§2. The K-theory associated to a ring. Let A be a ring with unit but not necessarily commutative. Let $GL(A)$ be the infinite general linear group (inductive limit of $GL_n A$) and let $E(A)$ be the subgroup generated by the elementary matrices (Bass [ ]). ~~[illegible] E(A) is~~ ~~perfect and~~ Let $BGL(A)$ be a classifying space for $GL(A)$ which is a pointed CW complex. Since $E(A) \subset GL(A) = \pi_1 BGL(A)$ is perfect we may kill it as in the preceding section obtaining a pointed CW ~~space~~ complex

$$BGL(A)^+ = BGL(A)/E(A)$$

unique up to homotopy type. ~~Define the~~ We define the algebraic K-groups of the ring A ~~K_iA [illegible] for $i \geq 0$~~ by setting

$$K_i A = \pi_i BGL(A)^+ \qquad i \geq 1$$

and ~~putting K_0A~~ taking $K_0 A$ to be the Grothendieck group of finitely-generated projective A-modules.

Since $E(A)$ is normal in $GL(A)$, 1.2(i) shows that $K_1 A = GL(A)/E(A)$, so our K_1 coincides with that of Bass [ ]. By 1.2(ii) there is a cartesian square

$$\begin{tikzcd} BE(A) \arrow[r, "f'"] \arrow[d] & BE(A)^+ \arrow[d, "g"] \\ BGL(A) \arrow[r, "f"'] & BGL(A)^+ \end{tikzcd}$$

where g is the universal covering map of $BGL(A)^+$ and where f' induces isomorphisms on homology. Hence ~~using~~ by the Hurewicz theorem

$$K_2 A = \pi_2\, BE(A)^+ = H_2\, BE(A)^+$$
$$= H_2\, BE(A)$$

showing that our K_2 coincides with the one defined by Milnor [ ].

The following ~~Next we consider the relation give an example~~ illustrates ~~illustrating~~ the relation of our K-groups to those of topological K-theory. Let $A = \underline{Hom}(X, \mathbb{R})$ be the ring of continuous real-valued functions on a compact space X where the (here and below) $\underline{Hom}$ denotes the function space with compact-open topology. Then $GL(A) = \varinjlim_n \underline{Hom}(X, GL_n \mathbb{R})$ is a topological group in a natural way (at least if we work in the category of compactly-generated spaces) and we let $B^{top} GL(A) = \varinjlim_n B^{top} \underline{Hom}(X, GL_n \mathbb{R})$ be its classifying space, e.g. as constructed by Segal [ ]. In analogy with the definition of $K_i A$ we define $K_i^{top} A$ for $i \geq 1$ to be the homotopy groups of a space $B^{top} GL(A)^+$ endowed with a universal arrow

$f : B^{top} GL(A) \longrightarrow B^{top} GL(A)^+$ killing $E(A)$, or more precisely the image of $E(A)$ in $\pi_1 B^{top} GL(A)$. But this image is zero since $E(A)$ is contained in the connected component of $GL(A)$, hence we can take f to be the identity and conclude that

$$K_i^{top} \underline{Hom}(X, \mathbb{R}) = \varinjlim_n \pi_{i-1} \underline{Hom}(X, \mathbb{R})$$

$$= KO^{-i}(X) \qquad i \geq 1.$$

Note that the identity map from $GL(A)$ with discrete topology to $GL(A)$ with the above topology induces a homomorphism

$$K_i^{\text{[redacted]}} \underline{Hom}(X, \mathbb{R}) \longrightarrow K_i^{top} \underline{Hom}(X, \mathbb{R}) = KO^{-i}(X)$$

from the algebraic to the topological K-groups. Similar things hold with $\mathbb{C}$ instead of $\mathbb{R}$.

Suppose now that A is a general ring, and set

$$\tilde{K}(X;A) = [X, BGL(A)^+]_0$$

for any pointed connected space X. The above example shows that $BGL(A)^+$ is the analogue in algebraic K-theory of the spaces BO and BU of topological K-theory, hence it is reasonable to think of an element of $\tilde{K}(X;A)$ as a <u>virtual</u> <u>vector</u> <u>bundle</u> for the ring A over X which is <u>reduced</u>, i.e. restricts over the basepoint to zero. We are now going to develop the properties of this algebraic K-functor following the example of topological ~~[illegible]~~ K-theory as closely as possible.

Let $\mathcal{P}(A)$ denote the additive category of finitely-generated projective (left) A-modules. By a <u>representations</u> of a group G over A we shall mean an object P of $\mathcal{P}(A)$ endowed with a linear action of G. By an A-<u>vector</u> <u>bundle</u> over a space X we mean a fibre bundle over X with A-module structures on the fibres which is locally isomorphic to $X \times P$ where P is an object of $\mathcal{P}(A)$ endowed with the discrete topology. For simplicity we suppose from now on that X is a pointed connected CW complex. Then associating to an A-vector bundle the natural action of $\pi_1 X$ on the fibre over the basepoint gives an equivalence of the category of A-vector bundles over X, with fibrewise A-linear maps for morphisms, with the category of representations of $\pi_1 X$ over A. So from now on we use this equivalence to identify A-vector bundles and representations of the fundamental group.

Call two representations E and E′ of G over A <u>stably isomorphic</u> if there are trivial representations P and P′, i.e. objects of $\mathcal{P}(A)$ with trivial G-action, such that $E \oplus P \simeq E' \oplus P'$.

Denote by $St(G;A)$ the set of stable isomorphism classes of representations of G and by $cl(E)$ the isomorphism class of E. $St(G;A)$ inherits an abelian monoid structure from the direct sum operation on representations. We call elements of $St(G;A)$ <u>stable representations</u> and elements of $St(\pi_1 X;A)$ <u>stable vector bundles</u> over X.

Let $E_n A$ be the subgroup of $GL_n A$ generated by elementary matrices. We claim that associating to a homomorphism of groups $G \to GL_n A$ the corresponding representation of G on A^n gives rise to a bijection

$$(1) \qquad \varinjlim_n \operatorname{Hom}_{gps.}(G, GL_n A)/E_n A \xrightarrow{\sim} St(G;A).$$

Indeed the map is surjective because given any representation E there is a trivial representation P such that $E \oplus P$ is a representation on a free A-module.

For injectivity suppose u, u' are two homomorphisms $G \longrightarrow GL_n A$ giving rise to stably isomorphic representations E and E'. We must show u and u' become conjugate in $GL_N A$ by an element of $E_N A$ for some $N \geq n$. It will suffice to show u and u' are conjugate by an element θ of $GL_N A$, because then they will be conjugate in $GL_{2N} A$ by $\theta \oplus \theta^{-1}$, which belongs to $E_{2N} A$ by the Whitehead lemma ([ ],). We are given that there exist trivial representations P and P' $E \oplus P$ and $E' \oplus P'$ are isomorphic. Adding a further trivial representation we can suppose P is free.

As E and E' are representations on A^n we have isomorphisms of A-modules $A^n \oplus P' \simeq A^n \oplus P \simeq A^m$ for some m, hence adding A^n to both P and P' yields isomorphism $E \oplus A^m \simeq E' \oplus A^m$, giving the desired element θ with $N = n+m$.

We now define a map

$$(2) \qquad \begin{aligned} St(\pi_1 X; A) &\longrightarrow [X, BGL(A)^+]_0 = \tilde{K}(X;A) \\ cl(E) &\longmapsto \eta[E] \end{aligned}$$

which should be thought of as the map associating to stable vector bundle the associated reduced virtual vector bundle. According to (1) an element $cl(E)$ of $St(\pi_1 X; A)$ determines a homomorphism $u: \pi_1 X \longrightarrow GL(A)$

which is unique up to conjugacy by elements of $E(A)$. Composing the map $X \longrightarrow BGL(A)$ in $\mathcal{H}_0$ associated to u with the canonical map $f : BGL(A) \longrightarrow BGL(A)^+$, which we recall is a quotient for the conjugation action of $E(A)$ on $BGL(A)$, we obtain a well-defined map $X \longrightarrow BGL(A)^+$ in $\mathcal{H}_0$ which will be denoted by $\eta[E]$.

Unlike topological K-theory where stable vector bundles and reduced virtual bundles are the same, at least over finte complexes, the map (2) in algebraic K-theory is not usually an isomorphism, e.g. X simply-connected. Instead we have the following universal property for the arrow

OKAY ~~necessary to add disjoint union of representable functors to get correct statement~~

(2). Denote by C_0 the category of pointed connected finite complexes and homotopy classes of basepoint-preserving maps, and call a functor $F: C^\circ \longrightarrow$ Sets representable if it is of the form $F(X) = [X, Z]_0$ for some pointed space Z.

Proposition 2.1: *The arrow (2) with X in C is a universal morphism of functors from $X \mapsto St(\pi_1 X; A)$ to a representable functor.*

Proof: Let A_5 be the alternating group on 5 letters and embed it in the natural way in $GL_5 A$. ~~The~~ Let N be the normal subgroup of $GL(A)$ generated by A_5.

Then N contains all even permutation matrices, hence $GL(A)/N$ is abelian since given $x, y \in GL(A)$ there is an even permutation matrix p such that x commutes with pyp^{-1}. Thus N contains $(GL(A), GL(A)) = E(A)$, hence $N = E(A)$ since $A_5 = (A_5, A_5) \subset E(A)$. Consequently $BGL(A)^+ = BGL(A)/A_5$.

~~If we take the classifying~~

If we take $BGL(A)$ to be the realization of the semi-simplicial set $\overline{W}(GL(A))$ (see [ ]) then BA_5 can be regarded as a sub-complex of

$BGL(A)$ in a natural way. Let Y_0 be the 2-skeleton of BA_5. Then $\pi_1 Y_0 = A_5$ is a simple non-abelian group so by attaching one 2-cell and one 3-cell we can construct an embedding $f': Y_0 \longrightarrow Y_0^+$ inducing an isomorphism on homology with Y_0^+ a finite simply-connected pointed complex, i.e. $Y_0^+ = Y_0/A_5$ in the notation of §1. Then we can take $BGL(A)^+ = BGL(A) \cup_{Y_0} Y_0^+$ because the inclusion map $f: BGL(A) \longrightarrow BGL(A)^+$ satisfies (i) and (ii)′ of 1.2. Let $\{Y_\nu\}$ be the lattice of finite subcomplexes of $BGL(A)$ containing Y_0 and set $Y_\nu^+ = Y_\nu \cup_{Y_0} Y_0^+$. Then $BGL(A) = \bigcup Y_\nu$ and $BGL(A)^+ = \bigcup Y_\nu^+$

so for any X in $\mathbf{C}$ we have

$$[X, BGL(A)^{\blacksquare}]_0 = \varinjlim_{\nu} [X, Y_\nu^{\blacksquare}]_0$$

$$[X, BGL(A)^{+}]_0 = \varinjlim_{\nu} [X, Y_\nu^{+}]_0 .$$

Using ( ) we have

$$St(\pi_1 X; A) = \mathrm{Hom}_{gps.}(\pi_1 X, GL(A))/E(A)$$

$$= [X, BGL(A)]_0 / E(A).$$

Therefore if Z is a pointed space

$$\mathrm{Hom}(St(\pi_1 ?; A), [?, Z]_0)$$

$$= \mathrm{Hom}([?, BGL(A)]_0/E(A), [?, Z]_0)$$

$$= \mathrm{Hom}([?, BGL(A)]_0/A_5, [?, Z]_0)$$

$$= \varprojlim_{\nu} [Y_\nu, Z]_0^{A_5}$$

$$= \varprojlim_{\nu} [Y_\nu^{+}, Z]_0$$

$$= \mathrm{Hom}([?, BGL(A)^{+}]_0, [?, Z]_0)$$

where Hom denotes morphisms in the category of functors from $\mathcal{C}^{o}$ to Sets. The proposition follows.

We use this proposition to extend operations on stable representations to operations on $\tilde{K}(X, A)$. For example if $u: A \to A'$ is a ring homomorphism, base extension by u : $E \mapsto A' \otimes_A E$ gives rise to a natural transformation $u^*: St(\pi_1 ?; A) \longrightarrow St(\pi_1 ?; A')$,

~~Composing with the canonical map $St(\pi_1 ?, A') \longrightarrow \tilde{K}(?; A')$ we obtain a natural transformation from $St(\pi_1 ?; A)$ to a representable functor, which by 2.1 is obtained from a unique natural transformation~~

~~$u^* : \tilde{K}(?; A) \longrightarrow \tilde{K}(?; A')$~~

and by 2.1 there is a unique natural transformation $u^*: \tilde{K}(?;A) \longrightarrow \tilde{K}(?;A')$ such that the square

$$\begin{tikzcd}
St(\pi_1 ?;A) \arrow[r] \arrow[d] & St(\pi_1 ?;A') \arrow[d] \\
\tilde{K}(?;A) \arrow[r] & \tilde{K}(?;A')
\end{tikzcd}$$

commutes. Similarly if A' is a finitely-generated projective A-module there is a "restriction of scalars" map

$$u_*: \tilde{K}(?;A') \longrightarrow \tilde{K}(?;A).$$

~~To handle binary operations we note that the products of the maps ( ) for~~

Applying 2.1 to the product $A \times A'$ of two rings we see that

$$St(\pi_1 ?; A) \times St(\pi_1 ?; A') \longrightarrow \tilde{K}(?, A) \times \tilde{K}(?, A')$$

is a universal map to a representable functor on $\mathcal{C}$, because this arrow is isomorphic to the arrow (2) for $A \times A'$ since there is an isomorphism in $\mathcal{H}_0$

$$\begin{aligned} BGL(A \times A')^+ &\simeq \{BGL(A) \times BGL(A')\}^+ \\ &\simeq BGL(A)^+ \times BGL(A')^+ \end{aligned}$$

by 1.5. Consequently any binary, tertiary, etc. operation on stable bundles extends uniquely to the $\tilde{K}$-functor. For example the sum operation on $St(\pi_1 ?; A)$ extends to define an abelian monoid structure on

$\tilde{K}(?;A)$, the associativity (etc.) of the extension resulting from the uniqueness. But one sees quite easily by induction on the number of cells of the finite complex X that the monoid $\tilde{K}(X;A)$ is in fact an abelian group, so we have proved

<u>Proposition</u> 2.2: <u>There is a unique abelian group structure on</u> the functor $\tilde{K}(?;A)$ <u>such that the canonical</u> arrow $St(\pi_1 ?;A) \longrightarrow \tilde{K}(?;A)$ <u>is a homomorphism of monoids.</u>

Another way of saying that the functor $\tilde{K}(?;A)$ has a monoid structure is to say that $BGL(A)^+$ is a weak H-space. One consequence is that $BGL(A)^+$ is a simple space (in the sense that) the fundamental group acts trivially on $[X, BGL(A)^+]_0$ for all X in $\mathcal{C}$, or equivalently that

$$[X, BGL(A)^+]_0 = [X, BGL(A)^+].$$

Another consequence is a formula for the groups $K_i A \otimes \mathbb{Q}$ in terms of the rational homology of $GL(A)$.

Proposition 2.3: *The Hurewicz homomorphism for $BGL(A)^+$ ~~[illegible]~~ induces an isomorphism of $K_i A \otimes \mathbb{Q}$ with the subspace of primitive elements ~~[illegible]~~ of degree i of the Hopf algebra* $H_*(BGL(A), \mathbb{Q})$.

~~[illegible]~~

~~[illegible]~~

~~[illegible] This would follow from [illegible] theorem of Milnor and Moore ([ ], appendix) [illegible]~~

This follows from a theorem of Milnor and Moore ([ ], appendix), or more precisely from their argument which one can check works for weak H-spaces.

<u>Remark</u> 2.4: Actually $BGL(A)^+$ is a homotopy commutative and associative H-space, in fact it is an infinite loop space as we shall prove later by relating it to Segal's theory [ ]. On a more elementary level one can define the product on $BGL(A)^+$ to be the one induced by the embedding of $GL(A) \times GL(A)$ into $GL(A)$ which one obtains from enumerating $\mathbb{N}^+ \amalg \mathbb{N}^+$, where $\mathbb{N}^+$ is the

set of positive integers. To prove the unitarity, associativity, and commutativity one can the following lemma which we state without proof.

~~Lemma: Let $u: BGL(A)^+ \to BGL(A)^+$ be the map in $\mathcal{H}_0$ induced by the embedding of $GL(A)$ into $GL(A)$ obtained from an injection $\mathbb{N}^+ \to \mathbb{N}^+$. Then~~

<u>Lemma</u>: <u>Let</u> $u: GL(A) \longrightarrow GL(A)$ <u>be the embedding</u> <u>associated to an injection of</u> $\mathbb{N}^+$ <u>into</u> $\mathbb{N}^+$. <u>Then the triangle in</u> $\mathcal{H}_0$

$$\begin{tikzcd}
BGL(A) \arrow[dr, "f"] \arrow[d, "B(u)"'] & \\
BGL(A) \arrow[r, "f"'] & BGL(A)^+
\end{tikzcd}$$

<u>is commutative</u>.