-1)\, Y_+(\sigma-1)\, g(A,\sigma)\, Y_-(\sigma)$$

$$\| \qquad\qquad \|$$

$$[A^2-(\sigma\mp\tfrac{1}{2})^2]\, id_\sigma = f(A,\sigma-1)\, g(A,\sigma)\, A^2\, id_\sigma$$

2nd condition

$$f(A, \sigma - 1)\, g(A, \sigma) = \frac{A^2 - (\sigma \mp \frac{1}{2})^2}{A^2}$$

now f and g are to be polys. in A^2.

Questions: 1

<u>Conclude:</u> Consider the following categories:

$\mathcal{A}$: objects: finite semi-simple k modules

morphisms: $\mathcal{A}(\Lambda_1, \Lambda_2) = \operatorname{Hom}_{\mathfrak{g}}(\mathcal{U}(\mathfrak{g}) \otimes_k \Lambda_1, \mathcal{U}(\mathfrak{g}) \otimes_k \Lambda_2)$

$\mathcal{B}$: objects: finite semi-simple k modules

morphisms $\mathcal{B}(\Lambda_1, \Lambda_2) = K \otimes_{\mathcal{U}(\alpha)^W} [\mathcal{U}(\alpha) \otimes \operatorname{Hom}_M(\Lambda_1, \Lambda_2)]^W$

where K is the quotient field of $\mathcal{U}(\alpha)^W$.

Then there is no functor $F: \mathcal{A} \to \mathcal{B}$ such that $F(\Lambda) = \Lambda$.

I am going to calculate this out very carefully for $\mathfrak{sl}(2, \mathbb{R})$. Let's work with $\mathcal{B}$. Then can restrict to Λ irreducible $\Lambda_1 = \sigma$. $\Lambda_2 = \tau$ where

$$H|_\sigma = \sigma|_\sigma$$

Now $M = \{e^{2\pi i n H} \mid n \in \mathbb{Z}\}$.

$\hat{M} = \{e^{\pi i n H} \mid n \in \mathbb{Z}\}$.

So $\operatorname{Hom}(\sigma, \tau) \simeq \tau - \sigma$

so $\operatorname{Hom}_M(\sigma, \tau) \neq 0 \iff \tau - \sigma \in \mathbb{Z}$.

Then $\hat{M}$ acts by sign whether $\tau - \sigma$ is odd or even.

On $\mathcal{U}(\alpha) \simeq \mathbb{C}[A]$ W acts by -1, hence

$$\left[\mathcal{U}(\alpha) \otimes \mathrm{Hom}_{\mathcal{M}}(\sigma, \sigma+n)\right]^W = \begin{cases} \mathbb{C}[A^2] \cdot A \otimes \varphi^{\sigma}_{\sigma+n} & n \text{ odd} \\ \mathbb{C}[A^2] \otimes \varphi^{\sigma}_{\sigma+n} & n \text{ even.} \end{cases}$$

In particular setting

$$Y_+(\sigma) = A \otimes \varphi^{\sigma}_{\sigma+1}$$
$$Y_-(\sigma) = A \otimes \varphi^{\sigma}_{\sigma-1}$$

we have that at least when rational functions $K \otimes_{\mathcal{U}(\alpha)^W}$ are allowed that

$$B(\sigma, \sigma+n) = \begin{cases} K(Y_+)^n & n \geq 0 \\ K(Y_-)^{-n} & n \leq 0 \end{cases}$$

and that

$$Y_-(\sigma+1)Y_+(\sigma) = A \otimes \varphi^{\sigma+1}_{\sigma} \circ A \otimes \varphi^{\sigma}_{\sigma+1}$$
$$= A^2 \, id_\sigma$$

$$\boxed{\begin{aligned} Y_-(\sigma+1)\, Y_+(\sigma) &= A^2 \, id_\sigma \\ Y_+(\sigma-1)\, Y_-(\sigma) &= A^2 \, id_\sigma \end{aligned}}$$

Now for $\mathfrak{a}$. Must calculate

$$\mathrm{Hom}_k(\Lambda_1, \mathcal{U}(\mathfrak{g}) \otimes_k \Lambda_2)$$
$$\cong$$
$$S(\mathfrak{p}) \otimes \Lambda_2$$
$$\|$$
$$\underbrace{\sum X^i Y^j \Lambda_2}_{\text{weight } \sigma} \qquad \text{center gen. by } H^2 - H - XY$$

Thus

$$\sigma = i - j + \tau$$

So again $\sigma - \tau \in \mathbb{Z}$

observe also that

$\mathrm{Hom}_k(\sigma, \mathcal{U}(\mathfrak{g}) \otimes_k (\sigma + n))$ free module over Z with generator

$$\cancel{\sigma \longrightarrow Y^n(\sigma+n)}$$

$$1_\sigma \longmapsto Y^n 1_{\sigma+n} \qquad n \geq 0$$

$$1_\sigma \longmapsto X^{-n} 1_{\sigma+n} \qquad n \leq 0$$

~~Hence~~ if we let $X_+(\sigma) : 1_\sigma \longmapsto Y 1_{\sigma+1}$

$$X_-(\sigma) : 1_\sigma \longmapsto X 1_{\sigma-1}$$

we have

$$\mathfrak{a}(\sigma, \sigma+n) = \begin{cases} Z \cdot X_+^n & n \geq 0 \\ Z X_-^{-n} & n \leq 0 \end{cases}$$

and the commutation relation

$$X_+(\sigma-1)X_-(\sigma) : 1_\sigma \overset{X(\sigma)}{\longmapsto} X1_{\sigma-1} \xrightarrow{X_+(\sigma-1)} XY\,1_\sigma$$

$$XY\,1_\sigma = \left[\left\{XY + \tfrac{1}{2}(H^2-H)\right\} - \tfrac{1}{2}(\sigma^2-\sigma)\right]1_\sigma$$

$$\therefore \quad X_+(\sigma-1)X_-(\sigma) = \tfrac{1}{2}\left\{C - \sigma^2+\sigma\right\}1_\sigma$$

$$C = 2YX + H^2 + H = H^2 - H + 2XY$$

$$X_-(\sigma+1)X_+(\sigma) : 1_\sigma \overset{X_+\sigma}{\longmapsto} Y1_{\sigma+1} \longrightarrow YX\,1_\sigma$$

$$YX\,1_\sigma = \left\{\left[YX + \tfrac{1}{2}(H^2+H)\right] - \tfrac{1}{2}(\sigma^2+\sigma)\right\}1_\sigma$$

$$X_-(\sigma+1)X_+(\sigma) = \tfrac{1}{2}\left\{C - \sigma^2-\sigma\right\}1_\sigma .$$

$$\boxed{\begin{aligned} X_+(\sigma-1)X_-(\sigma) &= \tfrac{1}{2}\left[C - \sigma^2+\sigma\right]\mathrm{id}_\sigma \\ X_-(\sigma+1)X_+(\sigma) &= \tfrac{1}{2}\left[C - \sigma^2-\sigma\right]\mathrm{id}_\sigma \end{aligned}}$$

Now define

$$F(X_+(\sigma)) = f(A^2;\sigma)\, Y_+(\sigma)$$
$$F(X_-(\sigma)) = g(A^2;\sigma)\cdot Y_-(\sigma)$$

Then for F to be a functor

$$F(X_+(\sigma-1)\cdot X_-(\sigma)) = F(X_+(\sigma-1))\, F(X_-(\sigma))$$
$$\Vert \qquad\qquad\qquad \Vert$$
$$F\left(\tfrac{1}{2}[C - \sigma^2 + \sigma]\, id_\sigma\right) \qquad f(A^2;\sigma-1)\, Y_+(\sigma-1)\, g(A^2;\sigma)\, Y_-(\sigma)$$
$$\Vert$$
$$\tfrac{1}{2}\varphi(C-\sigma^2+\sigma)\, id_\sigma = f(A^2;\sigma-1)\, g(A^2;\sigma)\, A^2\, id_\sigma$$

$$\varphi : Z = \mathbb{B}(C) \longrightarrow K = \mathbb{C}(A^2)$$ homom

$$F(X_-(\sigma+1)\, X_+(\sigma)) = F(X_-(\sigma+1))\, F(X_+(\sigma))$$
$$\Vert \qquad\qquad\qquad \Vert$$
$$F\left(\tfrac{1}{2}(C - \sigma^2 - \sigma)\, id_\sigma\right) \qquad g(A^2;\sigma+1)\, f(A^2;\sigma)\, id_\sigma$$

$$\tfrac{1}{2}\varphi(C - \sigma^2 - \sigma)\, id_\sigma =$$

$$\boxed{\begin{aligned} \tfrac{1}{2}\varphi(C-\sigma^2+\sigma) &= f(A^2,\sigma-1)\,g(A^2,\sigma)\,A^2 \\ \tfrac{1}{2}\varphi(C-\sigma^2-\sigma) &= g(A^2,\sigma+1)\,f(A^2,\sigma)\,A^2 \end{aligned}}$$

$$\|$$

$$\tfrac{1}{2}\varphi(C-\sigma^2-2\sigma-1+\sigma+1) = g(A^2,\sigma+1)\,f(A^2,\sigma)\,A^2$$

here's your contradiction:

$$f(A^2,\sigma)\,g(A^2,\sigma+1) = F\left(\tfrac{1}{2}\left[C-(\sigma+1)^2+(\sigma+1)\right]\right)$$

$$\|$$

$$F\left(\tfrac{1}{2}(C-\sigma^2-\sigma)\right)$$

$$C-\sigma^2-2\sigma-\cancel{1}+\sigma+\cancel{1}$$

$$C-\sigma^2-\sigma$$

NO CONTRADICTION

$$\cancel{f(A^2,\sigma)\,g(A^2,\sigma+1) = \frac{(A-\frac{1}{2})^2-(\sigma+\frac{1}{2})^2}{A^2} = \frac{(A+\sigma)(A-\sigma-1)}{A^2}}$$

see page 13

Feb. 10

Situation: You have no theorems of your own yet.

Reexamined structure of Ω_Λ:

Previously we defined a mapping

$$\star \qquad \mathrm{Hom}_{\mathfrak{g}}(U(\mathfrak{g})\otimes\Lambda_1, U(\mathfrak{g})\otimes_k\Lambda_2) \longrightarrow U(\mathfrak{a})\otimes \mathrm{Hom}_M(\Lambda_1,\Lambda_2)$$

using the functor $V \mapsto 1\otimes_{\mathfrak{n}} V$. We hoped that after the map $'$: $U(\mathfrak{a}) \to U(\mathfrak{a})$ given by $A - g$, the image would land in the Weyl group invariants. This is false for $sl(2,\mathbb{R})$.

The associated graded map of $\star$ is the map

$$[S(\mathfrak{p})\otimes \mathrm{Hom}(\Lambda_1,\Lambda_2)]^k \longrightarrow S(\mathfrak{a})\otimes \mathrm{Hom}_M(\Lambda_1,\Lambda_2)$$

induced by the map $S(\mathfrak{p}) = S(\mathfrak{g}/\mathfrak{k}) \longrightarrow S(\mathfrak{g}/\mathfrak{k}+\mathfrak{n}) = S(\mathfrak{a})$ which comes from the orthogonal projection $\mathfrak{p} \to \mathfrak{a}$ at least for $sl(2,\mathbb{R})$. One can show that we get a map

$$[S(\mathfrak{p})\otimes \mathrm{Hom}(\Lambda_1,\Lambda_2)]^k \longrightarrow [S(\mathfrak{a})\otimes \mathrm{Hom}_M(\Lambda_1,\Lambda_2)]^W$$

We hoped this was an isomorphism, but it turned out to be false: It is always injective, and surjective if tensored with $\mathbb{C}[\frac{1}{D}]$, where D is discriminant. As $S(\mathfrak{p})^k \xrightarrow{\sim} S(\mathfrak{a})^W$ both modules are the same rank so we derive Rallis's results on the multiplicities of H as a K representation. $\star$ always injective

Problem: Classify irreducible $\tilde{\mathfrak{g}}, k$ modules.

Proposition: Let Λ be a finite k-module. Then

$$- : (S(\mathfrak{p}) \otimes \Lambda)^k \otimes \mathbb{C}[D^{-1}] \longrightarrow (S(\mathfrak{a}) \otimes \Lambda)^{\tilde{M}} \otimes \mathbb{C}[\bar{D}^{-1}]$$

is an isomorphism.

Proof: $-$ is induced by the orthogonal projection $\mathfrak{p} \to \mathfrak{a}$; if we identify $\mathfrak{p}$ with $\mathfrak{p}'$ and $\mathfrak{a}$ with $\mathfrak{a}'$ via the Killing form, then we can think of $-$ as the restriction of functions on $\mathfrak{p}$ to functions on $\mathfrak{a}$. Recall that $- : S(\mathfrak{p}')^k \xrightarrow{\sim} S(\mathfrak{a}')^W$ and that D is the element of $S(\mathfrak{p})^k$ such that

$$\bar{D}(a) = \prod_{\alpha \in \Delta'} \alpha(a) \qquad a \in \mathfrak{a}.$$

Alternatively we have $(\operatorname{ad} X)^2 : \mathfrak{p} \to \mathfrak{p}$ for $X \in \mathfrak{p}$ and consider the polynomial

$$\det\left(T - (\operatorname{ad} X)^2\right) = T^g + p_1(X) T^{g-1} + \cdots + p_g(X)$$

where $p_j \in S(\mathfrak{p}')^k$. Note that if $x \in \mathfrak{a}$, then we can calculate the eigenvalues as follows: We have $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{q}$ where $\mathfrak{q}$ has the basis $e_\alpha - e_{\theta\alpha}$ $\alpha \in \Sigma'$. If $x \in \mathfrak{a}$, then $x \in \mathfrak{h}$, so

$$(\operatorname{ad} x)^2 (e_\alpha - e_{\theta\alpha}) = \alpha(x)^2 e_\alpha - (\theta\alpha)(x)^2 e_{\theta\alpha}$$

But $(\theta\alpha)(x) = \alpha(\theta x) = -\alpha(x)$

so $(\mathrm{ad}\, x)^2 (e_\alpha - e_{\theta\alpha}) = \alpha(x)^2 (e_\alpha - e_{\theta\alpha})$.

Thus $\det(T - (\mathrm{ad}\, x)^2) = \prod_{\alpha \in \Sigma'} (T - \alpha(x)^2) \cdot T^{\ell}$ where $\ell =$ dim $\mathfrak{a}$. Thus $p_{\mathfrak{g}-\ell}(x) = (-1)^{\mathrm{card}\,\Sigma'} \prod_{\alpha \in \Sigma'} \alpha(x)^2 = \bar{D}(x) = D(x)$.

We now proceed to the proof of the proposition. An element of $(S(\mathfrak{p}') \otimes \Lambda)^k \otimes \mathbb{C}[D^{-1}] = (S(\mathfrak{p}')_D \otimes \Lambda)^k$ is an algebraic function on $\mathfrak{p}_{reg} = \{x \in \mathfrak{p} \mid D(x) \neq 0\}$ with values in Λ ~~such that~~ which is k-equivariant. Similarly an element of $(S(\mathfrak{a}')_{\bar{D}} \otimes \Lambda)^N$ is an algebraic function on $\mathfrak{a}_{reg}$ with values in Λ which is N-equivariant. Suppose we can show that the quotient of $\mathfrak{p}_{reg}$ by k exists and is isomorphic to the quotient of $\mathfrak{a}_{reg}$ by N, and that these are are faithfully flat descents with descent data generated by the respective groups. Then over $\mathfrak{p}_{reg}/k$ I obtain a locally free sheaf whose sections are $(S(\mathfrak{p}')_D \otimes \Lambda)^k$ and similarly a locally free sheaf over $\mathfrak{a}_{reg}/N$ whose sections are $(S(\mathfrak{a}')_{\bar{D}} \otimes \Lambda)^N$. These two sheaves will be isomorphic hence so will their sections.

<u>Problem:</u> Classify irreducible $\mathfrak{g}, k$ modules.

$$(\mathfrak{g}, k) \longrightarrow (M, \alpha)$$
$$V \xrightarrow{} W$$

Exactly as before except now I know how to make the ~~Galois~~ Weyl group act!! so now I can define an isom.? of $I(\xi) \longrightarrow I(\xi^s)$?

$$\mathrm{Hom}_k(\Lambda, I(\xi)) = \mathrm{Hom}_M(\Lambda, \xi)$$

$$I(\xi) = k\text{fin}\mathrm{Hom}_{M,\alpha,n}(U(\mathfrak{g}), \xi) \quad !$$

Let $\alpha_s \in \bar{N}$, we want to define a map $\Gamma(\alpha_s, \xi): I(\xi) \longrightarrow I(\xi^{\alpha_s})$ somehow.

$$\mathrm{Hom}_{M,\alpha,n}(U(\mathfrak{g}), \xi) \longrightarrow \mathrm{Hom}_{M,\alpha,n}(U(\mathfrak{g}), \xi^{\alpha_s})$$

There is an obvious way of proceeding, namely to apply α_s all the way through i.e. send

f into $g \longmapsto f(\alpha_s g)$.

which is defined since $\alpha_s \in K$.

Then of course you get a map

$$\mathrm{Hom}_{M,\mathfrak{a},\mathfrak{n}}\left(\mathcal{U}(\mathfrak{g}), \xi^{\alpha_s}\right) \quad \text{ie}$$

$$\alpha_s f(xg) = f(xg\alpha_s) =$$

Unfortunately, this doesn't help much with the degeneracy. This clearly works + is correct relative to the map

$$\mathrm{End}_{\tilde{\mathfrak{g}}}(\mathcal{U}(\tilde{\mathfrak{g}})\otimes_k \Lambda) \longrightarrow \left[\mathcal{U}(\mathfrak{a})\otimes \mathrm{Hom}_M(\Lambda,\Lambda)\right]^W$$

Still how do you get irreducibility?

$$I(\xi)/\mathfrak{n}I(\xi) = ?$$

Go back: You want to decide when a principal series repn. is completely reducible, i.e. so have to examine the ~~operator maps~~ objects

$$\mathrm{Hom}_k(\Lambda, I(\xi)) = \mathrm{Hom}_M(\Lambda, \xi)$$

and the maps between them. One sees that if we are away from the singular locus there is no problem (Theorem of Bruhat) because then we map onto $\left[\mathcal{U}(\mathfrak{a})\otimes \mathrm{Hom}_M(\Lambda,\Lambda)\right]^W$

However if we are on the singular locus things may still be irreducible!!

We know that every irred module has its support ~~[illegible]~~ on an orbit closure. Suppose we are in complex case with adjoint action of k on k and that we have a module supported on the variety of nilpotent which is of invariant. There are several ~~nilpotent [illegible]~~ orbits of nilpotents in addition to the orbits of principal nilpotents and we have to make sure that these do not give rise to ~~reducible~~ quotients. Can you formulate geometrically?

X variety $Y \subset X$ subvariety, F coherent sheaf on X. Then have

$$\to F \to F \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to 0$$

Now suppose a group acts on X, Y, F.

$\mathfrak{p} \;\; \mathfrak{g}/\mathfrak{k} \longrightarrow \mathfrak{g}/\mathfrak{k}+\mathfrak{n} \xrightarrow{\sim} \mathfrak{a}$ is orth proj.?

$X = \frac{1}{\sqrt{2}}[-(N-\cancel{H}) + A] \longmapsto \frac{1}{\sqrt{2}} A$

$Y = \frac{1}{\sqrt{2}}[(N-H) + A] \longmapsto \frac{1}{\sqrt{2}} A$

$\underbrace{\frac{X+Y}{\sqrt{2}}}_{A}$; $\underbrace{i\frac{X-Y}{\sqrt{2}}}_{B}$

$\cancel{X =} (A - iB) = \frac{2X}{\sqrt{2}}$

$\begin{cases} \text{rank } M = \text{rank } \mathfrak{g} - \text{rank } \mathfrak{a} = \ell - 1 \\ \text{rank } K \leq \text{rank } \mathfrak{g} = \ell \end{cases}$

two case for rank 1

either rank K = rank $\mathfrak{g}$ $\Rightarrow$ rang M = rang G

rank K = rank $\mathfrak{g} - 1$ $\Rightarrow$ rang K = rang G [hermitian

Both Rallis + Hermann seems to imply that if G/K is of rank 1, then any irred rep. ν of M occurs at most once in an irred rep. of K. ~~Equivalently if~~ This is true if ν is 1-dim'l. Equivalently that if we take the sections of $G \times_B \mathbb{C}$ and decompose over K we get a ladder.

$$\mathrm{Hom}_B(G, \mathfrak{z}) = \{ f : G \to \mathfrak{z} \mid f(bg) = \mathfrak{z}(b) f(g) \}$$

$$\mathrm{Hom}_K(\Lambda, \underline{\mathrm{Hom}_B(G, \mathfrak{z})}) = \mathrm{Hom}_M(\Lambda, \mathfrak{z}).$$

~~idea is that such a representation~~
~~or is a /~~

Claiming that the rep. by <u>spherical harmonics</u> is used

$$\mathrm{Hom}_M(\Lambda, \nu) = \mathrm{Hom}_K(\Lambda, \cancel{S(\mathfrak{p})} \; \mathrm{Hom}_{M, \mathfrak{a}, \mathfrak{n}}(\overset{U(\bar{\mathfrak{g}})}{\cancel{S(\mathfrak{p})}}, \nu))$$

Feb. 11

Problems 1. Irreducibility of ~~[illegible]~~ Principal series for $\tilde{\mathfrak{g}}$.

2. Structure of Ω_Λ + the Weyl action

3. maximal ideals in $U(\mathfrak{g})$.

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(1)

Theorem 1: The irreducible $\tilde{\mathfrak{g}}, k$ modules V are as follows. As an $S(\mathfrak{p})$ module V has support on a closed orbit of K in $\mathfrak{p}'$, necessarily the orbit of a semi-simple element α. ~~[illegible]~~ V is the space of sections of a homogeneous vector bundle over ~~[illegible]~~ $K\alpha$ coming from some irreducible representation ν of the isotropy group M_α of α.

Proof: Let V be an irred. $\tilde{\mathfrak{g}}$ module so that V is ~~[illegible]~~ a quotient of $U(\tilde{\mathfrak{g}}) \otimes_k \Lambda \cong S(\mathfrak{p}) \otimes \Lambda$ for some k finite rep Λ. Then V is finite type over $S(\mathfrak{p})$ so defines a ~~[illegible]~~ coherent alg sheaf over $\mathfrak{p}'$. The support of V is closed as a subscheme and K stable, call it Z. If Y is a closed K stable subscheme ~~subvariety~~ of Z with ideal I, then $V/IV \neq 0$ so $IV = 0$ so $Y = Z$. Thus Z must be minimal closed + K stable, so Z must be ~~the~~ a closed orbit $K\alpha$. ~~Let M_α be the isotropy group of~~ The sheaf ~~[illegible]~~ defined by V ~~[illegible]~~ comes from ~~[illegible]~~ a coherent sheaf $\mathcal{F}$ on $Z \simeq K/M_\alpha$. As Z is reduced and the rank of $\mathcal{F}$ is constant $\mathcal{F}$ is a homogeneous vector bundle $K \times_{M_\alpha} \nu$ on Z. Clearly ν must be irreducible.

Conversely given a closed orbit $K\alpha$ and a homogeneous vector bundle $K \times_{M_\alpha} \nu$ on Z we get an irreducible $\tilde{\mathfrak{g}}$ module.

Why – how is this done? Suppose I give myself $\lambda \in \mathfrak{a}'$ with centralizer M in K and also an irred rep ν of M. Then I have to form the homogeneous bundle

$$K \times_{M_\lambda} \nu$$

$$K/M_\lambda \xrightarrow{\quad i \quad} \mathfrak{p}'$$

and take $i_*(K \times_{M_\lambda} \nu)$ and the representation is then sections of this bundle i.e.

$$V = \Gamma\left(i_*(K \times_{M_\lambda} \nu)\right) = \mathrm{Hom}_{M_\lambda}(K, \nu)$$

and where $\mathfrak{p}$ acts by restriction to the orbit, i.e. an element of $\mathfrak{p}$ is a function on $\mathfrak{p}'$ hence also on $K\lambda$, so if

$$f : K \xrightarrow[M_\lambda]{} \nu$$

and $X \in \mathfrak{p}$, then X defines $\tilde{X} : K \longrightarrow 1$ by

$$\tilde{X}(k) = \langle X, k^{-1}\lambda \rangle$$

$$\tilde{X}(mk) = \langle X, k^{-1}m^{-1}\lambda \rangle = \langle \mathrm{Ad}\,k(X), \lambda \rangle$$

and

$$(Xf)(k) = \langle k \cdot X, \lambda \rangle f(k)$$
$$= \lambda(k \cdot X) f(k).$$

ie

$$(Xf)(\delta) = \sum_i \lambda(\delta_i' \cdot X)\ f(\delta_i'')$$

if $\Delta\delta = \sum \delta_i' \otimes \delta_i''$.

~~Moreover~~ This tells ~~me~~ how p acts and I know how k acts so I am done!

$$Y \in k \qquad (Yf)(\delta) = f(\delta Y) \qquad \delta \in \mathcal{U}(k)$$

$$X \in p \qquad (Xf)(\delta) = \lambda(\delta_i' * X) f(\delta_i'')$$

$$(Y \cdot X \cdot f)(\delta) = (Xf)(\delta Y) = \lambda((\delta Y)' * X)\ f((\delta Y)'')$$

$$(XYf)(\delta) = \lambda(\delta' * X)(Yf)(\delta'')$$
$$= \lambda(\delta' * X) \cdot f(\delta'' Y)$$

$$([Y,X]f)(\delta) = \lambda(\delta' * [Y,X]) \cdot f(\delta'')$$

$$\Delta(\delta Y) = (\delta'_i \otimes \delta''_i)(Y \otimes 1 + 1 \otimes Y)$$

$$= \underline{\delta' Y \otimes \delta'' + \delta' \otimes \delta'' Y}$$

$$(YXf)(\delta) = \lambda(\delta' Y * X) f(\delta'') + \lambda(\delta' * X) f(\delta'' Y)$$

$$(XYf)(\delta) = \lambda(\delta' * X) f(\delta'' Y)$$

$$- \qquad = \underline{\lambda(\delta' Y * X) f(\delta'')}. \qquad = ([Y,X]f)(\delta)$$

$$\delta' * [Y,X] = \delta' * (Y * X) = \delta' Y * X.$$

so its OKAY

---

now see if you can describe this in the form $I(\zeta)$?

Mackey says to take ~~MP~~ the representation of $\underline{MP}$ and induce up to the whole group! λ is a representation of P ie $\lambda \in \mathfrak{p}'$, idea is that $\lambda \in \mathfrak{a}'$ but the inclusion $\mathfrak{a}' \hookrightarrow \mathfrak{p}'$ comes from choosing $\tilde{\mathfrak{n}} \ni \mathfrak{p} = \mathfrak{a} \oplus \mathfrak{n}$

Therefore

$$\begin{array}{ccc} \mathrm{Hom}_{M_\lambda \mathfrak{p}}(U(\mathfrak{g}), \zeta) & & \\ \| & \searrow & \\ \mathrm{Hom}_{M_\lambda, \mathfrak{p}}(U(\mathfrak{p})U(\mathfrak{k}), \zeta) & = & \underline{\mathrm{Hom}_{M_\lambda}(U(\mathfrak{k}), \zeta)} \end{array}$$

$$U(k) \xrightarrow{i} U(\mathfrak{g}).$$

$$\begin{tikzcd} U(\mathfrak{g}) \arrow[r, "\tilde{f}"] & Y \\ U(k) \arrow[u, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

Then $\underline{\tilde{f}(X^\alpha \delta) = X^\alpha \cdot \tilde{f}(\delta) = \langle X^\alpha, e^\lambda \rangle \tilde{f}(\delta)}$

then $(X\tilde{f})(X^\alpha \delta) = \tilde{f}(X^\alpha \delta X) = \tilde{f}(X^\alpha X \cdot \delta) + \tilde{f}(X^\alpha(\delta * X))$

$$= \langle X^\alpha, e^\lambda \rangle \lambda(X) \tilde{f}(\delta) + \lambda(\delta * X) \underbrace{\tilde{f}(1)}_{\langle X^\alpha, e^\lambda \rangle}$$

$$= \lambda(X) \tilde{f}(X^\alpha \delta)$$

no

$\cancel{\delta X = X \cdot \delta + \sum_i (\delta'_i * X) \cdot \delta''_i} \; ?$

$$\boxed{\delta X = \sum_i (\delta'_i * X) \cdot \delta''_i}$$

$\delta \in U(k)$
$X \in \mathfrak{p}$.

$\therefore (X\tilde{f})(X^\alpha \delta) = \tilde{f}(\sum_i \delta'_i * X \cdot \delta''_i)$

$$= \lambda(\delta'_i * X) f(\delta''_i).$$ OKAY

The problem remains: Suppose I define

$$I(\xi) = k\text{fin} \operatorname{Hom}_{M,\mathfrak{p}}(\mathcal{U}(\mathfrak{g}), \xi)$$

Why is $I(\xi)$ irreducible?

Theorem: Let $\lambda \in \mathfrak{p}'$ be such that $K\lambda$ is closed, ~~and~~ let M_λ be the isotropy group of λ, and let ν be an irred. rep. f.d. of M_λ. Then if $\xi = \lambda \otimes \nu$

$$I(\xi) = \left[k \text{ finite } \operatorname{Hom}(\mathcal{U}(\mathfrak{g}), \xi) \right]^{M,\mathfrak{p}}$$

is an irreducible $\mathfrak{g}, k$ module.

Proof: ~~ok~~ at present:

$$\underset{\tilde{f}}{I(\xi)} \simeq \left(k \text{ finite } \underset{\tilde{\tilde{f}}}{\operatorname{Hom}}(\mathcal{U}(k), \xi)\right)^{M}$$

with action given by

$$(Xf)(\delta) = \lambda(\delta' * X) \cdot f(\delta'') \qquad \text{if } X \in \mathfrak{p}$$
$$(Yf)(\delta) = f(\delta . Y) \qquad \text{if } Y \in k.$$

and latter is isomorphic to the sections of the homogeneous vector bundle $K \times_{M_\lambda} \nu$ ~~with its~~ over $K/M_\lambda \simeq K\lambda$ with its obvious $k, \mathfrak{p}$ structure.

The latter is clearly irreducible by alg. geometry.

Back to Nakayama:

Basic facts about K action ~~of~~ on $\mathfrak{p}$: TFAE for $x \in \mathfrak{p}$
(i) ~~Any closed orbits the orbit is closed~~ Kx closed
(ii) $\operatorname{ad} x$ is semi-simple
(iii) $Kx \cap \mathfrak{a} \neq \emptyset$.

Do for adjoint action so that $\mathfrak{a} = \mathfrak{h}$

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha .$$

$\mathfrak{g}$ semi-simple ie $\operatorname{tr} \operatorname{ad} x \operatorname{ad} y \; y = \langle x, y \rangle$ non-degenerate.
~~x regular~~

If $\mathfrak{h}$ nilpotent subalg. ~~in~~ $\mathfrak{g}$, get $\mathfrak{g} = \bigoplus_{\lambda \in \mathfrak{h}'} \mathfrak{g}^\lambda$. Say $\mathfrak{h}$ Cartan if $\mathfrak{h}$ nilpotent + $\mathfrak{h}$ = its normalizer.

Then x regular $\iff$ $\dim (x)^0$ minimal.
Look at char poly of $\operatorname{ad} x$

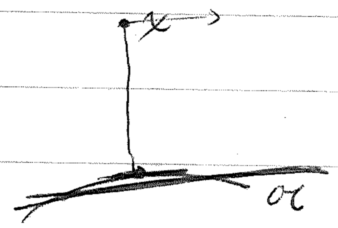
$$\det(T - \operatorname{ad} x) = T^n + \cdots + p_n(x)$$

so that for some ℓ $p_{n-\ell}(x) \neq 0$ and gives regular elements. It follows that

Examine proofs of conj. thm. for Cartan subalgs.

① alg. geom one: One consider Zariski open of regular elts and ~~the spaces they span~~ their centralizers. One consider the set of Cartan subalgs. in Grassmannian – image of reg elts. This set is connected. ~~and the group acts so we get that~~ One calculates that the orbits are open by Jacobian criterion hence must be all.

② compact one. One takes an element x and minimizes its distance to $\mathfrak{a}$, i.e.



$$\langle (\mathrm{Ad}\, X)\, x, H \rangle = 0 \quad \text{all } H \in \mathfrak{a},\ X \in K. \qquad x \in \mathfrak{p}.$$

$$\Longrightarrow \langle X, [x, H] \rangle = 0 \quad \text{all } X$$

$$\Longrightarrow [x, H] = 0 \Longrightarrow \mathfrak{a} \text{ not max abelian}$$

③ topological one. Let T be a ~~map~~ max. torus in K so that centralizer of $T = T$ at least for connected part, so one gets weight spaces and so taking a generic member of T one calculates the Lefschetz no. by fix pt. formula

$$K/T \qquad x\,kT = kT$$

$$\Leftrightarrow k x k^{-1} = T \Leftrightarrow x \in N.$$

One knows that N/T is finite and that all fixpts have $+1$ for mult. by Lef formula $\Longrightarrow$ Lefchetz no. = order of W. $\therefore$ every element has a fixed pt. $\Rightarrow$ every elt. conj. to an elt of T.

Let $\operatorname{ad} x$ be semi-simple. Why does x belong to a C.S.? Consider the weight spaces for $\operatorname{ad} x$ and consider the centralizer of x. ~~Killing form usual~~ Have usual decomp.

$$\mathfrak{g} = \sum_{\lambda \in \mathbb{C}} \mathfrak{g}^{\lambda} \qquad \mathfrak{g}^{\lambda}\mathfrak{g}^{\mu} \subset \mathfrak{g}^{\lambda+\mu}$$

hence $\mathfrak{g}^{0}$ is its own orth. etc. hence is reductive. Now choose a Cartan subalg. $\mathfrak{h}$ of $\mathfrak{g}^{0}$. Then $\mathfrak{h}$ red. in $\mathfrak{g}^{0}$ + $\mathfrak{g}^{0}$ red. in $\mathfrak{g}$ ($\mathfrak{g}^{0}$ red + center is s.s)

Let $\mathfrak{a}$ be a semi-simple max abelian subspace of $\mathfrak{g}$ and take the weight decomp

$$\mathfrak{g} = \sum_{\lambda \in \mathfrak{a}'} \mathfrak{g}^{\lambda}$$

Killing of $\mathfrak{g}$ restricts to a non-deg. form on $\mathfrak{g}^{0}$, so $\mathfrak{g}^{0}$ reductive, but its center is $\mathfrak{a}$ which acts semi-simply on the weight spaces. $\therefore$ $\mathfrak{g}^{0}$ reductive in $\mathfrak{g}$ ~~But [illegible]~~ So take a Cartan $\mathfrak{h}$ of $\mathfrak{g}^{0}$. $\mathfrak{a} \subset \mathfrak{h}$ + $\mathfrak{h}$ red. in $\mathfrak{g}^{0}$, $\mathfrak{g}^{0}$ red in $\mathfrak{g} \Rightarrow \mathfrak{h}$ red in $\mathfrak{g} \Rightarrow \mathfrak{h} = \mathfrak{a}$ by max. $\therefore$ $\mathfrak{a}$ = its own cent + is a Cartan subalg.

So have proved

Thm: $\mathfrak{g}$ reductive ~~[illegible]~~, $\mathfrak{a}$ abelian semi-simple subspace of $\mathfrak{g}$ $\Rightarrow$ $\mathfrak{a}$ contained in a C.S.

Using only weight spaces and basic properties of reductive algebras. (not that $\mathfrak{g}$ red $\Rightarrow$ $\mathfrak{g}$ has non-dg. inv. bilinear form)

~~Prop $\mathfrak{g}$ acts on V & [illegible] non-degenerate [illegible] an invariant non-degenerate bilinear form $\Rightarrow V$~~

$$\mathfrak{g} = k + \mathfrak{p}. \qquad \boxed{\mathfrak{a}}$$

$\mathfrak{b}$ max. abelian s.s. $\subset \mathfrak{p}$. Look at K action

$X \in \mathfrak{p}$ $\quad X = S + N$ in $\mathfrak{g} \Rightarrow S, N \in \mathfrak{p}$

$$(\mathrm{ad}\, s+n)^2 = (\mathrm{ad}\, s)^2 + \underbrace{2\,\mathrm{ad}\, s\, \mathrm{ad}\, n + (\mathrm{ad}\, n)^2}_{\text{nilp.}} \qquad \text{s.s.}$$

~~commutes~~

$$\therefore 2\,\mathrm{ad}\, s\, \mathrm{ad}\, n + (\mathrm{ad}\, n)^2 = 0$$

Symmetric space theory.

$\mathfrak{g}$ semi-simple, θ involution, $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$, $\mathfrak{a}$ max abelian subspace of $\mathfrak{p}$ consisting of semi-simple elements of $\mathfrak{g}$. The centralizer of $\mathfrak{a}$ is reductive + θ stable + meets $\mathfrak{p}$ in $\mathfrak{a}$ so $=\mathfrak{m}+\mathfrak{a}$ $\quad \mathfrak{m}\subset\mathfrak{k}$. If $\mathfrak{h}$ is a Cartan of $\mathfrak{g}$ containing $\mathfrak{a}$, then $\mathfrak{a}\subset\mathfrak{h}\subset\mathfrak{m}+\mathfrak{a}$ $\Rightarrow$ $\mathfrak{h}=\mathfrak{h}_k+\mathfrak{a}$. Now take roots of $\mathfrak{g}$ wrt to $\mathfrak{h}$, ~~[illegible]~~ call them Δ and θ acts on them; $\Delta''=$ invariant ones, $\Delta'=$ non-invariant ones. ~~Choosing simple roots carefully~~ ~~At this stage you have to choose Σ carefully so that $\Delta=\Sigma'\cup\Sigma''\cup-\Sigma'\cup-\Sigma''$. The ro~~ Let Σ be a set of positive roots, so that $\Delta=\Sigma\cup-\Sigma$, and let $\Sigma'=\Sigma\cap\Delta'$ $\Sigma''=\Sigma\cap\Delta''$. Then $\Sigma=\Sigma'\cup\Sigma''$. Claim that if $\alpha\in\Sigma'$ then $\theta\alpha\in-\Sigma'$ not true so have to be carful. So proceed as follows – ~~start with $\mathfrak{m}$, Δ'' and choose Σ'' a pos. root system for $\mathfrak{m}$ by ordering $\mathfrak{h}_k$, then extend the ordering to $\mathfrak{h}$~~ order $\mathfrak{a}$ and extend the ordering to $\mathfrak{h}'$ and take the positive roots relative to this ordering (choose a basis $X_1 \dots X_\ell$ for $\mathfrak{a}$ extend to $X_{\ell+1}\dots X_n$ and call $\mu=\sum a_i \hat{X}_i>0$ if first non-zero $a_i>0$). i.e. $\alpha>0 \Rightarrow \alpha|\mathfrak{a}>0$.

Any two max abelian subspaces $\mathfrak{a}$ are K conjugate by same differential argument.

Note that X in $sl(2,\mathbb{R})$ $\ni$ $(\mathrm{ad}X)^2 = 0$.

I understand why any semi-simple elt of $\mathfrak{p}$ is K conjugate to an element of $\mathfrak{a}$. Why does a closed orbit consist of semi-simple elements? Better, given $x \in \mathfrak{p}$ show that $\overline{Kx} \cap \mathfrak{a} \neq \phi$.

First try.

$$x = \sum_{\alpha \in \Sigma'} c_\alpha (e_\alpha - e_{\theta\alpha}) + u$$

$c_\alpha \in \mathbb{C}$, $u \in \mathfrak{a}$.

$$H \in h_k$$

$$e^{tH} * x = \sum_{\alpha \in \Sigma'} e^{t\alpha(H)} c_\alpha (e_\alpha - e_{\theta\alpha}) + u.$$

Thus if we can find an element $H \in \mathfrak{m}$ such that

$$\alpha(H) > 0 \quad \text{all} \quad \alpha \in \Sigma'$$

we let $t \to \infty$ and we are done. Not ~~usually~~ often the case e.g. in the complex case $\mathfrak{m} = \Delta h \subset h \times h$.

$$\Sigma = \left\{ \begin{matrix} \alpha, 0 \\ 0, -\alpha \end{matrix} \quad \alpha \in \Sigma_k \right\}.$$

thus have $\left\{ \begin{matrix} \alpha(H) \\ -\alpha(H) \end{matrix} \right.$ both appearing over $\mathfrak{m}$.

Fundamental problem: For any $x \in \mathfrak{p}$ show that $\overline{Kx} \cap \mathfrak{a} \neq \phi$.

Try adjoint case - for any $x \in \mathfrak{g}$ show that

$$\boxed{\overline{Gx} \cap \mathfrak{h} \neq \phi.}$$

Method I: ~~Gx~~ (alg. geometry) $\overline{Gx}$ will be a closed inv. subvariety of ~~G~~ $\mathfrak{g}$, hence ~~its~~ ideal I in $S(\mathfrak{g}')$ will be invariant. Let $\tilde{I}$ be a maximal invariant ideal in $S(\mathfrak{g}')$. By my theorem $\tilde{I}$ will intersect $S(\mathfrak{g}')^{\mathfrak{g}}$ in a maximal ideal. Thus any invariant element of I ~~will have a~~ when restricted to $\mathfrak{h}$ will have a zero since

$$-: S(\mathfrak{g}')^{\mathfrak{g}} \xrightarrow{\sim} S(\mathfrak{h})^W$$

Now suppose that $\overline{Gx} \cap \mathfrak{h} = \phi$ ie

$$I + J = S(\mathfrak{g}')$$

where $J =$ ideal of functions which vanish on $\mathfrak{h}$.

?

Kostant's proof: Write $x = s + n$ and work in the group G^s which is reductive; this reduces to case where s is 0. But any nilpotent embeds in a TDS ie $\exists\ y \in \mathfrak{g}^s \ni [y, n] = n$
$\Rightarrow e^{-ty} n \rightarrow 0$

assume $s \in \mathfrak{h}$

Back to Nakyama.

Conjecture: V f.t. $(\mathfrak{g}, \mathfrak{k})$ module $\overset{\neq 0}{\Longrightarrow}$ $V/\mathfrak{n}V \neq 0$.

First attempt: Assume V irred. Choose Λ so that $U(\mathfrak{g}) \otimes_{\mathfrak{k}} \Lambda \neq 0$. Then we know that

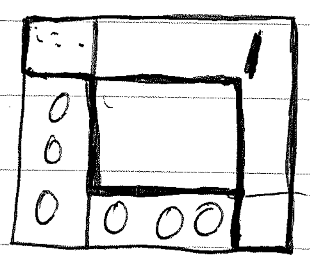
$$S(\mathfrak{p}) \otimes \Lambda \longrightarrow V \quad \text{onto.}$$

The problem is to get hold of the $\mathfrak{n}$ module structure.

Filter V in the obvious way get a graded module over $S(\mathfrak{p})$ whose support at ∞ intersects $\mathfrak{a}'$ quite nicely.

Let $\mathfrak{z}$ = center of $\mathfrak{n}$ and consider the ~~centralizer~~ normalizer of $\mathfrak{z}$ in $\mathfrak{g}$ include $\mathfrak{a}$, $\mathfrak{m}$ possibly a lot more.

Try normal form for $sl(3, \mathbb{R})$.



$$[e_{k1}, e_{1n}] = e_{kn}$$

$$[e_{1n}, e_{nk}] = -e_{1k}$$

$\therefore$ Norm $= \{ [a] \mid a_{nk} = 0 \quad k < n$

$a_{k1} = 0 \quad k > 1.$

Centralizer $= [a] \quad a_{nk} = a_{k1} = 0 \quad k = \ldots n$

Problem is that $\mathfrak{k} \cap$ Norm $\subset$ Cent

Can you form a nicer filtration than by $U(\mathfrak{g})$? In other words we know that ~~…~~ V finitely generated over V and $\mathfrak{n}$ ~~so~~ define a filtration using Z + powers of $\mathfrak{n}$.

Abelian case again

Another idea: Assume $V/\mathfrak{n}V = 0$.

V defines an irred. rep $\mathrm{Hom}_k(\Lambda, V)$ of $\mathfrak{L}_\Lambda$

have functor $F: (\mathfrak{g}, k) \longrightarrow$ ℓ modules

we know that inj.

$$U(\mathfrak{g})\otimes_k \Lambda_2 \longrightarrow U(\mathfrak{g})\otimes_k \Lambda_1 \longrightarrow U(\mathfrak{g})\otimes_k \Lambda_0 \longrightarrow V \longrightarrow 0$$

$$U(\mathfrak{a})\Lambda_2 \longrightarrow U(\mathfrak{a})\Lambda_1 \xrightarrow[\text{onto}]{} U(\mathfrak{a})\Lambda_0$$

Hom

Artin-Rees holds

$$\left(U(\mathfrak{g})\otimes_k \Lambda_1\right)^{\wedge} \longrightarrow \left(U(\mathfrak{g})\otimes_k \Lambda_0\right)^{\wedge} \longrightarrow \hat{V} \longrightarrow 0$$

$H_*(\mathfrak{n}, V)$

$\underline{V = IV} \Rightarrow W = IW$ all submodules

Suppose V irreducible $/R$ ie $\underline{V = U/\ell}$

Then $\underline{IV} = V \iff I + \ell = U \iff I \not\subset \ell$.

If commutative then we can proceed as follows: We obtain an element $1 - x$ in the annihilator of V.

Feb 12

maximal ideals in $U(\mathfrak{g})$.
prime ideals in $U(\mathfrak{g})$, $\mathfrak{g}$ semi-simple

Kostant theory:

~~every orbit of~~ $\quad u: \mathcal{O}_s \xrightarrow{\sim} \mathbb{C}^l$

~~every or~~ $\quad \mathcal{O}_h \xrightarrow{\sim} \mathbb{C}^l$

have trouble where bad values occur.
Using a Cartan subalgebra the bad values occur when $D = 0$.

Basic maps:

$$res : S(\mathfrak{g})^{\mathfrak{g}} \longrightarrow S(h)^W$$

$$\gamma: U(\mathfrak{g})^{\mathfrak{g}} \longrightarrow U(h)^W$$

recall γ defined by

$$U(\mathfrak{g}) \xrightarrow{\varepsilon_- \otimes 1 \otimes \varepsilon_+} U(h) \xrightarrow{+g} U(h)^W$$

This enables us to define a map from prime ideals in $U(\mathfrak{g})$ to prime ideals in $U(h)^W$, ie orbits of prime ideals in h.

The bad set in the latter case consists of those λ such that $2\lambda(H_\alpha) \in \mathbb{Z} - 0$ for some $\alpha \in \Delta$.

Check for $sl(2)$: Let Δ be Casimir, it has eigenvalue $|\lambda+g|^2 - |g|^2$ in the irred rep with dominant wgt λ, so with standard base H, X, Y for $sl(2)$ ~~$[H,X]=\alpha(H)X$~~,

the root is ~~$[\dots]$~~ $aH \mapsto a$ so $g = aH \to \frac{1}{2}a$.

What is Killing form. $\langle H, H \rangle = 2$

$$\langle X, Y \rangle = 2$$

$$\operatorname{ad} X \operatorname{ad} Y \, H = [X, Y] = H$$

$$\operatorname{ad} X \operatorname{ad} Y \, X = [X, -H] = [H, X] = X.$$

$$Y = 0.$$

so Casimir is $\frac{1}{2}(H^2 + XY + YX)$

$$\begin{matrix} H & X & Y \\ \frac{1}{2}H & \frac{1}{2}Y & \frac{1}{2}X \end{matrix}$$

$$\parallel$$

$$\frac{1}{2}(H^2 + H + 2YX)$$

$$\parallel$$

$$\frac{1}{2}(\lambda^2 + \lambda)$$ in dominant wgt sp of λ. $Hv = \lambda v$.

~~$= \frac{1}{2}[\lambda + \frac{1}{2}]^2 - [\frac{1}{2}]^2$~~

Conclude that $|\mu|^2 = \frac{1}{2}\mu^2$.

eigenvalues of Δ to avoid are $\frac{1}{2}\left[\left(\frac{\ell}{2}\right)^2 + \frac{\ell}{2}\right] = \frac{1}{8}(\ell^2 + 2\ell)$

$\ell = 0, 1, 2, \cdots$

In standard set up

$$X_\alpha = X$$

$$X_{-\alpha} = \frac{1}{2}Y \qquad \therefore \alpha(H_\alpha) = \frac{1}{2}$$

$$\alpha(H)X = [H, X] = X$$

$$H_\alpha = \frac{1}{2}H$$

Thus must avoid $\lambda(H) = \frac{\ell}{2}$

ie $\lambda(H_\alpha) = \frac{\ell}{4}$ $\qquad \ell = 0, 1, 2, \dots$

~~To make extremely clear!!!~~

One define isom. $\gamma : \mathcal{U}(\mathfrak{g})^{\mathfrak{g}} \longrightarrow \mathcal{U}(\mathfrak{h})^W$ in such a way that the Weyl char. fmla holds. ~~i.e.~~ ~~[illegible]~~ If V is the irred finite $\mathfrak{g}$ module with dominant wgt λ one has a character

$$\chi_\lambda : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathbb{C}$$

given by

$$\chi_\lambda(u) = \frac{1}{\dim V} \cdot tr\, \rho(u)$$

~~note that $\chi_\lambda(u)$~~ Idea is that $\mathcal{U}(\mathfrak{g}) = [\mathcal{U}(\mathfrak{g}), \mathcal{U}(\mathfrak{g})] + \mathcal{U}(\mathfrak{h})$ and χ_λ vanishes on $[\mathcal{U}(\mathfrak{g}), \mathcal{U}(\mathfrak{g})]$; thus want a formula for

$$\chi_\lambda(h) = \left\langle h, \prod_{\alpha > 0} \frac{\langle g, \alpha \rangle}{\langle \lambda + g, \alpha \rangle} \frac{\det e^{\lambda + g}}{\det e^{g}} \right\rangle \quad \text{if } h \in \mathcal{U}(\mathfrak{h})$$

This is the Weyl character formula as proved in Sophus Lie. proved first for f.d. reps, then for inf. reps by density as follows. One defines

$$V = \bigoplus V_{\sigma} = V_+ \oplus V_\lambda$$

Define $\beta : U(\mathfrak{g}) \longrightarrow U(\mathfrak{h})$ so that

$\overset{\shortparallel}{\varepsilon_- \otimes 1 \otimes \varepsilon_+}$

if $u \in U(\mathfrak{g})$, then ~~$\beta(u)\sigma_\lambda$~~

$$u\sigma_\lambda \equiv \beta(u)(\lambda)\sigma_\lambda \quad \text{mod} \quad V_+$$

Then

$$\begin{tikzcd} U(\mathfrak{g})^{\mathfrak{g}} \arrow[r, "\beta"] \arrow[dr, "\gamma"'] & U(\mathfrak{h}) \arrow[d, "+g"] \\ & U(\mathfrak{h}) \end{tikzcd}$$

shown to be inj.

$H \mapsto H + g(H)$

one observes that if $z \in U(\mathfrak{g})^{\mathfrak{g}}$ then

$$\beta(z)(\lambda) = \chi_\lambda(z) \qquad \text{ie}$$

$$\chi_\lambda(z) = \langle \beta(z), e^\lambda \rangle$$

$$= \langle \gamma(z), e^{\lambda+g} \rangle$$

$\beta(z)(\lambda)$

$\gamma(z)(\lambda) = \beta(z)(\lambda$

$P(H+g(H))(\lambda) = P(H)(\lambda+g)$

NOT CLEAR -

1. Want to describe ~~all~~ prime ideals in $U(\mathfrak{g})$
Understand $\mathfrak{g}$ situation – orbits in dual – Kostant's paper. Want the analogous situation ~~#~~, Features:

ⓐ ~~a~~ A bad set described by vanishing of a fn.

ⓑ Parameterization ~~of~~ of max ideals by h^*/W. ✓

ⓒ An analysis of the bad set, showing that there is a unique minimal prime (gen. by max ideal in center) and that there are only finitely many other primes always of length at most l.

ⓓ description of quotient fields of these prime ideals if they exist (~~...~~ Gelfand-Kirilov)

Point is that ~~max~~ prime ideals in $U(\mathfrak{g})$ may be "parameterized" by elements of h^*/W in the same way that orbits ~~...~~ in $\mathfrak{g}$ can be.

Bad set in the first

Answers for $sl(2)$.

We have a canonical isom

$$\gamma : \mathcal{U}(\mathfrak{g})^{\mathfrak{g}} \longrightarrow \mathcal{U}(\mathfrak{h})^{W}$$

$$\Delta \longmapsto \frac{1}{2}\left(H^2 - \frac{1}{4}\right)$$

$$\shortparallel$$

$$\frac{1}{2}(H^2 + H + YX) \mapsto \frac{1}{2}(H^2+H) \longrightarrow \frac{1}{2}\left(\left(H-\tfrac{1}{2}\right)^2 + H - \tfrac{1}{2}\right)$$

$$\shortparallel$$

$$\frac{1}{2}\left(H^2 - H + \tfrac{1}{4} + H - \tfrac{1}{2}\right)$$

$$\alpha(H) = 1 \qquad g = \tfrac{1}{2}\alpha$$

$$g(H) = \tfrac{1}{2}$$

eigenvalue of Casimir in dominant wgt corr λ is

$$\frac{1}{2}\left\{[\lambda(H)]^2 + \lambda(H)\right\} = |\lambda + g|^2 - |g|^2$$

where $\langle \lambda_1, \lambda_2 \rangle = \frac{1}{2}\lambda_1(H)\lambda_2(H)$

Check $\frac{1}{2}\left[\left(\lambda(H) + \tfrac{1}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2\right] = \frac{1}{2}\left[[\lambda(H)]^2 + \lambda(H)\right]$

This is bad iff $\lambda(H) = \ell/2 \quad \ell = 0, 1, 2, \dots$ $\qquad H_\alpha = \frac{1}{2}H$

$\therefore$ iff $\lambda(H) + \frac{1}{2} = \ell/2 \quad \ell = 1, 2, \dots$

$\therefore$ iff $(\lambda + g)(H_\alpha) = \frac{\ell}{4} \quad \ell = 1, 2, \dots$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

$\therefore$ iff $$\frac{\sin(\lambda+g)}{(\lambda+g)}(4H_\alpha) = 0.$$

how about

$$\chi_\lambda(h) = \left\langle h,\ \underline{\frac{e^{\lambda+g} - e^{-(\lambda+g)}}{e^{g} - e^{-g}} \cdot \frac{g(H_\alpha)}{\lambda+g(H_\alpha)}} \right\rangle$$

$$\frac{e^{\lambda+g} - e^{-(\lambda+g)}}{\lambda+g(H_\alpha)}.$$ function on $\mathfrak{H}$

$$\frac{e^{\lambda+g} - e^{-(\lambda+g)}}{e^{g} - e^{-g}}$$ a function on the group $\exp h$.

$$\therefore \frac{e^{\lambda+g} - e^{-(\lambda+g)}}{e^{g} - e^{-g}}(\exp H) \overset{?}{=} \frac{e^{(\lambda+g)(H)} - e^{-(\lambda+g)(H)}}{e^{g(H)} - e^{-g(H)}}$$

Problem: We have a map $\exp : \mathfrak{g} \to \mathrm{Aut}(\mathfrak{g})$

$\exp : h \to \mathrm{Aut}(\mathfrak{g})$

what is the kernel.

$$e^{\mathrm{Ad}(H)} X_\alpha = e^{\alpha(H)} X_\alpha = X_\alpha$$

$$\Longleftrightarrow \quad \alpha(H) \in 2\pi i \mathbb{Z}.$$

zero point for exp:

when $H = 4\pi i n H_\alpha$.

Want to consider the fn.

$$\frac{e^{\lambda+g}-e^{-(\lambda+g)}}{e^g-e^{-g}}\cdot\frac{g(H_\alpha)}{(\lambda+g)(H_\alpha)}$$

$$\frac{e^{\lambda+g}-e^{-(\lambda+g)}}{\lambda+g}\cdot\frac{g}{e^g-e^{-g}}$$

as a function on h. i.e.

$$\frac{e^{(\lambda+g)(H)}-e^{-(\lambda+g)(H)}}{(\lambda+g)(H)}\cdot\frac{g(H)}{e^{g(H)}-e^{-g(H)}} = F_\lambda(H)$$

when is this function singular? Answer: If λ generic, then this function is singular when

$$e^{g(H)}-e^{-g(H)}=0 \qquad H\neq 0$$

ie $$e^{2g(H)}=1$$

ie $$g(H)\in\pi i(\mathbb{Z}-0).$$

in my case ~~H~~ H is variable say

$$H = tH_\alpha \quad \text{and} \quad \boxed{g(H_\alpha)=\tfrac{1}{4}}$$

so get problems when

$$H = tH_\alpha \quad \& \quad t\in 4\pi i(\mathbb{Z}-0)$$

$$\boxed{\text{bad} \quad H = 4\pi i n\, H_\alpha \quad n\neq 0}$$

The badness disappears if λ such that

$$\frac{e^{(\lambda+g)(4\pi i n H_\alpha)} - e^{-(\lambda+g)(4\pi i n H_\alpha)}}{(\lambda+g)\, 4\pi i n H_\alpha}$$

$$= \sim \frac{\sin (\lambda+g)(4\pi i n H_\alpha)}{(\lambda+g)(4\pi n H_\alpha)} = 0.$$

It is important to use Γ functions rather than sin.

Lorentz gp $sl(2,\mathbb{C})$. In this case get pair (k_0, c), $k_0 = \frac{\ell}{2}$ $\ell = 0, 1, \ldots$ $c \in \mathbb{C}$. Principal series is when $c^2 \neq (k_0+n)^2$ $n = 1, 2, \cdots$. In case of finite rep with wgts $k_0, \cdots, k_1$ where $c^2 = (k_1+1)^2$ relate c to λ, ν. $sl(2) \times sl(2) = \mathfrak{g}$ k = diag.

$$\mathfrak{h} = \mathfrak{h}_k + \mathfrak{a} = \mathfrak{h}_1 \times \mathfrak{h}_2$$

if $\exists$ maximal wgt of finite $\mathfrak{g}$ module put

$$\rho = (\nu, \lambda) = (\rho_1 \times \rho_2)$$

as a k rep get weights

$$\lambda = \rho_1 + \rho_2$$
$$\nu = \rho_1 - \rho_2$$

as a k rep get weights ν λ

+ here $\nu + n = \lambda$. $\therefore$ $k_0 = \nu$, $k_1 = \lambda$. so

$$\boxed{c = (\lambda+1)}$$
$$\boxed{k_0 = \nu}$$

condition becomes $(\lambda+1)^2 \neq (\nu+n)^2$. $n = 1, 2, \ldots$

$$\lambda + 1 \neq \nu + n \qquad n = 1, 2, \ldots\ldots$$
$$\lambda + 1 \neq -\nu - n \qquad n = 1, 2, \ldots\ldots$$

use Γ function. $\frac{1}{\Gamma(z)}$ has zeroes $0, 1, 2, \ldots$

$\therefore$ ~~[illegible]~~

$$\lambda - \nu \neq$$

$$\nu - \lambda \neq 1 - n \qquad n = 1, 2, \ldots$$
$$0, -1, \ldots$$

$$\frac{1}{\Gamma(\nu-\lambda)} \neq 0$$

and

$$\lambda + \nu \neq -1 - n$$
$$\lambda + \nu + 2 \neq$$

$$\therefore \quad \frac{1}{\Gamma(\nu-\lambda)\Gamma(\lambda+\nu-2)} \neq 0.$$

Calculate ideals for $sl(2)$.

$$\Delta = \tfrac{1}{2}(H^2 + XY + YX) = \tfrac{1}{2}(H^2 - H + 2XY)$$

Try to determine when $\Delta - \alpha$ generates a maximal ideal! $\operatorname{gr} \mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{g})(\Delta - \alpha) = S(\mathfrak{g})/S(\mathfrak{g})\Delta$

$$= \mathbb{C}[X, Y, Z]/(H^2 + 2XY)$$

Let $f(Y, H, X)$ be a non-zero polynomial and assume that it is not divisible by Δ.

Let I be an ideal containing $\Delta - \alpha$ and let $f \in I$; we wish to show that $f \in \mathcal{U}(\mathfrak{g})\Delta$. So proceed as follows. If f has degree n consider $\bar{f} \in S_n(\mathfrak{g})$. If divisible by $\bar{\Delta} = H^2 + 2XY$ say

$$\bar{f} = \bar{\Delta}\bar{g} \quad \text{where} \quad \bar{g} \in S_{n-2}(\mathfrak{g})$$

then we have that $f - \Delta g$ is of degree $n-1$. Then we may assume that $\bar{f}$ is not divisible by $H^2 + 2XY$. So we take an element f of I of least degree not belonging to $\mathcal{U}(\mathfrak{g})\Delta$. Now we let the group act on f! All of these elements will belong to I. Really we are looking at what happens to $\bar{f} \in S_n(\mathfrak{g}) = \underline{J \otimes H}$, where $J = \mathbb{C}[\bar{\Delta}]$ and $H =$ functions on nilp. elements in $\mathfrak{g}$. Thus

$$\bar{f} = \sum_{i=0} \bar{\Delta}^i h_i \qquad \text{degree } h_i = n - 2i$$

now $h_0 \neq 0$ otherwise $\bar{f} \in S(\mathfrak{g})\bar{\Delta}$.

Now ~~this~~ the thing to observe is that in

$$S_n(\mathfrak{g}) = \underline{H_n} + \sum_{\substack{i+j=n \\ i>0}} J_i \otimes H_j$$

$$\mathrm{Hom}_k(\Lambda, H) = \mathrm{Hom}_M(\Lambda, 1)$$

mult. of 0 wgt space

$\therefore$ each H_n is a different irreducible repr. of $\mathfrak{g}$ so by applying a suitable projection op in $U(\mathfrak{g})$ we can arrange that I contains all of H_n and is therefore ~~[illegible]~~ of finite codimension.

What happens in general? We have $U(\mathfrak{g}) = Z \otimes H$
H = powers of nilpotent elements. So

$$\begin{tikzcd} U(\mathfrak{g})/I & U(\mathfrak{g})/\mathfrak{m}_Z \arrow[l] \arrow[d, phantom, "\cong" rotate=90] \\ & H \arrow[ul] \end{tikzcd}$$

H inherits a peculiar ring structure whose associated graded ring is an integral domain $\Longrightarrow$ H integral

<u>Question:</u> In nilpotent case H is A_n = diff ops. on affine space. Here H might be DO on a curved manifold

~~Take fibrations of orbit~~ Take orbit choose polarization

$$\overset{n}{B/H} \longrightarrow \boxed{\overset{2n}{G/H}} \longrightarrow \boxed{G/B}^{\,n}$$

G/B manifold homog. bundle

B acting on $b/n \simeq$ ~~h~~ . n affine.

So can you ~~[illegible]~~ interpret $U(\mathfrak{g})/m_Z$ as a ring of operators on a line bundle over G/B.

Thus you try to ~~interpret~~ construct an induced module representation over G/B with correct character.

Wild Conjecture: We know that for any ~~[illegible]~~ λ of the form

$$\lambda = \sigma(\lambda_0 + g) - g \qquad \sigma \in W$$

the induced rep $U(\mathfrak{g}) \otimes_{\mathfrak{b}} \lambda$ has correct character. Maybe by taking J-H components of 1 and hence all in the orbit and their annihilators we get all the primes this way!!!

Let $\lambda \in h'$ and consider

$$\mathcal{U}(g) \otimes_b \lambda \;\simeq\; \underline{\mathcal{U}(n^-) \otimes \lambda}. \qquad \text{as a } b^- \text{ module.}$$

Show that if λ is such that

$\lambda(H_\alpha)$ ~ integral

then for some λ' of the form

$$\lambda' + g = \sigma(\lambda + g)$$

~~the~~ we have?

The weights of $\mathcal{U}(g) \otimes_b \lambda$ are of the form

$$\lambda - \sum n_i \alpha_i \qquad \begin{array}{l} n_i \geq 0 \\ \{\alpha_i\} = \Pi \end{array}$$

Suppose $2\dfrac{\lambda(H_\alpha)}{\alpha(H_\alpha)}$ is a ~~positive~~ non-negative integer, α positive root.

Then

$$\sigma_\alpha(\lambda + g) = \cancel{\lambda + g}\ \lambda - 2\frac{\langle \alpha, \lambda\rangle}{\langle \alpha, \alpha\rangle}\alpha + g - \alpha$$

$$\sigma_\alpha(\lambda + g) - g = \lambda - 2\frac{\langle \alpha, \lambda\rangle}{\langle \alpha, \alpha\rangle}\alpha - \alpha$$

$$= \lambda - n\alpha$$

n positive integer.

Thus the obvious necessary condition holds. See if you can show that if

$$2\,\frac{\lambda(H_\alpha)}{\alpha(H_\alpha)} = \ell \quad \text{integer} \geq 0 \qquad \text{for some } \alpha \in \Sigma$$

then $\quad U(\mathfrak{g}) \otimes_{\mathfrak{b}} \lambda \quad$ is reducible.

Have to find an element in $\underline{U(\mathfrak{n}^-) \otimes \lambda}$ of weight $\quad \lambda - (\ell+1)\alpha$

~~This is a combination of~~ $\cancel{X_{\alpha_i} \cdots X_\alpha}$

Try the obvious, namely

$$X_{-\alpha}^{\ell+1} \otimes \lambda$$

so is $\quad X_{\alpha_i} X_{-\alpha}^{\ell+1} \otimes \lambda = 0 \;?$

Try $\ell = 0$.

$$X_{\alpha_i}(X_{-\alpha} \otimes \lambda) = \underbrace{[X_{\alpha_i}, X_{-\alpha}]}_{\text{not ass}} \otimes \lambda$$

If α_j is simple, then

$$X_{\alpha_i}\left(X_{-\alpha_j} \otimes \lambda\right) = [X_{\alpha_i}, X_{-\alpha_j}] \otimes \lambda = \begin{cases} 0 & j \neq i \\ \underline{H_{\alpha_i} \otimes \lambda} & j = i \end{cases}$$

if $j \neq i$

so if $\underline{1 \otimes H_{\alpha_i}\lambda = 1 \otimes \lambda(\alpha_i) = 0}$ since $l = 0$

$sl(2)$
rank 1 $\quad 2\dfrac{\lambda(H_\alpha)}{\alpha(H_\alpha)} = 1$

$\cancel{X_{2\alpha} \otimes X + \ldots}$

$\cancel{X_\alpha(X_{2\alpha} \otimes X)}$

$$X_\alpha\left(X_{-\alpha}^2 \otimes \lambda\right) = \left(H_\alpha X_{-\alpha} + X_{-\alpha} H_\alpha\right) \otimes \lambda$$

$$= \underbrace{\left(-\alpha(H_\alpha) + 2\lambda(H_\alpha)\right)}_{=0} X_{-\alpha} \otimes \lambda$$

So you need <u>some</u> technique at this point.

we have to consider $S(n^-) \otimes \lambda = S(n^*) \otimes \lambda$ as an n module.

$U(n^-)$ using b operations

so that I take $X \in n$ and given

$$Y^\xi \otimes \lambda$$

consider $X(Y^\xi \otimes \lambda) = (XY^\xi - Y^\xi X) \otimes \lambda + \underbrace{Y^\xi X \otimes \lambda}_{=0}$

$ad X$ carries $\underline{U(n^-)}$ into $\underline{U(n^-) \otimes h}$. $\quad$ ($U(n^-) \otimes 1$)

$ad X \; \underline{Y_{\alpha_1}^{\xi_1}} \cdots\cdots Y$

derivation + either gives a Y
or it gives an H which then must moved thru
to the end.

Problem: Show that if $\quad 2\dfrac{\lambda(H_\alpha)}{\alpha(H_\alpha)} = \ell$ int ≥ 0

that $\quad U(g) \otimes_b \lambda$ is reducible.

One method is to take

$$U(n^+) \otimes U(n^-) \longrightarrow U(g) \xrightarrow{\beta} U(h) \xrightarrow{\lambda} C$$

and ~~...~~ determine when non-singular. i.e. want

$$U(n^-) \otimes \lambda \longrightarrow \mathrm{Hom}\,(U(n^+), \lambda)$$

thus I have to take

$$X^P \, Y^Q \qquad\qquad \text{wgt } P = \text{wgt } Q$$

and write out in the form $\quad \sum \alpha_{IJK}^{PQ} \, Y^I X^J H^K$

and then consider the matrix

$$P, Q \longmapsto \alpha_{IJK}^{PQ} \; \lambda(H)^K$$

and calculate the determinant. If this determinant is non-zero there is no weight vector.

$sl(2)$

$$X^i Y^i = \sum a_{j} Y^j X^j$$

instead try

$$X^i Y^j = \sum a^{ij}_{klm} \frac{Y^k}{k!} \frac{X^l}{l!} H^m$$

$$\sum_{ij} \frac{s^i X^i}{i!} \frac{t^j Y^j}{j!} = \underline{e^{sX} e^{tY}} = e^{\varphi(s,t) Y} e^{\psi(s,t) X} e^{\delta(s,t) H}$$

$$XY = YX + H$$

$$X^2 Y^2 = X(YX + H)Y = (YX+H)(YX+H) + XHY$$

$$=$$

$\boxed{\text{hell of a mess}}$

Suppose you know that $2\frac{\lambda(H_\alpha)}{\alpha(H_\alpha)} = \ell$ int ≥ 0

then produce ~~an element~~

a dominant wgt vector.

Try geometry. If

$$2\frac{\lambda(H_\alpha)}{\alpha(H_\alpha)} = \ell \geq 0 \quad \text{all } \alpha \in \Delta,$$

it is known that the ~~induced~~ assoc-irreducible representation is finite dimensional and in fact explicit formula for the maximal ideal is known.

$$\mathfrak{M}_\lambda = \sum_{\alpha\in\Sigma} U\cdot(H_\alpha - \lambda(H_\alpha)) + \sum_{\alpha\in\Sigma} U\cdot X_\alpha + \sum U\cdot Y_\alpha^{\lambda(H_\alpha)+1}$$

where
$$[H_\alpha, X_\alpha] = 2X_\alpha$$
$$[H_\alpha, Y_\alpha] = -2Y_\alpha$$
$$[X_\alpha, Y_\alpha] = H_\alpha$$

and $\alpha(H_\alpha) = 2$, so that $s_\alpha(\lambda) = \lambda - \lambda(H_\alpha)\alpha$

Also seems to be true that

$$U(\mathfrak{n}^-) \cap \mathfrak{M}_\lambda = \sum U(\mathfrak{n}^-)\, Y_\alpha^{\lambda_i+1}$$

Proposition: Let $\lambda \in h'$ be such that $2\dfrac{\lambda(H_\alpha)}{\alpha(H_\alpha)} = \ell$

ℓ int ≥ 0 for some $\alpha \in \Sigma$. Then

$$s_\alpha(\lambda + g) - g = \lambda - (\ell+1)\alpha$$

Grand hope: If $\exists\, \alpha \in \Sigma \ni 2\dfrac{\lambda(H_\alpha)}{\alpha(H_\alpha)} = \ell$, then

$U(\mathfrak{g}) \otimes_{b_-} \lambda$ is reducible, and the annihilator the irred. rep. with dominant wgt. λ is not generated by the maximal ideal in the center.

This is true for $sl(2,\mathbb{R})$, since $\alpha(H_\alpha) = \frac{1}{2}$ so we have trouble with $\lambda \ni$

$$\lambda(H) = \frac{\ell}{2}$$

Geometric Idea: We know that $U(\mathfrak{g}) \otimes_b \lambda$ is irreducible $\Longleftrightarrow$ $U(\mathfrak{g}) \otimes_b \lambda \longrightarrow \boxed{Hom_{b_-}(U(\mathfrak{g}), \lambda)}$ is injective

Think of these as some kind of sections of the bundle $G \times_{B_-} \lambda \longrightarrow G/B_-$

now using the integrality define a differential equation. The idea is that if $2\dfrac{\lambda(H_\alpha)}{\alpha(H_\alpha)} =$ int ≥ 0 all α

then I know that the induced bundle is holom. + has sections. Note that if we have a zero it is easy because then get a larger group.