second part

§1. The groups $K_i(A)$

$E(A)$ perfect $\triangleleft\ GL(A) = \pi_1 BGL(A)$

hence $\exists$ acyclic $f: BGL(A) \longrightarrow BGL(A)^+$

with $\operatorname{Ker} \pi_1(f) = E(A)$. f unique up to

canon. isom. in homot. cat of ptd. spaces

~~We first show agrees with Bass. M.~~

Defn: $K_i(A) = \pi_i(BGL(A)^+)$ for $i \geq 1$.

~~We begin by showing this agrees with Bass-Milnor~~

Prop: $K_1 A =$

$K_2 A =$

Recall construction of $BGL(A)^+$:

$$\begin{tikzcd} BE(A) \arrow[r, "f'"] \arrow[d, "p'"'] & BE(A)^+ \arrow[d, "p"] \\ BGL(A) \arrow[r, "f"'] & BGL(A)^+ \end{tikzcd}$$

p' covering space. ~~other ways~~

$BE(A)^+$ obtained from $BE(A)$ by attaching

2 and 3 cells to kill $E(A)$ without changing

homology. ~~Rest of~~ Diagram cocartesian, so by

van Kampen

~~Recall map has~~ ~~$K_1(A)$~~ $\pi_1(BGL(A)^+) = GL(A)/E(A)$

Claim ~~$BE(A)$~~ canonical map $BE(A)^+ \longrightarrow$

$(BGL(A)^+)^\sim$ heq. whence ~~on replacing~~ we obtain

~~$BE(A)$~~

a cartesian square

$$\begin{tikzcd} BE(A) \arrow[r, "f'"] \arrow[d, "p'"'] & (BGL(A)^+)^\sim \arrow[d, "p"] \\ BGL(A) \arrow[r, "f"'] & BGL(A)^+ \end{tikzcd}$$

where vertical maps are principal coverings with group $GL(A)/E(A)$, such that f' induces isomo. on homology.

$\therefore$

$$\pi_2 BGL(A)^+ = \pi_2 (BGL(A)^+)^\sim$$
$$= H_2((BGL(A)^+)^\sim$$
$$= H_2(BE(A)).$$

Dror's approach to the K-groups.

~~Start with $X_0 = BGL(A)$ and~~ Construct a Postnikov system

$$X_2 \longrightarrow X_1 \longrightarrow X_0$$

~~as follows~~ with X_n n-acyclic:

$$\tilde{H}_i X_n = 0 \qquad i < n$$

as follows. Take $X_0 = BGL(A)$. ~~Given X_n~~ Having obtained X_{n-1}, we have

$$H^n(X_{n-1}, M) = \mathrm{Hom}_{\mathbb{Z}}(H_n X_{n-1}, M)$$

for all abelian groups M; in part, exists canonical class with $M = H_{n-1} M$, hence a canonical map up to homotopy

$$X_{n-1} \longrightarrow EM(H_n X_{n-1}, n).$$

Take X_n to be fibre of this map. Then

$$X_0 = BGL(A)$$
$$X_1 = BE(A)$$
$$X_2 = BSt(A)$$

where $St(A)$ is the Steinberg group. The limit

$$X_\infty = \varprojlim X_n$$

is acyclic and

$$\begin{cases} \pi_1 X_\infty = St(A) \\ \pi_n X_\infty = H_{n+1} X_n \qquad n \geq 2 \end{cases}$$

~~We may identify $BGL(A)^+$ as the cofibre of the map $X_\infty \longrightarrow BGL(A)$~~

We may identify the cofibre of the map $X_\infty \longrightarrow BGL(A)$ with $BGL(A)^+$. We know that X_∞ is then homotopy equivalent to the fibre of the map $f: BGL(A) \longrightarrow BGL(A)^+$, hence

$$\pi_n X_\infty = K_{n+1}(A) \qquad n \geq 2.$$

Logic: Let $f: X_0 \longrightarrow Z$ be cofibre of $X_\infty \to X$. Then f is acyclic with $\operatorname{Ker} \pi_1(f) = \operatorname{Im}\{St(A) \to GL(A)\} = E(A)$, so we know $Z = BGL(A)^+$. But also know that X_∞ is homot equiv. to fibre of f, hence long exact homotopy sequence gives

$$K_{n+1}(A) \xrightarrow{\sim} \pi_n X_\infty \qquad n \geq 2$$

$$0 \longrightarrow K_2(A) \longrightarrow St(A) \longrightarrow GL(A) \longrightarrow K_1(A) \to 0$$

~~All patterns~~ Therefore

$$K_n A \simeq H_n X_{n-1} \qquad \text{for } n \geq 1$$

---

present scheme:

§1. K-groups

~~E(A)~~ Defn.

Show ~~eqs~~ agrees Bass-Milnor

Prop. 1: $K_1 A = GL(A)/E(A)$

$K_2 A = H_2 \cancel{B}E(A)$

In the proof you recall construction

Dror's definition of ~~formula~~ the K-groups. Start by describing system

$$\longrightarrow X_1 \longrightarrow X_0 = BGL(A)$$

then Dror's formula

$$K_n A = H_n X_{n-1}$$

probably necessary to put in why process goes on + X_0, $X_1 = B\cancel{E}E(A)$, $X_2 = BSt(A)$.

§1. <u>Acyclic maps.</u>

§2. ~~[illegible]~~

<u>Perfect fundamental groups</u>

Here work with pointed conn. CW cxs.

(a) Observe $H_1(X) = 0 \Longleftrightarrow \pi_1(X)$ perfect.

<u>Prop 1</u>: If $H_1(X)=0$, $\exists$ map $f : X \to Y$ inducing isom. on homology with $\pi_1 Y = 0$.

<u>Proof</u>: Attach cells.

<u>Prop 2</u>: If $f : X \to Y$ as above, then ~~[illegible]~~ obstruction theory.

<u>Mention</u>: Dror & Bousfield-Kan constructions

Remark: <u>about R-perfect.</u>

---

§3. Classification of acyclic maps with fixed source.

~~So therefore we operate~~

acyclic maps.

$f : X \longrightarrow Y$

(i) for all local coefficient systems L on Y

Claim $\Updownarrow$ $\quad H_*(X, f^*L) \xrightarrow{\sim} H_*(Y, L)$

(ii) homotopy fibres of f are acyclic.

may assume Y connected
may assume f fibration

$$E^2_{pq} = H_p\left(Y, \mathcal{H}_q \cancel{(F, f^*L)}\right) \Longrightarrow H_{p+q}(X, f^*L)$$

$y \mapsto H_q(f^{-1}\{y\}, f^*L)$

~~assume~~ as f^*L const on $f^{-1}\{y\}$

$(ii) \Longrightarrow$

$$H_q(f^{-1}\{y\}, f^*L) = \begin{cases} L_y & q = 0 \\ 0 & q > 0 \end{cases}$$

so spec. seq. deg. $\Longrightarrow$ (i) for any L' on X

$$E^2_{pq} = H_p(Y, y \mapsto H_q(f^{-1}(y), L'))$$

$$\Longrightarrow H_{p+q}(X, L').$$

Now take

$$L = f^*L$$

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "p'"'] & \tilde{Y} \arrow[d, "p"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

universal covering

cartesian

$$H_*(\tilde{Y}, A) = H_*(Y, p_! A)$$

$$H_*(\tilde{X}, A) = H_*(X, p'_! A)$$

$$p'_! A = f^* p_! A$$

so hypothesis (i) $\Longrightarrow$ $H_*(\tilde{X}, A) \xrightarrow{\sim} H_*(\tilde{Y}, A)$

for all abelian groups A.

But $\pi_1 \tilde{Y} = 0$ so comparison thm. $\Longrightarrow$ fiber of $\tilde{f}$

As f and $\tilde{f}$ have same fibres get (ii).

Seems desirable to add

(i)′ If $p: \tilde{Y} \to Y$ is a universal covering of Y and $p' : \tilde{X} \to X$ induced covering, then $\tilde{f} : \tilde{X} \longrightarrow \tilde{Y}$ induces isom. on homology.

Corollary 1: Acyclic maps closed under

composition
products
homotopy base change

~~Now suppose f~~

Classification of acyclic maps with X fixed.

<u>**Assertion:**</u> X conn. with basepoint

~~(i) N perfect $\triangleleft\ \pi_1 X$, $\exists$ acyclic map $f: X \longrightarrow Y \ni \pi_1(f): \pi_1 X/N \xrightarrow{\simeq} Y$.~~

(i) $f: X \longrightarrow Y$ perfect $\Longrightarrow N = \operatorname{Ker} \pi_1(f)$ is perfect $\triangleleft$.

(ii) Given N perfect $\triangleleft\ \pi_1 X$, $\exists$ acyclic $f: X \longrightarrow Y \ni \operatorname{Ker} \pi_1(f) = N$

(iii) Universal property:

$$[Y, Z]_*^{(0)} \xrightarrow{\simeq} \{ g \in [X, Z] \mid \pi_1(g)(N) = 0 \}.$$

<u>Proof</u>: (i)

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "i"'] & \tilde{Y} \arrow[d, "p"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

$$\begin{tikzcd}
\pi_1 F \arrow[r] \arrow[d, "\cong"] & \pi_1 \tilde{X} \arrow[r] \arrow[d] & 0 \arrow[d] \\
\pi_1 F \arrow[r] & \pi_1 X \arrow[r] & \pi_1 Y
\end{tikzcd}$$

shows

$$\pi_1 \tilde{X} \twoheadrightarrow N$$

But

$$\begin{tikzcd}
H_1(\tilde{X}, \mathbb{Z}) \arrow[r, "\sim"] \arrow[d, equal] & H_1(\tilde{Y}, \mathbb{Z}) = 0 \\
\pi_1(\tilde{X})_{ab} &
\end{tikzcd}$$

and $N_{ab} = 0$.

(ii) ~~first suppose~~

(iii) obstruction theory. ~~May assume f inclusion cofibration~~

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & Z \\
Y \arrow[ur] &
\end{tikzcd}$$

May assume f cofibration.

~~Write Given~~ Uniqueness: Given

$u, v : Y \to Z \Rightarrow uf \simeq vf$

(ii) First suppose $\pi_1(X)$ perfect. Then construct $f: X \longrightarrow Y$ inducing isom on H_* with $\pi_1(Y) = 0$. This proves existence here. Then in general

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

cocartesian to get existence of f

(iii) First use obstruction theory to do when $\pi_1 Y' = 0$.

Now generalize to cocartesian square.

Next

§2. H-space structure

Start with $S \sqcup S \simeq S$ yields

$$GL(A) \times GL(A) \longrightarrow GL(A)$$

carries $E(A) \times E(A)$ into $E(A)$ clearly.

$$\begin{tikzcd}
BGL(A) \times BGL(A) \arrow[r] \arrow[d, "f \times f"] & BGL(A) \arrow[d, "f"] \\
BGL(A)^+ \times BGL(A)^+ \arrow[r, dashed, "\mu"'] & BGL(A)^+
\end{tikzcd}$$

μ exists because $f \times f$ is acyclic + universal property.

Prop: $u: S \hookrightarrow S$ injection induces homo:

$$GL(A) \longrightarrow GL(A)$$

The induced map

$$BGL(A)^+ \dashrightarrow BGL(A)^+$$

is homotopic (preserving basepoints) to identity.

Proof: ~~Must show u induces isomorphism~~

1) $BGL(A)^+$ simple

2) u induces isomorphisms on homology

So by Whitehead theorem, homotopy equivalence.

§2. ~~BGL(A)~~ The H-space structure on $BGL(A)^+$.

~~simplicity of $BGL(A)^+$~~.

$S = \mathbb{N} - \{0\}$. $u : S \hookrightarrow S$ induces

$$\begin{tikzcd} BGL(A) \arrow[r] \arrow[d, "f"'] & BGL(A) \arrow[d, "f"] \\ BGL(A) \arrow[r, dashed, "u^+"'] & BGL(A)^+ \end{tikzcd}$$

u^+ exists + unique up to a homot. pres. basepoints.

Prop: u^+ homotopic to identity

Proof: We first show u^+ is a homotopy equivalence. Claim u induces an isom on $H_*(BE(A))$. Indeed

$$H_*(BE(A)) = \lim H_*(BE_n(A))$$

~~and~~ so ~~you~~ suffices to show the two homos from $E_n(A) \longrightarrow E(A)$ ~~given by indu~~ given by inclusion i and ui have the same effect on homology. But recall that even permutation matrices belong to $E_n(A)$, hence homos. are conjugate in $E(A)$, so done. Similarly u induces ~~identity~~ on $H_*(BGL(A))$, in particular on $K_1 A$.

Follows that u induces isom on homology of $BE(A)^+$, and since latter 1-conn. it is a heq by Whitehead thm. Thus u induces heq on

__Defn:__ $\tilde{R}(X, A) = [X, BGL(A)^+]$.

Suppose to simplify that X is connected. Then

$$St(X, A) = \varinjlim_n [X, BGL_n(A)]$$

maps to $[X, BGL(A)]$ which in turn maps to $[X, BGL(A)^+]$, hence we obtain a canonical map

$$\tau : St(X, A) \longrightarrow [X, BGL(A)^+].$$

__Lemma:__ τ is a monoid homomorphism. ~~for the abel.~~

__Proof:__ Choose $\mathbb{N}' \sqcup \mathbb{N}' \longrightarrow \mathbb{N}'$ so that $\{1, \ldots, m\} \cup \{1', \ldots, n\} \simeq \{1, \ldots, m+n\}$

$$\begin{tikzcd}
BGL_m A \times BGL_n A \arrow[r] \arrow[d, "i_m \times i_n"] & BGL_{m+n} A \arrow[d, "i_{m+n}"] \\
BGL(A) \times BGL(A) \arrow[r] \arrow[d, "f \times f"] & BGL(A) \arrow[d, "f"] \\
BGL(A)^+ \times BGL(A)^+ \arrow[r, "\mu"'] & BGL(A)^+
\end{tikzcd}$$

One possibility to use ~~earlier tech~~ fact that H-space structure obtained from any way

$$\mathbb{N}' \sqcup \mathbb{N}' \longrightarrow \mathbb{N}'$$

univ. covering ~~A~~ $BGL(A)^+$ ~~…~~ and on its fund. group ~~$K_1 A$~~ so it is a heq as claimed.

~~Thus~~ Clear that ~~$u \mapsto u \epsilon$~~

$$\mathrm{Inj}(S,S) \longrightarrow \mathrm{Aut}_{\mathcal{H}_0}(BGL(A)^+)$$

is a homom. of

<u>Lemma</u>: group comp. of $\mathrm{Inj}(S,S)$ is trivial.

Choose

Discuss products very carefully
want [illegible] ultimately

(1) $$K_i A \otimes K_j B \longrightarrow K_{i+j}(A \otimes_{\mathbb{Z}} B)$$

enjoying habitual anti-commutativity properties.

This will follow from pairings

(2) $$\tilde{K}_{\blacksquare}(X, A) \otimes \tilde{K}(Y, B) \longrightarrow \tilde{K}(X \wedge Y, A \otimes B)$$

for X, Y pointed spaces, with appropriate comm. properties:

i) assoc. for X, Y, Z, A, B, C

ii) comm.

$$\begin{tikzcd}
\tilde{K}(X, A) \otimes \tilde{K}(Y, B) \arrow[r] \arrow[d] & \tilde{K}(X \wedge Y, A \otimes B) \arrow[d, "t"', "(-1)^{ij}"] \\
\tilde{K}(Y, B) \otimes \tilde{K}(X, A) \arrow[r] & \tilde{K}(Y \wedge X, B \otimes A)
\end{tikzcd}$$

$S^i \wedge S^j = S^{i+j}$

$S^j \wedge S^i = S^{i+j}$

$$\begin{tikzcd}
K_i(A) \otimes K_j(B) \arrow[r] & K_{i+j}(A \otimes B) \arrow[d, "(-1)^{ij} T_*"] \\
 & K_{i+j}(B \otimes A)
\end{tikzcd}$$

Things to check

$$R(G, A \times B) = R(G, A) \times R(G, B)$$

$$\tilde{K}(X; A \times B) = \tilde{K}(X; A) \times \tilde{K}(X; B)$$

Important idea possibly: Relate the trace structures on $K(X, A)$ in X and in A. Thus if E is a representation over A, then

$$E^{\oplus p}$$

is a representation over $A^{\oplus p}$ with $\mathbb{Z}/p\mathbb{Z}$ action

Splitting theorem:

$\boxed{\text{Thm: If } 0 \to E' \to E \to E'' \to 0 \text{ is an exact sequence of repns., then } [E] = [E'] + [E''] \text{ in } K(X, A)}$

~~Proof: Enough to worry about stable representations.~~

$$\boxed{\cancel{St(X; A) = \varinjlim_n [X, BGL_n(A)]}}.$$

Can assume X connected, to show image same in $\widetilde{K}(X, A)$. ~~Start~~ ~~with~~ Can form limit group

$$\varinjlim GL_{m,n}(A) = GL^{(2)}(A)$$

then point is that the elementary subgroup is perfect + = commutator group.

$$BGL^{(2)}(A)^+ \xrightarrow{\sim} BGL(A)^+ \times BGL(A)^+$$

Logical steps an exact sequence of representations up to stable isom map is a map

$$X \longrightarrow BGL^{(2)}(A)$$

Then have two maps

$$GL^{(2)}(A) \rightrightarrows GL(A)$$

corresponding to repn. $E' \oplus E''$ and E.

Idea: Start with $\mathcal{A}^{(2)}$ = category of exact sequences. Prove

$$St(X, \mathcal{A}^{(2)}) = \lim_{m,n} [X, BGL_{m,n}(A)]$$

$$= [X, BGL^{(2)}(A)].$$

$$GL^{(2)}(A) \overset{\text{defn.}}{=} \lim GL_{m,n} A$$

$$E_{m,n}(A) \overset{\text{defn.}}{=} \text{subgroup of } GL_{m,n}A \text{ gen. by elementary matrices}$$

Claim $E_{m,n}(A) = (E_m A \times E_n A) \tilde{\times} \mathrm{Hom}(A^m, A^n)$ perfect for $m,n \geq 3$

$$E^{(2)}(A) \overset{\text{defn.}}{=} \lim E_{m,n}(A)$$

Claim $E^{(2)}(A)$ perfect $= (GL^{(2)}(A), GL^{(2)}(A))$.

Can form

$BGL^{(2)}(A)^{+}$.

Claim this simple

Notation:

defn | $GL_{m,n}(A) \subset \mathrm{Aut}(A^m \oplus A^n)$ preserving A^m

$E_{m,n}(A)$ subgroup gen. by ~~elem~~ those $1 + a e_{ij}$ in $GL_{m,n}(A)$.

Claim $\quad GL_{m,n}\{A\} = (GL_m A \times GL_n A) \tilde{\times} \mathrm{Hom}_A(A^n, A^m)$

$E_{m,n}(A) = (\qquad) \tilde{\times}$ — —

~~Thus~~ Claim $\quad E_{m,n} A$ perfect if $m, n \geq 3$

$GL^{(2)}A = \lim$

$E^{(2)}A = \lim$

Claim $\quad E^{(2)}A = (GL^{(2)}A, GL^{(2)}A)$

Can form

$BGL^{(2)}A^+$. Claim $\boxed{\text{simple}}$.

And there are maps

$$BGL(A)^+ \times BGL(A)^+ \;\substack{\xleftarrow{\;j\;} \\ \xrightarrow[\;i\;]{}}\; BGL^{(2)}(A)^+$$

such that $ji \simeq id$ (preserving points)

<u>Theorem</u>: j, i are homotopy equivalences

<u>Proof</u>: They induce isoms on homology and both spaces simple, so can apply Whitehead thm.

~~Next point is to discuss~~

Suppose

$$0 \to E' \to E \to E'' \to 0$$

is an exact sequence of ~~representation~~ bundles over X. Then get a map

$$[X, BGL^{(2)}(A)^+]$$

such that j

$$0 \to p^2V \to V \xrightarrow{p^2} V \to V/p^2V \to 0$$

$p^2V = pV$ $\qquad$ $pV = p^2V$

i.e. pV is p-divisible

$p(px) = 0 \Rightarrow px = 0$

$\Rightarrow pV$ is p-torsion

$$\tilde{K}(X, A) = [X, BGL(A)^+]$$

Given E over X, to define an element of $\tilde{K}(X)$. Assume X conn. Then get

$$f : X \longrightarrow BGL_n(A)$$

$\ni$ $f^*(E_n)$ stably isom. to E. essentially unique. Compose with

$$BGL_n(A) \xrightarrow{i_n} BGL(A) \longrightarrow BGL(A)^+.$$

This defines a natural transf

$$\gamma : St(X, A) \longrightarrow \tilde{K}(X, A).$$

Claim $\boxed{\gamma(E) = \gamma(E') + \gamma(E'').}$

$GL_{m,n}(A)$ autos of

$$0 \longrightarrow A^m \longrightarrow A^m \oplus A^n \longrightarrow A^n \longrightarrow 0$$

Then this gives rise to a canonical exact sequence of bundles on $BGL_{m,n}(A)$. Now if

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

is any exact sequence of bundles over X say connected, then by adding trivial bundles can suppose stalks of E', E'' are isom. to A^n, whence the exact sequence is induced

Let $\{m_a, m_a'\}$ be a non-zero solution of this congruence such that $\sum (m_a + 2m_a')$ is minimal. Clearly ~~It is clear that~~ $m_a' = 0$ for all a. If $m_b \geq p$ for some b, then by replacing m_b by $m_b - p$ and m_{b+1} by $m_{b+1}+1$ (or m_0 by $m_0 + 1$ if $b = d-1$) and keeping the others the same, we would get a new solution, contradicting minimality. Thus $m_a < p$ for $0 \leq a < d$, so using the uniqueness of the p-adic expansion of a natural number, we see that the minimal non-zero solution is $m_a = 1$, $m_a' = 0$ for all a. The minimal degree is $\sum (m_a + m_a') = d$ showing that $H_t(N)$ ~~has no non-trivial~~ does not contain the trivial repn for $0 < t < d$. The proof of the lemma is complete.

$$\begin{tikzcd}
 & BGL^{(2)}(A)^+ \arrow[r] & (BGL(A)^+)^2 \\
X \arrow[ur, "\sigma(E)"] \arrow[urr, bend right, "(\sigma E', \sigma E'')"'] & &
\end{tikzcd}$$

so what becomes difficult is map

$GL^{(2)}(A)$.

§1. Killing a perfect subgroup of the fundamental group. Let $f: X \to Y$ be a map of pointed connected CW complexes and form the cartesian square

$X^{(1)}$ primes

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

where Y' is the universal covering of Y and X' is the induced principal covering of X with group $\pi_1 Y$. We assume that f' induces an isomorphism on integral homology, or equivalently that f induces isomorphisms on homology with coefficients in any $\pi_1 Y$-module. Then X' is connected, hence the homomorphism $\pi_1(f): \pi_1 X \longrightarrow \pi_1 Y$ is surjective. Its kernel N is the fundamental group of X', hence N is perfect, i.e. equal to its commutator subgroup, because by Poincaré's theorem $N^{ab} = H_1 X' = H_1 Y' = 0$.

$\pi_{(1)}$ one

If Z is a pointed space, we denote by $\underline{\mathrm{Hom}}(X,Z)_0$ the space of basepoint-preserving maps from X to Z and by $[X,Z]_0$ the homotopy classes of these maps.

Proposition 1.1: *The map $f^*: \underline{\mathrm{Hom}}(Y,Z)_0 \to \underline{\mathrm{Hom}}(X,Z)_0$ is a weak homotopy equivalence of the former space with the subspace of the latter consisting of those $g: X \to Z$ such that $\pi_1(g)(N) = 0$. In particular*

$$[Y,Z]_0 \xrightarrow{\sim} \{\alpha \in [X,Z]_0 \mid \pi_1(\alpha)(N) = 0\}.$$

We may assume f is the inclusion of X as a subcomplex of Y, in which case f^* is a fibration. To see that the fibres of f^* are weakly contractible it suffices to show that the inclusion

$$(X \times I) \cup (Y \times \dot{I}) \longrightarrow Y \times I$$

is a homotopy equivalence. ~~The fundamental group of the former space is~~

$$\cancel{\pi_1 Y *_{\pi_1 X} \pi_1 Y \simeq \pi_1 Y}$$

~~by the van Kampen theorem so this map induces an isomorphism on fundamental groups. Using the Mayer Vietoris theorem for homology with local coefficients we that this map induces isomorphisms on such homology provided f does.~~ By the Whitehead theorem in the form used by Artin-Mazur [ ] it suffices to show the map induces isomorphisms on fundamental groups and on homology with coefficients in any $\pi_1 Y$-module L. Writing the former space as the union of the complements of $Y \times \{0\}$ and $Y \times \{1\}$, its fundamental group is

$$\pi_1 Y *_{\pi_1 X} \pi_1 Y \simeq \pi_1 Y$$

by the van Kampen theorem, so the map on fundamental group is an isomorphism. Similarly the Mayer-Vietoris theorem shows the map on homology with coefficients in L is an isomorphism

It remains to show that a map $g: X \longrightarrow Z$ such that $\pi_1(g)(N) = 0$ can be extended to Y. This condition on g allows one to extend g to the 2-skeleton of X relative to Y. Given an extension to the relative n-skeleton, it can be modified without change on the $(n-1)$-skeleton so as to extend to the $(n+1)$-skeleton provided an obstruction in the cohomology group $H^{n+1}(Y, X; \pi_n Z)$ vanishes [ ], where $\pi_n Z$ is regarded as a $\pi_1 Y$-module by the map $\pi_1 Y \longrightarrow \pi_1 Z$ induced by g. But this cohomology group is zero by the hypothesis on f, so an extension of g to all of Y exists and the proposition is proved.

<u>Proposition 1.2:</u> <u>Given a perfect subgroup</u> E <u>of</u> $\pi_1 X$, <u>where</u> X <u>is a pointed connected CW complex, then there is a CW complex Y and a map</u> $f: X \longrightarrow Y$ <u>as above</u> <u>such that</u> $\operatorname{Ker} \pi_1(f) =$ <u>the normal subgroup generated by</u> E.

If N is the normal subgroup generated by E, then the commutator subgroup $[N, N]$ is a normal subgroup containing $[E, E] = E$, hence also N, so N is perfect and we may suppose $E = N$. Let $X' \longrightarrow X$ be the covering space of X with fundamental group N, whence $H_1 X' = N^{ab} = 0$. Choose generators for N as a normal subgroup of $\pi_1 X$ and form a space Z by attaching 2-cells to X' by means of maps $S^1 \longrightarrow X'$ representing these generators. Then Z is simply-connected and $H_i X' \xrightarrow{\sim} H_i Z$ except when $i=2$ when $H_2 Z$ is the direct sum of $H_2 X'$ and a free abelian group with generators coming from the attached 2-cells. By the Hurewicz theorem $\pi_2 Z \simeq H_2 Z$, hence we may attach 3-cells to Z to obtain a simply-connected CW complex Z' such that the inclusion map $X' \longrightarrow Z'$ induces isomorphisms on homology. Define Y and f by a cocartesian square

$$\begin{array}{ccc} X' & \longrightarrow & Z' \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & Y. \end{array}$$

~~Let $Y' \longrightarrow Y$ be the universal covering of Y, and let h be the unique map making the triangle commute. By the van Kampen and Mayer-Vietoris theorems one sees that $\pi_1(f)$ is surjective with kernel N and that~~

By the van Kampen theorem

$$\pi_1 Y = \pi_1 X *_{\pi_1 X'} \pi_1 Z' = \pi_1 X / N$$

so $\pi_1(f)$ is surjective with kernel N. If L is a $\pi_1 Y$-module there is a Mayer-Vietoris long exact sequence

$$\longrightarrow H_i(X', L) \longrightarrow H_i(X, L) \oplus H_i(Z', L) \longrightarrow H_i(Y, L) \longrightarrow \cdots$$

and $H_i(X', L) \xrightarrow{\sim} H_i(Z', L)$ because $\pi_1(X')$, $\pi_1(Z')$ act trivially on L as Z' is simply-connected and because X' and Z' have the same homology by construction. Thus f induces isomorphisms on homology with coefficients in all $\pi_1 Y$-modules, so the proof of the proposition is complete.

According to proposition 1.2 the pointed space Y constructed above is characterized up to homotopy equivalence by the fact that it comes with a universal arrow $f : X \longrightarrow Y$ killing the subgroup E of $\pi_1 X$. We shall use the notation X/E for the space Y. This notation is justified by the fact that Y as an object of the homotopy category of pointed spaces is the quotient of X by the action of E obtained from the natural action of $\pi_1 X$ on X.

Proposition 13: Let X_1 and X_2 be pointed connected CW complexes and let E_i be a perfect subgroup of $\pi_1 X_i$ for $i = 1, 2$. Then the canonical map in the pointed homotopy category

$$(X_1 \times X_2)/(E_1 \times E_2) \longrightarrow (X_1/E_1) \times (X_2/E_2)$$

is an isomorphism.

The follows from the fact that if Y_i' is the universal covering of $Y_i = X_i/E_i$ and X_i' the induced covering of X_i, then $Y_1' \times Y_2'$ is the universal covering of $Y_1 \times Y_2$ and $X_1' \times X_2' \longrightarrow Y_1' \times Y_2'$ induces isomorphisms on homology.

§2. <u>The K-theory associated to a ring.</u> Let A be a ring with unit but not necessarily commutative. Let $GL_n A$ be the group of invertible $n \times n$ matrices over A, let $E_n A$ be the subgroup generated by the elementary matrices, and let $GL(A)$, $E(A)$ be the respective unions of these groups for all n [ ]. Then $E(A)$ and $E_n(A)$ for $n \geq 3$ are perfect groups, hence by attaching 2-cells and 3-cells we can form the spaces

$$BGL_n A^+ = BGL_n A / E_n A$$

$$BGL(A)^+ = BGL(A)/E(A)$$

where BG denotes a classifying space for the group G which is a pointed CW complex. We define

$$K_i A = \pi_i(BGL(A)^+) \qquad i \geq 1.$$

Since $E(A)$ is the commutator subgroup of $GL(A)$ it is a normal subgroup so

$$K_1 A = GL(A)/E(A)$$

is the same as the K_1 of Bass [ ]. ~~The universal covering of $BGL(A)^+$ is a simply connected space with the same homology as $BE(A)$, hence~~

$$\cancel{K_2 A = \pi_2}$$

Moreover there is a cartesian square

$$\begin{tikzcd} BE(A) \arrow[r, "f'"] \arrow[d] & BE(A)^+ \arrow[d] \\ BGL(A) \arrow[r, "f"'] & BGL(A)^+ \end{tikzcd}$$

of covering spaces where $BE(A)^+$ is the universal covering of $BGL(A)^+$ and where f' induces isomorphisms on homology. Hence

$$\begin{aligned} K_2 A = \pi_2\, BGL(A)^+ &= \pi_2\, BE(A)^+ \\ = H_2(BE(A)^+) &= H_2(BE(A)) \end{aligned}$$

so this K_2 agrees with the K_2 of Milnor [ ].

The inclusions $GL_n A \longrightarrow GL_{n+1} A \longrightarrow \cdots \longrightarrow GL(A)$ give rise to maps unique up to homotopy

$$BGL_n A^+ \longrightarrow BGL_{n+1} A^+ \longrightarrow \cdots \longrightarrow BGL(A)^+$$

and hence to at least one map

$$\lim_n BGL_n(A)^+ \longrightarrow BGL(A)^+$$

where $\lim_n$ denotes the infinite mapping cylinder. This last map is a homotopy equivalence because both spaces have the same fundamental group and their universal coverings have the same homology. Therefore for any pointed finite complex X we have

$$[X, BGL(A)^+]_0 = \varinjlim [X, BGL_n A^+]_0.$$

One can pose the problem of whether the stability theorems of algebraic K-theory admit generalizations asserting that the map $[X, BGL_n A^+]_0 \longrightarrow [X, BGL(A)^+]_0$ is surjective (resp. an isomorphism) for $n \geq \dim X + \dim(\mathrm{Max}\, A)$ (resp. for $n > \dim X + \dim(\mathrm{Max}\, A)$).

The homomorphism $GL_m A \times GL_n A \longrightarrow GL_{m+n} A$ obtained from the direct sum of matrices carries $E_m A \times E_n A$ into $E_{m+n} A$, hence using proposition 3 it induces a map

$$\mu_{m,n} : BGL_m A^+ \times BGL_n A^+ \longrightarrow BGL_{m+n} A^+.$$

The collection of μ_{mn} is homotopy associative in an evident sense. Moreover if x and y are $m \times m$ and $n \times n$ matrices respectively, then the matrices

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \qquad \begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}$$

are conjugate by the matrix

$$\sigma = \begin{pmatrix} 0 & (-1)^{mn} I_n \\ I_m & 0 \end{pmatrix} \qquad I_n = \text{identity } n \times n \text{ matrix}$$

which belongs to $SL_{m+n}\mathbb{Z} = E_{m+n}\mathbb{Z} \subset E_{m+n}A$. Hence the diagram

$$\begin{tikzcd}
BGL_m A \times BGL_n A \arrow[r, "\oplus"] \arrow[d, "T"] & BGL_{m+n}A \arrow[d, "\tilde{\sigma}"] \arrow[dr] & \\
BGL_n A \times BGL_m A \arrow[r, "\oplus"] & BGL_{m+n}A \arrow[r] & BGL_{m+n}A^+
\end{tikzcd}$$

is commutative in the pointed homotopy category, where $T(u,v) = (v,u)$ and $\tilde{\sigma}$ is induced from the conjugation by σ, so $\mu_{m,n}$ and $\mu_{n,m}T$ are homotopic. Consequently $\coprod_n [X, BGL_n A^+]_0$ is an abelian monoid, ~~and~~ so $[X, BGL(A)^+]_0$ is an abelian monoid in a natural way for any pointed finite complex X, and in fact even an abelian group as one sees easily by ~~the~~ induction on the number of cells in X. In other words $BGL(A)^+$ is a homotopy commutative and associative weak H-space and in particular it is simple, i.e. the fundamental group acts trivially on the other homotopy groups. Actually with more work we shall show that $BGL(A)^+$ is an infinite loop space (§).

<u>Proposition 2.1:</u> <u>The Hurewicz homomorphism for $BGL(A)^+$ induces an isomorphism of $K_i A \otimes \mathbb{Q}$ with the primitive subspace of degree i of the Hopf algebra</u> $H_*(GL(A), \mathbb{Q})$.

This follows from a theorem of Milnor-Moore ([ ], appendix) ~~provided~~ once we know that $BGL(A)^+$

is an H-space. This follows from the weak H-space structure when $BGL(A)^+$ is a countable CW complex in virtue of the surjectivity of the natural map $[\lim_n X_n, Z]_0 \longrightarrow \operatorname{invlim} [X_n, Z]_0$, so the proposition is true for A countable, hence in general by passage to the limit.

Finally suppose A is commutative and let m be an integer which is a unit in A. Then Grothendieck [ ] has defined Chern classes

$$c_i \in H^{2i}(\operatorname{Spec} A, GL_n A ; \mu_m^{\otimes i})$$

taking values in the etale cohomology of the scheme Spec A with $GL_n A$ as operators acting trivially. If $I^\cdot$ is the cochain complex of global sections of an injective resolution of the etale sheaf $\mu_m^{\otimes i}$ on Spec A, then the class c_i may be identified with a cohomology class of $BGL_n A$ with coefficients in $I^\cdot$, or equivalently with a homotopy class of maps

$$BGL_n(A) \longrightarrow K(I^\cdot ; 2i)$$

where $K(I^\cdot; 2i)$ is the generalized Eilenberg-Maclane space whose j-th homotopy group is the $(2i-j)$-th cohomology group of $I^\cdot$. Since $K(I^\cdot; 2i)$ is a simple spaces the above map kills $E_n(A)$ hence induces a map unique up to homotopy

$$BGL_n(A)^+ \longrightarrow K(I;2i).$$

These maps are compatible as n goes to infinity since c_i is a stable class, hence passing to homotopy groups we get a well-defined homomorphism

$$c_i^{\#} : K_j A \longrightarrow H^{2i-j}(\mathrm{Spec}\, A, \mu_m^{\otimes i})$$

from K-groups to etale cohomology. Hopefully this homomorphism coincides with the ones constructed by Bass-Tate (see [ ]) when A is a field and i,j are both 1 or 2. In a similar way using Chern classes in Hodge cohomology [ ] one obtains homomorphisms

$$c_j^{\#} : K_j A \longrightarrow \Omega^j_{A/\mathbb{Z}}$$

for $j \geq 1$.

§3. Splitting exact sequences stably. Let $GL_{m,n}A$ be the group of automorphisms of the right A-module $A^m \oplus A^n$ which preserve the second factor, and let

$$p: GL_{m,n}A \longrightarrow GL_m A \times GL_n A$$

be the evident surjection. In the case of topological K-theory where A is a commutative Banach algebra and these linear groups are endowed with the topologies induced by the topology of A, one knows that the map on classifying spaces induced by p is a homotopy equivalence. But in the algebraic case under consideration here, this isn't the case. Aside from the fact that the fundamental groups are not the same, a problem which can be side-stepped by killing elementary subgroups, the two groups have different homology in general, e.g. when A is a field of finite characteristic and $m=n=1$. Nevertheless we shall see that these two groups have the same homology in the limit when $m,n=\infty$.

~~Using the embedding $A^n \to A^{n+1}$ with last coordinate zero we obtain injective homomorphisms~~

Let $GL_{(2)}A$ be the union of the $GL_{m,n}A$ with respect to the inclusions induced by the standard injection $A^n \longrightarrow A^{n+1}$ with last coordinate zero; $GL_{(2)}A$ is the group of

Automorphisms of the right A-module $A^\infty \oplus A^\infty$ which preserve the second factor and ~~[illegible]~~ whose matrix is almost everywhere the identity. The following will be proved in the next section.

Theorem 3.1: *The surjection $GL_{(2)}A \longrightarrow GL(A)^2$ induces isomorphisms on the homology of the classifying spaces.*

~~The [illegible] of this section will be involved with the proof of~~

~~Before giving the proof we deduce some consequences. Note that $GL_{m,n}A$ is the semi-direct product~~

$$\cancel{GL_{m,n}A = (GL_m A \times GL_n A) \tilde{\times} \mathrm{Hom}_A(A^m, A^n)}$$

~~Before giving the proof we deduce some consequences of the theorem.~~ Let $E_{m,n}A$ be the subgroup of $GL_{m,n}A$ generated by elementary matrices, i.e. matrices which becomes elementary matrices in $GL_{m+n}A$ under the obvious embedding. Then there are semi-direct product decompositions

$$GL_{m,n}A = (GL_m A \times GL_n A) \tilde{\times} \mathrm{Hom}_A(A^m, A^n)$$

$$E_{m,n}A = (E_m A \times E_n A) \tilde{\times} \mathrm{Hom}_A(A^m, A^n)$$

and it is easy to see that $E_{m,n}A$ is perfect for $m, n \geq 3$.

In the limit as m, n go to infinity we get a perfect subgroup $E_{(2)}A$ of $GL_{(2)}A$ which is in fact the commutator subgroup. Set

$$BGL_{m,n}A^+ = BGL_{m,n}A / E_{m,n}A$$
$$BGL_{(2)}A^+ = BGL_{(2)}A / E_{(2)}A .$$

<u>Corollary 3.2:</u> <u>The natural map</u> $BGL_{(2)}A^+ \longrightarrow (BGL(A)^+)^2$ <u>is a homotopy equivalence.</u>

In effect one shows that $BGL_{(2)}A^+$ is a weak H-space by the same argument used for $BGL(A)^+$ in the preceding section. Thus both spaces are simple and the map induces an isomorphism on homology by the theorem, hence the map is a homotopy equivalence by the Whitehead theorem.

~~If E is a representation of a group G over A, i.e. a projective finitely-generated A-module endowed with an action of G, then to E we can associate an element~~ ~~$(E) \in [BG, BGL(A)^+]_0$ as~~

~~[illegible]~~ By a representation of a group G over A we mean a finitely generated projective A-module endowed with a linear action of G. Call two representations stably ~~[illegible]~~ isomorphic if they become isomorphic ~~after~~ after adding trivial representations, and let $I_A G$ be the abelian monoid of ~~stably~~ stable isomorphism classes of representations. Then

$$I_A G = \varinjlim_n \mathrm{Hom}_{\mathcal{C}}(G, GL_n A)$$

where $\mathcal{C}$ the category of "groups up to inner automorphisms." Now ~~[illegible]~~ if $j: BGL_n A \longrightarrow BGL(A)^+$ is the canonical map and if $\tilde{\sigma}$ denotes the endomorphism of $BGL_n A$ produced by an inner automorphism σ of $GL_n A$, then $j\tilde{\sigma}$ is homotopic to j preserving basepoints, because σ can be effected by an inner automorphism in $GL(A)$ coming an element of $E(A)$ and $E(A)$ ~~gets~~ killed in $\pi_1 BGL(A)^+$. Consequently to each representation V of G over A is associated an element

$$(E) \in [BG, BGL(A)^+]$$

depending only on the stable isomorphism class of E.

<u>Corollary 3.3</u>: *If* $0 \to E' \to E \to E'' \to 0$ <u>is an exact sequence of representations of G over A,</u>

<u>then</u> $(E) = (E' \oplus E'')$.

It suffices to consider the case where $G = GL_{m,n}A$ acting in the natural way on $E = A^m \oplus A^n$ with $E' =$ the invariant submodule A^n. By using the isomorphism $\varepsilon_{mn} : A^m \oplus A^n \simeq A^{m+n}$ which one obtains by enumerating the standard basis for $A^m \oplus A^n$, we get two homomorphisms u_1, u_2 from $GL_{m,n}A$ to $GL_{m+n}A$ corresponding to the representations E and $E' \oplus E''$, and we must show that the effect of u_1 and u_2 on classifying spaces becomes the same after composing with the canonical map from $BGL_{m+n}A$ to $BGL(A)^+$. Extend ε_{mn} to an isomorphism $\varepsilon : A^\infty \oplus A^\infty \simeq A^\infty$ preserving the standard bases. Then u_1 and u_2 extend to two similarly defined homomorphisms $v_1, v_2 : GL_{(2)}A \longrightarrow GL(A)$ and it suffices to show that the two maps from $BGL_{(2)}A^+$ to $BGL(A)^+$ induced by v_1 and v_2 are homotopic. But this is clear because v_1 and v_2 agree on the subgroup $GL(A)^2$ of $GL_{(2)}A$ and because the map $(BGL(A)^+)^2 \longrightarrow BGL_{(2)}A^+$ induced is a homotopy equivalence since it is the inverse of the map of 3.2.

§1. *Killing a perfect subgroup of the fundamental group.* In this section we work only with pointed spaces and with maps preserving basepoints. Recall that a group is called *perfect* if it is equal to its commutator subgroup.

Let X be a pointed connected CW complex and let E be a perfect subgroup of $\pi_1 X$. Let $g' : X' \longrightarrow X$ be the covering space of X with $\pi_1 X' \xrightarrow{\simeq} E$. By Poincaré's theorem $H_1 X' \simeq E^{ab} = 0$. Choose generators for E, represent them by maps $u_i : S^1 \longrightarrow X'$, $i \in I$, and let X'' be the CW complex obtained by attaching 2-cells to X' with boundaries

u_i. By the van Kampen theorem $\pi_1 X'' = 0$; from the long exact (integral) homology sequence of the pair (X'', X') we see that $H_q X' \simeq H_q X''$ for $q \neq 2$ and that there is an exact sequence

$$0 \longrightarrow H_2 X' \longrightarrow H_2 X'' \longrightarrow F \longrightarrow 0$$

where F is a free abelian group with generators indexed by I. Since X'' is simply-connected, $\pi_2 X'' \simeq H_2 X''$ by the Hurewicz theorem, hence we may choose maps $v_i : S^2 \longrightarrow X''$, $i \in I$, whose images in $H_2 X''$ form a basis for a free abelian subgroup mapped isomorphically to F. Attaching 3-cells to X'' with boundaries v_i we obtain a simply-connected CW complex Y' containing X' such that the inclusion map $f' : X' \longrightarrow Y'$ induces isomorphisms on homology.

Define Y and f by a cocartesian diagram

(1)
$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} .$$

Clearly f kills E, i.e. $\pi_1(f)(E) = 0$. Suppose $u : X \longrightarrow Z$ kills E. Then $ug' : X' \longrightarrow Z$ lifts to the universal covering $\tilde{Z}$ of Z. The obstructions to extending this map $X' \longrightarrow \tilde{Z}$ to Y' lie in the groups $H^{n+1}(Y', X'; \pi_n Z)$ which are zero since f' induces isomorphisms on homology. Similarly by obstruction theory any two such extensions are homotopic relative to X'. Thus there exists an extension $v' : Y' \longrightarrow Z$ of ug'

unique up to homotopy relative to X', hence

also an extension $v: Y \longrightarrow Z$ of u which is unique up to homotopy relative to X.

Using the homotopy extension theorem it

~~perfect subgroup~~

follows that if $v_0, v_1 : Y \longrightarrow Z$ are two maps such that $v_0 f$ and $v_1 f$ are homotopic, then v_0 and v_1 are homotopic, so we have proved the following proposition.

Let E ~~be~~ a perfect subgroup of

Proposition 1.1: ~~If~~ E ~~is a~~ perfect subgroup

$\pi_1 X$, where X is a pointed connected CW complex. Then

of $\pi_1 X$, then there exists a map $f: X \longrightarrow Y$

there exists a map $f: X \longrightarrow Y$ with Y a

in the homotopy category of pointed

pointed connected CW complex which is

spaces which is "universal" for killing E, i.e.

$$f^*: [Y, Z]_0 \xrightarrow{\sim} \{ u \in [X, Z]_0 \mid \pi_1(u)(E) = 0 \},$$

Denote by $\mathcal{H}_0$ the homotopy category of pointed spaces, i.e. the category with pointed spaces for objects and with the set of morphisms from A to Z defined to be the set $[A, Z]_0$ of homotopy classes of basepoint-preserving maps.

Proposition 1.1: *Let E be a perfect subgroup of $\pi_1 X$, where X is a pointed connected CW complex. Then there exists a map $f: X \longrightarrow Y$ with Y a pointed connected CW complex which is "universal" in $\mathcal{H}_0$ for killing E, i.e.*

$$[Y, Z]_0 \xrightarrow{\sim} \{u \in [X, Z]_0 \mid \pi_1(u)(E) = 0\}$$

for all Z.

It follows from this universal property that the pair (Y, f) is unique up to isomorphism in

$\mathcal{H}_0$ and in particular is independent of the choices made in its construction. We use the notation X/E for Y, this notation being justified by the following considerations. Recall that there is a natural action of $\pi_1 X$ on X as an object of $\mathcal{H}_0$ which induces the conjugation action of $\pi_1 X$ on itself. Hence E acts on X and as E is perfect it is clear that a map $u : X \longrightarrow Z$ kills E iff $u\tilde{g} = u$ for all $g \in E$, where $\tilde{g} : X \longrightarrow X$ denotes the endomorphism of X associated to the element g of $\pi_1 X$. Thus X/E is the quotient in the category sense of X by the action of E.

We finish this section by deriving a

useful description of X/E.

Proposition 1.2: The pointed CW complex Y and the map $f: X \longrightarrow Y$ of 1.1 are characterized up to isomorphism in $\mathcal{H}_0$ by the following properties:

(i) f induces an isomorphism $\pi_1 X/N \xrightarrow{\sim} \pi_1 Y$ where N is the normal subgroup of $\pi_1 X$ generated by E.

(ii) If $\tilde{Y} \longrightarrow Y$ is the universal covering of Y and $\tilde{X} \longrightarrow X$ is the covering induced by f, then the map $\tilde{X} \longrightarrow \tilde{Y}$ induces isomorphisms on homology. Equivalently:

(ii)' f induces an isomorphism $H_*(X, L) \xrightarrow{\sim} H_*(Y, L)$ on homology with (twisted) coefficients in the $\pi_1 Y$-module $L = \mathbb{Z}[\pi_1 Y]$.

(The equivalence of (ii) and (ii)' results from the fact that $H_*(X, L) = H_*(\tilde{X}, \mathbb{Z})$ and similarly for $Y, \tilde{Y}$.)

If f is the map of 1.1, then the van Kampen theorem applied to the square (1) proves (i). Similarly as (1) is cocartesian, (ii)$'$ follows once we know $f': X' \longrightarrow Y'$ induces isomorphism on homology with twisted coefficients in L. But this is clear since Y' is simply-connected and since f' induces isomorphism on homology. Thus f satisfies (i) and (ii).

Conversely suppose $f: X \longrightarrow Y$ satisfies (i) and (ii), let $p: X \longrightarrow X/E$ be a universal map killing E, and let $h: X/E \longrightarrow Y$ be the unique map in $\mathcal{H}_0$ with $hp = f$. Then as p satisfies conditions analogous to

(i) and (ii) by what we've just proved, it follows that h induces isomorphisms on fundamental groups and isomorphisms on homology of the universal coverings. By the Whitehead theorem h is a homotopy equivalence, so f is isomorphic in $\mathcal{H}_0$ to p, completing the proof of the proposition.

<u>Remark</u> 1.3: The commutator subgroup (N,N) of N is a normal subgroup of $\pi_1 X$ containing $(E,E) = E$, hence (N,N) contains N and N is perfect. Since $X/E \xrightarrow{\sim} X/N$ in $\mathcal{H}_0$ by the universal property, we can restrict attention to the case where E is normal.

1.4: If E is normal, then the X' in

square (1) is equal to $\tilde{X}$, hence the lifting $Y' \to \tilde{Y}$ of g is a homotopy equivalence by the Whitehead theorem, as both spaces are simply-connected with the same homology as $\tilde{X}$. Thus when E is normal the square (1) is up to homotopy equivalence both cartesian and cocartesian.

Corollary 1.5: Let X_1 and X_2 be pointed connected CW complexes and let E_i be a perfect subgroup of $\pi_1 X_i$ for $i = 1, 2$. Then the canonical map in $\mathcal{H}_0$

$$(X_1 \times X_2)/(E_1 \times E_2) \longrightarrow (X_1/E_1) \times (X_2/E_2)$$

is an isomorphism.

This follows easily from 1.2.

Theorem: ~~[illegible]~~ Assume Γ has no non-trivial perfect subgroup in π_1 of any of its components. Then

$$\mathrm{Hom}(\tilde{K}(\,;A), [\ ,\Gamma]) \xrightarrow{\sim} \mathrm{Hom}(R(\,;A), [\ ,\Gamma])$$

Proof: ① First show true for stable reps:

$$\begin{tikzcd} BGL(A) \arrow[d] \arrow[r] & \Gamma \\ BGL(A)^+ \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

diagram of weak maps

by ~~[illegible]~~ writing
$$BGL(A) = \bigcup Z_i$$
$$BGL(A)^+ = \bigcup Z_i^+ .$$
where $Z_i^+ = Z_i \cup_{Z_0} Z_0^+$, and using

$$[Z_i^+, \Gamma] \xrightarrow{\sim} [Z_i, \Gamma].$$

② same true for $BGL(A)^n \longrightarrow [BGL(A)^+]^n$ (equivalent to ① for the ring A^n).

③ Now [illegible] do for $\tilde{R}$:

$$\begin{tikzcd} \tilde{R}(\,;A) \arrow[d] \arrow[r] & \Gamma \\ \tilde{R}(\,;A) = BGL(A)^+ \arrow[ur, dotted, "\exists!"'] & \end{tikzcd}$$

the proof here consists in ~~[illegible]~~

$\cancel{\tilde{K}(X,A)^3 \to \tilde{K}(X,A)^2 \to \tilde{K}(X,A) \to 0}$

the diagram

$$
\begin{tikzcd}
BGL(A)^3 \arrow[r, shift left, "\mu \times id"] \arrow[r, shift right, "id \times \mu"'] \arrow[d] & BGL(A)^2 \arrow[r, "surj."] \arrow[d] & \tilde{R} \arrow[d] \\
BGL(A)^{+3} \arrow[r, shift left, "\mu \times id"] \arrow[r, shift right, "id \times \mu"'] & BGL(A)^{+2} \arrow[r] & BGL(A)^{+}
\end{tikzcd}
$$

bottom row is exact. ~~the~~ Given the map $\tilde{R} \to \Gamma$ we get

$$
\begin{tikzcd}
BGL(A)^2 \arrow[r] \arrow[d] & \Gamma \\
BGL(A)^{+2} \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd}
$$

and the maps from $BGL(A^{+})^3$ are equalized by the dotted arrows, by the uniqueness part ②. Thus (by exactness) $\exists!$ map $\ni$

$$
\begin{tikzcd}
BGL(A)^2 \arrow[r, two heads, "surj"] \arrow[d] & \tilde{R} \arrow[r] & \Gamma \\
BGL(A)^{+} \arrow[r] & BGL(A)^{+} \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd}
$$

commutes. Surjectivity shows that $\triangledown$ commutes

<u>Remark</u>: shows any map $BGL(A) \longrightarrow \Gamma$ extends uniquely to $\tilde{R} \longrightarrow \Gamma$.

4 Now

$$\begin{tikzcd} R \arrow[r, "\text{given}"] & \Gamma \\ K_0A \times \tilde{R} \arrow[u, "(\cong \text{ if } X \text{ conn.})"] \arrow[r] & K_0A \times BGL(A)^+ \arrow[u, dotted, "\exists!"'] \end{tikzcd}$$

look at each $\alpha \in K_0 A$ separately.

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & \Gamma \\ & K_0A \times BGL(A)^+ \arrow[u] \end{tikzcd}$$

commutes for X connected, hence in general.

---

true for $\prod K(\; ; A_i)$

Atlantic City

$$\begin{tikzcd} Y \arrow[r, "\text{as in L.1}"] \arrow[d, "\substack{\text{covering} \\ \pi_1 Y = E}"'] \arrow[dr, phantom, "\text{cocart}"] & Y^+ \arrow[d] \\ X \arrow[r] & X^+ \end{tikzcd}$$

van Kampen $\to$ $\pi_1 X^+ = \pi_1 X *_{\pi_1 Y} \pi_1 Y^+ = \pi_1 X / N$

~~Mayer-Vietoris~~
excision $H_i(X^+, Y; L) \simeq H_i(X^+, X; L)$

~~$\to H_i(Y, L) \to H_i(X, L) \oplus H_i(Y^+, L) \to H_i(X^+, L) \xrightarrow{\partial}$~~

but L trivial over Y & Y^+ and $H_i(Y, L) \simeq H_i(Y^+, L)$
so (ii) holds.

Prop: ~~Corollary~~: Given $g : X \to Z$ $\pi_1(g)(E) = 0$
$\exists$ extension

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "f"'] & Z \\ & X^+ \arrow[u, dashed] \end{tikzcd}$$

and any two extensions are homotopic relative to X.

Obstructions lie in $H^*(X^+, X; \pi_* Z)$ and hypothesis implies $\pi_* Z$ are $\pi_1 X^+$ modules, so these are zero.

Proof: Choose elements $\alpha_i \in \pi_1 X$, $i \in I$ such that the normal subgroup of $\pi_1 X$ gen. by α_i is $\pi_1 X$ itself. Realize α_i by $u_i : S^1 \longrightarrow X$, let $u = (u_i) : \bigvee_I S^1 \longrightarrow X$, set

$$Y = X \cup_u \bigvee_I e_2$$

$$H_i X \xrightarrow{\sim} H_i Y \qquad i \geq 3$$

$$\begin{tikzcd} 0 \arrow[r] & H_2 X \arrow[r] & H_2 Y \arrow[r] & \bigoplus_I \mathbb{Z} \arrow[r] \arrow[l, bend left, dashed] & 0 \end{tikzcd}$$

$\pi_1 Y = 0$ by van Kampen. $\pi_2 Y = H_2 Y$ Hurewicz so attaching for each $i \in I$ a 3 cell to Y via a map $S^2 \longrightarrow Y$ giving lifting.

$$H_i Y \simeq H_i Z \qquad i \geq 4$$

$$\begin{tikzcd} & & & & 0 \arrow[d] & & \\ & & & & H_2 X \arrow[d] & & \\ 0 \arrow[r] & H_3 Y \arrow[r] & H_3 Z \arrow[r] & \bigoplus_I \mathbb{Z} \arrow[r, hook] & H_2 Y \arrow[r] \arrow[d] & H_2 Z \arrow[r] & 0 \\ & & & & \bigoplus_I \mathbb{Z} & & \end{tikzcd}$$

Prop: Let E be a perfect subgroup of $\pi_1 X$, N the normal subgroup generated by E. Then $\exists$ embedding $f : X \hookrightarrow X^+$ s.t.

(i) $\pi_1(X)/N \xrightarrow{\sim} \pi_1 X^+$

(ii) ~~$H_*(X;L) \to H_*(X^+;L)$~~ all $\pi_1 X^+$ modules L.

$H^*(X^+, X; L) = 0$

$BGL(\Lambda)^+$ (weak) H-space

theorem of Milnor – Moore $\Longrightarrow$

$$\pi_i\, BGL(\Lambda)^+ \otimes \mathbb{Q} = P H_i(BGL(\Lambda)^+, \mathbb{Q})$$

$$\boxed{K_i \Lambda \otimes \mathbb{Q} = P\, H_i(GL(\Lambda), \mathbb{Q}) \qquad i \geq 1}$$

<u>Theorem of Borel</u>: Let Λ be the ring of integers in a number field F with r_1 real places and r_2 complex places. Then

$$\left\{\begin{array}{ll} K_{2i} \Lambda \otimes \mathbb{Q} = 0 & i > 0 \\ \dim K_1 \Lambda \otimes \mathbb{Q} = r_1 + r_2 - 1 & \\ \dim K_{4i+1} \Lambda \otimes \mathbb{Q} = r_1 + r_2 & \\ \dim K_{4i-1} \Lambda \otimes \mathbb{Q} = r_2 & \end{array}\right.$$

	1	2	3	4	5	6	7	8	9
real place	$\mathbb{Q}$	0	0	0	$\mathbb{Q}$	0	0	0	$\mathbb{Q}$
cx place	$\mathbb{Q}$	0	$\mathbb{Q}$	0	$\mathbb{Q}$	0	$\mathbb{Q}$	0	$\mathbb{Q}$

take sum and reduce by 1 for K_1 according to Dirichlet.

~~3.3 + 3.1~~

Basic outline:

1. killing a perfect subgroup of the fundamental group.
2. (weak) H-space structures on $BGL(\Lambda)^+$

$$\boxed{K_*\Lambda \otimes \mathbb{Q} = P H_*(GL(\Lambda), \mathbb{Q}).}$$

Borel theorem.

3. finite field $\mathbb{F}_q$

filtration: $$BGL(\mathbb{F}_q)^+ \longrightarrow BU \xrightarrow{\Psi^q - 1} BU$$

$$\begin{cases} K_{2i}(\mathbb{F}_q) = 0 & i > 0 \\ K_{2i-1}(\mathbb{F}_q) \cong \mathbb{Z}/(q^i - 1)\mathbb{Z} & i > 0 \end{cases}$$

4. symmetric groups

$$\Sigma_\infty = \bigcup \Sigma_n$$

Thm: $B\Sigma_\infty^+ \sim (\varinjlim_n \Omega^n S^n)_0$

stable homotopy of symmetric groups = stable homotopy gps. of spheres

Λ ring with unit

$E_n(\Lambda) =$ subgroup of $GL_n(\Lambda)$ gen. by $\begin{pmatrix} 1 & & * \\ & 1 & \\ & & 1 \end{pmatrix}$

$$GL(\Lambda) = \bigcup_n GL_n(\Lambda)$$

$$E(\Lambda) = \bigcup_n E_n(\Lambda)$$

Recall $E_n(\Lambda)$ perfect $n \geq 3$, $E_n\Lambda = (E_n\Lambda, E_n\Lambda)$
and $(GL(\Lambda), GL(\Lambda)) = E(\Lambda) = (E(\Lambda), E(\Lambda))$.

$K_0\Lambda =$ Groth. gp. f. gen. proj. Λ-modules

$K_1\Lambda = GL(\Lambda)/E(\Lambda)$ Bass / Whitehead

$K_2\Lambda = H_2(E(\Lambda), \mathbb{Z})$ Milnor

I propose to extend these groups $K_n\Lambda$, $n \geq 0$.

§1. Killing all perfect subgroup of π_1

All spaces are connected CW cx with basepoint.

Lemma 1: Given X with $H_1(X) = 0$ ($\pi_1 X$ perfect)
then $\exists$ an ~~maps~~ embedding $f : X \hookrightarrow X^+$ such that

(i) $H_*(X^+, X; \mathbb{Z}) = 0$

(ii) $\pi_1 X^+ = 0$.

Prop

Given $E \subset \pi_1 X$, E perfect, $N =$ normal subgp gen. by E

$$\begin{tikzcd}
Y \arrow[r, "\text{lemma 1}"] \arrow[d, "\substack{\text{covering} \\ \pi_1 Y = E}"'] \arrow[dr, phantom, "\text{cocart.}"] & Y^+ \arrow[d] \\
X \arrow[r, "f"'] & X^+
\end{tikzcd}$$

Then (i) $\pi_1 X^+ = \pi_1 X /$ normal subgroup gen. by E (van Kampen)

(ii) $H_*(X, L) \xrightarrow{\sim} H_*(X^+, L)$ all $\pi_1 X^+$-modules L.

<u>Corollary</u>: Given $g \ni \pi_1(g)(E)$ $\exists$ extension

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "f"] & Z \\
X^+ \arrow[ur, dashed] &
\end{tikzcd}$$

and any two are homotopic relative f. Enough to do for $Y \to Y^+$

$$\begin{tikzcd}
 & & \tilde{Z} \arrow[d] \\
Y \arrow[urr] \arrow[rr] \arrow[d] & & Z \\
Y^+ \arrow[urr, dashed] & &
\end{tikzcd}$$

but obstructions lie in $H^*(Y^+, Y; \pi_* Z) = 0$.

~~maps~~ trivial as $\pi_1 Y \to 0$.

This shows f universal map in homotopy category ~~for~~ killing E, in particular independent of choices in its construction.

$\boxed{\text{Prop.}}$ If E normal then $Y^+ \sim \widetilde{X^+}$ universal covering.

Proof: ~~Enough to show $Y^+ \to \widetilde{X}^+$ homology isom.~~

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "p"'] & Y^+ \arrow[r] \arrow[d] & \widetilde{X}^+ \arrow[dl, "p'"] \\ X \arrow[r] & X^+ & \end{tikzcd}$$

p, p' normal coverings group $\pi_1 X^+ = \pi_1 X/E$.

$$H_*(Y) = H_*(X, \mathbb{Z}[\pi_1 X^+]) \xrightarrow{\sim} H_*(X^+, \mathbb{Z}[\pi_1 X^+])$$
$$= H_*(\widetilde{X}^+)$$

Thus $Y \longrightarrow \widetilde{X}^+$ homol isom $\Rightarrow$ $Y^+ \longrightarrow \widetilde{X}^+$ homology isom. $\Longrightarrow$ $Y^+ \sim \widetilde{X}^+$ (Whitehead)

Apply this construction to $E(\Lambda) \subset GL(\Lambda) = \pi_1 BGL(\Lambda)$:

~~[illegible]~~

$$\begin{tikzcd} BE(\Lambda) \arrow[r, "H_* \text{ isom}"] \arrow[d, "E(\Lambda)", "\substack{\text{normal}\\ \text{covering}\\ \text{group } GL(\Lambda)}"'] & BE(\Lambda)^+ \arrow[d, "\substack{\text{universal}\\ \text{covering}}"] \\ BGL(\Lambda) \arrow[r, "f"] & BGL(\Lambda)^+ \end{tikzcd}$$

$$\pi_1 BGL(\Lambda)^+ = GL(\Lambda)/E(\Lambda)$$

$$\pi_2 BGL(\Lambda)^+ = \pi_2 BE(\Lambda)^+ = H_2(BE(\Lambda)^+)$$
$$= H_2(BE(\Lambda))$$

<u>Definition:</u> $\boxed{K_i\Lambda = \pi_i \; BGL(\Lambda)^+ \qquad i \geq 1}$

---

§2: <u>H-space structure on</u> $BGL(\Lambda)^+$.

$E_n(\Lambda)$ perfect $n \geq 3$, so can define

$BGL_n(\Lambda) \hookrightarrow BGL_n(\Lambda)^+$ univ. killing $E_n(\Lambda)$.

Can take $$BGL_n(\Lambda)^+ = BGL_n(\Lambda) \bigcup_{BGL_3(\Lambda)} BGL_3(\Lambda)^+ \qquad 3 \leq n \leq \infty.$$

$\Longrightarrow$

$$BGL(\Lambda)^+ = \bigcup_{n \geq 3} BGL_n(\Lambda)^+ .$$

$$\Longrightarrow \quad [X, BGL(\Lambda)^+]_0 = \varinjlim [X, BGL_n(\Lambda)^+]_0 \qquad X \text{ finite cx.}$$

Whitney sum:
$$GL_m\Lambda \times GL_n\Lambda \longrightarrow GL_{m+n}\Lambda$$
$$A \,,\; B \longmapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

$$\mu_{mn} : BGL_m\Lambda^+ \times BGL_n\Lambda^+ \longrightarrow BGL_{m+n}\Lambda^+$$

<u>**Proposition:**</u> μ_{mn} define an abelian ~~[illegible]~~ group structure on $[X, BGL(\Lambda)^+]$

~~Proof like for $BO = \bigcup BO_m$ This classical argument doesn't~~

won't give proof - like for $BO = \bigcup BO_m$; would like to point why this argument works for $BGL(\Lambda)^+$ but not $BGL(\Lambda)$. Need to know that ~~the auto φ~~ conjugation ~~auto of BGL~~ by

$$\begin{pmatrix} 0 & I & \\ I & 0 & \\ & & \pm 1 \end{pmatrix} \qquad \det = 1$$

~~auto~~ produces an auto BO which is homotopic to id. ~~[illegible]~~ True for BO as this matrix can be joined by a path in O, not true for $BGL(\Lambda)$ as conj. non-trivial on π_1; true for $BGL(\Lambda)^+$ as this matrix is in $E(\Lambda)$.

$\Longrightarrow$ $BGL(\Lambda)^+$ (weak) H-space. (in particular simple)

~~[illegible]~~

~~[illegible]~~

Milnor-Moore: $\Longrightarrow$

$$\pi_i\, BGL(\Lambda)^+ \otimes \mathbb{Q} \xrightarrow{\sim} P\, H_i(BGL(\Lambda)^+; \mathbb{Q})$$

$$\boxed{K_i \Lambda \otimes \mathbb{Q} = P\, H_i(GL(\Lambda), \mathbb{Q}) \quad i \geq 1}$$