Lemma: To give a G-bundle over X is the same as associating to each G-torsor Q a bundle $E(Q)$ over $(X\times Q)/G$ in compatible fashion, i.e. to each map $Q' \longrightarrow Q$ must give $E(Q') \longrightarrow E(Q)$ $\ni$

$$\begin{tikzcd} E(Q') \arrow[r] \arrow[d] & E(Q) \arrow[d] \\ (X\times Q')/G \arrow[r] & (X\times Q)/G \end{tikzcd}$$

cartesian.

Proof: Given a G-bundle E/X set
$$E(Q) = (E\times Q)/G.$$

$$\begin{tikzcd} E & E\times Q \arrow[r] & (E\times Q)/G \\ X & X\times Q \arrow[l] \arrow[r] & (X\times Q)/G \end{tikzcd}$$

Conversely given $E \longmapsto E(Q)$ consider $E(G) \to (X\times G)/G$, $E(G) \to X$, $X \xrightarrow{s} (X\times G)/G$ on which G acts to the left

Gives a G-bundle over X.

$$X \longrightarrow (X\times G)/G$$
$$x \longmapsto (x,e)$$
$$xg^{-1} \longleftrightarrow (x,g)$$

Check these are inverse procedures.

<u>Lemma</u>: To give a G-manifold Z over X is the same as associating to each G-torsor Q a manifold $Z(Q)$ over $(X\times Q)/G$ in compatible fashion, e.g.

$$\begin{tikzcd} Z(Q') \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cartesian}"] & Z(Q) \arrow[d] \\ (X\times Q')/G \arrow[r] & (X\times Q)/G \end{tikzcd}$$

<u>Proof</u>: Given $Z \xrightarrow{\text{diag}} X$ set

$$Z(Q) = (Z\times Q)/G \xrightarrow{\text{diag}} (X\times Q)/G.$$

Conversely given $Q \mapsto Z(Q)$ have

$$\begin{tikzcd} & Z(G) \arrow[d] \\ X & (X\times G)/G \arrow[l, "\sim"'] \end{tikzcd}$$

Now I claim these processes are inverse to each other. Given Q consider ~~these fun~~

$$\begin{tikzcd} & Z(G) \arrow[d] \arrow[dr, phantom, "\text{cart}"] & Z(Q\times G) \arrow[l] \arrow[d] & \\ x,g \arrow[d, mapsto] & (X\times G)/G \arrow[d, "\wr"] & (X\times Q\times G)/G \arrow[l] \arrow[d, "\wr"] & (x,q,g) \arrow[d, mapsto] \\ xg^{-1} & X & X\times Q \arrow[l, "pr_1"] & (xg^{-1}, qg^{-1}) \end{tikzcd}$$

Thus get an isomorphism $\Phi : Z(Q\times G) \xrightarrow{\sim} Z(G)\times Q$

Make G act on $Q\times G$ and G by $g(x,g') = (x, gg')$ and $g(g') = gg'$.

~~One sees that~~

$$\Phi\left(Z(id_Q \times L_g)\right) = Z(L_g) \times R_{g^{-1}}$$

so this action is free and we may divide out by it getting

$$\begin{tikzcd}
& Z(Q\times G)/G \arrow[r, "\sim"] \arrow[d, phantom, "\wr"] & (Z(G)\times Q)/G \\
\text{need} \arrow[r] & Z(Q) &
\end{tikzcd}$$

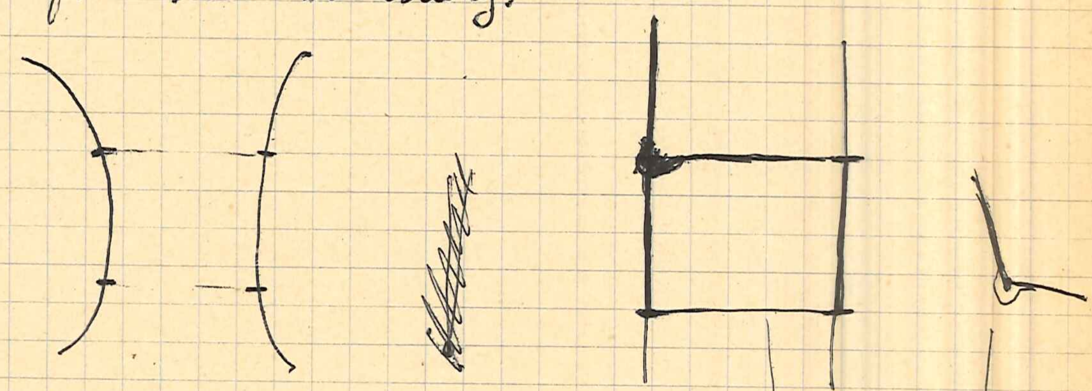
so I seem to need a descent result i.e. that ~~since~~

$$\begin{tikzcd}
Z(Q\times G) \arrow[r] \arrow[d] & Z(Q) \arrow[d] \\
(X\times Q\times G)/G \arrow[r] & (X\times Q)/G \\
X\times Q \arrow[u, "(id\times id\times e)", "\wr"'] \arrow[ur, "nat."'] &
\end{tikzcd}$$

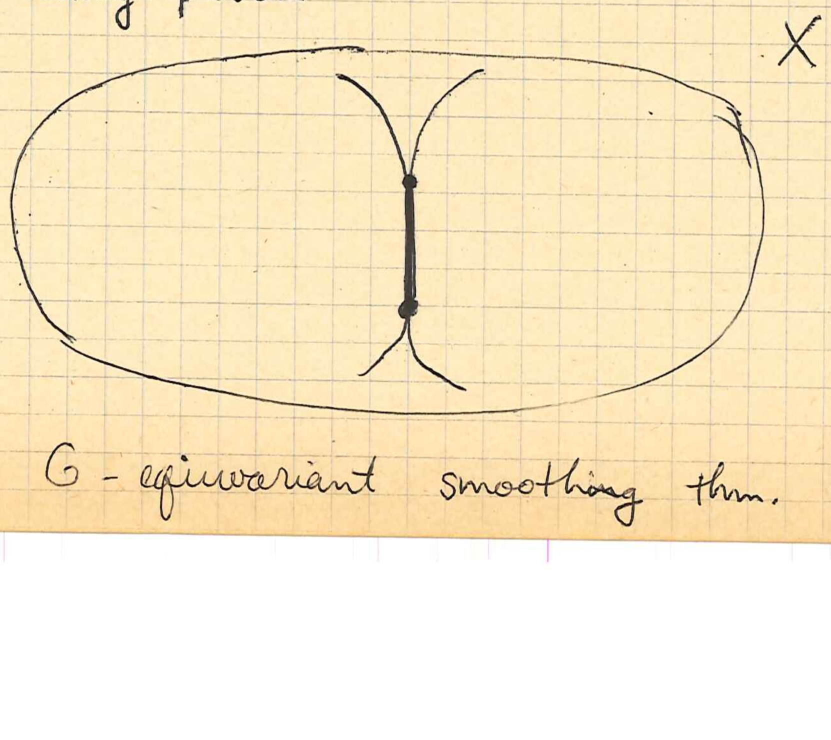
since G acts freely on the torsor $Q\times G$ with quotient Q, then G acts freely on $Z(Q\times G)$ with quotient $Z(Q)$. For manifolds at least when G is compact this follows from cartesianness of the square since then $Z(Q\times G)/G \longrightarrow Z(Q)$ will be bijective, hence an isomorphism.

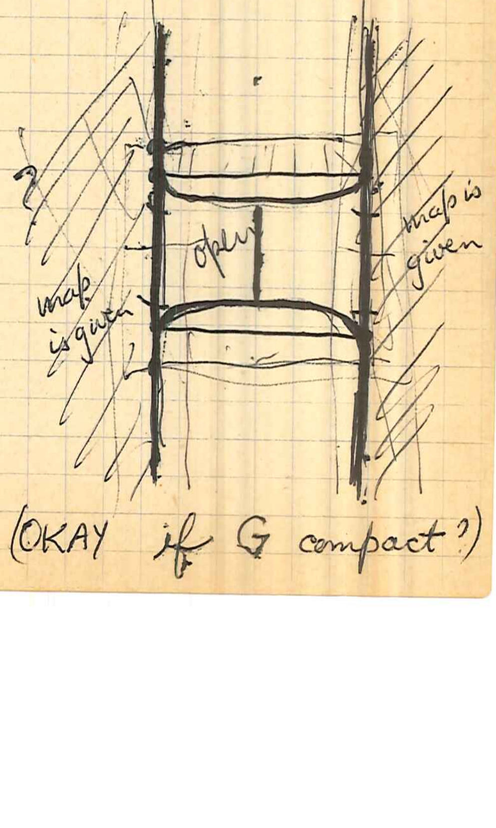
Two candidates for equivariant bordism theory:

A. Let X be a manifold on which a compact Lie group G acts. An equivariant bordism class in X of dimension q is represented by a G-map $f: Z \longrightarrow X$ where $Z \longrightarrow pt.$ is (proper) oriented of dim-q. (eventually may want G to act on the orientations). Usual bordism equivalence relation. Proof that it is an equivalence relation: Suppose given $f_1: Z_1^q \longrightarrow X$ and $f_2: Z_2^q \longrightarrow X$ and embeddings $\alpha: Q^{q-1} \hookrightarrow \partial Z_1$, $\beta: Q^{q-1} \hookrightarrow \partial Z_2$ over X, where Q is a manifold with boundary.



Usual smoothing problem





need G-equivariant smoothing thm. (OKAY if G compact?)

January 4, 1968.

Equivariant bordism theory:

Let G be a compact Lie group and let $\mathcal{V}_G$ be the category of C^∞ manifolds with G-action and equivariant maps.

Problems for non-compact manifolds:

1. Any G manifold has a closed embedding in a finite dimensional representation space of G.

1'. Any G-bundle over X is the quotient of a bundle of the form $X \times V$ where V is a finite dimensional repr. of G.

2. If Y is a G-submanifold of X, then Y has a tubular neighborhood isomorphic to the normal bundle of Y in X.

Will assume these to be true in the following. If $Y \xrightarrow{i} X$ is an embedding, then its normal bundle is a G-bundle on Y. By an orientation of i, we mean a reduction of the structure group of ν_i from $O(d)$ ($d =$ codim i) to $SO(d)$ as a G-bundle (i.e. want a G-equivariant principal $SO(d)$ bundle yielding ν_i). (Later it will be necessary to treat the case where G acts on the orientation). Actually we are only interested in the stable normal bundle of the map, so that we can speak of oriented maps $f: X \to Y$. Precisely, there is a stable Picard category of G-equivariant oriented bundles and for G-manifolds, a relative cotangent complex formalism.

Now we wish to classify homology functors F on $\mathcal{V}_G$, i.e. something associating an object $F(X)$ of a category $\mathcal{A}$ to each object X of $\mathcal{V}_G$ and to each morphism $f: X \to Y$, $f_*: F(X) \to F(Y)$ and to each oriented proper map $f: X \to Y$, $f^*: F(Y) \to F(X)$, such that the functorality, homotopy, and transversality axioms hold.

Tranversality thm. false in $\mathcal{V}_G$. Example: Take an isolated fixed point P in X, then $P \rightrightarrows X$ can not be moved apart.

However if G acts freely on Z, then $f: Z \to Y$ can be moved transversally to $g: X \to Y$. Letting $Y' = Z \times Y$, graph

$$\begin{tikzcd}
 & X' \arrow[r] \arrow[d] & X \arrow[d] \\
Z \arrow[r] & Y' \arrow[r] & Y
\end{tikzcd}$$

factorization may assume f an embedding, hence that Y is the normal bundle of f, and consequently that G acts freely on Z, X, Y. Now by usual transversality thm. can move $f/G: Z/G \to Y/G$ transversal to X/G to Y/G and by covering homotopy theorem can lift this motion of f/G to an equivariant motion of f. The new f is then transversal to g because locally Z, X, Y are $Z/G \times G$, etc. Same argument works relatively to show that if on some closed G-stable set A of Z we have transversality, then we can leave f fixed on A, at least if A/G is nice (neighborhood deformation retract, I guess works) in Z/G.

$\Omega_q^G(X) =$ bordism classes of equivariant maps $Z \longrightarrow X$ where Z is a principal G bundle with base Z/G a compact oriented q-manifold.

$= \Omega_q((X \times P)/G)$ $\qquad P \underbrace{\text{principal } G\text{-bundle}}_{}$ universal

$= \varinjlim_{Q} \Omega_q((X \times Q)/G)$

where Q runs over the category of principal G-bundles which are manifolds and homotopy classes of equivariant maps, a **filtering category.** By transversality theorem we can define a Gysin homomorphism $f^*: \Omega_*^G(Y) \longrightarrow \Omega_{*+q}^G(X)$ if $f: X \to Y$ is of dimension q (i.e. $\dim f = \dim X/Y$).

Example: Let $1 \in \Omega_0^G(pt)$ be represented by the principal bundle $G \to pt$. Then if X is compact and oriented of dim q we get $1_X = \pi^* 1 \in \Omega_q^G(X)$ ($\pi: X \to pt$ is the canonical map), the fundamental class of X. It is represented by the map $pr_1: X \times G \longrightarrow X$. where $(X \times G)/G$ is identified with X via the map $(x,g) \longmapsto xg^{-1}$.

~~$\Omega_q^G(X,Y) =$ bordism classes of equivariant maps $Z \xrightarrow{(f,g)} X \times Y$ where Z is ,~~

~~?~~

Note that $\Omega_*^G(X)$ does not seem to have products in the strict sense. Thus if we ~~[illegible]~~ ~~[illegible]~~ try to define the product of $Z \longrightarrow X$ and $Z' \longrightarrow Y$ to be $Z \times Z' \longrightarrow X \times Y$, then ⓐ it is not immediately clear how $(Z \times Z')/G$ is to be oriented, although perhaps one can orient G and then pass from orientations of Z/G (resp. Z'/G) to orientations of Z, Z' hence $Z \times Z'$ and then down to an orientation of $(Z \times Z')/G$. (seems to require an orientation of the map $BG \longrightarrow B(G \times G)$) ⓑ there is no element $1 \in \Omega_*^G(pt)$ such that ~~[illegible]~~ $1 \times Z$ is equivalent to Z under the isom $X \times pt = X$. In effect the product goes from $\Omega_k^G(pt) \times \Omega_\ell^G(X) \longrightarrow \Omega_{k+\ell+g}^G(X)$ so 1 would have to be of degree $-g$ and there is no such element.

G-manifolds Z correspond to manifolds over the classifying site of G which are fibre bundles by the correspondance $Q \longmapsto (Q \times Z)/G$ which is a fibre bundle over Q/G with fibre Z. Therefore if X, Y are compact oriented G-manifolds of dimension x, y respectively, then

$$\mathrm{Hom}^q(X,Y) = \Omega_G^{x+q}(X \times Y) \qquad \text{(by analogy with } H^*)$$

$$= \Omega^{x+q}((X \times Y)/G) \qquad \text{if } G \text{ acts freely on } X \times Y.$$

Therefore in order to define the motive category we need to have a definition of equivariant cobordism theory.

Remark 1: In the notes of January 3; page 4 we defined different bordism groups for a G-manifold X using all compact Oriented G-manifolds, not just the G-free ones. It appears that this is the ~~fiber bordism~~ (in the sense of Shih) of the classifying space B_G.

2: If G, H are compact Lie groups, is $\mathrm{Hom}(G,H)^H \xrightarrow{\sim} [B_G, B_H]$? True for $H = S^1$, but this is somewhat special as $\pi_q(S^1) = 0 \quad q \geq 2$.

3: An obvious candidate for $\Omega^*_G(X)$ is $\Omega^*((X\times P)/G)$, however we would like a homomorphism $\Omega^*_G(X) \longrightarrow K_G(X)$ and a Conner-Floyd thm, and one knows that $K_G(X) \neq K((X\times P)/G)$ in general. Thus want a diagram

$$\begin{tikzcd}
\Omega^*_G(X) \arrow[r] \arrow[d] & \Omega^*((X\times P)/G) \arrow[d] \\
K_G(X) \arrow[r] & K((X\times P)/G)
\end{tikzcd}$$

and maybe even this should be cartesian, for then one could define equivariant Chern classes and produce the Conner-Floyd section.

Program: Try to translate everything into homology language.

~~Let X Y~~ Let $X \xrightarrow{f} Y$ be a map of smooth manifolds. To each open set $\boxed{U \text{ of } Y}$ consider $H_\delta(f^{-1}U)$. This is a ~~sheaf~~ ψ

Let ~~[crossed out]~~ X be a manifold over S and for each manifold U consider

$$U \longmapsto H_*(U \times_S X) = F_*(U)$$

this should be a cohomology theory on manifolds U over S provided X smooth over S. ~~But~~ Assume $S =$ point. Then

$$F_*(pt) = H_*(X)$$

Example:

Write down axioms which allow one to eliminate transversality + boundaries.

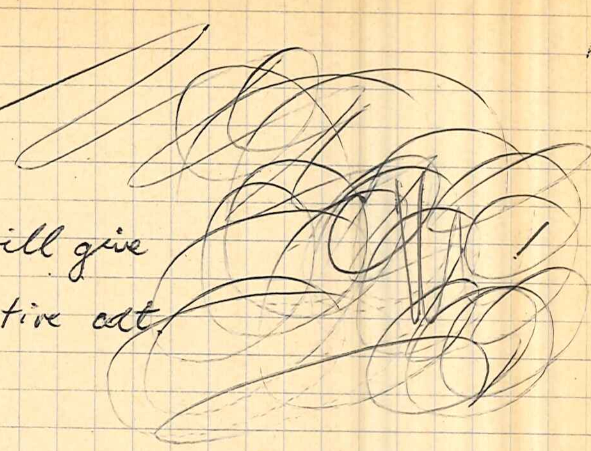
over a point we expect M-module spectra.

Conjecture: ① Suppose that F_* is a homology theory ~~with products~~ on $\mathcal{V}$. Then F_* is representable by an M-module spectrum. Gives us $D_+(pt.)$ at least

② ~~Then~~ Similarly $D_+(S)$ ~~is the category~~ should be the homology theories ~~with products~~ on $\mathcal{V}/S$.
add below

Problems

1. Equivariant cobordism theory
   model: $K_G(X)$ of Atiyah
   you want properties which will give
   $\Omega_G(X,Y)$ homo in motive cat.
   problem with transversality.



candidate $\Omega^*_G(X,Y)$ = bordism classes $Z \longrightarrow X \times Y$
with G acting freely on Z.

then $\Omega_{G*}(X) = \Omega_*((X \times E_G)_G)$.

Observe like if $X \times Y$ is G free, a kind of X, Y being transversal wrt G

and same for $*$.

Unfortunately it is not clear how to replace an equivariant $Z \longrightarrow X$ with one on which G acts freely.

Next idea: equivariant Chern classes should define a transformation from $K_G(X) \longrightarrow \Omega_G(X)$.

maybe how Conner constructs his section of the map

$\boxed{\Omega^0(X) \longrightarrow K^0(X)}$ This ~~really~~ should be fascinating

a Riemann-Roch-Conner thm. for $K(X)$ versus $\Omega(X)$.

2.

$\Omega^0(X) \longrightarrow K^0(X)$ has a section

i.e. get. $\Omega^0(BU) \longrightarrow K^0(BU)$

$\alpha \longmapsto id$

But α is a linear combination (possibly infinite) of $\Omega^*(pt)[c_1, \dots, \dots]$.
so you should be able to write down a (simple formula for α.

$\boxed{\text{basic problem:}}$

$$\begin{tikzcd} X & Y \arrow[d, "g"] \\ S \arrow[r, "f"'] & T \end{tikzcd}$$

~~If~~

Check carefully that if $\begin{tikzcd} Z \arrow[dr, "h"'] & X' \arrow[r, "f'"] \arrow[d, "g'"] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"] & S \end{tikzcd}$ are transversal, then

$$\mathrm{Hom}_X((Z), (X \times_S Y)) = \mathrm{Hom}_S((Z), (Y)).$$

From my point of view

it is obvious if Z transversal to Y over S since

then $$B(Z \times_X (X \times_S Y)) = B(Z \times_S Y).$$

From G's point of view

$$\mathrm{Hom}_X(g'_* \mathcal{O}_{X'}, h_* \mathcal{O}_Z) \overset{?}{=} \mathrm{Hom}_S(g_* \mathcal{O}_Y, \cancel{g_*} f_* h_* \mathcal{O}_Z)$$

to be true for all $h_* \mathcal{O}_Z$ seems to mean that

$$g'_* f'^* \mathcal{O}_Y = f^* g_* \mathcal{O}_Y$$

which is false by standard example

Example: $K(\mathbb{Z}, n) = \mathrm{Sym}_\infty(S^n)$. = set of all [redacted] 0-cycles $\sum_{i \in I} x_i$ of points on S^n almost all at ∞.

~~the nk-skeleton is $(S^n)^{\times k}/\Sigma(k)$ in the senses that~~ X

$$(S^n)^{\times(k-1)}/\Sigma(k-1) \hookrightarrow (S^n)^k/\Sigma(k) \longrightarrow S^{nk}/\Sigma(k)$$

where $\underbrace{S^n \wedge \cdots \wedge S^n}_{k} = S^{nk}$.

which shows that it isn't the nk-skeleton since

$$\tilde{H}_q\{S^{nk}/\Sigma(k)\} = \pi_q\{\bar{\mathbb{Z}}(S^{nk}/\Sigma(k))\}$$

$$= \pi_q\left[\mathrm{Sym}_k(\mathbb{Z}S^n)\right]$$

(pretending spaces behave like simplicial sets.)

Connectivity is $\sim n+2k$

is definitely not zero for all $q < nk$. ~~I was hoping to consider~~ ~~the~~ Hoping that given $f: X \longrightarrow K(\mathbb{Z}, n)$ X smooth, (compact) f generic then after moving f into $Y = (S^n)^k/\Sigma(k)$, then $f^{-1} Y_{sing}$ is a subvariety of X "representing" $f^*u \in H^n(X, \mathbb{Z})$. CONTRAVARIANT APPROACH.

Program:

I. Conjecture: $M(S)$ has objects X_Φ where $X \longrightarrow S$ is smooth and Φ is a family of supports of X relative to S (i.e. $\forall\, U$ in S get $\Phi(U)$ a family of supports in $X \times_S U$.)

Problem is to verify conjecture determining the maps from X_Φ to Y_Ψ and to show existence of $f^*, f^!, f_*, f_!$ with correct properties. At the moment I think I know how to define full subcategory of $M(S)$ consisting of objects $f_! \mathcal{O}_X$ for each $f: X \longrightarrow S$ (not nec. sm.)

II. Construction of motive cat. uses only ~~the~~ bordism ~~&~~ constructions and is independent of nature of orientation and of the cycles used. This means that $\exists$ axiomatic approach.

Problem: Isolate ~~[illegible]~~ properties of the cohomology theory that you need and then construct universal gadget.

III. Equivariant bordism theory, power operations.

---

① $H_*^\Phi(X)$

② What is $\pi_*(M \wedge X)$ where X is a pointed compact space? $\subset \mathbb{R}^n$.

③ axiomatic construction

Notes, January 10, 1968

Conner-Floyd section of $\Phi : \Omega^{ev}(X) \longrightarrow K^0(X)$:

Define $c_1 : \mathrm{Pic}(X) \longrightarrow \Omega^2(X)$. By splitting principle it extends to an additive homomorphism $K(X) \longrightarrow \Omega^2(X)$. Now note that if

$$\alpha(E) = rg\,E - [\mathbb{P}^1]\, c_1(E') \qquad rg : K(X) \to \Omega^0(X) \text{ rank.}$$

Thus $\Phi\alpha = id$ since ~~...~~ both sides are additive and coincide for line bundles.

note that $\Phi(c_t(E)) = (1+t)^{g(E)} \lambda_{-\frac{t}{1+t}}(E')$ by same argument

---

Conjectural formula $\quad c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2) - a\, c_1(L_1) c_2(L_2) \qquad a = [\mathbb{P}^1]$

has following agreeable consequence that if one defines

$$ch\, L = (1 - a c_1(L))^{-1/a}$$
$$= 1 + c_1(L) + \frac{1+a}{2!} c_1(L)^2 + \frac{(1+a)(1+2a)}{3!} c_1(L)^3 + \cdots$$

then $ch(L_1 \otimes L_2) = ch\, L_1 \cdot ch\, L_2$, so additive extension is a homomorphism from $K(X) \longrightarrow \Omega^{ev}(X) \otimes \mathbb{Q}$. Among other virtues

$a \to 1 \qquad \Phi(ch\, L) = L.$

$a \to 0$ from $\Omega^* \longrightarrow H^*$, then ~~...~~ $ch\, L \longmapsto$ usual character.

After conjectural formula is proved, we then face problem of a Riemann-Roch thm. with ch.

Actually a better character is given by

$$ch\, L = 1 - a c_1(L^{-1})$$

for if conjectural formula true, it follows that ch is a multiplicative section of Φ.

Cohomology of a blow-up

$Y \hookrightarrow X$ embedding, normal bundle V has a complex structure, $\tilde{X}$ the blow-up of X along Y, $\pi : \tilde{X} \to X$ the canonical map, $\tilde{Y} = \pi^{-1} Y$. Then $\tilde{Y} = \mathbb{P}V$ and $\pi : \tilde{X} - \tilde{Y} \xrightarrow{\sim} X - Y$.

$$\begin{tikzcd}
{} \arrow[r] & H^*(\tilde{X}, \tilde{Y}) \arrow[r] & H^*(\tilde{X}) \arrow[r] & H^*(\tilde{Y}) \arrow[r, "\delta"] & {} \\
{} \arrow[r] & H^*(X, Y) \arrow[r] \arrow[u, "\cong"'] & H^*(X) \arrow[r] \arrow[u] & H^*(Y) \arrow[r, "\delta"] \arrow[u] & {}
\end{tikzcd}$$

gives rise to Mayer-Vietoris

$$\cdots \longrightarrow H^*(X) \longrightarrow H^*(\tilde{X}) \oplus H^*(Y) \longrightarrow H^*(\tilde{Y}) \longrightarrow \cdots$$

$$H^*(\tilde{Y}) \cong H^*(Y)[\xi] / \left(\xi^n + c_1(V)\xi^{n-1} + \cdots + c_n(V) \right)$$

Also

$$\begin{tikzcd}
{} \arrow[r] & H^*_{\tilde{Y}}(\tilde{X}) \arrow[r] & H^*(\tilde{X}) \arrow[r] & H^*(\tilde{X} - \tilde{Y}) \\
{} \arrow[r] & H^*_Y(X) \arrow[r] \arrow[u] & H^*(X) \arrow[r] \arrow[u] & H^*(X - Y) \arrow[u, "\wr"']
\end{tikzcd}$$

gives rise to M-V

$$\longrightarrow H^*_Y(X) \longrightarrow H^*_{\tilde{Y}}(\tilde{X}) \oplus H^*(X) \longrightarrow H^*(\tilde{X}) \longrightarrow \cdots$$

$$H^{*+2n}_Y(X) \xleftarrow{\sim} H^*(Y)$$

What is map $H^*_Y(X) \longrightarrow H^*_{\tilde{Y}}(\tilde{X})$?

$$H^{*+2}_{\tilde{Y}}(\tilde{X}) \xleftarrow{\sim} H^*(\tilde{Y})$$

To calculate the map $H^*_Y(X) \longrightarrow H^*_{\tilde Y}(\tilde X)$ we may by excision assume that $X \simeq V$, ~~this special case~~ in which case

$$\tilde Y = \mathbb{P}V$$

$$\tilde X = \{(v, \ell) \mid \ell \in \mathbb{P}V = \tilde Y,\ v \in \ell\} = L \quad \text{subbundle whose}$$

sheaf of sections is $\mathcal{O}(-1)$. ~~In particular~~

$$\begin{array}{ccc} H(X) & \longrightarrow & H(\tilde X) \\ \downarrow\wr & & \downarrow\wr \\ H(Y) & \longrightarrow & H(\tilde Y) \end{array}$$

~~is injective so in this case we have an exact sequence~~

$$\cancel{0 \longrightarrow H^*_Y(X) \longrightarrow H^*_{\tilde Y}(\tilde X) \oplus H^*(X) \longrightarrow H^*(\tilde X) \longrightarrow 0}$$

$$H^*(X) = H^*(Y) \qquad H^*(\tilde X) = H^*(\tilde Y)$$

So we get a long exact sequence

$$\begin{array}{ccccc} H^*_Y(X) & \longrightarrow & H^*_{\tilde Y}(\tilde X) \oplus H^*(X) & \longrightarrow & H^*(\tilde X) \\ \uparrow\cong & & \uparrow\cong \quad \downarrow\cong & & \downarrow\cong \\ H^{*-2n}(Y) & & H^{*-2}(\tilde Y) \quad H^*(Y) & & H^*(\tilde Y) \end{array}$$

which one sees is in fact short exact since ~~second~~ map is onto. In effect

$$H^*_{\tilde Y}(\tilde X) \text{ has basis } \xi^i \eta \text{ over } H^*(Y) \qquad 0 \le i < n$$

$$\boxed{\text{won't work for } \Omega^* \text{ necessarily}}$$

$$\xi = c_1(L^{-1}) \in H^2(\tilde Y) \qquad \longmapsto \xi \in H^2(\tilde Y)$$

$$\eta = [\tilde Y] \in H^2_{\tilde Y}(\tilde X) \qquad \longmapsto -\xi \in H^2(\tilde X) = H^2(\tilde Y)$$

Thus middle has basis $\xi^i\eta$ $i<n$ and 1 over $H(Y)$, hence the kernel of 2nd map is ~~generated by~~ free over $H(Y)$ with basis element

$$\sum_{i=1}^{n} f^* c_{n-i}(V)\,\xi^{i-1}\eta \oplus c_n(V) \qquad f: \tilde X \to X.$$

On the other hand the map

$$H(Y) \longrightarrow H_Y(X) \longrightarrow H(X) \longrightarrow H(Y)$$

is $i^* i_*$ where $i : Y \to X$ is inclusion, hence $i^* i_*(y) = c_n(V)\cdot y$.

Thus first map given by

$$u \longmapsto \sum_{i=1}^{n} f^* c_{n-i}(V)\, \xi^{i-1} \eta \oplus c_n(V)$$

where $u \in H^{2n}_Y(X)$ is the Thom class. Conclude that

$$H(Y) \xrightarrow{\sim} H_Y(X) \longrightarrow H_{\tilde{Y}}(\tilde{X}) \xrightarrow{\sim} H(\tilde{Y})$$

is given by

$$1 \longmapsto \sum_{i=1}^{n} f^*(c_{n-i}(V))\, \xi^{i-1} = c_{n-1}(Q)$$

where Q is the quotient bundle on $\mathbb{P}V$.
$0 \to \mathcal{O}(-1) \to f^*\tilde{V} \to \tilde{Q} \to 0.$

This is formula valid for a general X, so conclude injective always.

$\therefore$

$$0 \longrightarrow H(Y) \xrightarrow{(f^*,\, i_*)} H(\tilde{Y}) \oplus H(X) \xrightarrow{f_* - g^*} H(\tilde{X}) \longrightarrow 0$$

$$\begin{tikzcd} \tilde{Y} \arrow[r, "j"] \arrow[d, "f"] & \tilde{X} \arrow[d, "g"] \\ Y \arrow[r, "i"] & X \end{tikzcd}$$

$g^* : H(X) \hookrightarrow H(\tilde{X})$ hence

$$0 \longrightarrow H(X) \xrightarrow{(g^*,\, i^*)} H(\tilde{X}) \oplus H(Y) \xrightarrow{j^* - f^*} H(\tilde{Y}) \longrightarrow 0$$

Additively as $H(Y)$ modules we have

$$H(\tilde{Y}) = \bigoplus_{i=0}^{n-1} H(Y)\xi^i \qquad \xi = c_1(\mathcal{O}(1))$$

$$H(\tilde{X}) = H(X) \oplus \bigoplus_{i=1}^{n-1} H(Y)\xi^{i-1}\eta \qquad \eta = j_*1$$

How to prove $\quad K_G(\mathbb{P}V) \simeq K_G(X)[T]/(\lambda_{-T}(V))$.

Start with $\pi_! : K_G(V^+) \longrightarrow K_G(X)$

($V^+ = V \cup \{\infty\}$ with proper/X supports)

ⓐ compatible with base change

ⓑ $\pi_! \pi^! = id$

Then by Atiyah's trick conclude $\pi^! \pi_! = id$

$$\begin{tikzcd} V \times_X V \arrow[r, "p_1"] \arrow[d, "p_2"'] & V \arrow[d, "\pi"] \\ V \arrow[r, "\pi"'] & X \end{tikzcd}$$

$$\pi^! \pi_! = p_{2!}\, p_1^! \qquad \text{by ⓐ}$$

but $\quad p_1^! = p_2^! \quad$ since $\quad p_1 \sim p_2$

$$= p_{2!}\, p_2^! = id \qquad \text{by ⓑ}.$$

Now using $\pi_!$ one defines $f_!$ in general for a proper oriented map having the good properties

Next step is to calculate $\quad f_! : K_G(\mathbb{P}V) \longrightarrow K_G(X)$
on element T^i, $T = [\mathcal{O}(1)]$; in particular one obtains a formula

$$\alpha \stackrel{?}{=} a_0 + a_1 T + \cdots + a_{n-1} T^{n-1}$$

where the $a_i(\alpha)$ are computed from $f_!(\xi^j \alpha)$ by a definite formula. To see if formula holds in general one may pull up to flag manifold since the map $f^!$ is injective. But now bundle splits.

Problem: Let $s: X \to L$ be a section of a line bundle and let $A = s^{-1}O$. Can you define $c_1(L)$ as an element of $H^2_A(X)$?

Recall an element of $H_A(X)$ is a triple

$$\left\{\begin{array}{ll} Z \longrightarrow X & \text{pr, or} \\ W \longrightarrow X-A & \text{pr, or} \\ \varphi: Z|X-A \xrightarrow{\sim} \partial W. & \end{array}\right.$$

How to construct such an element: Let $h: X \times I \xrightarrow{\text{proper}} L$ be such that
(i) $h_0 = s$
(ii) h_1 transversal to $i: X \to L$, zero section
(iii) h transversal to $X-A \xrightarrow{i} L$

Then set

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d, "h_1"] \\ X \arrow[r, "i"'] & L \end{tikzcd} \qquad \begin{tikzcd} W \arrow[r] \arrow[d] & X \times I \arrow[d, "h"] \\ X-A \arrow[r, "i"] & L \end{tikzcd}$$

Then Z proper $/X$, W proper$/X-A$ and $\partial W = \overset{\emptyset}{\overset{\|}{h_0^{-1}(O(X-A))}} \cup h_1^{-1}(O(X-A)) = Z|X-A$

Question: Can one always construct such a homotopy h?

The problem comes with condition (iii) since $X-A$ isn't closed.

How to construct such an element h: Choose an $\bar{h}$ satisfying (i) and (ii) and consider the set $\mathcal{H}$ of all h's with fixed $h_0 = s$ and $h_1 = \bar{h}_1$. For each compact set K_n

in $X - A$, the set $\mathcal{U}_n$ of all $h \in \mathcal{H}$ transversal to $X \to L$ on K_n is open and dense. Thus choosing an exhaustion of $X - A$ one finds $h \in \bigcap \mathcal{U}_n$.