

second part

§1. The groups  $K_i(A)$

$E(A)$  perfect  $\triangleleft GL(A) = \pi_1 BGL(A)$

hence  $\exists$  acyclic  $f: BGL(A) \rightarrow BGL(A)^+$

with  $\text{Ker } \pi_1(f) = E(A)$ .  $f$  unique up to

canon. isom. in homot. cat. of ptcl. spaces

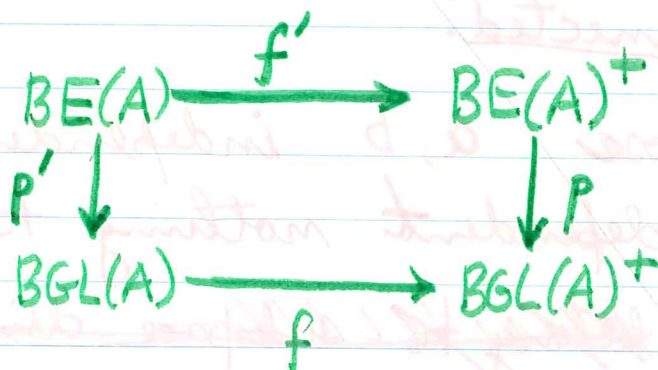
~~first show agrees with Bass m.~~

Defn:  $K_i(A) = \pi_i(BGL(A)^+)$  for  $i \geq 1$ .

We begin by showing this agrees with Bass-

Recall construction of  $BGL(A)^+$ :

Milnor  
Prop:  $K_1 A =$   
 $K_2 A =$



$P'$  covering spaces ~~isom.~~

$BE(A)^+$  obtained from  $BE(A)$  by attaching

2 and 3 cells to kill  $\pi_1(BE(A))$  without changing

homology. ~~Diagram~~ Diagram cocartesian, so by

van Kampen

~~Claim~~  $\pi_1(BGL(A)^+) = GL(A)/E(A)$

Claim ~~isom.~~ canonical map  $BE(A)^+ \rightarrow$

$(BGL(A)^+)^{\sim}$  heq. whence ~~on replacing~~ we obtain

~~BE(A)~~

a cartesian square

$$\begin{array}{ccc} BE(A) & \xrightarrow{f'} & (BGL(A)^+)^{\sim} \\ p' \downarrow & & \downarrow p \\ BGL(A) & \xrightarrow{f} & BGL(A)^+ \end{array}$$

where vertical maps are principal coverings with group  $GL(A)/E(A)$ , such that  $f'$  induces isom. on homology.

$$\begin{aligned} \pi_2 BGL(A)^+ &= \pi_2 (BGL(A)^+)^{\sim} \\ &= H_2((BGL(A)^+)^{\sim}) \\ &= H_2(BE(A)). \end{aligned}$$

## Dro's approach to the K-groups.

~~Start with  $X_0 = BGL(A)$  and~~ Construct a Postnikov system

$$X_2 \longrightarrow X_1 \longrightarrow X_0$$

~~with~~ with  $X_n$   $n$ -acyclic: ~~\*~~

$$\tilde{H}_i X_n = 0 \quad i < n$$

as follows. Take  $X_0 = BGL(A)$ . ~~Given  $X_{n-1}$~~   
Having obtained  $X_{n-1}$ , we have

$$H^n(X_{n-1}, M) = \text{Hom}_{\mathbb{Z}}(H_n X_{n-1}, M)$$

for all abelian groups  $M$ ; in part, exists canonical class with  $M = H_{n-1} M$ , hence a canonical map up to homotopy

$$X_{n-1} \longrightarrow EM(H_n, X_{n-1}, n).$$

Take  $X_n$  to be fibre of this map. Then

$$X_0 = BGL(A)$$

$$X_1 = BE(A)$$

$$X_2 = BSt(A)$$

where  $St(A)$  is the Steinberg group. The limit

$$X_\infty = \varprojlim X_n$$

is acyclic and

$$\begin{cases} \pi_1 X_\infty = \text{St}(A) \\ \pi_n X_\infty = H_{n+1} X_n \quad n \geq 2 \end{cases}$$

~~We may identify  $BGL(A)^+$  as the cofibre of the map  $X_\infty \rightarrow BGL(A)$ .~~

We may identify the cofibre of the map  $X_\infty \rightarrow BGL(A)$  with  $BGL(A)^+$ . We know that  $X_\infty$  is then homotopy equivalent to the fibre of the map  $f: BGL(A) \rightarrow BGL(A)^+$ , hence

$$\pi_n X_\infty = K_{n+1}(A) \quad n \geq 2.$$

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Logic: Let  $f: X_0 \rightarrow Z$  be cofibre of  $X_\infty \rightarrow X$ . Then  $f$  is acyclic with  $\text{Ker } \pi_1(f) = \text{Im} \{ \text{St}(A) \rightarrow GL(A) \} = E(A)$ , so we know  $Z = BGL(A)^+$ . But also know that  $X_\infty$  is homot equiv. to fibres of  $f$ , hence long exact <sup>homotopy</sup> sequence gives

$$K_{n+1}(A) \xrightarrow{\sim} \pi_n X_\infty \quad n \geq 2$$

$$0 \rightarrow K_2(A) \rightarrow \text{St}(A) \rightarrow GL(A) \rightarrow K_1(A) \rightarrow 0$$

~~Therefore~~ Therefore

$$K_n A \cong H_n X_{n-1} \quad \text{for } n \geq 1$$

present scheme:

§1. K-groups

~~(A)~~ Defn.

Show ~~as~~ agrees Bass-Milnor

Prop. 1:  $K_1 A = GL(A)/E(A)$

$$K_2 A = H_2 B E(A)$$

In the proof you recall construction

Oron's ~~formula~~ <sup>definition of</sup> the K-groups. Start by describing system

$$\longrightarrow X_1 \longleftarrow X_0 = BGL(A)$$

then

Oron's formula

$$K_n A \cong H_n X_{n-1}$$

probably necessary to put in why process goes on

$$X_0, X_1 = B E(A), X_2 = BSt(A).$$

§ 1. Acyclic maps.

§ 2. ~~Perfect fundamental groups~~

Perfect fundamental groups

Here work with pointed con. CW cks.

① Observe

$$H_1(X) = 0 \iff \pi_1(X) \text{ perfect.}$$

Prop 1: If  $H_1(X) = 0$ ,  $\exists$  map  $f: X \rightarrow Y$  inducing iso. on homology with  $\pi_1 Y = 0$ .

Proof: Attach cells.

Prop 2: If  $f: X \rightarrow Y$  as above, then ~~use~~ obstruction theory.

Mention: Drove + Bousfield-Kan constructions

Remark: about R-perfect.

§ 3. Classification of acyclic maps with fixed source.

~~so therefore we operate~~

acyclic maps.

$$f: X \rightarrow Y$$

(i) for all local coefficient systems  $L$  on  $Y$

Claim  $\Downarrow$

$$H_*(X, f^*L) \xrightarrow{\sim} H_*(Y, L)$$

(ii) homotopy fibres of  $f$  are ~~acyclic~~ acyclic.

may assume  $Y$  connected

may assume  $f$  fibration

$$E_{pq}^2 = H_p(Y, \overset{y \mapsto H_0(f^{-1}\{y\}, f^*L)}{H_q(F, f^*L)}) \Rightarrow H_{p+q}(X, f^*L)$$

~~assume~~ as  $f^*L$  const on  $f^{-1}\{y\}$

(ii)  $\Rightarrow$   ~~$H_0(F, f^*L) = 0$~~

$$H_0(f^{-1}\{y\}, f^*L) = \begin{cases} L_y & q=0 \\ 0 & q>0 \end{cases}$$

so spec. seq. deg.  $\Rightarrow$  (i)

~~for any  $L$  on  $X$~~

$$E_{pq}^2 = H_p(Y, y \mapsto H_q(f^{-1}(y), L))$$

$$\Rightarrow H_{p+q}(X, L).$$

Now take

$$L = f^* L$$

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ p' \downarrow & & p \downarrow \text{universal covering} \\ X & \xrightarrow{f} & Y \end{array}$$

cartesian

$$H_* (\tilde{Y}, A) = H_* (Y, p_! A)$$

$$H_* (\tilde{X}, A) = H_* (X, p'_! A)$$

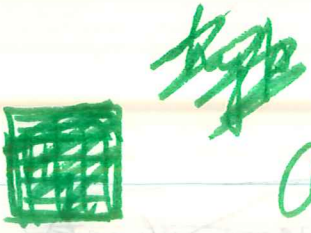
$\stackrel{f^*}{=} p_! A$

so hypothesis <sup>(i)</sup>  $\Rightarrow H_* (\tilde{X}, A) \cong H_* (\tilde{Y}, A)$

for all abelian groups  $A$ .

But  $H_1 \tilde{Y} = 0$  so comparison thm.  $\Rightarrow$  fibers of  $\tilde{f}$  are





As  $f$  and  $\tilde{f}$  have same fibres get (ii).

seems desirable to add

(i)' If  $p: \tilde{Y} \rightarrow Y$  is a universal covering of  $Y$  and  $p': \tilde{X} \rightarrow X$  induced covering. then  $\tilde{f}: \tilde{X} \rightarrow \tilde{Y}$  induces isom. on homology.

Corollary 1: Acyclic maps closed under

- composition
- products
- homotopy base change

$$\theta(E) = \theta(E) \theta(E) \dots \theta(E)$$

proof: ...

$$H^*(B\mathbb{Z}/2) \otimes H^*(B\mathbb{Z}/2) \cong H^*(B\mathbb{Z}/2)$$

~~Now suppose f~~

Classification of acyclic maps with X fixed.

Assertion: X conn. with basepoint

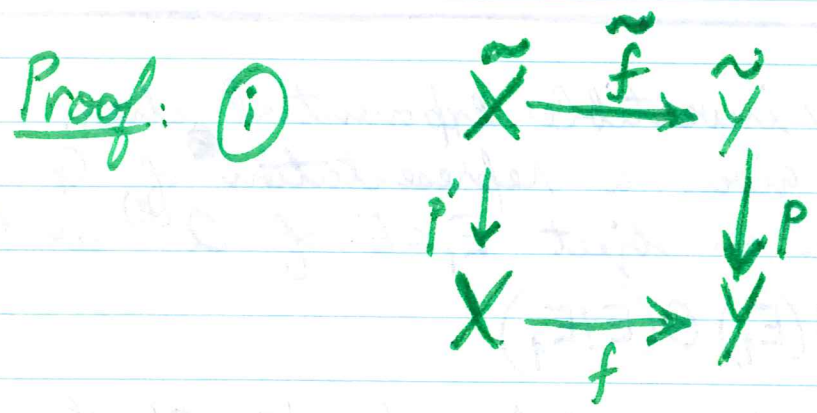
~~(i) N perfect  $\triangleleft \pi_1 X, \exists$  acyclic map  $f: X \rightarrow Y \ni \pi_1(f): \pi_1 X/N \xrightarrow{\sim} Y.$~~

(i)  $f: X \rightarrow Y$  perfect  $\implies N = \text{Ker } \pi_1(f)$  is perfect  $\triangleleft$ .

(ii) Given N perfect  $\triangleleft \pi_1 X, \exists$  acyclic  $f: X \rightarrow Y \ni \text{Ker } \pi_1(f) = N$

(iii) Universal property:

$$[Y, Z]_*^{(0)} \xrightarrow{\sim} \{g \in [X, Z] \mid \pi_1(g)(N) = 0\}.$$



$$\begin{array}{ccccc} \pi_1 F & \longrightarrow & \pi_1 \tilde{X} & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow & & \downarrow \\ \pi_1 F & \longrightarrow & \pi_1 X & \longrightarrow & \pi_1 Y \end{array}$$

shows

$$\pi_1 \tilde{X} \twoheadrightarrow N$$

But

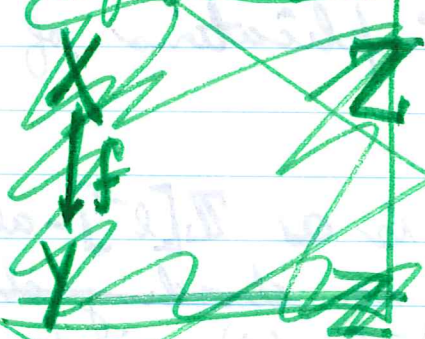
$$H_1(\tilde{X}, \mathbb{Z}) \xrightarrow{\cong} H_1(\tilde{Y}, \mathbb{Z}) = 0$$

$$\pi_1(\tilde{X})_{ab}$$

and ~~so~~  $N_{ab} = 0$ .

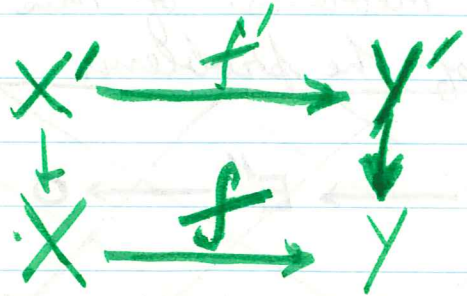
~~first step~~

iii) obstruction theory. ~~May assume f~~  
 inclusion ~~cofibration~~ ~~cofibration~~  
~~May assume f~~  
~~cofibration.~~



~~What is given~~ ~~g~~ Uniqueness: Given  
~~h~~  $u, v: Y \rightarrow Z \Rightarrow u \circ f \circ v \circ f$

(ii) First suppose  $\pi_1(X)$  perfect. Then construct  $f: X \rightarrow Y$  inducing isom on  $H_*$  with  $\pi_1(Y) = 0$ . This proves existence here. Then in general



cocartesian to get existence of  $f$

(iii) ~~Suppose  $\pi_1(X)$  perfect,  $H_*$  ...~~ theory ~~...~~ First use obstruction theory to do when  $\pi_1(Y') = 0$ .

Now generalize to cocartesian square.

Next

~~Thm~~

## §2. H-space structure.

~~Start~~ Start with  $S \# S \cong S$  yields

$GL(A) \times GL(A) \longrightarrow GL(A)$   
carries  $E(A) \times E(A)$  into  $E(A)$  clearly.

$$BGL(A) \times BGL(A) \longrightarrow BGL(A)$$

$$\downarrow f \times f$$

$$\downarrow f$$

$$BGL(A)^+ \times BGL(A)^+ \xrightarrow{\mu} BGL(A)^+$$

$\mu$  exists because  $f \times f$  is acyclic + universal property.

Prop:  $u: S \hookrightarrow S$  injection induces

homo.  $GL(A) \longrightarrow GL(A)$

~~Thm~~ The induced map

$$BGL(A)^+ \dashrightarrow BGL(A)^+$$

is homotopic (preserving basepoints) to identity.

Proof: ~~Must show  $u$  induces isomorphism~~

~~on  $h$~~  1)  $BGL(A)^+$  simple

2)  $u$  induces isomorphism on homology

So by Whitehead theorem, homotopy equivalence.

§2. ~~The~~ H-space structure on  $BGL(A)^+$ .

~~simplicity of  $BGL(A)^+$~~

$$S = N = \{0\}. \quad u: S \hookrightarrow S \text{ (inclusion)}$$

$$BGL(A) \longrightarrow BGL(A)$$

$$f \downarrow \qquad \qquad f \downarrow$$

$$BGL(A) \xrightarrow{u^+} BGL(A)^+$$

$u^+$  exists + unique up to a homot. pres. basepoints.

Prop:  $u^+$  homotopic to identity

Proof: We first show  $u^+$  is a homotopy equivalence. Claim  $u$  induces an isom in  $H_*(BE(A))$ . Indeed

$$H_*(BE(A)) = \lim H_*(BE_n(A))$$

so ~~it~~ suffices to show the <sup>two</sup> homos from  $E_n(A) \rightarrow E(A)$  given by inclusion  $i$  and  $u_i$  have the same effect on homology. But recall that even permutation matrices belong to  $E_n(A)$ , hence homos. are conjugate in  $E(A)$ , so done. Similarly  $u$  induces identity on  $H_*(BGL(A))$ , in particular on  $H_*$ . Follows that  $u$  induces isom on homology of  $BE(A)^+$ , and since latter 1-con. it is a hom. by Whitehead thm. Thus  $u$  induces hom. on

Defn:  $\tilde{R}(X, A) = [X, BGL(A)^+]$ .

Suppose to simplify that  $H$  is connected.

Then

$$St(X, A) = \varinjlim_n [X, BGL_n(A)]$$

maps to  $[X, BGL(A)]$  which in turn maps to  $[X, BGL(A)^+]$ , hence we obtain a canonical map

$$\tau : St(X, A) \longrightarrow [X, BGL(A)^+].$$

Lemma:  $\tau$  is a monoid homomorphism.

~~for the sake~~

Proof: Choose  $N' \sqcup N' \longrightarrow N'$  so that  $\{1, \dots, m\} \cup \{1, \dots, n\} \cong \{1, \dots, m+n\}$

$$BGL_m A \times BGL_n A \longrightarrow BGL_{m+n} A$$

$$\downarrow i_m \times i_n$$

$$\downarrow i_{m+n}$$

$$BGL(A) \times BGL(A) \longrightarrow BGL(A)$$

$$\downarrow f \times f$$

$$\downarrow f$$

$$BGL(A)^+ \times BGL(A)^+ \xrightarrow{\mu} BGL(A)^+$$

One possibility to use ~~earlier~~ fact that  $H$ -space structure obtained from any  $i_j$

$$N' \sqcup N' \longrightarrow N'$$

univ. covering ~~of~~  $BGL(A)$  and on ~~its~~ fund. groups so it is a hex as claimed.

Clear that  ~~$\alpha \mapsto \alpha \in$~~

$$\text{Inj}(S, S) \longrightarrow \text{Aut}(BGL(A)^+)$$

is a homom. of

Lemma: group comp. of  $\text{Inj}(S, S)$  is trivial.

Choose

$$\begin{array}{ccc} (M, (A) \circlearrowleft H) & \xrightarrow{\cong} & (M, (A) \circlearrowleft H) \\ \uparrow & & \uparrow \\ (M, (A) \circlearrowleft H) & \xrightarrow{\cong} & (M, (A) \circlearrowleft H) \end{array}$$

$$(A) \circlearrowleft H \cong (A) \circlearrowleft H$$

$$\cong \cong$$



Discuss products very carefully  
 want ~~the~~ ultimately

$$(1) \quad K_i A \otimes K_j B \longrightarrow K_{i+j}(A \otimes_{\mathbb{Z}} B)$$

enjoying habitual anti-commutativity properties.

This will follow from pairings

$$(2) \quad \tilde{K}(X, A) \otimes \tilde{K}(Y, B) \longrightarrow \tilde{K}(X \wedge Y, A \otimes B)$$

for  $X, Y$  pointed spaces, with appropriate comm. properties:

i) assoc. for  $X, Y, Z, A, B, C$

ii) comm.

$$\tilde{K}(X, A) \otimes \tilde{K}(Y, B) \longrightarrow \tilde{K}(X \wedge Y, A \otimes B)$$

$\downarrow$

$\tilde{K}(Y, B) \otimes \tilde{K}(X, A) \longrightarrow \tilde{K}(Y \wedge X, B \otimes A)$

$\downarrow$

$K_i(A) \otimes K_j(B) \longrightarrow K_{i+j}(A \otimes B)$

$\downarrow (-1)^{ij} T_*$

$K_{i+j}(B \otimes A)$

Things to check

$$R(G, A \times B) = R(G, A) \times R(G, B)$$

$$\tilde{K}(X; A \times B) = \tilde{K}(X; A) \times \tilde{K}(X; B)$$

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Important idea possibly: Relate the trace structures on  $K(X, A)$  in  $X$  and in  $A$ . Thus if  $E$  is a representation over  $A$ , then  $E^{\oplus p}$

is a representation over  $A^{\oplus p}$  with  $\mathbb{Z}/p\mathbb{Z}$  action

## Splitting theorem:

Thm: If  $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$   
is an exact sequence of reps., then  
 $[E] = [E'] + [E'']$  in  $K(X, A)$

Proof: ~~Enough to worry about stable representations.~~

$$\boxed{St(X, A) = \varinjlim_n [X, BGL_n(A)]}$$

Can assume  $X$  connected, to show  
image same in  $\tilde{K}(X, A)$ . ~~Start~~

~~with~~ Can form limit group

$$\varinjlim GL_{m,n}(A) = GL^{(2)}(A)$$

then point is that the elementary  
subgroup is perfect + = commutator  
group.

$$BGL^{(2)}(A)^+ \xrightarrow{\sim} BGL(A)^+ \times BGL(A)^+$$

Logical steps an exact sequence of representations up to stable coin map is a map ~~map~~

$$X \longrightarrow \text{BGL}^{(2)}(A)$$

Then have two maps

$$\text{GL}^{(2)}(A) \rightrightarrows \text{GL}(A)$$

corresponding to repr.  $E' \oplus E''$  and  $E$ .

Idea: Start with  $\mathcal{A}^{(2)}$  = category of exact sequences. Prove

$$\begin{aligned} \text{St}(X, \mathcal{A}^{(2)}) &= \lim_{m,n} [X, \text{BGL}_{m,n}(A)] \\ &= [X, \text{BGL}^{(2)}(A)]. \end{aligned}$$

$$\text{GL}^{(2)}(A) \stackrel{\text{defn.}}{=} \lim \text{GL}_{m,n} A$$

$$E_{m,n}(A) \stackrel{\text{defn.}}{=} \text{subgroup of } \text{GL}_{m,n} A \text{ gen. by elementary matrices}$$

Claim  $E_{m,n}(A) = (E_m A \times E_n A) \tilde{\times} \text{Hom}(A^m, A^n)$   
perfect for  $m, n \geq 3$

$$E^{(2)}(A) \stackrel{\text{defn.}}{=} \lim E_{m,n}(A)$$

Claim  $E_{\bullet}^{(2)}(A)$  perfect  $= (\text{GL}^{(2)}(A), \text{GL}^{(2)}(A))$ .

Can form

$$BGL^{(2)}(A)^+ \cong H_*(\mathbb{C}P^\infty) \leftarrow H_*(\mathbb{C}P^\infty) \cong H_*(\mathbb{C}P^\infty)$$

~~Claim this simple~~ Claim this simple

(ii) The identity of  $H_*(\mathbb{C}P^\infty)$  is the generator of  $H_0(\mathbb{C}P^\infty)$  corresponding to the component of  $\mathbb{C}P^\infty$  consisting of the null objects of  $\mathcal{A}$ .

(iii) If  $E_*$  and  $E'_*$  are representations of  $\mathcal{G}$ , then  $(E \oplus E')_* = \eta(E_* \otimes E'_*) \Delta_{\mathcal{G}}$

(ii) and (iii) imply  $\theta(E \oplus E') = \theta(E) \otimes \theta(E')$ . Therefore  $\theta$  is an exponential class map.

Let  $\theta$  is an exponential class map, we wish to show that  $\eta(\theta) = 1$  and that  $\eta(\eta) = \eta$ . The former follows from  $\theta(0) = 1$  and (ii) above. ~~It is clear that  $\eta(\eta) = \eta$  follows from the latter it will suffice to establish the formulae  $\eta(E_* \otimes E'_*) \Delta_{\mathcal{G}} = \eta(\eta(E_* \otimes E'_*)) \Delta_{\mathcal{G}}$~~

Notation:

defn  $\left\{ \begin{array}{l} GL_{m,n}(A) \subset \text{Aut}(A^m \oplus A^n) \text{ preserving } A^m \\ E_{m,n}(A) \text{ subgroup gen. by } \text{ those} \\ 1 + a e_{ij} \text{ in } GL_{m,n}(A). \end{array} \right.$

Claim  $GL_{m,n}(A) = (GL_m A \times GL_n A) \times \text{Hom}(A^n, A^m)$

$E_{m,n}(A) = ( \quad ) \times \text{---}$

~~Claim~~  $E_{m,n} A$  perfect  $\text{ if } m, n \geq 3$

$GL^{(2)} A = \text{lim}$

$E^{(2)} A = \text{lim}$

Claim  $E^{(2)} A = (GL^{(2)} A, GL^{(2)} A)$

~~Claim~~

Can form

$BGL^{(2)} A^+$

Claim simple.

And there are maps

$BGL(A)^+ \times BGL(A)^+ \begin{array}{c} \xleftarrow{j} \\ \xrightarrow{i} \end{array} BGL^{(2)}(A)^+$

such that  $ji \cong id$  (preserving points)

Theorem:  $j, i$  are homotopy equivalences

Proof: They induce isoms on homology and both spaces simple, so can apply Whitehead thm.

~~Next point is to discuss~~

~~Suppose~~

~~$$0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$$~~

~~is an exact sequence of ~~representation~~ bundles over  $X$ . ~~Then~~ get a map~~

~~$$[X \rightarrow BGL^{(2)}(A)^+]$$~~

~~such that~~

~~$j$~~

$$0 \rightarrow p^2V \rightarrow V \xrightarrow{p^2} V \rightarrow V/p^2V \rightarrow 0$$

$$p^2V = pV$$

$$pV = p^2V$$

i.e.,  $pV$  is  $p$ -divisible

$$p(p^2x) = 0 \Rightarrow p^3x = 0$$

$\Rightarrow p^2V$  is  $p$ -torsion

$$\tilde{K}(X, A) = [X, BGL(A)^+]$$

Given  $E$  over  $X$ , to define an element of  $\tilde{K}(X)$ . Assume  $X$  conn. Then get

$$f: X \longrightarrow BGL_n(A)$$

$\Rightarrow f^*(E_n)$  stably isom. to  $E$ .

essentially unique. Compose with

$$BGL_n(A) \xrightarrow{ln} BGL(A) \longrightarrow BGL(A)^+$$

This defines a natural transf

$$\gamma: St(X, A) \longrightarrow \tilde{K}(X, A).$$

Claim  $\boxed{\gamma(E) = \gamma(E') + \gamma(E'')}.$

~~$0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$~~   
 $GL_{m,n}(A)$  autos of

$$0 \rightarrow A^m \rightarrow A^m \oplus A^n \rightarrow A^n \rightarrow 0$$

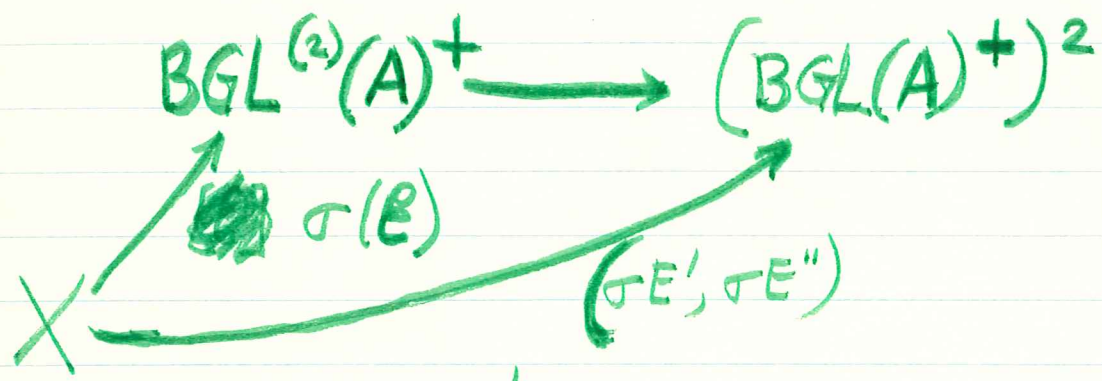
Then this gives rise to a canonical exact sequence of bundles on  $BGL_{m,n}(A)$ . Now if

$$0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$$

is any exact sequence of bundles over  $X$  say conn. then by adding trivial bundles can suppose ~~that~~ stalks of  $E', E''$  are isom. to  $A^n$ , whence the exact sequence is induced



Let  $\{m_a, m'_a\}$  be a non-zero solution of this congruence such that  $\sum (m_a + 2m'_a)$  is minimal. Clearly ~~it is clear that~~  $m'_a = 0$  for all  $a$ . If  $m_b \geq p$  for some  $b$ , then by replacing  $m_b$  by  $m_b - p$  and  $m_{b+1}$  by  $m_{b+1} + 1$  (or  $m_0$  by  $m_0 + 1$  if  $b = d-1$ ) and keeping the others ~~the~~ the same, we would get a new solution, contradicting minimality. Thus  $m_a < p$  for  $0 \leq a < d$ , ~~and~~ so <sup>using</sup> the uniqueness of the  $p$ -adic expansion of a natural number, we see that the minimal non-zero solution is  $m_a = 1, m'_a = 0$  for all  $a$ . The minimal degree is  $\sum (m_a + m'_a) = d$  showing that  $H_f(N)$  ~~has no non-trivial~~ does not contain the trivial repn for  $0 < \frac{1}{2} < d$ . The proof of the lemma is complete.



so what becomes difficult is map  $GL^{(2)}(A)$ .

§1. Killing a perfect subgroup of the fundamental group. Let  $f: X \rightarrow Y$  be a map of pointed connected CW complexes and form the cartesian square

$X'$  primes

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ g' \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

where  $Y'$  is the universal covering of  $Y$  and  $X'$  is the induced principal covering of  $X$  with group  $\pi_1 Y$ . We assume that  $f'$  induces an isomorphism on integral homology. Then  $X'$  is connected, hence

~~the~~ the homomorphism  $\pi_1(f): \pi_1 X \rightarrow \pi_1 Y$  is surjective. Its kernel  $N$  is the fundamental group of  $X'$ , hence  $N$  is perfect, i.e. equal to its commutator subgroup, because by Poincaré's theorem  $N^{ab} = H_1 X' = H_1 Y' = 0$ .

If  $Z$  is a pointed space, we denote by  $\underline{\text{Hom}}(X, Z)_0$  the space of ~~maps~~ basepoint-preserving maps from  $X$  to  $Z$  and by  $[X, Z]_0$  the homotopy classes of these maps.

Proposition 1.1: The map  $f^*: \underline{\text{Hom}}(Y, Z)_0 \rightarrow \underline{\text{Hom}}(X, Z)_0$  is a weak homotopy equivalence of the former space with the subspace of the latter consisting of those  $g: X \rightarrow Z$  such that  $\pi_1(g)(N) = 0$ . In particular

$\pi_1$  one  
or equivalently that  $f$  induces isomorphisms on homology with coefficients in any  $\pi_1 Y$ -module.

$$\bullet [Y, Z]_0 \xrightarrow{\sim} \{ \alpha \in [X, Z]_0 \mid \pi_1(\alpha)(N) = 0 \}.$$

We may assume  $f$  is the inclusion of  $X$  as a subcomplex of  $Y$  in which case  $f^*$  is a fibration. To see that ~~the~~ the fibres of  $f^*$  are weakly contractible it suffices to show that the inclusion

$$(X \times I) \cup (Y \times I) \longrightarrow Y \times I$$

is a homotopy equivalence.

~~group of the former space is~~

~~The fundamental~~

$$\pi_1 Y *_{\pi_1 X} \pi_1 Y \cong \pi_1 Y$$

~~by the van Kampen theorem so this map induces an isomorphism on fundamental groups. Using the Mayer-Vietoris theorem for homology with ~~coefficients~~ local coefficients we that this map induces isomorphisms on such homology provided  $f$  does.~~

By the Whitehead theorem ~~in~~ in the form used by Artin-Mazur [ ] it suffices to show the map induces isomorphisms on fundamental groups and on ~~homology~~ homology with coefficients in any  $\pi_1 Y$ -module  $L$ . Writing the former space as the union of the complements of  $Y \times \{0\}$  and  $Y \times \{1\}$ , its fundamental group ~~is~~ is

$$\pi_1 Y *_{\pi_1 X} \pi_1 Y \cong \pi_1 Y$$

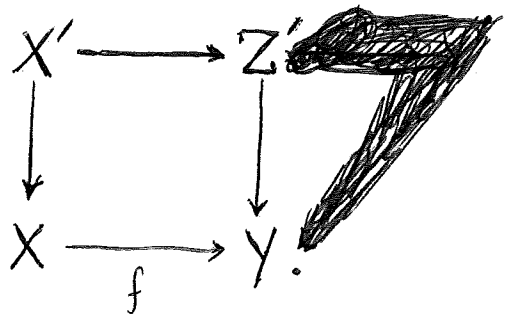
by the van Kampen theorem, so the map on fundamental group is an isomorphism. Similarly the Mayer-Vietoris theorem shows the map on homology with coefficients in  $L$  is an isomorphism.

It remains to show that a map  $g: X \rightarrow Z$  such that  $\pi_1(g)(N) = 0$  can be extended to  $Y$ . This condition on  $g$  allows one to extend  $g$  to the 2-skeleton of  $X$  relative to  $Y$ . Given an extension to the ~~relative~~ relative  $n$ -skeleton, it can be modified without change on the  $(n-1)$ -skeleton so as to extend to the  $(n+1)$ -skeleton provided an obstruction in the ~~relative~~ cohomology group  $H^{n+1}(Y, X; \pi_n Z)$  vanishes, where  $\pi_n Z$  is ~~regarded~~ regarded as a  $\pi_1 Y$ -module by the map  $\pi_1 Y \rightarrow \pi_1 Z$  induced by  $g$ . But this cohomology group ~~is~~ is zero by the hypothesis on  $f$ , so an extension of  $g$  ~~to~~ to all of  $Y$  exists and the proposition is proved.

Proposition 1.2: Given a perfect subgroup  $E$  of  $\pi_1 X$ , where  $X$  is a pointed connected CW complex, then there is a map  $f: X \rightarrow Y$  as above ~~with~~ ~~where~~ such that  $\text{Ker } \pi_1(f) = \text{the normal subgroup generated by } E$ .

If  $N$  is the normal subgroup generated by  $E$ , then ~~the~~ the commutator subgroup  $[N, N]$  ~~is~~ is a normal subgroup containing  $[E, E] = E$ .

hence also  $N$ , so  $N$  is perfect and we may suppose  $E=N$ . Let  $\square \quad \blacksquare \quad X' \rightarrow X$  be the covering space of  $X$  with fundamental group  $N$ , whence  $H_1 X' = N^{ab} = 0$ . Choose generators for  $N$  as a normal subgroup of  $\pi_1 X$  and form a space  $Z$  by attaching 2-cells to  $X'$  by means of maps  $S^1 \rightarrow X'$  representing these generators. Then  $Z$  is ~~simply~~ simply-connected and  $H_i X' \xrightarrow{\sim} H_i Z$  except when  $i=2$  when  $H_2 Z$  is the direct sum of  $H_2 X'$  and a free abelian group with generators coming from the attached 2-cells. By the Hurewicz theorem  $\pi_2 Z \cong H_2 Z$ , hence we may attach 3-cells to  $Z$  to obtain a simply-connected ~~space~~ <sup>CW complex</sup>  $Z'$  ~~such that the inclusion map  $X' \rightarrow Z'$  induces isomorphisms on homology.~~ such that the inclusion map  $X' \rightarrow Z'$  induces isomorphisms on homology. Define  $\gamma$  and  $f$  by a cocartesian square



~~Let  $\tilde{X} \rightarrow X$  be the universal covering of  $X$  and  $h$  be the unique map making the triangle commute. By the van Kampen and Mayer-Vietoris theorems one sees that  $\pi_1(f)$  is surjective with kernel  $N$  and that~~

By the van Kampen theorem

$$\pi_1 Y = \pi_1 X *_{\pi_1 X'} \pi_1 Z' = \pi_1 X / N$$

so  $\pi_1(f)$  is surjective with kernel  $N$ . If  $L$  is a  $\pi_1 Y$ -module there is a Mayer-Vietoris long exact sequence

$$\rightarrow H_i(X', L) \rightarrow H_i(X, L) \oplus H_i(Z', L) \rightarrow H_i(Y, L) \rightarrow \dots$$

and  $H_i(X', L) \xrightarrow{\sim} H_i(Z', L)$  because  $\pi_1(X')$ ,  $\pi_1(Z')$  act trivially on  $L$  as  $Z'$  is simply-connected and because  $X'$  and  $Z'$  have the same homology by construction. Thus  $f$  induces isomorphisms on homology with coefficients in all  $\pi_1 Y$ -modules, so the proof of the proposition is complete.

According to proposition 1.2 the <sup>(pointed space)</sup> ~~space~~  $Y$  constructed above is characterized up to homotopy equivalence by the fact that it comes with a universal arrow  $f: X \rightarrow Y$  killing the subgroup  $E$  of  $\pi_1 X$ . We shall use the notation  $X/E$  for the space  $Y$ . This notation is justified by the fact that  $Y$  ~~is~~ as an object of the ~~pointed space~~ homotopy category of pointed spaces is the quotient of  $X$  by the action of  $E$  obtained from the natural action of  $\pi_1 X$  on  $X$ .

Proposition 13: Let  $X_1$  and  $X_2$  be pointed connected CW complexes and let  $E_i$  ~~be a normal subgroup~~ be a perfect subgroup of  $\pi_1 X_i$  for  $i=1,2$ . Then the canonical map in the pointed homotopy category

$$(X_1 \times X_2)/(E_1 \times E_2) \longrightarrow (X_1/E_1) \times (X_2/E_2)$$

is ~~an isomorphism~~ an isomorphism.

It follows from the fact that if  $Y_i'$  is the universal covering of  $Y_i = X_i/E_i$  and  $X_i'$  the induced covering of  $X_i$ , then  $Y_1' \times Y_2'$  is the universal covering of  $Y_1 \times Y_2$  and  $X_1' \times X_2' \rightarrow Y_1' \times Y_2'$  induces isomorphisms on homology.

§2. The K-theory associated to a ring. Let  $A$  be a ring with unit but not necessarily commutative. Let  $GL_n A$  be the group of invertible ~~matrices~~  $n \times n$  matrices over  $A$ , let  $E_n A$  be the subgroup generated by the elementary matrices, and let  $GL(A)$ ,  $E(A)$  be the respective unions of these groups for all  $n \in \mathbb{N}$ . Then ~~the~~  $E(A)$  and  $E_n(A)$  for  $n \geq 3$  are perfect groups, hence <sup>by attaching 2-cells and 3-cells</sup> we can form the spaces

$$BGL_n A^+ = BGL_n A / E_n A$$

$$BGL(A)^+ = BGL(A) / E(A)$$

where  $BG$  denotes a classifying space for the group  $G$  which is a pointed CW complex. We define

$$K_i A = \pi_i(BGL(A)^+) \quad i \geq 1.$$

Since  $E(A)$  ~~is~~ is the commutator subgroup of  $GL(A)$  it is a normal subgroup so

$$K_1 A = GL(A) / E(A)$$

is the same as ~~the~~ the  $K_1$  of Bass [ ]. ~~The universal covering of  $BGL(A)^+$  is a simply connected space with the same homology as  $BE(A)$ , hence~~

$$K_2 A = \pi_2$$



Moreover there is a cartesian square

$$\begin{array}{ccc} BE(A) & \xrightarrow{f'} & BE(A)^+ \\ \downarrow & & \downarrow \\ BGL(A) & \xrightarrow{f} & BGL(A)^+ \end{array}$$

of covering spaces where  $BE(A)^+$  is the universal covering of  $BGL(A)^+$  and where  $f'$  induces isomorphisms on homology. Hence

$$\begin{aligned} K_2 A &= \pi_2 BGL(A)^+ = \pi_2 BE(A)^+ \\ &= H_2(BE(A)^+) = H_2(BE(A)) \end{aligned}$$

so this  $K_2$  agrees with the  $K_2$  of Milnor [ ].

The inclusions  $GL_n A \rightarrow GL_{n+1} A \rightarrow \dots \rightarrow GL(A)$  give rise to maps unique up to homotopy

$$BGL_n A^+ \rightarrow BGL_{n+1} A^+ \rightarrow \dots \rightarrow BGL(A)^+$$

and hence to at least one map

$$\varinjlim_n BGL_n(A)^+ \rightarrow BGL(A)^+$$

where  $\varinjlim_n$  denotes the infinite mapping cylinder. This last map ~~is~~ is a homotopy equivalence because both spaces have the same fundamental group and their universal

coverings have the same homology! ~~Therefore~~ Therefore for any ~~pointed~~ pointed finite complex  $X$  we have

$$[X, BGL(A)^+]_0 = \varinjlim [X, BGL_n A^+]_0.$$

~~One can pose the~~ One can pose the problem of whether the stability theorems of algebraic K-theory ~~admit~~ admit generalizations asserting that the map  $[X, BGL_n A^+]_0 \rightarrow [X, BGL(A)^+]_0$  is surjective (resp. an isomorphism) for  $n \geq \dim X + \dim(\text{Max } A)$  (resp. for  $n > \dim X + \dim(\text{Max } A)$ ).

The ~~map~~ <sup>homomorphism</sup>  $GL_m A \times GL_n A \rightarrow GL_{m+n} A$  obtained from the direct sum of matrices carries  $E_m A \times E_n A$  into  $E_{m+n} A$ , hence using proposition 3 it induces a map

$$\mu_{m,n} : BGL_m A^+ \times BGL_n A^+ \rightarrow BGL_{m+n} A^+.$$

The collection of  $\mu_{m,n}$  is homotopy associative in an evident sense. ~~Moreover~~ <sup>moreover</sup> if  $x$  and  $y$  are  $m \times m$  and  $n \times n$  matrices ~~respectively~~ respectively, then the matrices

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \quad \begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}$$

are conjugate by the matrix

$$\sigma = \begin{pmatrix} 0 & (-1)^{mn} I_n \\ I_m & 0 \end{pmatrix}$$

$I_n = \text{identity } n \times n \text{ matrix}$

which belongs to  $SL_{m+n} \mathbb{Z} = E_{m+n} \mathbb{Z} \subset E_{m+n} A$ . Hence the diagram

$$\begin{array}{ccc}
 BGL_m A \times BGL_n A & \xrightarrow{\oplus} & BGL_{m+n} A \\
 \downarrow T & & \downarrow \tilde{\sigma} \\
 BGL_n A \times BGL_m A & \xrightarrow{\oplus} & BGL_{m+n} A \longrightarrow BGL_{m+n} A^+
 \end{array}$$

is commutative in the pointed homotopy category, where  $T(u,v) = (v,u)$  and  $\tilde{\sigma}$  is induced from the conjugation by  $\sigma$ , so  $\mu_{m,n}$  and  $\mu_{n,m} T$  are homotopic. Consequently  $\coprod_n [X, BGL_n A^+]_0$  is an abelian monoid, ~~so~~  $[X, BGL(A)^+]_0$  is an abelian monoid in a natural way for any pointed finite complex  $X$ , and in fact even an abelian group as one sees easily by ~~the~~ induction on the number of cells in  $X$ . In other words  $BGL(A)^+$  is a weak <sup>homotopy commutative and associative</sup> H-space and in particular it is simple, i.e. the fundamental group acts trivially on the other homotopy groups. Actually with more work we shall show that  $BGL(A)^+$  is an infinite loop space (§).

Proposition 2.1: The Hurewicz homomorphism for  $BGL(A)^+$  induces an isomorphism of  $K_i A \otimes \mathbb{Q}$  with the primitive subspace of degree  $i$  of the Hopf algebra  $H_*(GL(A), \mathbb{Q})$ .

This follows from a theorem of Milnor-Moore ([ ], appendix) ~~provided~~ once we know that  $BGL(A)^+$

is an H-space, ~~which is not a CW complex~~ This follows from the weak H-space structures when  $BGL(A)^+$  is a countable CW complex in virtue of the surjectivity of the natural map  $[\lim_n X_n, Z]_0 \rightarrow \text{invarlim} [X_n, Z]_0$ , so the proposition is true for  $A$  countable, hence in general by passage to the limits.

Finally suppose  $A$  is commutative and let  $m$  be an integer ~~prime to~~ which is a unit in  $A$ . Then Grothendieck [ ] has defined Chern classes

check  $\mu$

$$c_i \in H^{2i}(\text{Spec } A, GL_n A; \mu_m^{\otimes i})$$

~~the~~ taking values in the étale cohomology of the scheme  $\text{Spec } A$  with  $GL_n A$  as operators acting trivially. If  $I^\bullet$  is ~~the~~ the cochain complex of global sections of an injective resolution of the étale sheaf  $\mu_m^{\otimes i}$  on  $\text{Spec } A$ , then the class  $c_i$  may be identified with a cohomology class of  $BGL_n A$  with coefficients in  $I^\bullet$ , or equivalently with a homotopy class of maps

$$BGL_n(A) \longrightarrow K(I^\bullet, 2i)$$

where  $K(I^\bullet, 2i)$  is the generalized Eilenberg-MacLane space whose  $j$ -th homotopy group is the  $(2i-j)$ -th cohomology group of  $I^\bullet$ . Since  $K(I^\bullet, 2i)$  is a

simple spaces <sup>the above map</sup> kills  $E_n(A)$  hence induces a ~~map~~ a map unique up to homotopy.

$$\langle \text{scribble} \rangle BGL_n(A)^+ \longrightarrow K(\mathbb{Z}; 2i).$$

These maps are compatible as ~~...~~  $n$  goes to infinity since  $c_i$  is a stable class, hence passing to homotopy groups we get a well-defined homomorphism

$$c_i^\# \langle \text{scribble} \rangle : K_j A \longrightarrow H^{2i-j}(\text{Spec } A, \mu_m^{\otimes i})$$

from  $K$ -groups to étale cohomology. ~~...~~ Hopefully this homomorphism ~~...~~ coincides with the ones constructed by Bass-Tate (see [1]) when  $A$  is a field and  $i, j$  are both 1 or 2. In a similar way using Chern classes in Hodge cohomology, <sup>[1]</sup> one obtains homomorphisms

$$c_j^\# : K_j A \longrightarrow \Omega_{A/\mathbb{Z}}^j$$

for  $j \geq 1$ .

§3. Splitting exact sequences stably. Let  $GL_{m,n}A$  be the group of automorphisms of the right  $A$ -module  $A^m \oplus A^n$  which preserve the second factor, and let

$$p: GL_{m,n}A \longrightarrow GL_m A \times GL_n A$$

be the evident surjection. In the case of topological  $K$ -theory where  $A$  is a commutative Banach algebra and these linear groups are endowed with the topologies induced by the topology of  $A$ , one knows that the map on classifying spaces induced by  $p$  is a homotopy equivalence. But in the algebraic case under consideration here, this isn't the case. Aside from the fact that the fundamental groups are ~~different~~ not the same, a problem which can be side-stepped by killing elementary subgroups, the two groups have different homology in general, e.g. when  $A$  is of characteristic  $p$  and  $m=n=1$ . Nevertheless we shall see that these two groups have the same homology in the limit when  $m, n = \infty$ .

~~Using the embedding  $A^n \rightarrow A^{n+1}$  with last coordinate zero we obtain injective homomorphisms  $GL_{m,n}A \rightarrow GL_{m,n+1}A$~~

Let  $GL_{(\infty)}A$  be the union of the  $GL_{m,n}A$  with respect to the inclusions induced by the standard injection  $A^n \rightarrow A^{n+1}$  with last coordinate zero;  $GL_{(\infty)}A$  is the ~~stabilizer~~ group of

Automorphisms of the right  $A$ -module  $A^\infty \oplus A^\infty$  which preserve the second factor and ~~whose~~ whose matrix is almost everywhere the identity. The following will be proved in the next section.

Theorem 3.1: The surjection  $GL_{(2)}A \rightarrow GL(A)^2$  induces isomorphisms on the homology of the classifying spaces.

~~Before giving the proof we deduce some consequences of the theorem.~~  
 Before giving the proof we deduce some consequences. Note that  $GL_{m,n}A$  is the semi-direct product  

$$GL_{m,n}A = (GL_m A \times GL_n A) \tilde{\times} Hom_A(A^m, A^n)$$

~~Before giving the proof we deduce some consequences of the theorem.~~ Let  $E_{m,n}A$  be the subgroup of  $GL_{m,n}A$  generated by elementary matrices, i.e. ~~which~~ <sup>matrices</sup> which becomes elementary matrices in  $GL_{m+n}A$  under the obvious embedding. Then there are semi-direct product decompositions

$$GL_{m,n}A = (GL_m A \times GL_n A) \tilde{\times} Hom_A(A^m, A^n)$$

$$E_{m,n}A = (E_m A \times E_n A) \tilde{\times} Hom_A(A^m, A^n)$$

and it is easy to see that  $E_{m,n}A$  is perfect for  $m, n \geq 3$ .

In the limit as  $m, n$  go to infinity we get a perfect subgroup  $E_{(2)}A$  of  $GL_{(2)}A$  which is in fact the commutator subgroup. ~~Let~~

$$BGL_{m,n}A^+ = BGL_{m,n}A / E_{m,n}A$$

$$BGL_{(2)}A^+ = BGL_{(2)}A / E_{(2)}A.$$

Corollary 3.2: The natural map  $BGL_{(2)}A^+ \rightarrow (BGL(A)^+)^2$  is a homotopy equivalence.

In effect one ~~shows~~ shows that  $BGL_{(2)}A^+$  is a weak H-space ~~by~~ by the same argument used for  $BGL(A)^+$  in the preceding section. Thus both spaces are ~~simple and the map induces an isomorphism on homology~~ simple and the map induces an isomorphism on homology by the theorem, hence the map is a homotopy equivalence by the Whitehead theorem.

~~If  $\rho$  is a representation of a group  $G$  over  $A$ , i.e. a projective finitely-generated  $A$ -module endowed with a linear action of  $G$ , then to  $\rho$  we can associate an element  $(\rho) \in [BG, BGL(A)^+]$  as~~



~~By a representation~~

By a representation of a group  $G$  over  $A$  we mean a finitely generated projective  $A$ -module endowed with a linear action of  $G$ . Call two representations stably ~~isomorphic~~ isomorphic if they become isomorphic ~~after~~ after adding trivial representations, and let  $I_A G$  be the abelian monoid of ~~stable~~ stable isomorphism classes of representations. Then

$$I_A G = \varinjlim_n \text{Hom}_{\mathcal{C}}(G, GL_n A)$$

where  $\mathcal{C}$  the category of "groups up to inner automorphisms". Now ~~the isomorphism of  $BGL_n A$  to  $BGL(A)^+$~~  if  $j: BGL_n A \rightarrow BGL(A)^+$  is the canonical map and if  $\tilde{\sigma}$  denotes the endomorphism of  $BGL_n A$  produced by an inner automorphism  $\sigma$  of  $GL_n A$ , then  $j\tilde{\sigma}$  is homotopic to  $j$  preserving basepoints, because  $\sigma$  can be effected by an inner automorphism in  $GL(A)$  coming an element of  $E(A)$  and  $E(A)$  gets killed in  $\pi_1 BGL(A)^+$ . Consequently to each representation  $V$  of  $G$  over  $A$  is associated an element

$$(E) \in [BG, BGL(A)^+]$$

depending only on the stable isomorphism class of  $E$ .

Corollary 3.3: If  $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$  is an exact sequence of representations of  $G$  over  $A$ ,

then  $(E) = (E' \oplus E'')$ .

It suffices to consider the case where  $G = GL_{m,n} A$  acting in the natural way on  $E = A^m \oplus A^n$  with  $E'$  = the invariant submodule  $\circlearrowleft A^n$ . By using ~~the~~ the isomorphism  $\overset{\varepsilon_{mn}}{A^m \oplus A^n} \simeq A^{m+n}$  which one obtains by enumerating the <sup>standard</sup> basis for  $A^m \oplus A^n$ , we get two homomorphisms  $u_1, u_2$  from  $GL_{m,n} A$  to  $GL_{m+n} A$  corresponding to the representations  $E$  and  $E' \oplus E''$ , and we must show that the effect of  $u_1$  and  $u_2$  on classifying spaces becomes the same after composing with the canonical map from  $BGL_{m+n} A$  to  $BGL(A)^+$ . ~~Now this situation~~ Extend  $\varepsilon_{mn}$  to an isomorphism  $\varepsilon: A^\infty \oplus A^\infty \xrightarrow{\sim} A^\infty$  ~~preserving the standard bases.~~ preserving the standard bases. Then  $u_1$  and  $u_2$  extend to two similarly defined homomorphisms ~~it~~  $v_1, v_2: GL_{(2)} A \rightarrow GL(A)$  and ~~it~~ it suffices to show that the two maps from  $BGL_{(2)} A^+$  to  $BGL(A)^+$  induced by  $v_1$  and  $v_2$  are homotopic. But this is clear because  $v_1$  and  $v_2$  agree on the subgroup  $GL(A)^2$  of  $GL_{(2)} A$  and because the map  $(BGL(A)^+)^2 \rightarrow BGL_{(2)} A^+$  induced is a homotopy equivalence since it is the inverse of the map of 3.2.

51. Killing a perfect subgroup of the

fundamental group. In this section we work only

with pointed ~~connected~~ spaces and with maps

preserving basepoints. Recall that a group is

called perfect if it is equal to its commutator

subgroup.

Let  $X$  be a pointed connected CW complex

and let  $E$  be a perfect subgroup of  $\pi_1 X$ . Let

$g': X' \rightarrow X$  be the covering space <sup>of  $X$</sup>  with  $\pi_1 X' \cong E$ .

By Poincaré's theorem  ~~$H_1 X' \cong E^{ab}$~~

$H_1 X' \cong E^{ab} = 0$ . Choose generators for  $E$ ,  ~~$E$~~

represent them by maps  $u_i: S^1 \rightarrow X'$ ,  $i \in I$ ,

and let  ~~$X''$~~   $X''$  be the ~~subgroup~~ CW complex

obtained by attaching 2-cells to  $X'$  with boundaries


$u_i$ . By the van Kampen theorem  $\pi_1 X'' = 0$ ; ~~from~~ from the long exact <sup>(integral) homology</sup> sequence of the pair  $(X'', X')$  we see ~~that~~ that  $H_g X' \cong H_g X''$  for ~~g ≠ 2~~  $g \neq 2$  and that there is an exact sequence

$$0 \rightarrow H_2 X' \rightarrow H_2 X'' \rightarrow F \rightarrow 0$$

where  $F$  is a free abelian group with generators indexed by  $I$ . Since  $X''$  is simply-connected,  ~~$\pi_2 X'' \cong H_2 X''$~~   $\pi_2 X'' \cong H_2 X''$  by the Hurewicz theorem, hence we may choose maps  $v_i : S^2 \rightarrow X''$ ,  ~~$v_i$~~   ~~$i \in I$~~   $i \in I$ , whose images in  $H_2 X''$  form a basis for a free abelian <sup>subgroup</sup> ~~group~~ mapped isomorphically to  $F$ . Attaching 3-cells to  $X''$  with boundaries  $v_i$  we obtain a simply-connected CW complex  $Y'$  containing  $X'$  such that the

inclusion map  $f': X' \rightarrow Y'$  induces isomorphisms on homology.

Define  $Y$  and  $f$  by a cocartesian diagram

(1)  
$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ g' \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

Clearly  $f$  kills  $E$ , i.e.  $\pi_1(f)(E) = 0$ . Suppose

$u: X \rightarrow Z$  kills  $E$ . Then  $ug': X' \rightarrow Z$

lifts to ~~the universal covering~~ the universal covering  $\tilde{Z}$  of  $Z$ . The

obstructions to extending this map  $X' \rightarrow \tilde{Z}$  <sup>(to  $Y'$ )</sup> lie in

the groups  $H^{n+1}(Y', X'; \pi_n Z)$  which are zero

since  $f'$  ~~induces isomorphisms on homology~~ induces isomorphisms on homology.

~~by obstruction theory any~~ <sup>by obstruction theory any</sup> ~~relative to  $X'$~~  <sup>relative to  $X'$</sup>  Similarly two such extensions are homotopic.

Thus ~~there exists~~ <sup>there exists</sup> an extension  $v': Y' \rightarrow Z$  of  $ug'$

unique up to homotopy relative to  $X'$ , hence ~~also~~

also an extension  $v: Y \rightarrow Z$  of  $u$  <sup>which is</sup> unique up to homotopy relative to  $X$ . ~~also~~

~~Using~~ Using the homotopy extension theorem it

~~is a perfect subgroup~~

~~follows~~ follows that if  $v_0, v_1: Y \rightarrow Z$  are two maps such that  $v_0 \circ f$  and  $v_1 \circ f$  are homotopic,

then  $v_0$  and  $v_1$  are homotopic, ~~so~~ <sup>so</sup> we have

proved the following proposition.

~~Proposition 1.1: Let  $E$  be a perfect subgroup of  $\pi_1 X$ , where  $X$  is a pointed connected CW complex. Then if  $E$  is a perfect subgroup of  $\pi_1 X$ , then there exists a map  $f: X \rightarrow Y$  ~~such that~~ there exists a map  $f: X \rightarrow Y$  with  $Y$  a ~~CW complex~~ ~~in the homotopy category of pointed~~ ~~pointed connected CW complex which is~~ spaces which is "universal" for killing  $E$ , i.e.~~

■  $f^*: [Y, Z]_0 \xrightarrow{\cong} \{u \in [X, Z]_0 \mid \pi_1(u)(E) = 0\}$ ,

Denote by  $\mathcal{H}_0$  the homotopy category of pointed spaces, i.e. the category with pointed spaces for objects and with the set of morphisms from  $A$  to  $Z$  defined to be the set  $[A, Z]_0$  of homotopy classes of basepoint-preserving maps.

Proposition 1.1: Let  $E$  be a perfect subgroup of  $\pi_1 X$ , where  $X$  is a pointed connected CW complex. Then there exists a map  $f: X \rightarrow Y$  with  $Y$  a pointed connected CW complex which is "universal" in  $\mathcal{H}_0$  for killing  $E$ , i.e.

$$[Y, Z]_0 \xrightarrow{\sim} \{u \in [X, Z]_0 \mid \pi_1(u)(E) = 0\}$$

for all  $Z$ .

It follows from this universal ~~property~~ property that the pair  $(Y, f)$  is unique up to isomorphism in

$\mathcal{H}_0$  and in particular is independent of the choices made in its construction. We use the notation  $X/E$  for  $Y$ , this notation being justified by the following considerations. Recall that there is a natural action of  $\pi_1 X$  on  $X$  as an object of  $\mathcal{H}_0$  which induces the conjugation action of  $\pi_1 X$  on itself.

Hence  $E$  acts on  $X$  and as  $E$  is perfect it is clear that a map  $u: X \rightarrow Z$  kills  $E$  iff

~~u \tilde{g} = u~~  $u \tilde{g} = u$  for all  $g \in E$ ,

where  $\tilde{g}: X \rightarrow X$  ~~denotes~~ denotes the endomorphism of  $X$  associated to the element  $g$  of  $\pi_1 X$ . Thus  $X/E$  is

the quotient in the category sense of  $X$  by the action of  $E$ .

We finish this section by ~~deriving~~ deriving a



useful description of  $X/E$ .

Proposition 1.2: The map  $f: X \rightarrow Y$  of 1.1 ~~are~~ are

characterized up to isomorphism in  $\mathcal{H}_0$  by the

following properties:

(i)  $f$  induces an isomorphism  $\pi_1 X/N \xrightarrow{\cong} \pi_1 Y$

where  $N$  is the normal subgroup of  $\pi_1 X$  generated

by  $E$ .

(ii) If  $\tilde{Y} \rightarrow Y$  is the universal covering of  $Y$  and

$\tilde{X} \rightarrow X$  is the covering induced by  $f$ , then the map

$\tilde{X} \rightarrow \tilde{Y}$  induces isomorphisms on homology. Equivalently:

(ii)'  $f$  induces an isomorphism  $H_*(X, L) \xrightarrow{\cong} H_*(Y, L)$

~~on homology with (twisted) coefficients~~ on homology with (twisted) coefficients

in the  $\pi_1 Y$ -module  $L = \mathbb{Z}[\pi_1 Y]$ .

(The equivalence of (ii) and (ii)' results from the

fact that  $H_*(X, L) = H_*(\tilde{X}, \mathbb{Z})$  and similarly for  $Y, \tilde{Y}$ .)

If  $f$  is the map of 1.1, then ~~the~~ the van Kampen theorem applied to the square (1) ~~proves~~ ~~proves~~ (i). Similarly as (1) is cocartesian, ~~the~~ (ii)' follows once we know  $f': X' \rightarrow Y'$  induces isomorphism on homology with twisted coefficients in  $L$ . But this is clear since  $Y'$  is simply-connected and since  $f'$  induces isomorphism on homology. Thus  $f$  satisfies (i) and (ii).<sup>□</sup>

Conversely suppose  $f: X \rightarrow Y$  satisfies (i) and (ii), ~~then~~ ~~then~~ let  $p: X \rightarrow X/E$  be a universal map killing  $E$ , and let  $h: X/E \rightarrow Y$  be the ~~unique map in  $\mathcal{H}_0$~~  unique map in  $\mathcal{H}_0$  with  $hp = f$ . Then as  $p$  satisfies conditions analogous to

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(i) and (ii) by what we've just proved, it follows that  $h$  induces isomorphisms on fundamental groups and isomorphisms on homology of the universal coverings. By the Whitehead theorem  $h$  is a homotopy equivalence, ~~so~~ so  $f$  is isomorphic in  $\mathcal{H}_0$  to  $p$ , completing the proof of the proposition.

Remark 1.3: The commutator subgroup  $(N, N)$  of  $N$  is a normal subgroup of  $\pi_1 X$  containing  $(E, E) = E$ , hence  $(N, N)$  contains  $N$  and  $N$  is perfect. Since  $X/E \cong X/N$  in  $\mathcal{H}_0$  ~~by the universal property~~ by the universal property, we ~~can construct~~ ~~the map~~ ~~can restrict~~ attention to the case where  $E$  is normal.

1.4: If  $E$  is normal, then ~~the~~ the  $X'$  in

square (1) is equal to  $\tilde{X}$ , hence the lifting  $Y' \rightarrow \tilde{Y}$  of  $g$  is a homotopy equivalence by the Whitehead theorem, as both spaces are simply-connected with the same homology as  $\tilde{X}$ . Thus when  $E$  is normal the square (1) is up to homotopy equivalence both cartesian and cocartesian.

~~Corollary~~ Corollary 1.5: Let  $X_1$  and  $X_2$  be pointed connected CW complexes and let  $E_i$  be a perfect subgroup of  $\pi_1 X_i$  for  $i=1,2$ . Then the canonical map in  $\mathcal{H}_0$

$$(X_1 \times X_2) / (E_1 \times E_2) \longrightarrow (X_1 / E_1) \times (X_2 / E_2)$$

is an isomorphism.

This follows easily ~~by the same argument~~ from 1.2.

Theorem: ~~Assume~~ Assume  $\Gamma$  has no non-trivial perfect subgroup in  $\pi_1$  of any of its components. Then

$$\text{Hom}(\tilde{R}(;A), [L, \Gamma]) \xrightarrow{\sim} \text{Hom}(R(;A), [L, \Gamma])$$

Proof: ① First show true for stable reps:

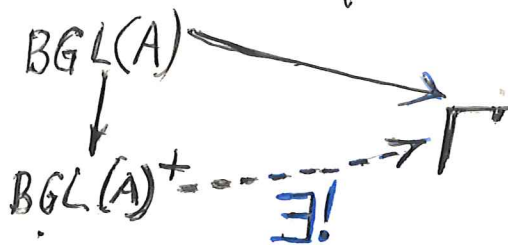


diagram of weak maps

by ~~writing~~ writing

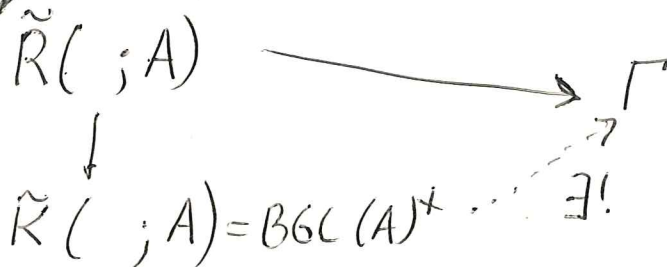
$$\begin{aligned} BGL(A) &= \cup Z_i \\ BGL(A)^+ &= \cup Z_i^+ \end{aligned}$$

where  $Z_i^+ = Z_i \cup_{Z_0} Z_0^+$ , and using

$$[Z_i^+, \Gamma] \xrightarrow{\sim} [Z_i, \Gamma]$$

② same true for  $BGL(A)^n \rightarrow [BGL(A)^+]^n$  (equivalent to  $A^n$  for the  $\Gamma$ ).

③ Now ~~do~~ do for  $\tilde{R}$ :



the proof here consists in ~~writing~~

~~$\tilde{R}(;A) \xrightarrow{\sim} \tilde{R}(;A) \xrightarrow{\sim} \tilde{R}(;A) \xrightarrow{\sim} \tilde{R}(;A)$~~

the diagram

$$\begin{array}{ccccc}
 BGL(A)^3 & \begin{array}{c} \xrightarrow{\mu \times id} \\ \xrightarrow{id \times \mu} \end{array} & BGL(A)^2 & \xrightarrow{\text{surj.}} & \tilde{R} \\
 \downarrow & & \downarrow & & \downarrow \\
 BGL(A)^3 & \begin{array}{c} \xrightarrow{\mu \times id} \\ \xrightarrow{id \times \mu} \end{array} & BGL(A)^2 & \longrightarrow & BGL(A)^+
 \end{array}$$

bottom row is exact. ~~the~~ given <sup>the</sup> map  $\tilde{R} \rightarrow \Gamma$   
 we get

$$\begin{array}{ccc}
 BGL(A)^2 & \longrightarrow & \Gamma \\
 \downarrow & & \nearrow \\
 BGL(A)^2 & \xrightarrow{\exists!} & \Gamma
 \end{array}$$

and the maps from  $BGL(A)^3$  are equalized by the dotted arrows, by the uniqueness part

(2). Thus <sup>(by exactness)</sup>  $\exists!$  map  $\rightarrow$

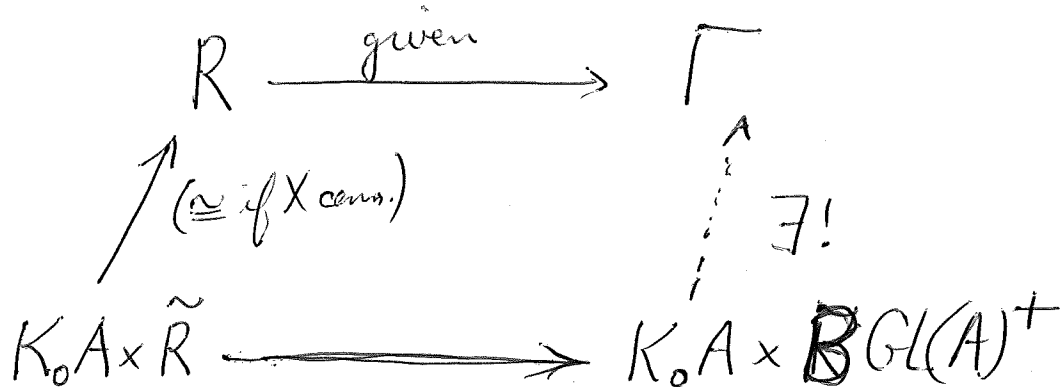
$$\begin{array}{ccc}
 BGL(A)^2 \xrightarrow{\text{surj.}} \tilde{R} & \longrightarrow & \Gamma \\
 \downarrow & & \nearrow \\
 BGL(A)^+ \longrightarrow BGL(A)^+ & & \exists!
 \end{array}$$

commutes. surjectivity shows that  $\triangleright$  commutes

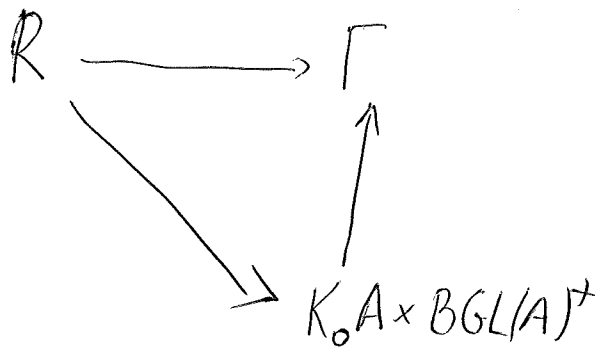
Remark: shows any map  $BGL(A) \rightarrow \Gamma$  extends uniquely to  $\tilde{R} \rightarrow \Gamma$ .

(4) Now ~~Apply~~

~~$K_0 A \times BGL(A)^+$~~   
 ~~$K_0 A \times \mathbb{R}$~~



look at each  $\alpha \in K_0 A$  separately.  ~~$\mathbb{R}$~~

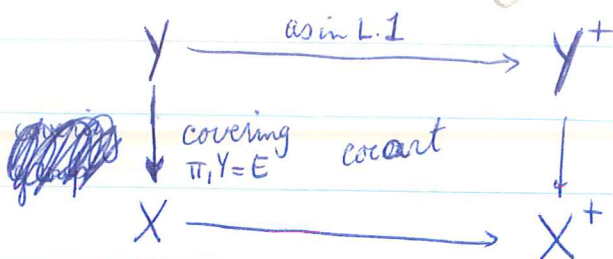


commutes for  $X$  connected, hence in general.

---

true for  $\prod K(\ ; \mathbb{R} A_i)$

# Atlantic City



van Kampen  $\rightarrow \pi_1 X^+ = \pi_1 X *_{\pi_1 Y} \pi_1 Y^+ = \pi_1 X / N$

~~Mayer-Vietoris~~  
excision

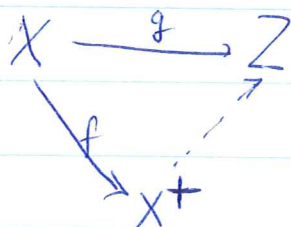
$$H_i(X^+, Y; L) \simeq H_i(X^+, X; L)$$

~~$$\rightarrow H_i(Y, L) \rightarrow H_i(X, L) \oplus H_i(Y^+, L) \rightarrow H_i(X^+, L) \rightarrow 0$$~~

but  $L$  trivial over  $Y \cup Y^+$  and  $H_i(Y, L) \simeq H_i(Y^+, L)$   
so (ii) holds.

Prop:

~~Prop:~~ Given  $g: X \rightarrow Z$   ~~$\pi_1(g)(E) = 0$~~   
 $\exists$  extension



and any two extensions are homotopic relative to  $X$ .

Obstructions lie in  $H^*(X^+, X; \pi_* Z)$  and hypothesis implies  $\pi_* Z$  are  $\pi_1 X^+$  modules, so these are zero.



Proof: Choose elements  $\alpha_i \in \pi_1 X$ ,  $i \in I$  such that ~~the~~ the normal subgroup of  $\pi_2 X$  gen. by  $\alpha_i$  is  $\pi_1 X$  itself. Realize  $\alpha_i$  by  $u_i: S^1 \rightarrow X$ , ~~and~~ let  $u_i^{(u)}: \bigvee_I S^1 \rightarrow X$ , set

$$Y = X \cup_{\bigvee_I} \bigvee_I e_2$$

$$H_i X \xrightarrow{\sim} H_i Y \quad i \geq 3$$

$$0 \rightarrow H_2 X \rightarrow H_2 Y \rightarrow \bigoplus_I \mathbb{Z} \rightarrow 0$$

~~Since~~  $\pi_1 Y = 0$  by van Kampen.  $\pi_2 Y = H_2 Y$  Hurewicz so attaching for each  $i \in I$  a 3 cell to  $Y$  via a map  $S^2 \rightarrow Y$  giving lifting.

$$H_i Y \xrightarrow{\sim} H_i Z \quad i \geq 4$$

$$0 \rightarrow H_3 Y \rightarrow H_3 Z \rightarrow \bigoplus_I \mathbb{Z} \hookrightarrow H_2 Y \rightarrow H_2 Z \rightarrow 0$$

$\begin{array}{c} 0 \\ \downarrow \\ H_2 X \\ \downarrow \\ \bigoplus_I \mathbb{Z} \end{array}$

~~Prop:~~ Let  $E$  be a perfect subgroup of  $\pi_1 X$ ,  $N$  the normal subgroup generated by  $E$ . Then  $\exists$  <sup>embedding</sup> map  $f: X \hookrightarrow X^+$  s.t.

(i)  $\pi_1(X)/N \xrightarrow{\sim} \pi_1 X^+$

(ii)  ~~$H^*(X; L) \xrightarrow{\sim} H^*(X^+; L)$~~  all  $\pi_1 X^+$  modules  $L$ .

$$H^*(X^+, X; L) = 0$$

~~$BGL(\Lambda)^+$  (weak) H-space~~

~~theorem of Milnor - Moore  $\Rightarrow$~~

~~$$\pi_i BGL(\Lambda)^+ \otimes \mathbb{Q} = \mathcal{P} H_i(BGL(\Lambda)^+, \mathbb{Q})$$~~

$$K_i \Lambda \otimes \mathbb{Q} = \mathcal{P} H_i(GL(\Lambda), \mathbb{Q})$$

$i \geq 1$

Theorem of Borel: Let  $\Lambda$  be the ring of integers in a number field  $F$  with  $r_1$  real places and  $r_2$  complex places. Then

$$K_{2i} \Lambda \otimes \mathbb{Q} = 0 \quad i > 0$$

$$\dim K_{2i-1} \Lambda \otimes \mathbb{Q} = r_1 + r_2 - 1$$

$$\dim K_{4i+1} \Lambda \otimes \mathbb{Q} = r_1 + r_2$$

$$\dim K_{4i-1} \Lambda \otimes \mathbb{Q} = r_2$$

	1	2	3	4	5	6	7	8	9
real place	<del><math>\mathbb{Q}</math></del>	0	0	0	<del><math>\mathbb{Q}</math></del>	0	0	0	<del><math>\mathbb{Q}</math></del>
cx place	$\mathbb{Q}$	0	$\mathbb{Q}$	0	$\mathbb{Q}$	0	$\mathbb{Q}$	0	$\mathbb{Q}$

take sum and reduce by 1 for  $K_1$  according to Dirichlet,

~~3348+~~

Basic outline:

1. killing a <sup>(perfect)</sup> subgroup of the fundamental group.
2. (weak) H-space structures on  $BGL(\Lambda)^+$

$$\boxed{K_* \Lambda \otimes \mathbb{Q} = PH_*(GL(\Lambda), \mathbb{Q})}$$

Borel theorem.

3. finite field  $\mathbb{F}_q$

fibration:  $BGL(\mathbb{F}_q)^+ \longrightarrow BU \xrightarrow{\mathbb{F}_q - 1} BU$

$$\begin{cases} K_{2i}(\mathbb{F}_q) = 0 & i > 0 \\ K_{2i-1}(\mathbb{F}_q) \cong \mathbb{Z}/(q^i - 1)\mathbb{Z} & i > 0 \end{cases}$$

4. symmetric groups

$$\Sigma_\infty = \bigcup \Sigma_n$$

Thm:  $B\Sigma_\infty^+ \sim \left( \varinjlim_n \Omega^n S^n \right)_0$

stable homotopy of symmetric groups = stable homotopy  
gps. of spheres



~~scribbled out text~~

~~Let~~  $\Lambda$  ring with unit

$E_n(\Lambda) =$  subgroup of  $GL_n(\Lambda)$  gen. by  $\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$

$$GL(\Lambda) = \bigcup_n GL_n(\Lambda)$$

$$E(\Lambda) = \bigcup_n E_n(\Lambda)$$

Recall  $E_n(\Lambda)$  perfect  $n \geq 3$ ,  $E_n \Lambda = (E_n \Lambda, E_n \Lambda)$   
and  $(GL(\Lambda), GL(\Lambda)) = \text{~~GL}(\Lambda)~~ E(\Lambda) = (E(\Lambda), E(\Lambda))$ .

$K_0 \Lambda =$  Groth. gp. f.gen. proj.  $\Lambda$ -modules

$K_1 \Lambda = GL(\Lambda)/E(\Lambda)$  Bass/Whitehead

$K_2 \Lambda = H_2(E(\Lambda), \mathbb{Z})$  Milnor

I propose to extend these groups  $K_n \Lambda$ ,  $n \geq 0$ .

§1. Killing all perfect subgroup of  $\pi_1$

All spaces are connected CW cx with basepoint.

Lemma 1: Given  $X$  with  $H_1(X) = 0$  ( $\pi_1 X$  perfect)

then  $\exists$  an ~~embedding~~ <sup>embedding</sup>  $f: X \hookrightarrow X^+$  such that

(i)  ~~$H_*(X; \mathbb{Z}) = 0$~~   $H_*(X^+, X; \mathbb{Z}) = 0$

(ii)  $\pi_1 X^+ = 0$ .

Prop  
 Given  $E \subset \pi_1 X$ ,  $E$  perfect,  $N =$  normal subgrp gen. by  $E$

$$\begin{array}{ccc} Y & \xrightarrow{\text{lemma 1}} & Y^+ \\ \text{covering} \downarrow & & \downarrow \\ \pi_1 Y = E & \text{cocart.} & \\ X & \xrightarrow{f} & X^+ \end{array}$$

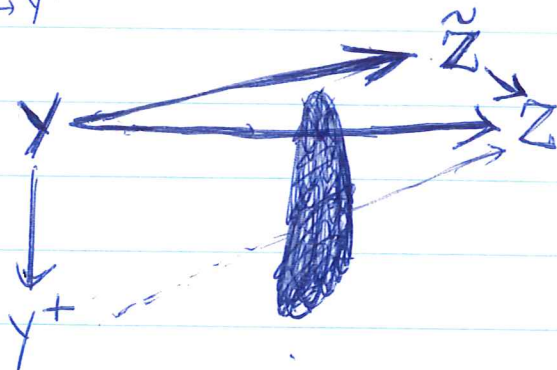
Then (i)  $\pi_1 X^+ = \pi_1 X / \text{normal subgroup gen. by } E$  (van Kampen)

(ii)  $H_*(X, L) \cong H_*(X^+, L)$  all  $\pi_1 X^+$ -modules  $L$ .

Corollary: Given  $g \in \pi_1(g)(E) \exists$  extension

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ \downarrow f & \nearrow & \uparrow \\ X^+ & & \end{array}$$

and any two are homotopic relative  $f$ . Enough to do for  $Y \rightarrow Y^+$

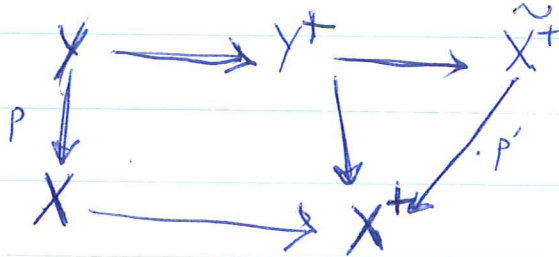


but obstructions lie in  $H^*(Y^+; Y; \pi_1 X^+ Z) = 0$ .

This shows  $f$  universal in homotopy category ~~for~~ killing  $E$ , in particular independent of choices in its construction.

**Prop:** If  $E$  normal then  $Y^+ \sim \tilde{X}^+$  universal covering.

Proof: ~~Enough to show  $Y^+ \rightarrow \tilde{X}^+$  homology isom.~~



$p, p'$  normal covering group  $\pi_1 X^+ = \pi_1 X / E$ .

$$H_*(Y) = H_*(X, \mathbb{Z}[\pi_1 X^+]) \xrightarrow{\sim} H_*(X^+, \mathbb{Z}[\pi_1 X^+]) = H_*(\tilde{X}^+)$$

Thus  $Y \xrightarrow{\sim} \tilde{X}^+$  homol isom  $\Rightarrow Y^+ \rightarrow \tilde{X}^+$  homology isom.  $\Rightarrow Y^+ \sim \tilde{X}^+$  (Whitehead)

Apply this <sup>construction</sup> FO  $E(\Lambda) \subset GL(\Lambda) = \pi_1 BGL(\Lambda)$ :

~~normal covering~~

$$\begin{array}{ccc} BE(\Lambda) & \xrightarrow{H_x \text{ isom}} & BE(\Lambda)^+ \\ \downarrow \text{normal covering group } GL(\Lambda)/E(\Lambda) & & \downarrow \text{universal covering} \\ BGL(\Lambda) & \xrightarrow{f} & BGL(\Lambda)^+ \end{array}$$

$$\pi_1 BGL(\Lambda)^+ = GL(\Lambda)/E(\Lambda)$$

$$\begin{aligned} \pi_2 BGL(\Lambda)^+ &= \pi_2 BE(\Lambda)^+ = H_2(BE(\Lambda)^+) \\ &= H_2(E(\Lambda)) \end{aligned}$$

Definition:

$$K_i \Lambda = \pi_i \text{BGL}(\Lambda)^+ \quad i \geq 1$$

§ 2: H-space structure on  $\text{BGL}(\Lambda)^+$ .

$E_n(\Lambda)$  perfect  $n \geq 3$ , so can define

$$\text{BGL}_n(\Lambda) \hookrightarrow \text{BGL}_n(\Lambda)^+ \quad \text{univ. killing } E_n(\Lambda).$$

Can take

$$\text{BGL}_n(\Lambda)^+ = \text{BGL}_n(\Lambda) \cup \text{BGL}_3(\Lambda)^+ \quad 3 \leq n < \infty$$

~~case (i)  $n < 3$~~

$\Rightarrow$

$$\text{BGL}(\Lambda)^+ = \bigcup_{n \geq 3} \text{BGL}_n(\Lambda)^+$$

$$\Rightarrow [X, \text{BGL}(\Lambda)^+]_0 = \varinjlim [X, \text{BGL}_n(\Lambda)^+]_0 \quad X \text{ finite ex.}$$

Whitney sum:

$$\begin{array}{ccc} GL_m \Lambda \times GL_n \Lambda & \longrightarrow & GL_{m+n} \Lambda \\ A, B & \longmapsto & \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \end{array}$$

$$\mu_{mn} : \text{BGL}_m \Lambda^+ \times \text{BGL}_n \Lambda^+ \longrightarrow \text{BGL}_{m+n} \Lambda^+$$

Proposition:  $\mu_{mn}$  define ~~an abelian~~ ~~group structure~~ ~~on~~  $[X, \text{BGL}(\Lambda)^+]_0$ .

~~Proof like for  $\text{BO} = \bigcup \text{BO}_m$ . This classical argument~~

won't give proof - like for  $BO = UBO_m$ ; would like to point why this argument works for  $BGL(N)^+$  but not  $BGL(N)$ . Need to know that ~~the auto is~~ conjugation ~~into  $BGL$~~  by

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad \det = \pm 1$$

~~auto~~ produces an auto  $BO$  which is homotopic to id. True for  $BO$  as this matrix can be joined by a path in  $O$ , not true for  $BGL(N)$  as conj. non-trivial on  $\pi_1$ ; true for  $BGL(N)^+$  as this matrix is in  $E(N)$ .



$\Rightarrow BGL(N)^+$  (weak) H-spaces. in particular (simple)

~~auto~~

~~Problem:  $\pi_1$  simple~~

Milnor-Moore:  $\Rightarrow$

$$\pi_i BGL(N)^+ \otimes \mathbb{Q} \xrightarrow{\sim} \mathbb{P} H_i(BGL(N)^+, \mathbb{Q})$$

$$K_i \Lambda \otimes \mathbb{Q} = \mathbb{P} H_i(GL(N), \mathbb{Q}) \quad i \geq 1$$