late January

Concerning Nakyama's lemma:

Example showing that the support of a finite type module needn't be closed. Take $\mathfrak{g}$ = Heisenberg alg (X, Y, Z), $[X,Y]=Z$, $[X,Z]=[Y,Z]=0$ and let $V = U(\mathfrak{g})/U(\mathfrak{g})(1+YZ)$. For each $\lambda \neq 0$ the ~~left~~ ideal $U(1+YZ)+U(Z-\lambda)$ is not ~~of finite~~ the unit ideal, for this would mean

$$\alpha(1+YZ)+\beta(Z-\lambda)=1$$

where $\alpha, \beta \in U$. Writing polys as sums of monomials $X^i Y^j Z^k$ one sees that the above relation must hold in the poly ring $\mathbb{C}[X,Y,Z]$ which is impossible if $\lambda \neq 0$. Thus for each $\lambda \neq 0$ we can find an irred. quotient of V such that the center Z acts as ~~scalar~~ the scalar λ. Clearly Z cannot act as 0 unless the quotient is 0.

Can one define the support of a module.

<u>Gabriel</u>: Take M form its injective hull I, write $I = \oplus I_j$ indecomposable and let $I(\mathfrak{p}) = \sum' I_j$ those $I_j \ni I_j$ belonging to the prime $\mathfrak{p}$. Then $I = \oplus I(\mathfrak{p})$ $\quad M = \cap M_{\mathfrak{p}}$

where each $M_{\mathfrak{p}}$ ~~is associated to~~ has only one associated prime ideal. The basic problem is whether there is more than one indecomposable injective associated to a prime $\mathfrak{p}$.

Nouazé and Gabriel reduce this to whether A_n has more than 1 indecomposable injective object for $\mathfrak{g}$ nilpotent!!

<u>Problem A</u>: Show $hd\, A_n = n$ and Krull dim $A_n = n$.

A_n = algebra of polynomial diff. operators in n variables. Let M be a f.t. A_n modules. If $\operatorname{gr} M$ is flat over $k[x_1, \ldots, x_n]$ then we can construct a resolution by Spencer sequences of length $\leq n$.

Nakayama's lemma in the comm. case, see if it generalizes.

V f.t. $\mathfrak{g}, k$ module : finite type $\mathfrak{a}+\mathfrak{n}$ module

$$\begin{array}{ccc} U(\mathfrak{g}) \otimes_k \Lambda & \xrightarrow{\text{onto}} & V \\ \cong & & \\ \underline{U(\mathfrak{n}+\mathfrak{a}) \otimes \Lambda} & & \end{array}$$

Problem: In the case of $sl(2,\mathbb{R})$ can you show $V/\mathfrak{n}V \neq 0$ without using the fact that V is f.t. over $\mathfrak{n}$.

Better – how do you show V f.t. over $\mathfrak{n}$?

Subproblem: Show that V f.t. over $Z \otimes U(\mathfrak{n})$.

: Use decomposition $\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + k$

$$U(\mathfrak{g}) = U(\mathfrak{n}) \otimes U(\mathfrak{a}) \otimes U(k).$$

And now shift somehow from $U(\mathfrak{a})$ to Z. The obvious way is to consider the mapping

$$\mathrm{Hom}\,(\Lambda, U(\mathfrak{g}) \otimes_k \Lambda) = U(\mathfrak{g}) \otimes_k \underline{\mathrm{Hom}\,(\Lambda,\Lambda)}$$

$$= U(\mathfrak{g}) \otimes_k U(k)/\mathfrak{n}(\Lambda).$$

$\mathfrak{h} = \mathfrak{a} + \mathfrak{h}_k$

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}) \otimes \mathcal{U}(\mathfrak{a}) \otimes \mathcal{U}(k) = \mathcal{U}(\mathfrak{n}) \otimes \mathcal{U}(\mathfrak{a}) \otimes \mathcal{U}(\mathfrak{m}) \otimes \mathcal{U}(\mathfrak{q}).$$

Z

$$z \in \sum_{\alpha \in \Sigma'} X_\alpha \mathcal{U} + P(H)$$

$\mathcal{U}(\mathfrak{n}) \otimes Z \quad \mathcal{U}($

How do you prove that

$$\mathcal{U}(\mathfrak{g}) / \mathcal{U}(\mathfrak{g}) \mathcal{J} \simeq \mathcal{U}(\mathfrak{n} + \mathfrak{a}) \mathcal{U}(k)/\mathcal{J}$$

is finite over $\mathcal{U}(\mathfrak{n}) Z$? By ~~showing that $\mathcal{U}(k)$~~ reducing to the case $\mathcal{J} = k\mathcal{U}(k)$. Observe if $\mathcal{J} = k\mathcal{U}(k)$, then have ~~that~~ a chance — HOW.

Ω_Λ finite over Z

$\mathcal{J} = k\mathcal{U}(k)$. $\quad \mathcal{U}(\mathfrak{n})\mathcal{U}(\mathfrak{a})\Lambda$ finite over $\mathcal{U}(\mathfrak{n}) Z$

Show that $\quad \mathcal{U}(\mathfrak{a})\Lambda \quad$ mod $\mathcal{U}($

Why is $U(\mathfrak{g}) \otimes_k \Lambda$ f.t. over $Z \otimes U[\mathfrak{a}]$?

Suppose $\Lambda = 1$. Have to produce elements $u_1, \dots, u_N \in U(\mathfrak{g}) \otimes_k \Lambda$ which generate. So proceed as follows. ~~Given something in $U(\mathfrak{n})U(\mathfrak{a})\Lambda$, if it's~~ Take something α in $U(\mathfrak{a})\Lambda$, we have to modify it so that it lands in **Z**. The idea is to make it symmetric under W.

$$\begin{array}{c} \alpha \in U(\mathfrak{a}) \\ \cup \\ U(\mathfrak{a})^W \end{array}$$

Choose $\alpha_1, \dots, \alpha_r$ to ~~be~~ generate $U(\mathfrak{a})$ over $U(\mathfrak{a})^W$. Then

Try $sl(2, \mathbb{R})$. Then h

Review the functorial situation.

$$\begin{array}{ccc} \mathfrak{V} & & \mathfrak{W} \\ (\mathfrak{g}, k) & \begin{array}{c} \xrightarrow{I_!} \\ \xleftarrow{I} \\ \xrightarrow[I_*]{} \end{array} & (\mathfrak{b}, \mathfrak{n}) \end{array} \qquad (M\mathfrak{a}, M)$$

$$\mathrm{Hom}_{\mathfrak{V}}(V, IJ) = \mathrm{Hom}_{\mathfrak{W}}(H_0(\mathfrak{n}, V), J)$$

$$\boxed{\mathrm{Ext}^{p+q}_{\mathfrak{V}}(V, IJ) \Longleftarrow \mathrm{Ext}^{p}_{\mathfrak{W}}(H_q(\mathfrak{n}, V), J)}$$

Proof: Let $J^\cdot$ be an inj. resolution of J $\quad V^\cdot$ proj. res. of V. Then consider double complex $\quad \mathrm{Hom}_{\mathfrak{V}}(V^\cdot, IJ^\cdot) = \mathrm{Hom}_{\mathfrak{W}}(H_0(\mathfrak{n}, V^\cdot), J^\cdot)$

$H^q_V = 0 \quad p > 0$ since I exact.

But
$$\mathrm{Ext}^{p}_{\mathfrak{W}}(H_q(\mathfrak{n}, V), J) = \mathrm{Ext}^{p}_{\mathfrak{a}}(H_q(\mathfrak{n}, V), J)^M$$

$$\mathrm{Hom}_{\mathcal{V}}(I\mathfrak{J}, V) = \mathrm{Hom}_{\mathcal{W}}(\mathfrak{J}, \mathrm{Hom}_{\mathcal{V}}(\bar{J}, V)).$$

$$\boxed{E_2^{pq} = \mathrm{Ext}^p_{\mathcal{W}}(\mathfrak{J}, \mathrm{Ext}^q_{\mathcal{V}}(\bar{J}, V)) \Longrightarrow \mathrm{Ext}^{p+q}_{\mathcal{V}}(I\mathfrak{J}, V)}$$

Proof: Let $\mathfrak{J}^\bullet$ be a $\mathcal{W}$ proj res of $\mathfrak{J}$ $V^\bullet$ a $\mathcal{V}$ inj. res. of V and consider double cx

$$\mathrm{Hom}_{\mathcal{V}}(I\mathfrak{J}^\bullet, V^\bullet) = \mathrm{Hom}_{\mathcal{W}}(\mathfrak{J}^\bullet, \mathrm{Hom}_{\mathcal{V}}(\bar{J}, V^\bullet))$$

~~now taking horizontal homology first get 0.~~ ✓

Recall defn of $\bar{J}$ ~~Hom...~~

$$I(\mathfrak{J}) = k\,\mathrm{fin\,hom}_{\mathfrak{b}}(U(\mathfrak{g}), \mathfrak{J}) = \bar{J} \otimes_{\mathcal{W}} \mathfrak{J}$$

$$\bar{J} = k\,\mathrm{fin\,hom}_{\mathfrak{b}}(U(\mathfrak{g}), U(\mathfrak{n}+\mathfrak{a}))$$

$$= k\text{-}\mathrm{fin}\text{-}\mathrm{hom}_{\mathfrak{b}}(U(\mathfrak{g}), U(\mathfrak{n}+\mathfrak{a})).$$

$$I(\mathfrak{J}) = k\,\mathrm{fin\,hom}_{\mathfrak{b}}(U(\mathfrak{g}), \mathfrak{J}) \qquad \mathfrak{b} = \mathfrak{k}+\mathfrak{a}+\mathfrak{n} = \mathfrak{b} \ \cancel{\mathfrak{k}}$$

$$= k\,\mathrm{fin\,hom}_{\mathfrak{n}}(U(\mathfrak{k}), \mathfrak{J})$$

$$= R(K) \otimes_M \mathfrak{J} \qquad \therefore \bar{J} = R(K) \otimes U(\mathfrak{a})$$

as a right $M, \mathfrak{a}$ mod. left K mod.

<u>Board calculations:</u>

If M is a symplectic manifold with form Ω and if L is a line bundle with connection form η having curvature Ω, then we can make the Poisson algebra of functions act on the sections of L by the formula

$$f * s = (\nabla_{X_f} + f)\, s .$$

[We have ignored the $2\pi i$]

Relation between the baby Weyl and W^1.

$G_{\mathbb{C}}/B_{\mathbb{C}}$ has a Bruhat decomposition in terms of Schubert cells ($N_{\mathbb{C}}$ orbits) which are parameterized by elements of $W^1 \simeq \mathrm{Weyl}\,G_{\mathbb{C}}/\mathrm{Weyl}\,B_{\mathbb{C}}$.

G/B has a Bruhat decomposition in terms of cells (N orbits) parameterized by elements of the baby Weyl group: $W(G,K)$. Thus the map

$$G/B \longrightarrow G_{\mathbb{C}}/B_{\mathbb{C}}$$

gives a map $\quad W(G,K) \longrightarrow W^1$

In the complex case this map is the diagonal

$$W(G,K) = \Delta W \hookrightarrow W^1 = W(G) \cong W(K) \times W(K).$$

Any relation between $\mathrm{Im}\left\{ K/M_{\mathbb{C}} \longrightarrow G_{\mathbb{C}}/B_{\mathbb{C}} \right\}$ and the position of $W(G,K)$ in W^1.

~~[illegible]~~ We therefore obtain two spectral sequences

$$\begin{cases} E_2^{pq} = \mathrm{Ext}^p_{\mathcal{W}}(\mathcal{J}, \mathrm{Ext}^q_{\mathfrak{r}}(\bar{\mathcal{J}}, V)) \Longrightarrow \mathrm{Ext}^{p+q}_{\mathfrak{r}}(I(\mathcal{J}), V) \\ E_2^{pq} = \mathrm{Ext}^p_{\mathcal{W}}(H_q(\mathfrak{n}, V), \mathcal{J}) \Longrightarrow \mathrm{Ext}^{p+q}_{\mathfrak{r}}(V, I(\mathcal{J})) \end{cases}$$

where $\bar{\mathcal{J}} = k\text{-fin-}\mathrm{Hom}_{\mathfrak{b}}(U(\mathfrak{g}), U(\mathfrak{g}))$

$= k\text{-fin-}\mathrm{Hom}_{\mathfrak{g}_1}(1 \otimes_{\mathfrak{n}} U(\mathfrak{g}), U(\mathfrak{g}))$?

$= I(U(\mathfrak{g}))$.

Some observations.

(i) $\mathcal{W}$ = category of $M, \mathfrak{a}$ modules which are indfinite semi-simple over M; hence the $\mathrm{Ext}_{\mathcal{W}}$ is really ~~and~~ an $(\mathrm{Ext}_{\mathfrak{a}}\ )^M$.

Question: It seems as if

$$\mathrm{Ext}^*_{\mathfrak{r}}(V, V') \cong \mathrm{Ext}^*_{\mathfrak{a}+\mathfrak{n}}(V, V')^M . \qquad \underline{\text{NO.}}$$

ie.

$$\mathrm{Hom}_{\mathfrak{r}}(U(\mathfrak{g}) \otimes_k \Lambda, V') = \mathrm{Hom}_k(\Lambda, V')$$

$$\cap$$

$$\mathrm{Hom}_{M+\mathfrak{a}+\mathfrak{n}}(U(\mathfrak{a}+\mathfrak{n})\Lambda, V') = \mathrm{Hom}_M(\Lambda, V').$$

$$\mathrm{Ext}^*_{\mathfrak{v}}(I(\mathfrak{I}_1), I(\mathfrak{I}_2)) = \mathrm{Ext}^*_{\mathfrak{b}\times\mathfrak{b}}(J, \mathrm{Hom}(\mathfrak{I}_1, \mathfrak{I}_2))$$

$$= \mathrm{Ext}^*_{\mathfrak{b}\times\mathfrak{g}_1}(\bar{J}, \mathrm{Hom}(\mathfrak{I}_1, \mathfrak{I}_2))$$

$$\Uparrow$$

$$\mathrm{Ext}^*_{\mathfrak{g}_1\times\mathfrak{g}_1}(H_*(\mathfrak{n}, \bar{J}), \mathrm{Hom}(\mathfrak{I}_1, \mathfrak{I}_2)).$$

$$\mathrm{Ext}^p_{\mathfrak{g}}(H_q(\mathfrak{n}, \bar{J}), \mathfrak{I}) \Rightarrow \mathrm{Ext}^{p+q}_{\mathfrak{g}}(\bar{J}, \mathfrak{I}).$$

Go back: Why is $U(\mathfrak{g}) \otimes_k \Lambda$ finite type over $Z \otimes U(\mathfrak{n})$? Suppose $\Lambda = 1$. Why is $U(\mathfrak{g})/U(\mathfrak{g})k$ f. type over $Z \otimes U(\mathfrak{n})$? ~~Thus the idea is the following: Thus~~

$$\begin{matrix} Z \otimes U(\mathfrak{n}) \\ U(\mathfrak{n}) \otimes U(\mathfrak{a}) \end{matrix} \quad (\mathrm{mod}\ U(\mathfrak{g})k).$$

<u>method</u>. It is proved that if $\alpha \in Z$ then $\exists\ \beta \in U(\mathfrak{a}) \ni$

$$\alpha - \beta \in \mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})k.$$

How about proving that given $\beta \in U(\mathfrak{a})^W$ ~~we can move~~ that for some translate $\tau\beta \equiv U(\mathfrak{n})Z$ mod $U(\mathfrak{g})k$.

Take an element of $\mathcal{U}(\mathfrak{a})$ say and see what one needs.

rank 1:

Casimir operator

$$C = C_{\mathfrak{m}} + \Delta_{\mathfrak{a}} \pm \rho \qquad \text{modulo} \quad \mathfrak{n}\,\mathcal{U}(\mathfrak{g}).$$

$$2C = A^2 - A - N^2 + 2NH$$

look at the associated graded ring!

$$\mathrm{gr}\ \mathcal{U}(\mathfrak{g}) \otimes_k \Lambda \simeq S(\mathfrak{g}) \otimes_{S(k)} \Lambda \simeq S(\mathfrak{g}/k) \otimes \Lambda$$

So $S(\mathfrak{n})$ acts via map $\mathfrak{n} \longrightarrow \mathfrak{g}/k$. and Z acts naturally also.

look at filtration induced on Z so that $\mathrm{gr}\, Z \simeq S(\mathfrak{g})^{\mathfrak{g}}$. In this case we are looking at $\mathrm{gr}\, Z$ acting via the map

$$\begin{array}{ccc} S(\mathfrak{g})^{\mathfrak{g}} & \longrightarrow & S(\mathfrak{g}/k)^{k} \\ \wr\wr & & \wr\wr \\ S(\mathfrak{h})^{W(G)} & \longrightarrow & S(\mathfrak{a})^{W(G/K)} \end{array}$$

which Harish-Chandra knows is finite. One is thus reduced to proving that $S(\mathfrak{g}/k) \otimes \Lambda$ is fin. gen. over $S(\mathfrak{n}) \otimes S(\mathfrak{g})^{\mathfrak{g}}$ which is now clear.

<u>Theorem</u>: Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ as usual. Suppose that $\mathfrak{n}$ is abelian and that V is a non-zero finite type $(\mathfrak{g}, \mathfrak{k})$ module. Then $H_0(\mathfrak{n}, V) \neq 0$.

<u>Proof</u>: May assume V irreducible.

Then may find a surjection $U(\mathfrak{g}) \otimes_{\mathfrak{k}} \Lambda \longrightarrow V$.

Claim that $U(\mathfrak{g}) \otimes_{\mathfrak{k}} \Lambda$ is a finitely generated $U(\mathfrak{n}) \otimes Z$ module. In effect filter $U(\mathfrak{g}) \otimes_{\mathfrak{k}} \Lambda$ by $F_n(U(\mathfrak{g}) \otimes_{\mathfrak{k}} \Lambda) = F_n U(\mathfrak{g}) \cdot \Lambda$ so that $gr\{U(\mathfrak{g}) \otimes_{\mathfrak{k}} \Lambda\} \simeq S(\mathfrak{g}/\mathfrak{k}) \otimes \Lambda$ with $gr\, U(\mathfrak{g}) \simeq S(\mathfrak{g})$ module structure coming from the map $S(\mathfrak{g}) \longrightarrow S(\mathfrak{g}/\mathfrak{k})$ deduced from $\mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{k}$. Filter $U(\mathfrak{n}) \otimes Z$ by the product filtration so that

$$\begin{tikzcd} gr(U(\mathfrak{n}) \otimes Z) \arrow[r, phantom, "\simeq"] \arrow[d] & S(\mathfrak{n}) \otimes S(\mathfrak{g})^{\mathfrak{g}} \arrow[d, "\theta"] \\ gr(U(\mathfrak{g})) \arrow[r, "\sim"] & S(\mathfrak{g}) \end{tikzcd}$$

commutes where θ is the obvious map. ~~Clearly~~ Claim that $S(\mathfrak{g}/\mathfrak{k})$ is a fin. gen. $S(\mathfrak{n}) \otimes S(\mathfrak{g})^{\mathfrak{g}}$ module. In effect since we are in a graded situation Nakyama holds, thus $1 \underset{\mathfrak{n}}{\otimes} S(\mathfrak{g}/\mathfrak{k}) \simeq S(\mathfrak{a})$ which we know is fin gen over image of $S(\mathfrak{g})^{\mathfrak{g}}$ (Harish-Chandra)

As Z becomes a scalar in V we find that V is finitely generated over $U(\mathfrak{n})$. <u>Corollary</u>: the weight spaces of V under $\mathfrak{h}$ are all finite dimensional. Next get grading of V by weight spaces under $\mathfrak{h}$ which shows that Nakyama holds in general.

On the Mackey double coset formula:

Suppose $j : B \to G$ is injective. If M is a B module there is a canonical injection

$$\begin{array}{ccc} \Phi : j_! M & \hookrightarrow & j_* M \\ \| & & \| \\ G \times_B M & & \mathrm{Hom}_B(G, M) \\ (g_1, m) & \longmapsto & (g \longmapsto \chi_B(gg_1)\, gg_1 m) \end{array}$$

where χ_B is the characteristic function of $j(B)$. Φ is an isomorphism iff B is of finite index in B.

Proposition 1: Suppose that $[G : B] < \infty$ and that N is a G module. Then the following is commutative

$$\begin{tikzcd} j_! j^* N \arrow[r, "\Phi", "\sim"'] \arrow[d, "\text{adjunction}"] & j_* j^* N \arrow[dl, "\text{trace}"] \\ N & \end{tikzcd}$$

where $tr : j_* j^* N \to N$ is the transfer or trace homomorphism $\quad tr\, f = \sum g_i^{-1} f(g_i) \qquad G = \coprod_i B g_i\,, \; f \in \mathrm{Hom}_B(G, M)$

Remarks: This gives a defintion of trace for traceable elements in general i.e. those in $\mathrm{Im}\ \Phi$.

Inverse of Φ is given by $\quad \Psi f = \sum_i' (g_i^{-1}, f(g_i)) \qquad f \in \mathrm{Hom}_B(G, M)$

~~Structure of Λ_λ ... $U(\mathfrak{g})^k/(U(\mathfrak{g})\mathfrak{k})^k$~~

$$\mathrm{Hom}_{k}(\Lambda, \mathrm{Hom}_{\mathfrak{b}_+}(U(\mathfrak{g}), \lambda_0)) \qquad \mathfrak{b} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$$

$$= \mathrm{Hom}_{\mathfrak{b}\times k}(U(\mathfrak{g}), \mathrm{Hom}(\Lambda, \lambda_0)) \qquad \mathfrak{b}\cap k = \mathfrak{m}$$

now the point is that

$$\boxed{k/\mathfrak{m} \xrightarrow{\sim} \mathfrak{g}/\mathfrak{b}}$$

Consequently I want to prove that

$$U(\mathfrak{b})\otimes \mathfrak{m} \otimes U(k) \longrightarrow U(\mathfrak{b})\otimes U(k) \longrightarrow U(\mathfrak{g}) \longrightarrow 0$$

is exact. But this is completely clear by filtering

$$\bigoplus_{i+j=n-1} F_i U(\mathfrak{b})\otimes \mathfrak{m}\otimes F_j U(k) \longrightarrow \bigoplus_{i+j=n} F_i U(\mathfrak{b})\otimes F_j U(k) \longrightarrow F_n U(\mathfrak{g})$$

$$S(\mathfrak{b})\otimes \mathfrak{m}\otimes S(k) \longrightarrow S(\mathfrak{b})\otimes S(k) \longrightarrow S(\mathfrak{g}) \longrightarrow 0$$

completely clear. Therefore

$$\boxed{\mathrm{Hom}_{k}(\Lambda, \mathrm{Hom}_{\mathfrak{b}}(U(\mathfrak{g}), \lambda_0)) \simeq \mathrm{Hom}_{\mathfrak{m}}(\Lambda, \lambda_0)}$$

Composition in the Mackey double coset formula:

Problem: $B \overset{j}{\hookrightarrow} G$ of finite index we get an isomorphism

$$\mathrm{Hom}_G(j_*L, j_*M) \xrightarrow{\sim} \mathrm{Hom}_G(j_!L, j_*M)$$

$$\cong$$

$$\mathrm{Hom}_{B\times B}(G, \mathrm{Hom}(L,M))$$

$$\|$$

$$\left\{ g \mapsto \varphi_g \in \mathrm{Hom}(L,M) \;\middle|\; \begin{array}{l} \varphi_{bg} = b\varphi_g \\ \varphi_{gb} = \varphi_g \cdot b \end{array} \right\}$$

How does this depend on composition?
The relation of $\alpha \in \mathrm{Hom}_G(j_*L, j_*M)$ and φ is

$$\alpha(\Phi_L(g,\ell))\,\tilde{g} = \varphi(\tilde{g}g)\,\ell$$

If $\beta \in \mathrm{Hom}_G(j_*M, j_*N)$ and ψ are similarly related then $\beta\circ\alpha$ is related to $\psi * \varphi$ where

$$\boxed{(\psi * \varphi)_g = \sum_i \psi_{gg_i^{-1}}\varphi_{g_i}} \qquad G = \coprod Bg_i$$

Remark: In the symmetric space case, $B =$ max. compact subgp. of G, $L = M = 1$, then $\mathrm{Hom}_{B\times B}(G, 1)$ is the biinvariant under K functions on G and

$$(\psi * \varphi)_g = \int_G \psi_{gx^{-1}}\varphi_x \qquad \text{same as above provided } \int_K 1 = 1$$

$$= \int_{B\backslash G} \psi_{gx^{-1}}\varphi_x$$

~~except for normalization replacing $\sum_i$ by $\int_{G/B}$~~

K is an algebraically closed field of char. 0 and all Lie algs. are finite dimensional over K. finite means finite dimensional

Definition: Let $\mathfrak{b} \subset \mathfrak{g}$ be a subalgebra. By a $\mathfrak{g}, \mathfrak{b}$ module we mean a $\mathfrak{g}$ module M which is the union of its finite ~~dimensional~~ $\mathfrak{b}$-submodules.

These form an abelian category $\mathcal{M}(\mathfrak{g}, \mathfrak{b})$ full abelian subcat of $\mathcal{M}(\mathfrak{g})$, hence $\mathcal{M}(\mathfrak{g}, \mathfrak{b})$ is locally noetherian.

~~Proposition 1: $\mathcal{M}(\mathfrak{g}, \mathfrak{b})$ has suff. many injectives and is of homological dimension dim $(\mathfrak{g}/\mathfrak{b})$.~~

adjoint functors:

$$\mathcal{M}(\mathfrak{g}, \mathfrak{b}) \underset{H^0_{\mathfrak{b}}}{\overset{\subset}{\rightleftarrows}} \mathcal{M}(\mathfrak{g})$$

$$H^0_{\mathfrak{b}}(M) = \varinjlim_{I} \operatorname{Hom}_{\mathfrak{g}} \left(U(\mathfrak{g})/\cancel{U(\mathfrak{g})}I, M \right)$$

I runs over all ideals in $U(\mathfrak{b})$ of finite codimension

$$\boxed{H^q_{\mathfrak{b}}(M) = \varinjlim_{I} \operatorname{Ext}^q_{\mathfrak{g}} \left(U(\mathfrak{g})/U(\mathfrak{g})I, M \right).}$$

Conjecture that $H^q_{\mathfrak{b}}(M) = 0$ for $q > \dim \mathfrak{b}$.

$$H^0_{\mathfrak{b}}(M) = \varinjlim_{I} \operatorname{Hom}_{\mathfrak{b}} \left(U(\mathfrak{b})/I, M \right)$$

if M injective over $U(\mathfrak{g}) \overset{?}{\Longrightarrow}$ injective over $U(\mathfrak{b})$?

$$\operatorname{Hom}_{\mathfrak{b}}(?, M) = \operatorname{Hom}_{\mathfrak{g}}(\underline{U(\mathfrak{g}) \otimes_{\mathfrak{b}}} ?, M)$$

exact since $\mathfrak{b} \subset \mathfrak{g}$.

Thus

$$\boxed{H^q_{\mathfrak{b}}(M) = \varinjlim_{I} \operatorname{Ext}^q_{\mathfrak{b}}(\mathcal{U}(\mathfrak{b})/I, M)}$$

where I runs over all ideals of $\mathcal{U}(\mathfrak{b})$. Hence $=0$ for $q>0$.

~~Proof of Prop 1~~

Relation with equations on a homogeneous space:

Let $B \subset G$ ~~and B~~ be Lie groups over $\mathbb{R}$ with complexified LA's $\mathfrak{b} \subset \mathfrak{g}$. ~~If V is a finite B modules~~ Assume B s.c. so the finite (complex) B modules are same as finite $\mathfrak{b}$ modules. Then if V is a finite B module we have the homogeneous vector bundle $G \times_B V$ over G/B and

$$\Gamma(\mathcal{U}, G\times_B V) = \{ \varphi : \pi^{-1}(\mathcal{U}) \to V \;/\; \varphi(gb) = b^{-1}\varphi(g) \}$$

$\mathcal{U}$ open in G/B, $\pi: G \to G/B$ natural projection. ~~Thus~~ Let $C^\infty(\pi^{-1}\mathcal{U})$ be the smooth fns on $\pi^{-1}\mathcal{U}$ with B action $(bf)(g) = f(gb)$. Then

$$\operatorname{Hom}_{\mathfrak{g}}\left(\mathcal{U}(\mathfrak{g}) \otimes_{\mathfrak{b}} V', C^\infty(\pi^{-1}\mathcal{U})\right) = \operatorname{Hom}_{\mathfrak{b}}\left(V', C^\infty(\pi^{-1}\mathcal{U})\right)$$

$$= \left\{ \varphi : \pi^{-1}\mathcal{U} \times V' \to \mathbb{C} \;/\; \begin{array}{l} \varphi(g,\cdot) \text{ linear} \\ \varphi(gb,\lambda) = \varphi(g,b\lambda) \end{array} \right\}$$

$$= \{ \psi : \pi^{-1}\mathcal{U} \to V \;/\; \psi(gb) = b^{-1}\psi(g) \}$$

$$= \Gamma(\mathcal{U}, G\times_B V).$$

~~Thus we get a sheaf of a contra~~

Clearly if

$$\alpha : U(\mathfrak{g}) \otimes_{\mathfrak{b}} V_1' \longrightarrow U(\mathfrak{g}) \otimes_{\mathfrak{b}} V_0'$$

is a $\mathfrak{g}$ module map we get induced maps

$$\mathcal{S}(G \times_B V_0) \xrightarrow{\alpha^\#} \mathcal{S}(G \times_B V_1)$$

of sheaves which means $\alpha^\#$ is a differential operator (Petrie) but this can be checked anyway. Thus get a functor contravariant ~~functor~~

$$\# : \mathcal{M}_{ft}(\mathfrak{g}, \mathfrak{b}) \longrightarrow G \text{ sheaves on } G/B$$

$$M \longmapsto \left(U \mapsto \mathrm{Hom}_{\mathfrak{g}}(M, C^\infty(\pi^{-1}U)) \right).$$

~~Which is~~ and it is "clear" ~~not hard to show~~ that $\#$ is an anti-equivalence of $\mathcal{M}_{f.t.}(\mathfrak{g}, \mathfrak{b})$ with G sheaves on G/B ~~defined by~~ which are solutions of invariant DE's.

~~The following is the~~

Conjecture: If G real semi-simple and K comes from a max. compact subalgebra of $\mathfrak{g}_0 = $ L.A. of G, then ~~for an~~ for any convex open subset U of G/K, $M \in \mathcal{M}_{ft}(\mathfrak{g}, \mathfrak{k})$

$$\mathrm{Ext}^q_{\mathfrak{g}}(M, C^\infty(\pi^{-1}U)) = 0 \qquad q > 0$$

(i.e. Spencer sequence is exact) Note convex makes sense since G/K neg. curv.

Save for a possible generalization of Schur's lemma

<u>Proposition:</u> Let B be a semi-simple subring of $\mathrm{End}\, V$ and let C be the commutant of B. Then any irreducible ~~right~~ C module is ~~of the form~~ isomorphic to

$$\mathrm{Hom}_B(\mu, V)$$

where μ is an irreducible B sub-module of V.

<u>Proof:</u> V is a semi-simple B module, so

$$V \xleftarrow{\sim} \bigoplus_{\mu} \mu \otimes \mathrm{Hom}_B(\mu, V)$$

where μ runs over the inequivalent B submodules of V.

The above isomorphism is compatible with C action on V. Claim $\mathrm{Hom}_B(\mu, V)$ irreducible under C. ~~Clear because $\mathrm{Hom}_B(\mu, V) \cong$~~ Clear from Wedderburn i.e.

$$\mathrm{End}_B(V) \xleftarrow{\sim} \bigoplus_{\mu} \mathrm{End}_C(\mathrm{Hom}_B(\mu, V)).$$

January

~~Somehow this isn't compatible with~~

Problem: Define a composition on

$$U(\mathfrak{a})^W \otimes \mathrm{Hom}_M(\Lambda_1, \Lambda_2)$$

of convolution type

Method is to choose the fundamental repn.

$$\Lambda \longrightarrow \mathrm{Hom}_k(U(\mathfrak{g}), \Lambda)$$

$$\lambda \longmapsto (\alpha \longmapsto \varepsilon_k(\alpha)\cdot\Lambda)$$

where $\varepsilon_k(\alpha)$ is the proj.

$$U(\mathfrak{g}) = U(k) \otimes U(\mathfrak{a}+\mathfrak{n}) = \cancel{U(\mathfrak{g})(\mathfrak{a}+\mathfrak{n})}$$

Then

$$\mathrm{Hom}_{\mathfrak{g}}(j_!\Lambda_1, j_!\Lambda_2) \longrightarrow \mathrm{Hom}_k(\Lambda_1, \mathrm{Hom}_k(U(\mathfrak{g}), \Lambda))$$

$$\|$$

$$\mathrm{Hom}_{k\times k}(U(\mathfrak{g}), \mathrm{Hom}(\Lambda_1, \Lambda))$$

$$\left\{ \varphi : U(\mathfrak{g}) \longrightarrow \mathrm{Hom}(\Lambda_1, \Lambda) \ni \begin{array}{l} \varphi(kx) = k\,\varphi(x) \\ \varphi(xk) = \varphi(x)\,k \end{array} \right\}.$$

By expo

$$\mathrm{Hom}_{k\times k}\left(U(\mathfrak{g}), \mathrm{Hom}(\Lambda_1, \Lambda_2)\right)$$

$$\|$$

$$\left\{\begin{array}{c}\varphi : U(\mathfrak{g}) \longrightarrow \mathrm{Hom}(\Lambda_1, \Lambda_2) \\ \varphi(k_1 x k_2) = k_1 \varphi k_2\end{array}\right\}$$

$$\left\{\begin{array}{c}\psi : U(\mathfrak{a}) \longrightarrow \mathrm{Hom}_M(\Lambda_1, \Lambda_2) \\ \psi(a^w) = \psi(a)^w\end{array}\right\}.$$

$$\mathrm{Hom}_K\left(\Lambda_1, \mathrm{Hom}_K(G, \Lambda_2)\right)$$

$$\|$$

$$\mathrm{Hom}_{K\times K}\left(G, \mathrm{Hom}(\Lambda_1, \Lambda_2)\right)$$

$$\|$$

$$\left\{\begin{array}{c}\varphi : G \longrightarrow \mathrm{Hom}(\Lambda_1, \Lambda_2) \\ \varphi(k_1 x k_2) = k_1 \varphi(x) k_2\end{array}\right\}$$

$$\|$$

$$\left\{\begin{array}{c}\psi : A \longrightarrow \mathrm{Hom}(\Lambda_1, \Lambda_2) \\ \psi(nan^{-1}) = n\psi(a)\end{array}\right\}$$

$$\|$$

$$\left\{\begin{array}{c}\psi : A \longrightarrow \mathrm{Hom}_M(\Lambda_1, \Lambda_2) \\ \psi(a^w) = \psi(a)^w\end{array}\right\}$$

~~Method~~ Calculate ~~note~~ again the Mackey coset formula!

$$
\begin{aligned}
&\mathrm{Hom}_{\mathfrak{g}}\left(U(\mathfrak{g})\otimes_{\mathfrak{k}}\Lambda_1,\ U(\mathfrak{g})\otimes_{\mathfrak{k}}\Lambda_2\right)\\
&\qquad\downarrow\\
&\mathrm{Hom}_{\mathfrak{k}}\left(\Lambda_1,\ U(\mathfrak{g})\otimes_{\mathfrak{k}}\Lambda_2\right) \longrightarrow \mathrm{Hom}_{\mathfrak{k}}\left(\Lambda_1,\ \mathrm{Hom}_K(G,\Lambda_2)\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\parallel\\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathrm{Map}_{K\times K}\left(G,\ \mathrm{Hom}(\Lambda_1,\Lambda_2)\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\downarrow\\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathrm{Map}_{N_A}\left(A,\ \mathrm{Hom}(\Lambda_1,\Lambda_2)\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\parallel\\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathrm{Map}_{W}\left(A,\ \mathrm{Hom}_{M}(\Lambda_1,\Lambda_2)\right).
\end{aligned}
$$

$$\underline{\mathrm{Hom}_K(\Lambda_1, \mathrm{Hom}_K(G, \Lambda_2))} = \mathrm{Hom}_K(\Lambda_1, \Gamma(G \times_K \Lambda_2)).$$

$$\|$$

$$\mathrm{Hom}(\mathcal{D}(G) \otimes_{\mathcal{D}(K)} \Lambda_1, \mathrm{Hom}_K(G, \Lambda_2))$$

$\mathcal{D}$ = dist comp. support under convolution.

$$\|$$

$$\mathrm{Hom}_{\mathfrak{g}}(U(\mathfrak{g}) \otimes_k \Lambda_1, \underline{\mathrm{Hom}_K(G, \Lambda_2)}).$$

since K conn.

Define $\quad U(\mathfrak{g}) \otimes_k \Lambda_2 \longrightarrow \mathrm{Hom}_K(G, \Lambda_2)$

ie $\quad \Phi : \Lambda_2 \longrightarrow \mathrm{Hom}_K(G, \Lambda_2)$

Easy: ~~make~~ write $G = KAN$ and make

$$\Phi(\lambda)(k\underline{an}) = k \cdot \lambda$$

Return to Mackey formula.

~~Again have $j: K \to G$ only this time have a subgroup S of G such that were~~

Assume everything finite + proceed formally

$$\boxed{G = KAN}$$

Define $\Phi : G \times_K \Lambda \longrightarrow Hom_K(G, \Lambda)$ by

$$\Phi(1, \lambda) = (g \longmapsto \pi(g)\lambda)$$

where $\pi : G \longrightarrow K$ projection on first factor.

$$\therefore \quad \Phi(g_1, \lambda) = (g \longmapsto \pi(gg_1)\lambda)$$

$$\Phi(g_1 k, \lambda)(g) = \pi($$

?

Define $\Lambda \longrightarrow Hom_K(G, \Lambda)$ by

$$\cancel{Hom(AN)}$$

$$\lambda \longmapsto (\overset{ks}{\cancel{g}} \longmapsto k\lambda).$$

$$\lambda \longmapsto (ks \longmapsto k\lambda)$$

$$\Phi(\lambda)(g) = \pi(g)\lambda$$

$$\Phi(\lambda)(kg) = \pi(kg)\lambda = k\,\pi(g)\lambda = k\,\Phi(\lambda)(g).$$

$\Phi(\lambda) \in \mathrm{Hom}_K(G, \Lambda)$ ✓.

Next

$\Phi(g_1, \lambda)(g) = \Phi(\lambda)(g g_1) = \pi(g g_1)\lambda.$

$\Phi(k\lambda)(g) = \pi(g) k \lambda$

I need a map

$$\boxed{\Lambda \longrightarrow \mathrm{Hom}_K(G, \Lambda)}$$

$G \times_K \Lambda \longrightarrow \Lambda.$ Any ideas.

need fundamental repn.

Choose $\Phi : \Lambda \xrightarrow[K]{} \mathrm{Hom}_K(G, \Lambda)$

$$\Phi \in \mathrm{Hom}_K(\Lambda, \mathrm{Maps}_K(G,\Lambda)) = \mathrm{Maps}_{K\times K}(G, \mathrm{Hom}(\Lambda,\Lambda))$$

$$= \mathrm{Maps}_W(A, \mathrm{Hom}_M(\Lambda,\Lambda)).$$

Obvious candidate is to map A to identity. Clearly

Let $\varphi : G \longrightarrow \mathrm{Hom}(\Lambda,\Lambda)$

be given by

$$\varphi(k_1 a k_2) = k_1 \circ k_2$$

have to check well-defined. Suppose that

$$k_1 a k_2 = \tilde{a}$$

If a regular this means that $k_2 = k_1^{-1} \in M$ hence clear.

Define to be identity for regular elements of a and 0 elsewhere.

Best Choose a fn. ~~W on A~~ τ on A invariant under W vanishing on the irregular elements and set

$$\varphi(k_1 a k_2) = k_1 \tau(a) k_2$$

~~If $k_1 a k_1' = k_2 a^* k_2'$ then~~

If $\quad k_1 a k_2 = \tilde{k}_1 \tilde{a} \tilde{k}_2$, then

$$\tilde{k}_1^{-1} k_1 a k_2 \tilde{k}_2^{-1} \equiv \tilde{a}$$

if a hence $\tilde{a}$ irregular both give 0 otherwise clear.

Thus

$\cancel{\Phi(\lambda)(g)}$

$$\underline{\Phi(\lambda)(k_1 a k_2) = k_1 \tau(a) k_2 \lambda}$$

$$\Phi(k\lambda)(k_1 a k_2) = k_1 \tau(a) k_2 k \lambda$$

$$\parallel$$

$$[k\Phi(\lambda)](k_1 a k_2) = \Phi(\lambda)(k_1 a k_2 k).$$

$$\Phi(\lambda)(kg) = k\Phi(\lambda)(g) \quad \checkmark$$

$\Phi(g,\lambda)(g_1) = \Phi(\lambda)(g_1 g)$ It's hard to analyze and gives a map

$$\Phi : G \times_K \Lambda \longrightarrow \mathrm{Hom}_K(G, \Lambda)$$

Is this injective? ie $\underline{\Phi(g,\lambda)}(g_1) = 0$ all g_1

$$\parallel$$

$$\Phi(\lambda)(g_1 g) \Rightarrow \Phi(\lambda) = 0$$

$$\Rightarrow \underline{\lambda = 0}.$$

$$\begin{tikzcd} & \mathrm{Hom}_K(G,\Lambda) & \\ G\times_K\Lambda \arrow[r, "\varphi"] & G\times_K\Lambda \arrow[u, "\Phi"] \arrow[r, "\psi"] & G\times_K\Lambda \end{tikzcd}$$

$$\left[(\Phi\varphi)(\lambda)\right](k_1 a k_2) = \Phi(\underline{\varphi(\lambda)})(k_1 a k_2) = \cancel{k_1\,\sigma(a)\,k_2\,\varphi(\lambda)}. \quad X$$

$$\underline{\varphi(\lambda) = \quad (na, \mu).}$$

$$\begin{tikzcd} G\times_K\Lambda \arrow[r, "\alpha"] \arrow[dr, "\varphi"'] & G\times_K\Lambda \arrow[r, "\beta"] \arrow[d, "\Phi"] \arrow[dr, "\psi"] & G\times_K\Lambda \arrow[d, "\Phi"] \\ & \mathrm{Hom}_K(G,\Lambda) \arrow[r, dashed] & \mathrm{Hom}_K(G,\Lambda) \end{tikzcd}$$

We want

$$\psi\,\Phi^{-1}\varphi.$$

need $\Phi^{-1} = \Phi$. given $f: G \xrightarrow[K]{} \Lambda$ ie

$$\Phi(g,\lambda)(g_1) = \Phi(\lambda)(g_1 g)$$

$$\Phi(an,\lambda)(k_1 a_1 n_1) = \Phi(\lambda)(k_1\underline{a_1 n_1 an})$$

$$= \Phi($$

$$\Phi(an,\lambda)(g) = \Phi(\lambda)(ang$$

kak kak

Thus τ defines an ~~injection~~ isomorphism

$$\underline{\Phi} \quad \boxed{\#: \; G\times_K \Lambda \xrightarrow{\sim} \mathrm{Hom}_K(G,\Lambda).}$$

with this function calculate the ~~ring~~ algebra

$$\begin{array}{ccc} \varphi \in \mathrm{Hom}_G(G\times_K\Lambda_1, G\times_K\Lambda_2) & = & \mathrm{Hom}_K(\Lambda_1, G\times_K\Lambda_2) \\ \downarrow\wr\ \underline{\Phi} & & \downarrow \underline{\Phi} \\ \mathrm{Hom}_G(G\times_K\Lambda_1, \mathrm{Hom}_K(G,\Lambda_2)) & & \mathrm{Hom}_K(\Lambda_1, \mathrm{Hom}_K(G,\Lambda_2)) \\ \| \wr & & \\ \mathrm{Map}_W(A, \mathrm{Hom}_M(\Lambda_1,\Lambda_2)) . & & \end{array}$$

$$\varphi : G\times_K\Lambda_1 \longrightarrow G\times_K\Lambda_2$$

$\cancel{\varphi(g,\lambda)}$

$$(\underline{\Phi}\circ\varphi)(\lambda_1)\ (k_1 a k_2) = k_1\tau(a)k_2\ \varphi(\lambda).$$

$f : G \longrightarrow \Lambda$

$$\Phi(\lambda)(kp) = \Phi(\lambda)(k k_1 a k_1^{-1})$$

$$= \tau(a)\, k\lambda$$

~~so~~ if we extend τ to P in the obvious way ~~it~~ we get

$$\Phi(\lambda)(kp) = \tau(p)\, k\lambda$$

and $\Phi(\lambda)(pk) = \tau(p)\, k\lambda$

Thus $\Phi(\lambda)(kp) = \Phi(\lambda)(pk) = k\,\Phi(\lambda)(p)$

$= k\,\tau(p)\lambda = \tau(p)\, k\lambda.$

$$\begin{tikzcd}
G \times_K \Lambda \arrow[r, "\Phi"] \arrow[d, phantom, "\cong" rotate=90] & \boxed{\mathrm{Hom}_K(G, \Lambda)} \arrow[d, phantom, "\cong" rotate=90] \\
P \times \Lambda & \mathrm{Hom}(P, \Lambda)
\end{tikzcd}$$

$$\Phi(p, \lambda)(p_1) = \Phi(\lambda)(p_1 p)$$

$$=$$

$$\Psi(\Phi(\lambda)) = (e, \lambda)$$

Given $\quad \tilde{\varphi} : A \longrightarrow \mathrm{Hom}_M(\Lambda, \Lambda)$ comp. with W.

what is $\quad \varphi(\lambda)(k_1 a k_2) = \underline{k_1 \tilde{\varphi}(a) k_2 \lambda}$.

I need $\quad \xi :$ ~~[illegible]~~ $P \longrightarrow \Lambda \quad$ so that

$$\sum_p{}' \Phi(p, \xi(p^{-1})) = \varphi(\lambda)$$

ie.

$$\sum_p{}' \Phi(\xi(p^{-1}))(k_1 a k_2 p) = k_1 \tilde{\varphi}(a) k_2 \lambda.$$

ie $\quad \displaystyle\sum_p \tau(\pi_P(a k_2 p)) \pi_K(a k_2 p) \xi(p^{-1}) = \tilde{\varphi}(a) k_2 \lambda$

$$\sum_p{}' \tau(\pi_P(a k_2 p)) \; \pi_K(a k_2 p) \; \xi(p^{-1}) = \tilde{\varphi}(a) k_2 \lambda.$$

$$\tilde{\psi} : A \longrightarrow \mathrm{Hom}_M(\Lambda, \Lambda)$$

$$\tilde{\psi}(\lambda)(k_1 a k_2) = k_1 \tilde{\varphi}(a) k_2$$

$$\sum_p{}' \psi(p, \xi(p^{-1}))(k_1 a k_2) =$$

Given $\tilde{\varphi} : A \longrightarrow \mathrm{Hom}_M(\Lambda, \Lambda)$

$$\varphi(\lambda)(k_1 a k_2) = k_1 \tilde{\varphi}(a) k_2 \lambda$$

Let $\quad \xi : P \longrightarrow \Lambda$ be so that

$$\sum_{p \in P}' \Phi(p, \xi(p)) = \varphi(\lambda)$$

i.e.

$$\sum_p \Phi(\xi(p))(k_1 a k_2 p) = k_1 \tilde{\varphi}(a) k_2 \lambda$$

i.e.

$$\cancel{\sum_p \tau\, \pi_p(ak_2 p)\, \pi_K(ak_2 p)\, \xi = \tilde{\varphi}(a) k_2 \lambda}$$

$$\boxed{\sum_p{}' \tau(\pi_p(akp))\, \pi_K(akp)\, \xi(p) = \tilde{\varphi}(a) k \lambda}$$

Also given $\tilde{\psi} : A \longrightarrow \mathrm{Hom}_M(\Lambda, \Lambda) \qquad \tilde{\psi}(\lambda)(k_1 a k_2) = k_1 \tilde{\psi}(a) k_2 \lambda$

$$\left(\sum_p \psi(p, \xi(p))\right)(g) = \sum_p \psi(\xi(p))(gp)$$

$$\cancel{= \sum_p{}' \tilde{\psi}(\pi_P(gp))\, \pi_K(gp)\, \xi(p)} \quad ?$$

$$= \sum_p{}' k_1(gp) \tilde{\psi}(a(gp))\, k_2(gp)\, \xi(p) = (\tilde{\psi} * \tilde{\varphi})(\lambda)(g)$$

$\tilde{\varphi}: \; G \longrightarrow \mathrm{Hom}_M(\Lambda, \Lambda)$

$\tilde{\psi}: \; G \longrightarrow \mathrm{Hom}_M(\Lambda, \Lambda)$

$$\begin{tikzcd}
G \times_K \Lambda \arrow[r, "\alpha"] \arrow[dr, "\varphi"'] & G \times_K \Lambda \arrow[d, "\Phi"] \arrow[dr, "\psi"] & \\
& \mathrm{Hom}_K(G, \Lambda) & \mathrm{Hom}(G, \Lambda)
\end{tikzcd}$$

$$\varphi(g, \lambda)^{(g_1)} = \underline{\tilde{\varphi}(g_1 g)\lambda}$$

$$\Phi(g, \lambda)^{(g_1)} = \tau(\pi_P g_1 g)\, \pi_K g_1 g \cdot \lambda$$

want $\left[\Phi \sum_p (p, \xi(p))\right]^{(g_1)} = \tilde{\varphi}(g_1 g)\lambda$

$\qquad\qquad ||$

$$\sum_p \tau(\pi_P(g_1 p)) \cdot \pi_K(g_1 p) \cdot \xi(p) = \tilde{\varphi}(g_1 g)\lambda$$ defined $\xi(p)$

$$\left[\psi \sum_p (p, \xi(p))\right](g_1) = \sum_p \tilde{\psi}(g_1 p)\, \xi(p)$$

Answer is the function

$$g_1 \longmapsto \sum_p \tilde{\varphi}(g_1 p)\, \xi(p)$$

where

$$\sum_p \tau(\pi_p(g_2 p))\, \pi_k(g_2 p)\, \xi(p) = \tilde{\varphi}(g_2 g)\, \lambda \qquad \text{all } g_2$$

This is $\underline{\rho(g,\lambda)}(g_1) = \sum_p \tilde{\varphi}(g_1 p)\, \xi(p)$.

Looks hopeless.