January 22, Reorganization

A. $\mathfrak{g}, \mathfrak{h}, \mathfrak{n}^+ = \mathfrak{n}$. When is $U(\mathfrak{g}) \otimes_{\mathfrak{b}} \lambda_0$ irred? This is equivalent to injectivity of the map

$$U(\mathfrak{g}) \otimes_{\mathfrak{b}} \lambda_0 \longrightarrow \mathrm{Hom}_{\mathfrak{b}_-}(U(\mathfrak{g}), \lambda_0)$$

or to the map

$$U(\mathfrak{g}) \otimes_{\mathfrak{b}} \lambda_0 \longrightarrow \mathrm{Hom\,cont}_{\mathfrak{b}_-}(U(\mathfrak{g}), \lambda_0)$$

being an isomorphism. In effect one sees that

$$\mathrm{Hom}_{\mathfrak{n}}(1, \mathrm{Hom}_{\mathfrak{b}_-}(U(\mathfrak{g}), \lambda_0))$$

$$= \mathrm{Hom}_{\mathfrak{b}_- \times \mathfrak{n}}(U(\mathfrak{g}), \mathrm{Hom}(1, \lambda_0))$$

$$= \mathrm{Hom}_{\mathfrak{b}_- \times \mathfrak{n}}(U(\mathfrak{b}_-) \cdot U(\mathfrak{n}), \mathrm{Hom}(1, \lambda_0))$$

$$= \mathrm{Hom}(1, \lambda_0).$$

so the left hand side is irreducible.

<u>Necessary condition</u>: If $V = U(\mathfrak{g}) \otimes_{\mathfrak{b}} \lambda_0$ not irred, then it has a vector $v \neq 1 \otimes \lambda_0$ killed by $\mathfrak{n}$ and hence another dominant weight λ. But ~~the~~ as V generated by $1 \otimes \lambda_0$, Z has same scalar value on V. Thus $\chi_{\lambda_0} = \chi_\lambda$ so we find that $\exists\ \sigma \in W$, $\lambda \in \mathfrak{h}'$ $\ni$

$\sigma \neq 1$, $\quad \sigma(\lambda_0 + g) = \lambda + g \quad$ and $\quad \lambda_0 - \lambda = \sum m_i \alpha_i$ $\quad m_i \in \mathbb{Z}\ m_i \geq 0.$

~~By use of~~

This necessary condition may also be derived by using the vanishing of the Laplacean for $H^0(\mathfrak{n}, V)$ and given that

$$|\lambda_0+g| = |\lambda+g| \qquad \lambda_0-\lambda = \sum m_i\alpha_i \qquad m_i \in \mathbb{Z},\ m_i \geq 0$$

which by a lemma in Kostant's paper $\Rightarrow \exists \sigma,\ \sigma(\lambda_0+g) = \lambda+g$.

I was not able to determined whether this nec. condition is sufficient or whether it is equivalent to a more manageable one.

Tried to do for $sl(n,\mathbb{C})$ but didn't succeed.
Killing form for ~~sl~~ $gl(n,\mathbb{R})$ is

$$tr(\mathrm{ad}X\ \mathrm{ad}Y) = 2n\ tr(XY) - 2(trX)(trY).$$

as one sees for $X=Y=$ diagonal, then polarization + density.

Obviously extremely important to determine

$$1 \otimes_{\mathfrak{n}} I(\xi) = 1 \otimes_{\mathfrak{n}} J \otimes_{\mathfrak{n}} \xi$$

perhaps even better to obtain

$$\boxed{1 \otimes_{\mathfrak{n}} J \otimes_{\mathfrak{n}} 1}$$

Face up to the structure of J.

$$J \simeq \mathrm{kfinhom}_{\mathfrak{b}}(U(\mathfrak{g}), U(\mathfrak{b})) \simeq \underline{R(K) \otimes_{\mathfrak{m}} U(\mathfrak{b})}$$

It Remains to determine the left action of $\mathfrak{g}$ on the lie algebra level.

<u>Introduce lots of notation:</u>

Suppose $\varphi : U(\mathfrak{g}) \to U(\mathfrak{b})$ is K-finite !!!!!!

be more precise about structure of $R(K)$ namely

$$R(K) \simeq \bigoplus_{\Lambda} \underline{\Lambda \otimes \Lambda^*} \qquad ?$$

<u>how</u>?

$$R(K) \simeq \bigoplus_{\Lambda} \underline{\mathrm{Hom}(\Lambda, \Lambda)}.$$

with what ring <u>structure</u> — not important here!

~~Proof:~~

$$R \simeq \bigoplus_{\Lambda} \operatorname{Hom}(\Lambda,\Lambda) \underset{M}{\otimes} U(\mathfrak{g}^{b})$$

question: how do you interpret $\operatorname{Hom}(\Lambda,\Lambda)$ as functions on K? by means of the trace ie.

$$\Lambda \otimes \underbrace{\operatorname{Hom}_K(\Lambda, C^\infty(K))}_{\Lambda'} \xrightarrow{ev} C^\infty(K)$$

$$\operatorname{Hom}_K(\Lambda, C^\infty(K)) = \operatorname{Hom}_K(\Lambda, \operatorname{Hom}(K,1)) = \operatorname{Hom}(K \times_K \Lambda, 1) = \Lambda'.$$

$\omega \in \Lambda'$

$$\varphi : \Lambda \otimes \Lambda' \longrightarrow C^\infty(K)$$

$$\lambda \otimes \omega \longmapsto \langle k\lambda, \omega \rangle$$

Check: $\varphi(k_1 \lambda \otimes \omega)(k) = \langle k k_1 \lambda, \omega\rangle = \varphi(\lambda,\omega)(kk_1)$
$= [k_1 \varphi(\lambda,\omega)](k).$

OKAY

$$\begin{array}{ccc} \mathrm{Hom}(\Lambda,\Lambda) & & (\lambda_1 \mapsto \lambda\langle \lambda_1, \omega\rangle) \\ \Uparrow & & \uparrow \\ & & \lambda\otimes\omega \end{array}$$

$$\bigoplus_{\Lambda} \Lambda \otimes \Lambda' \longrightarrow C^\infty(K)$$

$$\lambda\otimes\omega \longmapsto (k \mapsto \langle k\lambda, \omega\rangle).$$

$$\Lambda \otimes \Lambda' \longrightarrow \mathrm{Hom}(\Lambda,\Lambda)$$

$$\lambda \otimes \omega \longmapsto (\lambda_1 \to \lambda\langle\lambda_1,\omega\rangle).$$

$$\sum_i A e_i \otimes \hat{e}_i \longleftarrow\!\!\!\!\!\mid \; A$$

Thus

$$\mathrm{Hom}(\Lambda,\Lambda) \longrightarrow C^\infty(K)$$

$$A \longmapsto \sum_i \langle k A e_i, \hat{e}_i\rangle.$$

Thus

$$J = \bigoplus_{\Lambda} \Lambda \otimes \Lambda' \otimes_{\mathfrak{m}} U(\mathfrak{b})$$

$$\boxed{J = \bigoplus_{\Lambda} \Lambda \otimes \mathrm{Hom}_{\mathfrak{m}}(\Lambda, U(\mathfrak{b}))}$$

The problem is to determine how $\mathfrak{b}$ acts to the left

A typical element of $\mathfrak{a}, \mathfrak{m}$ (H_α, X_α) $\quad \underline{\alpha\in\Sigma'}$.

Try

$$H_*(\mathfrak{n}, \underline{J \otimes_{\mathfrak{n}} 1})$$

$$R(K)\otimes_{\mathfrak{n}} U(\mathfrak{b})\otimes_{\mathfrak{n}} 1 = \boxed{R(K)\otimes U(\alpha)}$$

The idea is to understand structure over $\cancel{\ }$ $\underline{\mathfrak{b}}$.

$I(\mathfrak{z}) = R(K) \otimes_{\mathfrak{n}} \mathfrak{z}$ this should be free over $\underline{\mathfrak{n}}$. also examine $\overset{\text{left}}{\underline{\ }}$ $\mathfrak{n}$ or $\underline{\mathfrak{m}}$ structure.

$$R(K) \otimes_{\mathfrak{n}} \mathfrak{z}$$

Somehow you should prove that if $\sigma(\mathfrak{z}_1 + \rho) = \mathfrak{z}_2 + \rho$ then there is a map

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$J \otimes_{\mathfrak{n}} 1 = R(K) \otimes_{\mathfrak{n}} \mathfrak{U}(\alpha)$ conjecture this perhaps is free over $\mathfrak{n}$

Suppose so Then

$H_+(\mathfrak{n}, J\otimes_{\mathfrak{n}} 1) = 0$ so one gets

$$\mathrm{Ext}^*_{\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}}(1 \otimes_{\mathfrak{n}} J \otimes_{\mathfrak{n}} 1, \mathrm{Hom}(\mathfrak{z}_1, \mathfrak{z}_2))$$

General fact

$$\mathrm{Ext}^*_{\mathfrak{g}}(M,N) = H^*(\mathfrak{g}, \mathrm{Hom}(M,N)).$$

<u>Proof:</u> Take injective resolution $I^\cdot$ of N. Note that $I = \mathrm{Hom}_\circ(U(\mathfrak{g}), Q)$ Q v.s. is injective since

$$\mathrm{Hom}_{\mathfrak{g}}(?, \mathrm{Hom}(U(\mathfrak{g}), Q)) = \mathrm{Hom}(?, Q) \quad \text{is exact}$$

and that any $\mathfrak{g}$ module Q may be anbedded into the injective

$$Q \longrightarrow \mathrm{Hom}(U(\mathfrak{g}), Q).$$

This shows that $\mathrm{Hom}(M, I^\circ)$ is an injective resolution of $\mathrm{Hom}(M,N)$ since

$$\mathrm{Hom}(M, I^8) \text{ direct sum of } \mathrm{Hom}(M, \mathrm{Hom}(U(\mathfrak{g}), I^8))$$
$$\| $$
$$\mathrm{Hom}(U(\mathfrak{g}), \mathrm{Hom}(M, I^\circ))$$

is injective. Thus

$$\mathrm{Ext}^*_{\mathfrak{g}}(M,N) = H^*(\mathrm{Hom}(M, I^\circ)^{\mathfrak{g}}) = H^*(\mathfrak{g}, \mathrm{Hom}(M,N)).$$

<u>Kostants theorem:</u>

$\mathfrak{g}$, ~~b~~ $\mathfrak{b}$ parabolic

$\mathfrak{n}$ nilradical

$$\mathfrak{g}_1 = \mathfrak{m} + \mathfrak{a}$$
$$= \mathfrak{h}_k + \mathfrak{h}_p + \sum_{\alpha \in \Sigma''} (e_\alpha) + (e_{-\alpha}) \quad = \text{centralizer of } \mathfrak{h}_p$$

have W Weyle group of $\mathfrak{g}$

W_1 —————— $\mathfrak{g}_1$, subgp. of W

W_b baby Weyl group = elements of W preserving θ decomp. and in K

observe that we have

$$\cancel{W_K \subset W}$$
$$\cancel{\parallel}$$
~~autos. of h coming from elements of K~~
$$\cancel{\parallel}$$
$$\cancel{\mathrm{Norm}\, h}$$

$$\mathrm{Norm}_{\mathfrak{g}_1} h \subset \mathrm{Norm}_{\mathfrak{g}} h$$
$$\cup$$
$$\mathrm{Norm}_K h \subset (\mathrm{Norm}_{\mathfrak{g}} h)^\theta$$

$$W_1 \subset \mathrm{Norm}_{\mathfrak{g}} h / h$$
$$\cup$$
$$(\mathrm{Norm}_{\mathfrak{g}} h)^\theta / h_k$$
$$\cup\ ?$$
$$\mathrm{Norm}_K h / h_k$$

The cx case:

$\mathfrak{g} = \bar{\mathfrak{g}} \times \bar{\mathfrak{g}}$

$\mathfrak{k} = \Delta$

$\mathfrak{n} = \bar{\mathfrak{n}}_+ \times \bar{\mathfrak{n}}_-$

normal form

baby $W = W$

$W_1 = 0$

Compact form

$W_1 = W.$

Since $\mathfrak{g}_1 = \mathfrak{g}$

$\mathfrak{h} = \bar{\mathfrak{h}} \times \bar{\mathfrak{h}} = \mathfrak{h}_{\mathfrak{k}} + \mathfrak{a} = \mathfrak{g}_1$

$W = \bar{W} \times \bar{W}$

baby Weyl group $= \Delta \bar{W}$

$$\frac{\mathrm{Norm}_K A}{\mathrm{Cent}_K A} = \frac{\mathrm{Norm}_G A}{\mathrm{Cent}_G A} \; ?$$

$W_1 = 0.$

$\alpha \in \mathrm{Norm}_G A$ set

$\alpha = kp$

$W' = W$. not the ~~baby Weyl group~~ !!!!!!!!!!

$$kpAp^{-1}k^{-1} = A$$

$$pAp^{-1} = k^{-1}Ak \subset P.$$

$$\Downarrow$$

$$(\mathrm{Ad}\, p)\, \mathfrak{a} \subset \mathfrak{p}.$$

W_1 comes from Σ''

baby W ——————— Σ'

<u>Bott-Borel-Weil</u>

Think of $\mathfrak{g}$ as the complexification of a complex Lie alg. $\mathfrak{g}_0$, J with compact form $\mathfrak{k}_0$, torus $\mathfrak{t}_0$, roots $\Delta_+ \subset i\mathfrak{t}_0'$. Better think of $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C}) + i\,\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{g}_0 = \mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{k}_0 = \mathfrak{su}(n,\mathbb{C})$. Hopeless. Think of everything before as being real.

~~Need a holomorphic character on b~~

$\mathfrak{k}_0 = \mathfrak{su}(n,\mathbb{C})$, $\mathfrak{p}_0 =$ hermitian matrices, $\mathfrak{a}_0 =$ real diags of tr 0, $\mathfrak{m}_0 = J\mathfrak{a}_0$, baby $W =$ permutation gp., $\mathfrak{n}_0 =$ cx upper triangular matrices.

Botts starts with a holomorphic character ξ on $\mathfrak{b}$, i.e. J linear. Thus it means that $\xi = \lambda + \nu$, $\lambda \in \mathfrak{h}_k'$ $\nu \in \mathfrak{h}_p'$ and we have $\lambda = \nu J$ where $J : \mathfrak{k} \xrightarrow{\sim} \mathfrak{p}$

(up to sign)

$$\begin{tikzcd} \mathfrak{k} \arrow[rr, "\sim"] \arrow[dr, "pr_1"'] & & \mathfrak{p} \arrow[dl, "pr_1"] \\ & \mathfrak{g}_1 & \end{tikzcd}$$

The basic point is that one can define the complex

$$\Lambda\, \mathfrak{g}^{01} \otimes F$$

where $F = \mathrm{Hom}_{\mathfrak{b}}(U(\mathfrak{g}), \mathbb{C}_\xi)$

$$P_k = U(\mathfrak{g}) \otimes_k \Lambda \otimes U(\mathfrak{b}).$$

$$= U(\mathfrak{b}) \otimes \Lambda \otimes U(\mathfrak{b}).$$

$$\parallel$$

$$\mathrm{Ext}^*_{\mathfrak{b}\times\mathfrak{b}}\left(J, \mathrm{Hom}(\mathfrak{I}_1, \mathfrak{I}_2)\right).$$

$$\boxed{\text{Proposition: } \quad \mathrm{Ext}^*_{\mathfrak{g}}(I(\mathfrak{I}_1), I(\mathfrak{I}_2)) \simeq \mathrm{Ext}^*_{\mathfrak{b}\times\mathfrak{b}}(\underbrace{J, \underbrace{\mathrm{Hom}(\mathfrak{I}_1, \mathfrak{I}_2)}}) }$$

$$\begin{tikzcd}
\mathcal{M}(\mathfrak{g}, k) \arrow[r, bend left, "1 \otimes_{\mathfrak{n}} (?)"] \arrow[r, bend right, "\mathrm{Hom}_{\mathfrak{g}}(J, ?)"'] & \mathcal{M}(M \times \mathfrak{a}) \arrow[l]
\end{tikzcd}$$

$$\mathrm{Hom}_{B\times B}(G, \mathrm{Hom}(L, M)) \overset{?}{=} \prod_{\substack{\text{double} \\ \text{cosets}}}$$

at some stage you should connect with the Borel–Weil–Bott–Kostant theorems concerning when need to know structure of J as a $B\times B$ module.

$$I(\mathfrak{J}) = k\text{ fin. } \mathrm{Hom}_{\mathfrak{b}}(U(\mathfrak{g}), \mathfrak{J}) \simeq J \otimes_{\mathfrak{b}} \mathfrak{J}.$$

$$\mathrm{Hom}_{\mathfrak{g}}(I(\mathfrak{J}_1), I(\mathfrak{J}_2)) \simeq \mathrm{Hom}_{\mathfrak{b},\mathfrak{b}}(J, \mathrm{Hom}(\mathfrak{J}_1, \mathfrak{J}_2))$$

What is J?

$$\mathrm{Ext}^*_{\mathfrak{g}}(I(\mathfrak{J}_1), I(\mathfrak{J}_2))$$
$$\cong$$
$$\mathrm{Ext}^*_{(\mathfrak{g},k)}(J \otimes_{\mathfrak{b}} \mathfrak{J}_1, I(\mathfrak{J}_2))$$

need and $\mathrm{Ext}^*_{\mathfrak{g}}(\ \ , I(\mathfrak{J}_2))$ acyclic resolution of

$U(\mathfrak{g}) \otimes_k \Lambda \otimes U(\mathfrak{b})$

free $k, \mathfrak{b}$ mod.

$P_\cdot$ free over $\mathfrak{g}, \mathfrak{b}$ and k finite?

$$H^* \mathrm{Hom}_{\mathfrak{g}}(P_\cdot \otimes_{\mathfrak{b}} \mathfrak{J}_1, \overset{\mathrm{Hom}_{\mathfrak{b}}(U(\mathfrak{g}), \mathfrak{J}_2)}{I(\mathfrak{J}_2)})$$

k finite ✓

$$\mathrm{Ext}^*_{(\mathfrak{g},k)}(U(\mathfrak{g}) \otimes_k \Lambda, I(\mathfrak{J})) = \mathrm{Ext}^*_{k\,f.}(\Lambda, I(\mathfrak{J})) \cong 0.$$

k semi-simple

$$H^* \mathrm{Hom}_{\mathfrak{b},\mathfrak{b}}(P_\cdot, \mathrm{Hom}(\mathfrak{J}_1, \mathfrak{J}_2)).$$

What is J?

$$k \text{ finite } \mathrm{Hom}_{\mathfrak{b}}(U(\mathfrak{g}), \mathfrak{I}) \simeq J \otimes_{\mathfrak{b}} \mathfrak{I}.$$

If this holds for all $\mathfrak{I}$ set $\mathfrak{I} = U(\mathfrak{b})$ and get.

$$k \text{ finite } \mathrm{Hom}_{\mathfrak{b}}(U(\mathfrak{g}), U(\mathfrak{b})) \simeq J.$$

$U(\mathfrak{b}) \otimes_{\mathfrak{m}} U(k)$

$$\wr\wr \; k.$$

$$k \text{ finite } \mathrm{Hom}_{\mathfrak{m}}(U(k), 1)$$

$$\wr\wr$$

$R(K)^M$ representative functions on K ~~right~~ left invariant under M!

$$f(mx) = f(x)$$

Therefore as a left k module $J \simeq R(K) =$ regular fns on K.

~~so that~~ Problem now is to define a left $\mathfrak{g}$ and right B module structure. ~~So consider functions on G values in $U(\mathfrak{b})$~~

~~So given~~ try a ~~left~~ right B structure ie. given $f : U(k) \to 1$

~~set $(fd)(k) = f(k)d$~~

~~then $(fd_1d_2)(k) = fd_1$~~

define $\tilde{f} : U(\mathfrak{g}) \to U(\mathfrak{b})$ by $f(bx) = b(f(x))$. $x \in U(k)$

note that $\exists\ \varphi : K \to 1$ and $f(x) = (x\varphi)(e)$.

then $(\tilde{f}d)(bx) = b\, f(x) \cdot d$ $\quad d \in U(\mathfrak{b})$, $b \in U(\mathfrak{b})$.

Therefore given φ

~~$f(bx) = b\cdot(x\varphi)(e)$~~

~~$\therefore\ f(z) = \int_K z\varphi$?~~

real situation

functions on $\boxed{K/M \simeq G/B}$

J functions on K/M K finite.

$G = KAN$ real

$G/B = G/MAN = K/M$ real.

Look at complex group $\underline{K_{\mathbb{C}}/M_{\mathbb{C}}}$ algebraic fns. here.

$sl(2,\mathbb{R})$. $\quad K_{\mathbb{C}}/M_{\mathbb{C}} \simeq \mathbb{C}^*/\ Z \simeq \mathbb{C}^* = \mathbb{G}_m$

$$G_{\mathbb{C}}/B_{\mathbb{C}} = \begin{pmatrix} * & * \\ * & * \end{pmatrix} \Big/ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \simeq \mathbb{P}_1(\mathbb{C})$$

$$\begin{tikzcd} K_{\mathbb{C}} \arrow[r] \arrow[d] & G_{\mathbb{C}} \arrow[d] \\ K_{\mathbb{C}}/M_{\mathbb{C}} \arrow[r] & G_{\mathbb{C}}/B_{\mathbb{C}} \end{tikzcd}$$

projective variety

~~affine~~

compactification

tangent space to identity is $k/m \simeq g/b$.

cx case

$$\begin{tikzcd} K_{\mathbb{C}} \arrow[r, "\Delta"] & K_{\mathbb{C}} \times K_{\mathbb{C}} \arrow[d] \\ & K_{\mathbb{C}}/B_{\mathbb{C}} \times K_{\mathbb{C}}/B_{\mathbb{C}} \end{tikzcd}$$

?

$\mathfrak{g} = \mathfrak{n} \times \mathfrak{n}$.

$\mathfrak{k} = \Delta \mathfrak{n}$.

$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 = \mathfrak{h}_k \oplus \mathfrak{h}_p$.

$\Sigma' = \Sigma' \cup \underline{\Sigma''}$, $\Sigma'' = 0$

since $\mathfrak{n} = \mathfrak{h}_k$.

$$\begin{array}{ccc} K_{\mathbb{C}} & & K_{\mathbb{C}} \times K_{\mathbb{C}} \\ & & \downarrow \\ K_{\mathbb{C}}/T & \longrightarrow & K_{\mathbb{C}}/B_{\mathbb{C}} \times K_{\mathbb{C}}/U_{\mathbb{C}} \end{array}$$

$$\begin{array}{ccc} Sl(2) & & Sl(2) \times Sl(2) \\ & & \downarrow \\ Sl(2)/\text{diag} & & P^1 \times P^1 \end{array}$$

Clear that if you are careful ~~in your choice of~~ to twist things up by the right non-baby Weyl group elt, then get an embedding into a proj. variety.

$$(\varphi d)(x) = \varphi(x) \cdot d = \overset{U(\mathfrak{b})}{d} \cdot \overset{U(\mathfrak{k})}{\varphi(x)} = \varphi(d \cdot x).$$

$d \in U(\mathfrak{b})$ $x \in U(\mathfrak{k})$

$\cancel{\varphi(d\,x)}$

$\varphi : U(\mathfrak{k}) \xrightarrow[\mathfrak{n}]{} 1$

$\tilde{\varphi} : U(\mathfrak{g}) \xrightarrow[\mathfrak{b}]{} U(\mathfrak{b})$

$(\tilde{\varphi} d)(z) = \tilde{\varphi}(z) \cdot d$

$\cancel{\varphi(U(\mathfrak{k})) \; \varphi}$

no simple way of going from φ to $\tilde{\varphi}$

Status

$$I(\mathcal{Y}) = J \otimes_{MA} \mathcal{Y}$$

$$\mathrm{Hom}_{\mathfrak{g}}(I\mathcal{Y}_1, I\mathcal{Y}_2) = \mathrm{Hom}_{\mathfrak{g}}(J \otimes_{MA} \mathcal{Y}_1, \mathrm{Hom}_B(U(\mathfrak{g}), \mathcal{Y}_2))$$

$$= \mathrm{Hom}_{\cancel{MA}\ B}(J \otimes_{MA} \mathcal{Y}_1, \mathcal{Y}_2)$$

$$= \mathrm{Hom}_{B, MA}(J, \mathrm{Hom}(\mathcal{Y}_1, \mathcal{Y}_2)).$$

$$= \mathrm{Hom}_{MA, MA}(1 \otimes_{\mathfrak{n}} J, \mathrm{Hom}(\mathcal{Y}_1, \mathcal{Y}_2)).$$

$$\mathrm{Ext}^*_{\mathfrak{g}}(I\mathcal{Y}_1, I\mathcal{Y}_2) = \mathrm{Ext}^*_{\mathfrak{b}, \mathfrak{ma}}(J, \mathrm{Hom}(\mathcal{Y}_1, \mathcal{Y}_2))$$

$$= \mathrm{Ext}^*_{\mathfrak{b}, \mathfrak{b}}(\tilde{J}, \mathrm{Hom}(\mathcal{Y}_1, \mathcal{Y}_2))$$

here $\tilde{J} = J \otimes_{MA} U(\mathfrak{b})$ so

$$\tilde{J} \otimes_{\mathfrak{n}} 1 = J \otimes_{MA} U(MA)U(\mathfrak{n}) \otimes_{\mathfrak{n}} 1 = J.$$

At this stage I need a $\mathfrak{b} \otimes \mathfrak{b}$ resolution of $\tilde{J}$. From

$$I(\mathfrak{j}) = k\,\mathrm{fin}\ \mathrm{Hom}_{\mathfrak{b}}(U(\mathfrak{g}), \mathfrak{j}) \simeq \tilde{J} \otimes_{\mathfrak{b}} \mathfrak{j}$$
$$\simeq J \otimes_{\mathfrak{m}\mathfrak{a}} \mathfrak{j}$$

Set $\mathfrak{j} = U(\mathfrak{n}\mathfrak{a})$ get

$$J \simeq k\,\mathrm{fin}\ \mathrm{Hom}_{\underline{\mathfrak{b}}}(U(\mathfrak{g}), \underline{U(\mathfrak{n}\mathfrak{a})})$$
$$\cong$$
$$k\,\mathrm{fin}\ \mathrm{Hom}_{\mathfrak{b}}(U(\mathfrak{b})\,\underset{\mathfrak{m}}{}\,U(\mathfrak{k}), U(\mathfrak{m}\mathfrak{a}))$$
$$\cong$$
$$\mathrm{Hom}_{\mathfrak{b}}(U(\mathfrak{n})U(\mathfrak{a})U(\mathfrak{k}), U(\mathfrak{n}\mathfrak{a}))$$
$$\cong$$

$$\tilde{J} = k\mathrm{fin}\,\mathrm{Hom}_{\mathfrak{b}}(U(\mathfrak{b}) \otimes_{\mathfrak{n}} U(\mathfrak{k}), U(\mathfrak{b})) = k\mathrm{fin}\,\mathrm{Hom}_{\mathfrak{n}}(U(\mathfrak{k}), U(\mathfrak{b}))$$
$$= \cancel{R(K)}\ R(K) \otimes_{\mathfrak{n}} U(\mathfrak{b})$$
$$= \underline{R(K) \otimes U(\mathfrak{a}+\mathfrak{n})}$$

Loosely $\tilde{J} = \mathrm{Hom}_B(G, B)$.

$$\tilde{J} = R(K) \otimes_M U(\mathfrak{b}) \qquad \text{as a } K \text{ module + right } \mathfrak{b}$$

$$J = R(K) \otimes_M U(M\mathfrak{a}) \qquad K \underline{\qquad} \underline{\qquad}$$

$J\mathfrak{n} = 0.$

$$= \underline{R(K) \otimes U(\mathfrak{a})}$$

(i) left K action is clear $\qquad k(f \otimes d) = f R_k \otimes d.$
(ii) right $\mathfrak{a}$ action is clear
(iii) right M action is by

$$(f \otimes d) \cdot m = (f L_m) \otimes m^{-1} d m.$$

<u>Remains</u> to determine left $\mathfrak{a}$ and left $\mathfrak{n}$ action.

<u>Questions</u>: ① V f.t. $\in (\mathfrak{g}, k) \Longrightarrow \{ H_0(\mathfrak{n}, V) = 0 \Rightarrow V = 0 \}$.

② In irreducible case $H_+(\mathfrak{n}, I(\xi)) = 0$ and $I(\xi)$ <u>is a free $\mathfrak{n}$ module of rank w.</u>

J should almost be a free left $\mathfrak{b}$ module with w generators. Can you find W inside of J somehow?

Let $s \in W$ represent it by an element $k \in K$

~~Attempt~~ Attempts at Nakyama's lemma

$H_*(\mathfrak{n}, V^{\mathfrak{I}})$. $V(\mathfrak{I})$.

This ~~###~~ Nakyama's lemma seems to be absolutely crucial

Let G be real semi-simple, $G = KAN$ and let $\tilde{V}$ be an ~~####~~ irreducible, unitary representation of G with K finite subspace $V = \bigoplus_{\Lambda} V^{\Lambda}$ Λ runs over irred. reps. of K. Now if we know that $V/\mathfrak{n}V \neq 0$, then we get a map

$$V \longrightarrow I^{\mathfrak{I}}$$

which must be injective by irred. It follows that V appears as part of the induced representation.

Be more precise: Suppose we find a linear function ~~########~~ on V annihilated by $\mathfrak{n}$.

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{n}$$
$$= (\mathfrak{m} \oplus \mathfrak{a}) + \sum_{\alpha \in W'} \mathfrak{n}^{\alpha}$$

$$\mathfrak{n} F_q V \overset{?}{\subseteq} F_{q+1} \mathfrak{n} V$$

$\cancel{\mathfrak{n} F_q V}$

we're assuming that $NV = V$
and we'd like to show this implies that the support of V doesn't intersect $N^{\perp}$. i.e that

$N \, gr \, V$ is of finite codimension in $gr \, V$.

. either true or false

$$\cancel{N F_q V / F_q V}$$

$$N F_q V + F_q V / F_q V \overset{?}{=} F_{q+1} V / F_q V.$$

ie

$$N F_q V + F_q V = F_{q+1} V \quad \text{for large } q.$$

$$V \xrightarrow{N} V \rightarrow$$

get a spectral sequence

$$0 \quad gr V \xrightarrow{N} gr V \longrightarrow 0$$

$$gr V \xrightarrow{N} gr V$$

idea is that there is a map

$$F_q V / F_{q-1}V + NF_{q-1}V \xleftarrow{N} F_q V \cap N^{-1} F_q V / F_{q-1} V$$

and from the fact that N is onto we can conclude that this map is onto ~~for la~~ all q.

N (1) Prop: N onto $\Longleftrightarrow$ $F_q V \cap N^{-1} F_q V \xrightarrow{N} F_q V / F_{q-1} V + N F_{q-1} V$ onto all q.

nope

Proof $\Longrightarrow$: Let $\alpha \in F_q V$ then $\alpha = N\beta$ some β say $\beta \in F_r V$. Choose r least.

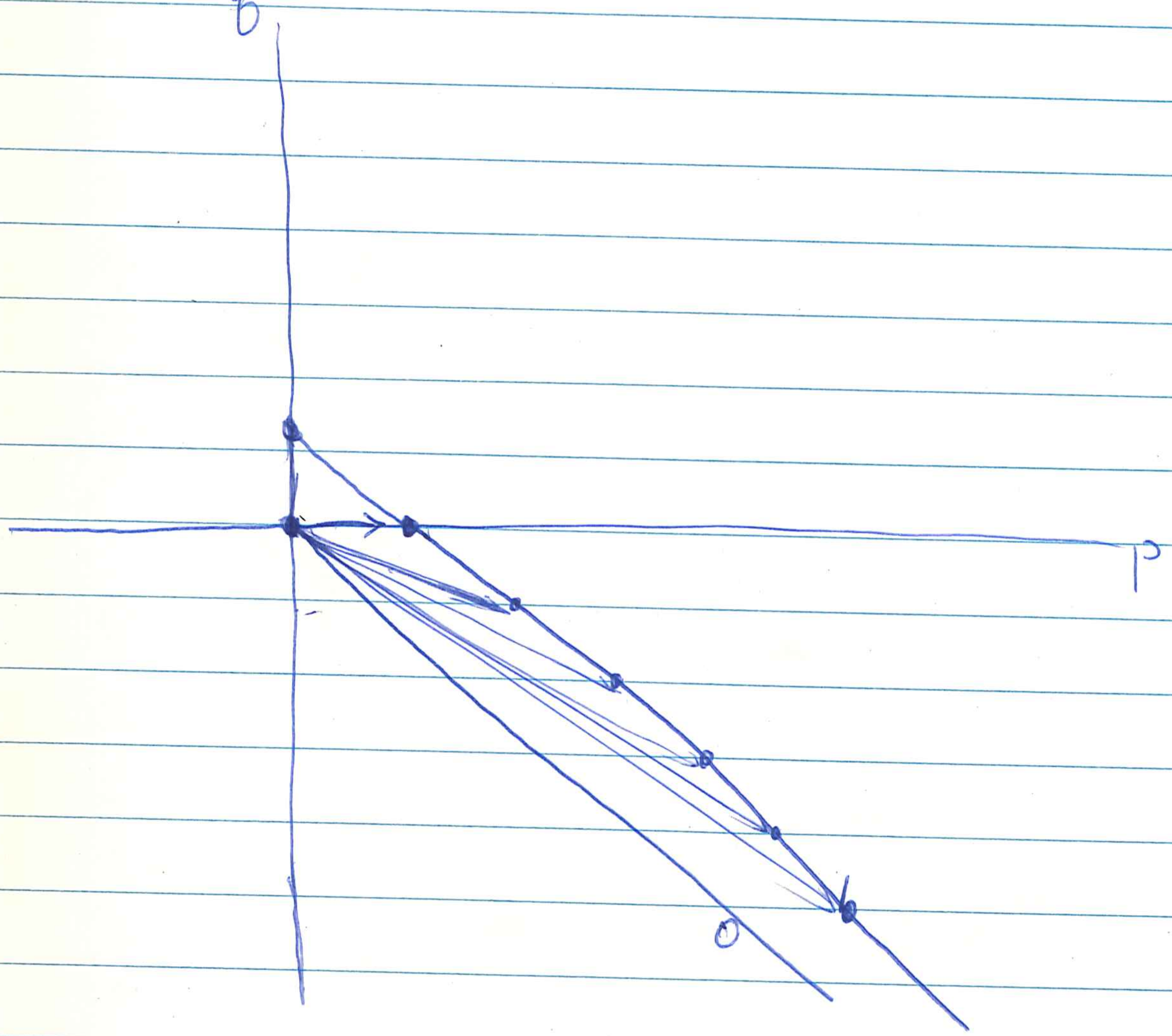
$$E^2_{pq} = H_{p+q}(gr_q V) \Longrightarrow H(V).$$

homology + increasing filtration.

$$E^1_{pq} = H_{p+q}(F_p/F_{p-1}) \xrightarrow{d_1} H_{p+q-1}(F_{p-1}/F_{p-2})$$

$$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \| $$

$$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad E^1_{p-1,q}.$$



Suppose M f.t. module over $K[X_1, \ldots, X_n]$ and that $XM = M$. Is it true that

$$X \, grM \quad \text{~~fin~~ codim. in } grM?$$

$$(1 - aX)M = 0.$$

NO if $X = 1$ on M then $X = 0$ on grM.

~~Let Then Suppose ... multip~~

Special case of Nakyama's lemma Assume $\mathfrak{n}$ abelian ~~and~~ that $\mathfrak{a}$ acts in such a way that any non-zero $\mathfrak{a}$ invariant closed subset of $\mathfrak{n}'$ meets zero and that M is finitely generated over $\mathfrak{n}$. Then $H_0(\mathfrak{n}, M) = 0 \Rightarrow M = 0$

Proof: Support M is a closed subset of $\mathfrak{n}'$. Would like to show it's invariant. Let $I \subset S(\mathfrak{n})$ be the annihilator of M. Then if $f \in I$, $A \in \mathfrak{a}$

$$[A, f]m = Afm - fAm = 0$$

so $[A, I] \subset I$. Question: Does this mean that rad I is stable under $\exp \mathfrak{a}$? ~~Suppose ... $f^n \in I$~~ Assume $f \in I$

~~expt~~ Action of $\mathfrak{a}$ on $S(\mathfrak{n})$ certainly a sum of finite reps. Thus given f, $U(\mathfrak{a}) f^n$ is a finite vector space ~~and~~

~~we know that~~ $\cancel{\sum_{k\geq 0}^{n} \frac{t^k}{k!} (\operatorname{Ad} A)^k A^k}$ which we know to be contained in I. But then $(\exp A) f^m \in I$ since $(\exp A) f^m \in U(\mathfrak{a}) f^m$ and so $\exp A\, I \subset I$. ~~$\exp A(f^n) = [\exp A(f)]^n$~~
Thus $\sqrt{I}$ is $\mathfrak{a}$ stable, so $\sqrt{I} \subset \mathfrak{n} S(\mathfrak{n}) \Longrightarrow H_0(\mathfrak{n}, M) \neq 0$.

This situation holds for action of $\mathfrak{a}$ on $\mathfrak{n}$. In effect $\mathfrak{n} = \sum_{\alpha\in\Sigma'} (e_\alpha)$ and $\alpha/\mathfrak{a}$ $\alpha \in \Sigma'$ are distinct (? in any case have $\mathfrak{n} + \mathfrak{a}$ acting and this contains h). Thus given $f \in \sum_{\alpha\in\Sigma'} (e_{-\alpha}) a_\alpha$, not all $a_\alpha = 0$ we have

$$Af = \sum a_\alpha (-\alpha(A))\, e_{-\alpha}$$

So

$$e^{tA} f = \sum a_\alpha \left(e^{-t\alpha(A)}\right) e_{-\alpha}.$$

So let $A \in$ dominant Weyl chambre and $t \to \infty$ and $0 \in$ closure of the orbit.

<u>Question:</u> Can this argument be generalized to the case where $\mathfrak{n}$ is nilpotent?

The idea is to generalize the support of a f.t. module M over $U(\mathfrak{n})$; should be a closed invariant subset of $\mathfrak{n}'$.

Novarz-Gabriel: If n nilpotent they calculate $\operatorname{Spec} U(n)$. This is set of indecomposable injectives and set of primes ideals in $U(n)$.

Inductive approach: Given n module M f.t. over $U(n)$ we must show that $H_0(n, M) \neq 0$, if $M \neq 0$.

Let $\mathfrak{z}$ be the center of n. Assume $\mathfrak{z} = (z)$. Then To study

$$0 \to zM \to M \to M/zM \to 0$$

If $M \neq zM$ then we are done by induction as $n/\mathfrak{z}$ is smaller. Thus if $z: M \to M$ is ~~onto~~ by noetherianness it is an isomorphism.

Otherwise M becomes a module over $U(n)[\frac{1}{z}]$.

So try old trick. Choose generators for M and filter over $\mathbb{C}[z] \otimes_{\mathbb{C}} U(n)$ in the obvious way. trouble is that its not n stable. However in any case we get a polynomial $f(z)$ such that M_f is free over $\mathbb{C}[z]_f$. If $f(z) \neq 0$ done. But if $f(z) = z$ i.e M_z free over $\mathbb{C}[z]_z$ what can we say? In this case $M_z = M$. which might be free over $\mathbb{C}[z, \frac{1}{z}]$. NO GOOD

Support should be a closed inv. subset F of n'. We want To show that it cannot be that under $n' \xrightarrow{z} \mathbb{C}$, $z(F)$ not closed.

Can it be that z isn't proper on invariant closed sets. I think this is unlikely in effect once the center is rep. by scalars it's only for exceptional values of the scalar that the representation ~~its~~ isn't uniquely det.

Calculated at the board: If $\mathfrak{z}$ 1-dimensional then $\{f \mid f(z)=\lambda\}$ for $\lambda \neq 0$ is a G orbit in $\mathfrak{g}'$. Reason: this set is of codimension 1 in $\mathfrak{g}'$. If f_0 fixed then $\{u \in \mathfrak{g} \mid (\mathrm{ad}\,u)f = 0\}$ is $\mathfrak{z}$ because given $u \notin \mathfrak{z}$ $\exists\, x \ni$ $[(\mathrm{ad}\,u)f](x) = f([x,u]) = f(z) \neq 0$. Thus Gf_0 is of dimension $n-1$ in $\{f \mid f(z)=\lambda\}$ so all orbits are open $\Rightarrow$ $Gf_0 = \{f \mid f(z)=\lambda\}$.

Consequently: In case $\mathfrak{z}$ 1-dimensional $\mathfrak{z}=(z)$ we have that if $F \subset \mathfrak{n}'$ is closed invariant under $\mathfrak{n}$, then $z(F)$ closed in $\mathbb{C}$.

Another attempt to derive that $M/\mathfrak{n}M \neq 0$ was the following. Assume

$$\mathrm{Supp}\, M = \{\text{primes } \mathfrak{p} \text{ in } U(\mathfrak{n}) \mid \mathfrak{p} \supset \mathrm{Ann}\, M\}$$

But by Gabriel-etc primes in $U(\mathfrak{n})$ are same as $\mathfrak{n}$ inv. primes in $S(\mathfrak{n})$. ~~containing~~ But by assumption all contain 0 ie $\mathfrak{n} \in \mathrm{Supp}\, M$. so by some kind of Nakyama $M/\mathfrak{n}M \neq 0$.

The point is that

$$\{\mathfrak{p} \mid M/\mathfrak{p}M \neq 0\} \qquad \{\mathfrak{p} \mid \mathfrak{p} \supset \mathrm{Ann}\, M\}$$

maybe equal in nilpotent case

are not necessarily equal. In effect for $sl(2,\mathbb{R})$ have M ~~tired~~ $\mathrm{Ann}\, M = U\cdot(\Delta - \alpha)$ yet finite codimensional $\mathfrak{p}$ and $\mathfrak{p}M = M$.

Iwasawa formulas.

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n} = \mathfrak{k} + \mathfrak{p} \qquad \mathfrak{h} = \mathfrak{h}_{\mathfrak{k}} + \mathfrak{a}$$

$$\Sigma' = \{\alpha \in \Sigma \mid \alpha\theta \in -\Sigma\}$$

$$\Sigma'' = \{\alpha \in \Sigma \mid \alpha\theta = \theta \text{ or } \alpha/\mathfrak{a} = 0\}.$$

$$\mathfrak{m} = \mathfrak{h}_{\mathfrak{k}} + \sum_{\alpha \in \Sigma''} (e_\alpha) + (e_{-\alpha}) \qquad \mathfrak{n} = \sum_{\alpha \in \Sigma'} (e_\alpha)$$

$$\mathfrak{k} = \mathfrak{m} + \mathfrak{q}$$

$$\mathfrak{q} = \sum_{\alpha \in \Sigma'} (e_\alpha + e_{\alpha\theta}).$$

<u>Problem:</u> Let V be a $\mathfrak{g}, \mathfrak{k}$ module, calculate $H_*(\mathfrak{n}, V)$.

$\mathfrak{sl}(2, \mathbb{R})$

$$\mathfrak{k} = (H) \qquad A = \frac{1}{\sqrt{2}}(X+Y) \qquad \langle A, A\rangle = 2$$

$$\mathfrak{a} = (A) \qquad N = H - \frac{1}{\sqrt{2}}(X-Y) \qquad \langle N, N\rangle = 0.$$

$$\mathfrak{n} = (N)$$

$$[A, N] = \frac{1}{\sqrt{2}}(-X+Y) + \frac{1}{2}[X, Y] - \frac{1}{2}[Y, X]$$

$$= N.$$

$$N_- = H + \frac{1}{\sqrt{2}}(X-Y) = 2H - N$$

$$[A, N_-] = \frac{1}{\sqrt{2}}[-X+Y] - H = -N_-$$

~~Let V be a (g, K) module of sl(2,R), then we want~~

minimum model:

$e_\alpha \in \mathfrak{n}$ | N

$H_\alpha \in \mathfrak{a}$ | A

K

$\boxed{e_\alpha}$ $\alpha \in \Sigma'$.

x, e, f $e+f \in \mathfrak{a}$

$\boxed{x \in \mathfrak{k}}$

$$\boxed{H_*(\mathfrak{n}, I(\xi)) = ?}$$

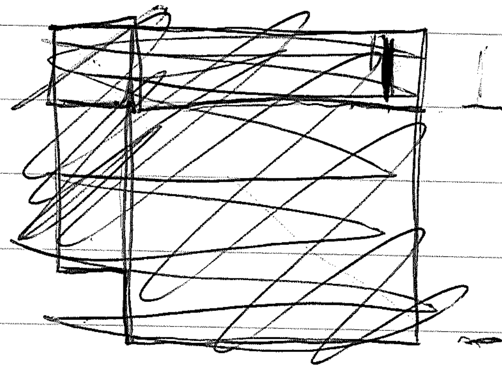
<u>Lemma:</u> $\mathfrak{n}$ nilpotent Lie algebra, V finitely generated $\mathfrak{n}$ module ∋ Nakayama's lemma holds. Then eq cond.

ⓐ $H_+(\mathfrak{n}, V) = 0$

ⓑ $H_1(\mathfrak{n}, V) = 0$

ⓒ regular sequence.

To make this effective I need to be able to proceed by induction on dim $\mathfrak{n}$, so have to cut down of somehow. Thus take e_α α largest in Σ' and it's normalizer.

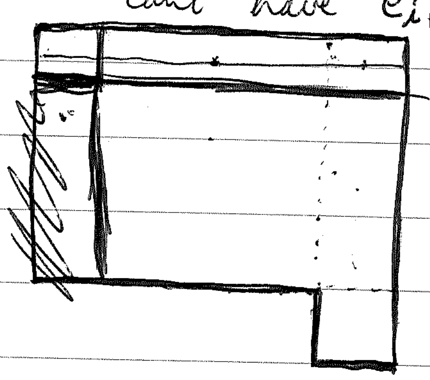


what is centralizer of e_{1n} $\quad [(d_i), e_{1n}] = (\lambda_1 - \lambda_n) e_{1n}$

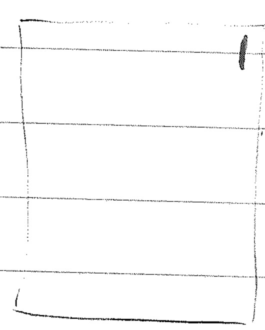
$$[e_{1n}, e_{nk}] = e_{1k} \qquad [e_{1n}, e_{k1}] = -e_{kn}$$

∴ can't have e_{ni} unless $i = n$.

can't have e_{i1} unless $i = 1$.



e_{1n}) = normalizer of



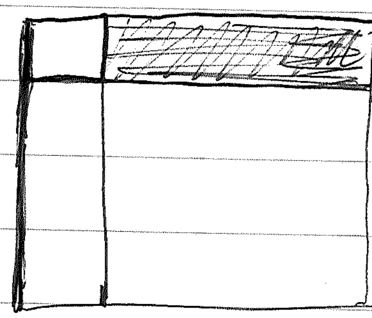
do normal form for $sl(n,\mathbb{R})$. $\theta = -{}^t$.

$K = SO$. $\mathfrak{n} = $ [illegible]

Take off the first line

$\mathfrak{n}' = (e_{1j}\ , \quad j \geq 1)$ ideal in $\mathfrak{n}$.

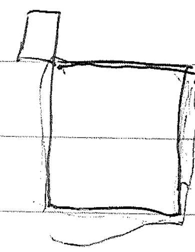
$[e_{1j}, e_{jk}] = \quad j<k$



$[e_{1j}, e_{lm}] = \begin{cases} e_{1m} & l=j \\ - \end{cases} \quad l<m$

Then $\mathfrak{n}'$ is an abelian ideal in $\mathfrak{n}$, so get spectral sequence

Then examine $H(\mathfrak{n}', V)$ should be nice over $\mathfrak{g}$,



V K finite K normal

idea is that K orbits should intersect ~~at ideal~~

If $\mathfrak{n} \subset \mathfrak{p}$ then want K cl. orbits in $\mathfrak{p}$ to intersect $\mathfrak{n}^{\perp} \subset \mathfrak{p}'$.

To understand the principal series.

There should be two defns.

a) induced | these coincide when
b) coinduced | irreducible

Idea that there should be a 1-1 correspondence between irred $(\mathfrak{g}, k)$ reps and $(\tilde{\mathfrak{g}}, k)$ reps. except where certain exp fns. have zeroes. How to make precise?

need **exp**: $\circ$

$\mathfrak{g}$ $\quad k_0 + \mathfrak{p}_0$

various possibilities
(o) comp
(i) vector gp.

~~x+iy~~

b)

$$\Gamma_k \operatorname{Hom}_{Marr_-}(U(\mathfrak{g}), \lambda\nu) = V$$

We know that $\operatorname{Hom}_k(\Lambda, V) \cong \operatorname{Hom}_M(\Lambda, \nu)$.
so V has the correct k structure.

a) $\bigoplus_{\Lambda} \operatorname{Hom}_k(U(\mathfrak{g}) \otimes_{Marr_+} (\lambda\nu), \Lambda^*) \otimes \Lambda) \longleftarrow U(\mathfrak{g}) \otimes_{Marr_+} (\lambda\nu)$

how to define correct $\mathfrak{g}$ module structure

$$U(\mathfrak{g}) \otimes_{\mathrm{Morn}_+} (\lambda\nu) \longrightarrow \prod_{\Lambda} \mathrm{Hom}(\mathrm{Hom}_k(U(\mathfrak{g}) \otimes_{\mathrm{Morn}_+} (\lambda\nu)), \Lambda), \Lambda).$$

not too clear!

$\cancel{\otimes}$

Does $\exists$ natural map

$$U(\mathfrak{g}) \otimes_{\mathrm{Morn}_+} (\lambda\nu) \longrightarrow \mathrm{Hom}_{\mathrm{Morn}_-}(U(\mathfrak{g}), (\lambda\nu)) \quad ?$$

i.e.

Recall irred reps. of semi-direct product $P \times_\sigma K = \tilde{G}$

Given V irred repn of $\tilde{G}$ have as a P module

$$\cancel{V \simeq \bigoplus_{j \in K\lambda} \mathbb{C}_j \otimes}$$

support of V in $\hat{P}$ is a single orbit $K\lambda$. Thus taking W to be the λ piece we get a m

$$V \longrightarrow W_\lambda = W \quad \text{proj. on } V_\lambda$$

is a rep of MP.

$$V \xrightarrow{\sim} \mathrm{Hom}_{MP}(\tilde{G}, W)$$

$$\wr\!\wr$$

$$\mathrm{Hom}_{M}(K, W).$$

$\tilde{G} = PK.$

<u>Borel-Weil problem.</u> $\mathfrak{g}$ semi-simple. We have a canonical isom. of $Z \simeq S(\mathfrak{g})^{\mathfrak{g}}$ hence the closure of an orbit in $\mathfrak{g}'$ corresponds 1-1 to a maximal ideal in Z. Such a closure contains a unique semi-simple orbit. The problem is to construct ~~a ...~~ for each such semi-simple orbit a simple $\mathfrak{g}$ module with the correct character.

This simple $\mathfrak{g}$ module is not unique, however, we still want a ~~...~~ reasonably canonical method.

<u>Example:</u> **Kostant constructs** a cross-section of the orbits of $\mathfrak{g}$ on $\mathfrak{g}$.

Suppose $\mathfrak{g} = \mathfrak{gl}(n)$. Let A be a matrix with char. poly.

$$\det(\lambda - A) = \lambda^n - (\mathrm{tr} A)\lambda^{n-1} + \cdots$$
$$= f(\lambda)$$

Then the matrix

$$f(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n$$

$$\begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ -a_n & & -a_2 & -a_1 \end{bmatrix}$$

Companion matrix

has char. poly $f(\lambda)$. In effect this is the matrix of mult. by X on $k[X]/(f(X))$.

This gives a nice cross-section of map $\mathfrak{g} \xrightarrow{u} \mathbb{C}^n$ meeting each generic orbit once and only once.

<u>Observe</u>: There is a uniform method for the cross-section, however, it misses the non-reg. semi-simple elts. So expect that the Borel Weil will only give ~~an~~ simple modules generically.

<u>The method</u>: Choose a Borel $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$. Then any orbit of a regular elt. x will meet $\mathfrak{b}$ at W different places in $\mathfrak{h}$. Thus the orbit O in $\mathfrak{g}'$ hits $\mathfrak{h}'$ in exactly W places.

Seems impossible. What is possible is:

$$\begin{tikzcd}
\mathfrak{h}' \arrow[r, "\text{induced}", "\substack{\text{rep.}\\ \text{relative to } \mathfrak{b}}"'] \arrow[d] & \left(\mathfrak{g}\ \substack{\text{irred-}\\ \text{modules}}\right) \arrow[d, "\text{character}"] \\
\mathfrak{h}'/W & Z^{\wedge} \\
\mathfrak{g}'/G \arrow[u, phantom, "\cong" rotate=90]
\end{tikzcd}$$

$\mathfrak{g}'/G$

Conclusion: It is probably not possible to select a maximal left ideal containing a given max ideal of Z in a uniform way. However probably possible up to a covering by W.

λ $\qquad$ top part $U(\mathfrak{g}) \otimes_{\mathfrak{b}} (\lambda + \mathfrak{g})$.

$$
\begin{tikzcd}
\mathfrak{h}' \arrow[r, "\sim"] \arrow[d, "W\ \cancel{\text{to } 1}\ \text{covering}"] & \left(\begin{array}{c}\text{irred } \mathfrak{g}\text{-}\\ \text{modules}\end{array}\right) \arrow[d, "\text{character}"', "W\ \cancel{\text{to } 1}\ \text{covering}"] \\
(S(\mathfrak{g})^{\mathfrak{g}})^\wedge \arrow[r, dash, "\sim"] & Z^\wedge
\end{tikzcd}
$$

Fdl thm: of Sophus Lie seminar.

Work out details of maximal weight representations.

Does there exist a way of assigning a representation to $\lambda \in h'_{reg.}$ with character

$$\begin{tikzcd} Z \arrow[d, "\gamma", "\wr"'] & S(g)^{g} \arrow[d, "\text{orth projection}"] \\ U(h)^{W} & S(h)^{W} \arrow[l, "\simeq"'] \end{tikzcd}$$

Assume same as restriction to $h' \subset g'$.

Then given $\lambda \in h'$ get a ~~linear functional~~ max ideal in Z by

$$z \longmapsto \langle \gamma(z), e^{\lambda} \rangle$$

Suppose Z is casimir ie. $\gamma(c) = \sum_{\alpha \in \Sigma} e_\alpha e_{-\alpha} + e_{-\alpha} e_\alpha + \sum H_i K_i$

$$= \sum_{\alpha \in \Sigma'} \underbrace{2 e_\alpha e_{-\alpha}}_{\in \mathfrak{n}} + \underbrace{[e_{-\alpha}, e_\alpha]}_{-H_\alpha}$$

$$\cancel{\gamma(c) = |\lambda|^2 - \langle g, \lambda \rangle = |\lambda - g|^2 - |\lambda|^2}$$

$$\beta(c) = -\sum_{\alpha \in \Delta} H_\alpha + \sum H_i K_i$$

$$g = \tfrac{1}{2}$$

$$\langle \gamma(z), e^{\lambda} \rangle = \langle \beta(z), e^{\lambda + g} \rangle$$

$$= |\lambda + g|^2 - 2\langle g, \lambda + g \rangle$$

$$= |\lambda|^2 \cancel{+ 2\langle g, \lambda \rangle} - |g|^2 \quad \cancel{|\lambda + g|^2 - |\lambda|^2}$$

$$= |\lambda|^2 - |g|^2$$